Urban Interactions and Spatial Structure

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Abstract

This paper considers equilibrium and optimum spatial development patterns in a model of urban interactions. In this model, agents choose to visit a particular location to interact with others. Interactions are therefore both endogenous and spatial. The interaction center can be interpreted as a traditional downtown or as a subcenter within a city. The agents can be interpreted as firms or households.

The model generates several key results. First, the equilibrium fails to achieve first-best levels of visits and population density when interaction cannot be contracted, a likely situation. Second, a construction subsidy can restore second-best efficiency. The subsidy fails to implement the first-best, however, because it does not operate on the visit margin directly. Third, a transportation subsidy that varies by location can achieve the first best. This result -- which contrasts with some earlier work -- comes from treating interaction as a choice variable, rather than focusing on population density, a correlate. Fourth, since a developer cannot control visits and is likely to be limited in the sense of controlling only a fraction of a city's area, the presence of a developer reduces inefficiency, but does not eliminate it, even compared to a second-best standard. Fifth, the degree of inefficiency in an edge city is likely to be smaller than for a downtown because developer limits are likely to be weaker there.

I. Introduction

Interactions between individual agents are fundamental to cities. Recognition of this goes back to Marshall (1920), at least, whose discussion of the "secrets of the trade" depends crucially on informal contacts between an industry's workers. This sort of interaction is central to Jacobs (1969) as well, who attributes the creation of "new work" to unplanned --actually, unplannable -- interactions between a city's residents. Similarly, Jacobs (1961) attributes neighborhood vitality to street level interactions among neighbors. More recently, Putnam (1993) discusses the impact of social capital on a city's prosperity. This involves a wide range of interactions, from participation in civic organizations to informally imposing order on urban life.

This paper begins by presenting a simple spatial model of urban interactions. The model applies to both a traditional "downtown" city and to newer "edge cities." A critical component of the model is the decision taken by a city's firms or households to visit a particular location to interact with others. The greater the aggregate number of visits, the greater is the value derived from any given visit. Visits involve transportation costs, however, and this generates downward sloping equilibrium housing rent, land rent, and population density functions. In equilibrium, all of these must be consistent with the interactions that take place in the center.

The model has natural applications to interactions among both households and firms. Suppose, for instance, that an agent wants to hear live music. This sort of consumption is directly enhanced by the presence of others in the audience, a conviviality effect. It is also enhanced indirectly, since the performance requires multiple agents having similar tastes in order to be economical, and so will take place only if there exists a critical mass. This example illustrates a more general phenomenon: the vitality of an urban center, and thus its value to the households and individuals who make use of it, is produced as an aggregate outcome of individual decisions.

The model also has applications to interactions among businesses. Suppose that the city's business service sector occupies a central location and that firms visit the center to obtain inputs. This remains a reasonably accurate characterization of business services sourcing in North American cities (Schwartz (1993)). When there is more activity, the business service sector becomes deeper and broader. This increases the value of a visit for any particular business. Alternatively, suppose that the interaction center is the heart of a given industry, like Wall Street in New York. Firms visit in order to meet others in the industry, which presumably allows labor pooling, facilitates knowledge spillovers, and more generally makes possible mutually beneficial commerce.

In either of these instances – agents as households or as firms -- it seems reasonable to consider the endogenous choice of the intensity of interaction. Households make choices about whether to go downtown or stay home. Businesses make choices about whether to actively pursue business partners (input suppliers, potential employees, or possible participants in deals). Our model captures this sort of endogenous interaction.

The analysis builds on the large body of work on spatial interactions in cities (see Fujita and Thisse (2002)). Beckmann (1976) presents a model where each agent interacts with each other agent. This interaction is costly in the sense that transportation costs are incurred. Agents are willing to pay more for space in the center in order to economize on these transportation costs. The model thus generates a "monocentric" configuration. Borukhov and Hochman (1977) and O'Hara (1977) consider interactions among firms. They also generate monocentric configurations under the assumption that each agent interacts with each other agent. Ogawa and Fujita (1980) and Fujita and Ogawa (1982) solve a more general model that includes both firms and households. They establish conditions under which a single central business district emerges surrounded by residential areas. In a similar vein, Berliant and Wang (1993) consider a model with endogenous marketplace formation. Our approach is intended to be complementary to this body of work. Instead of assuming agents' mutual interactions to be completely determined by location decisions and density, we allow agents to choose how intensely they will interact with others. In order that the analysis of endogenous interaction be tractable, we suppose that there exists an exogenous center of interaction.

It is not surprising that the existence of these urban interactions leads to a situation where equilibrium is inefficient. A number of instances of this sort of inefficiency are presented by Fujita and Thisse (2002). In contrast, Henderson (1974) and others have argued that large agents – city developers or local governments – can coordinate individual interactions and either lead to an efficient spatial allocation of resources or at least improve on the equilibrium. In the case of urban interactions, however, the control is likely to be incomplete. This paper will consider urban development in this sort of uncontrolled situation.

The rest of the paper deals with various ways that the inefficiency might be mitigated. In our model, subsidizing construction can realize a second-best efficient allocation. By raising density, more value accrues from interactions between agents. Of course, controlling only density fails to achieve first-best efficiency, since the number of visits is too low. On the other hand, subsidizing transportation costs can realize the first-best allocation, since the visits margin is directly addressed. This improves on the equilibrium by encouraging greater interaction. This result is exactly opposite the elegant result in Fujita and Thisse (2002) that doubling of transportation costs turns the equilibrium into an optimum by making a consumer of urban interactions internalize the transportation costs associated with the interactions that he or she transmits. A weaker version of this would be that an increase in transportation costs is welfare enhancing. The source of the result is that each agent is assumed to interact with every other agent. Without the corrective transportation tax, an agent would ignore the effects of his or her own location choice on the interaction costs of rivals. In contrast, in our model interactions are primitive. Transportation costs discourage visits to the center, hence the result that subsidies enhance efficiency.[1] This result is somewhat related to Glaeser-Kahn (2004), who argue that sprawl is fundamentally about cars, and that cars are simply a different technology of spatial interaction. Our result suggests that decreases in transport costs encourage interaction, whether the decrease is in mass transit costs that are oriented to downtown interaction or automobile costs that are oriented to suburban interaction. We will return to the downtown-edge city issue below.

The paper will also consider the ability of developers to improve on resource allocation. The paper's initial result is that a limited developer improves on resource allocation. The degree of improvement depends on the extent of the developer's control. A corollary to this result is that since developer control is likely to be more limited in central cities than in edge cities -- because of both land assembly issues and borrowing constraints -- there is likely to be greater inefficiency in central cities than in edge cities.[2] In addition, a well-known characteristic of edge cities, almost a defining characteristic, is that they tend to locate at or near intersections of major ring highways. To the degree that this results in lower transportation (interaction) costs, the equilibrium for edge cities will be closer to the optimum than for central cities.

The remainder of the paper is organized as follows. Section II presents the model of urban interactions. Sections III and IV show the equilibrium to be inefficient and respectively consider the effects of transportation and construction subsidies. Section V looks at the ability of a limited developer to improve on resource allocation in downtowns and edge cities. Section VI concludes.

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[1] It is worth pointing out that Fujita and Smith (1990) show a large class of situations where models of spatial interactions involving exogenous and endogenous interaction have identical properties. This case is an exception.

[2] See Helsley and Strange (1997) for a model where the span of the developer's control is an endogenous outcome of a land assembly game. Developers are shown to be limited in equilibrium.

II. A model of urban interactions and spatial structure

A. Overview

This section develops a model of urban spatial structure where agents visit a particular location to interact with others. They choose the number of visits, where the value of a visit depends in part on the number of other agents congregating there. Interactions in this model are thus endogenous and spatial. In this setting the aggregate level of interaction within a city is determined in part by the city's spatial structure. In this section, we characterize equilibrium spatial structure. The next section will show that atomistic urban development is inefficient in this context, with too little interaction in equilibrium.

B. Model

There are three types of decision-makers in the model. The first are those who engage in interactions. We will refer to them as “consumers” (of interactions). The model is sufficiently general that these agents may be households who benefit from consumption externalities or firms who benefit from agglomeration economies. As discussed in the Introduction, households value the sort of urban environment that requires broad participation. Firms value interactions with other firms. The second type of agent are the competitive builders, who determine building heights throughout the city. The third, appearing later in the paper, is a developer, a large builder who internalizes some of the externalities arising from interactions. In the absence of a developer, we assume that the markets for space and land are perfectly competitive. We also assume that consumers migrate freely in a competitive system of cities. This implies that in equilibrium every consumer in a given city must earn a payoff equal to the exogenous payoff that is available in other cities.

The location space is a long, narrow strip of land. There is one unit of land at each location. All interactions occur at a single location, located at the left edge of the strip. Locations are completely characterized by their distance from this "center," given by the variable x. To begin, it is best to think of the center as a metaphor for a traditional downtown. We will later discuss a different interpretation, with the center representing a peripheral "edge city."

Consumers are identical and derive utility from residential or commercial space s, other goods g (the numeraire), and interaction according to the additively separable utility function

$$U(s,g,v,K) = s + g + u(v,K), \tag{1}$$

where v is the number of visits to the center, and K measures the quality of interactions there.[3] We will assume that the subutility function for interactions $u(\cdot)$ is increasing and strictly quasi-concave, with $\partial^2 u/\partial v \partial K > 0$. This last assumption means that the marginal value of a visit to the center is increasing in the quality of the interactions there.

[3] In this specification there are no income effects.

There are two costs associated with a visit to the center: a fixed cost T and transportation cost tx, $t > 0$. T includes all costs that do not depend on distance traveled. These include the costs of arranging the visit (e.g., searching for an interaction partner, scheduling a meeting, preparing presentations, and so on) and any fixed cost of travel (e.g., buying a car). The assumption of a positive fixed cost of a visit is necessary in order that agents at $x = 0$ not choose an infinite quantity of visits. For a consumer with income y at location x, expenditure on other goods is

$$g = y - rs - (T + tx)v, \tag{2}$$

where r is rent per unit of space. We assume that each consumer occupies one unit of space.

C. Urban spatial structure

The consumer's choice problem is to choose v to maximize

$$U(1, y - r - (T + tx)v, v, K) = 1 + y - r - (T + tx)v + u(v,K). \tag{3}$$

The first-order condition for this problem is

$$-(T + tx) + \partial u/\partial v = 0. \tag{4}$$

(4) implicitly defines v(K,x), the optimal number of visits for a consumer at located at x. We suppose that, for all K, $\lim_{v\to 0+} \partial u/\partial v \to \infty$ and that that $\lim_{v\to\infty} \partial u/\partial v \to 0$. These conditions together with the continuity of $\partial u/\partial v$ ensure that v(K,x) is well-behaved. See the Mathematical Appendix for details. By the implicit function theorem,

$$\partial v/\partial K = -(\partial^2 u/\partial v \partial K)/(\partial^2 u/\partial v^2) > 0, \tag{5}$$

$$\partial v/\partial x = t/(\partial^2 u/\partial v^2) < 0. \tag{6}$$

Thus, the number of visits made to the center increases with the quality of interactions and decreases with distance. The latter is consistent with some of the empirical findings of Glaeser and Sacerdote (2000).

Letting u* + 1 represent the utility level available in other cities, free migration implies that the equilibrium rent per unit of building space is the maximum of zero and the consumer's bid rent for space,

$$r(K,x) = y - u^* - (T + tx)v(K,x) + u(v(K,x),K), \tag{7}$$

where

$$\partial r/\partial K = (\partial v/\partial K)(-(T + tx) + \partial u/\partial v) + \partial u/\partial K = \partial u/\partial K > 0, \tag{8}$$

$$\partial r/\partial x = -tv(K,x) + (\partial v/\partial x)(-(T + tx) + \partial u/\partial v) = -tv(K,x) < 0. \tag{9}$$

Thus, the rent on space increases with the quality of interactions and decreases with distance.

Builders choose how many units of space to build at each location, n, to maximize profit. Since each consumer occupies one unit of space, and there is one unit of land at each location, n also equals population, population density (persons per unit land) and structural density (units of residential or commercial space per unit land). The profit of a builder at location x is

$$\pi(K,x) = r(K,x)n - c(n) - R, \tag{10}$$

where c(n) is construction cost and R is land rent. We assume that $c(\cdot)$ is increasing and strictly convex with $c(0) = 0$ and $c'(0) = 0$. The first-order condition for profit maximization implies

$$r(K,x) - c'(n) = 0, \tag{11}$$

and this implicitly defines the maximizing density n(K,x).[4] Here r(K,x) is the private benefit of adding another unit of space at x, while c'(n) is the marginal cost. Implicitly differentiating (11) and using (8) and (9) gives

$$\partial n/\partial K = (\partial r/\partial K)/c''(n) = (\partial u/\partial K)/c''(n) > 0, \tag{12}$$

[4] For x where $r(K,x) = 0$, $n(K,x) = 0$.

$$\partial n/\partial x = (\partial r/\partial x)/c''(n) = -tv(K,x)/c''(n) < 0. \quad (13)$$

Thus, structural density increases with the quality of interactions and decreases with distance.

Competition ensures that the maximum profit of a builder equals zero, and this condition defines the bid rent for land:

$$R(K,x) = r(K,x)n(K,x) - c(n(K,x)). \quad (14)$$

Like the rent on space and structural density, the bid rent on land increases with the quality of interactions and decreases with distance. Using equations (8) and (9):

$$\partial R/\partial K = (\partial r/\partial K)n + (r - c'(n))(\partial n/\partial K) = (\partial u/\partial K)n(K,x) > 0, \quad (15)$$

$$\partial R/\partial x = (\partial r/\partial x)n + (r - c'(n))(\partial n/\partial x) = -tv(K,x)n(K,x) < 0. \quad (16)$$

Consumers will occupy all locations where the bid rent on land exceeds its opportunity cost, which we normalize to zero. Thus, using equation (14), the boundary of the city x^b satisfies

$$r(K,x^b)n(K,x^b) - c(n(K,x^b)) = 0. \quad (17)$$

Equation (17) implicitly defines $x^b(K)$, the boundary of the city as a function of the quality of interactions at the center. It is immediate that

$$dx^b/dK = - (\partial R/\partial K)/(\partial R/\partial x) = (\partial u/\partial K)/(tv(K,x^b) > 0. \tag{18}$$

Thus, the size of the occupied area increases with the quality of interactions. In sum, land rent, rent on space (structures), and density decline with distance to the center and rise with the quality of interactions. These conclusions recapitulate standard results from the monocentric model of a city.

D. Interaction quality

To close the model, we must specify how interaction quality, K, is determined. It is conventional to suppose that interaction quality is somehow an aggregate of individual interactions. There are many ways to do this. Our specific approach is to assume that the equilibrium level of interaction quality satisfies

$$K = \int_0^{x^b(K)} F\big(v(K,x)\big)n(K,x)dx, \tag{19}$$

where $F(\cdot)$ is increasing and strictly concave, and $F(0) = 0$. Implicit in this specification is the idea that each agent has the potential to benefit from interacting with any other agent. However, the value of interacting with any particular agent exhibits a diminishing marginal impact, captured by the concavity of $F(\cdot)$. This specification is in the spirit of

Jacobs' (1969) treatment of diversity in the sense that a small group of consumers who participate intensely creates an environment that is inferior to a larger group of consumers who participate more moderately. Glaeser et al (1992) and Rosenthal and Strange (2003) both find evidence consistent with the importance of diversity.

Our approach to urban interaction is quite different than models where agents interact with each other over space in exogenous ways (e.g., Ogawa and Fujita (1980), Fujita and Ogawa (1982)). In those models, interdependence arises from exogenous spatial interactions. In our model, the interdependence arises from the endogeneity of interactions: agents choose jointly both how much to contribution to a location and how much to make use of that location.

Our approach is broadly consistent with a range of urban interactions. For instance, it can capture many of the proposed microfoundations for agglomeration. As discussed in the Introduction, businesses choose how intensely to interact with each other. A firm located near the center will interact to a greater degree that will one located farther away. The nearby firm will thus both benefit more from and contribute more to the aggregate of interactions. This can allow greater input sharing (Helsley and Strange (2002)) or labor market pooling (Helsley and Strange (1990)) from thick market effects. It can also allow more knowledge spillovers of the sort modeled by Glaeser (1999). An important difference from some urban models is that in our model the group can be beneficial only to an agent who contributes to it, a setup parallel to the Cohen and Levinthal (1990) idea of "absorptive capacity." Our model is also consistent with consumption externalities (Glaeser et al (2001)) where the availability of consumer goods depends on city size. The model is also consistent with recent research on the vitality of downtowns. For example, in his book on New Haven, Rae (2003) argues that the gradual loss of social capital was both a cause and effect of the city's decline. The absence of a vital downtown left households less reason to go there, and this in turn contributed to the loss of vitality. Our model formalizes this argument.

The equilibrium level of interaction quality is a solution to (19). Letting Φ(K) represent the mapping on the right side of equation (19), sufficient conditions for the existence of a unique, stable solution to (19) are $\Phi(0) > 0$ and $\Phi'(K) < 1$ for all K, implying

$$1-\int_0^{x^b(K)} F'(\cdot)\frac{\partial v}{\partial K}n(\cdot)+F(\cdot)\frac{\partial n}{\partial K}dx - F(v(K,x^b(K))n(K,x^b(K))x^{b\prime}(K) > 0 \ . \tag{20}$$

These conditions ensure that the mapping Φ(K) is a contraction and therefore has a unique and stable fixed point by Banach's theorem; we will assume that these conditions are satisfied. Sufficient conditions for $\Phi(0) > 0$ are that consumers who live near the center are willing to visit even if interaction quality is zero and these visits generate a bid rent that induces housing supply. Our assumption that the marginal utility of a visit is positive for all K guarantees that this condition will be met. $\Phi'(K) < 1$ means that an increase in interaction does not increase interaction quality by so much that it would induce a subsequent unbounded increase in interaction. See the Mathematical Appendix for a formal proof.

III. Urban interactions and transportation policy

A. First-best

The previous section presented a model where urban interactions and spatial structure were jointly determined in equilibrium. This section considers efficiency. The focus of the section will be on how transportation policy can be designed to achieve efficiency, both in spatial structure and on the interaction margin.

An efficient allocation maximizes aggregate land rent subject to reservation levels of utility and profit, and the definition of the quality of urban interactions. We take the reservation payoff levels to be the open city utility level u* for consumers and zero profit for builders. This section considers a first-best optimum in which a planner can control the visit choices of consumers directly. The next section considers a constrained optimum where visits are determined by the utility maximizing choices of individual consumers.

Formally, an efficient allocation consists of a functions v(x) and n(x) and values for K and x^b that maximize aggregate land rent,

$$\int_0^{x^b} \big(y - u^* - (T + tx)v(x) + u(v(x), K)\big)n(x) - c(n(x))dx\,, \tag{21}$$

subject to

$$K = \int_0^{x^b} F(v(x))n(x)dx. \tag{22}$$

The key difference between (19) and (22) is that in the latter the number of visits, density at each location, and the border of the city are chosen by a planner, rather than arising from the equilibrium choices described in the previous section.

The first-order conditions for this problem are:

$$v(x): \left(-(T+tx)+\frac{\partial u}{\partial v}\right)n(x)+\lambda F'(v(x))n(x)=0, \text{ for all } x \in [0,x^b], \tag{23}$$

$$n(x): \; r(x) - c'(n(x)) + \lambda F(v(x)) = 0, \text{ for all } x \in [0,x^b], \tag{24}$$

$$K: \int_0^{x^b} \frac{\partial u}{\partial K} n(x)dx - \lambda = 0, \tag{25}$$

$$x^b: \; r(x^b)n(x^b) - c(n(x^b)) + \lambda F(v(x^b))n(x^b) = 0, \tag{26}$$

where, for notational simplicity, we let $r(x) = y - u^* - (T + tx)v(x) + u(v(x),K)$. The Mathematical Appendix sets out sufficient conditions for the existence of a unique efficient allocation.

Equations (23) and (24) are the Euler equations that govern the optimal visit and density functions. $\lambda > 0$ by equation (25). In economic terms, λ is the shadow price of interaction quality, the positive impact of a marginal increase in K on the maximized value of aggregate land rent (and aggregate utility). $\lambda F'(v(x))n(x)$ is then the external benefit of another visit by a consumer at location x. Equation (23) says that the optimal number of visits at each location equates the social marginal benefit of a visit, $\partial u/\partial v + \lambda F'(v(x))$, to the private marginal cost, $(T + tx)$. Comparing equations (23) and (4) shows that consumers make too few visits in equilibrium: individual consumers ignore the impact of their visits on aggregate interaction quality. Similarly, comparing equations (24) and (11) and (26) and (17) shows that densities are too low and the boundary is too near in equilibrium. Correcting the externality in visits would cause consumers to bid more for space, making it economical to build to higher densities and bring additional land into the urban use.

B. Transportation subsidies

In this model, agents incur transportation costs as a result of their interaction choices. Since they fail to account for the social benefit accruing from these costly interactions, the equilibrium city involves too little interaction. It is natural, therefore, to consider whether it would be possible to address the inefficiency by altering the costs of transportation.

From equations (23) and (4), the transportation subsidy per consumer required to generate the first-best number of visits at each location is

$$z(x) = \lambda F'(v(x)), \tag{27}$$

where

$$z'(x) = \lambda F''(\cdot)\frac{\partial v}{\partial x} > 0. \tag{28}$$

The required transportation subsidy per consumer is higher for consumers near the periphery than for consumers near the interaction center. This is a consequence of the concavity of F(·). A subsidy to visits by peripheral consumers has a large effect since they visit less, and so the marginal impact of an increase in their visits on the production of interaction quality is higher.[5]

(28) implies that the subsidy to transportation required to implement the first-best must vary across locations. It is certainly possible to take steps in this direction, for instance by subsidizing long transit trips. In practice, the geographic variation in transit pricing is typically discrete, with the marginal distance cost of a multi-zone trip being lower than for a shorter trip. Implementation of a continuously varying subsidy, as defined in (28), would require a relatively sophisticated electronic pricing system. In many situations, it seems more likely that a policymaker would be forced to subsidize all transportation. Thus, we see the primary implication of this result as being that a marginal reduction in transportation costs encourages interaction. Since interaction is inefficiently low in the equilibrium, this marginal change will raise aggregate welfare.

The result that transportation subsidies can improve resource allocation has an interesting relationship to a result in Fujita and Thisse (2002) on efficiency and equity in a model of spatial interactions. They adapt the Beckman (1976) model to consider a

[5] By the construction of the efficiency program, the subsidy raises aggregate land rents. It therefore could potentially be financed by taxing rent increases.

situation where agents choose locations and interact with all other agents, a fixed "interaction field." In their model, individual location choices generate a density profile which in turn determines the aggregate cost of interaction. An agent considers his or her own interaction costs, but the costs of other agents' interactions are external and so are ignored. This means that the agent choosing between two locations considers exactly half of the benefits of a particular interaction (his or her own). A transportation surcharge, by accounting for the costs of the other agent involved in the interaction, could correct this externality. In the special case considered by Fujita and Thisse, the required transportation surcharge is exactly 100%.[6] In our situation, in contrast, a subsidy to transportation costs would be welfare enhancing. This difference arises from our model's emphasis on the visit as the primitive atom of urban interactions. In this setting, a subsidy encourages interaction, and this is efficiency enhancing.[7]

This result has some interesting implications for the relative efficiency of interactions in downtowns and edge cities. A well-known characteristic of edge cities is their growth around intersections of suburban expressways, like New Jersey's 278 and 78. Relative to the decaying transportation capital of central cities, both roads and transit, this suggests that transport costs are likely to be considerably lower near edge cities. This suggests a stronger interaction potential for edge cities. The interaction takes place, of course, in malls and industrial parks rather than along avenues and in high-rises, but, as pointed out by Glaeser and Kahn (2004), it is still interaction.

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[6] A parallel result would hold for a model with an interaction center. If all agents were required to go to the center to interact, then densities would be too low near the center because individuals would ignore the extra transportation costs incurred by others. As in the model without an interaction center, this could be addressed by taxing transportation.

[7] Although we have not modeled the case of endogenous visits to dispersed locations, as long as the value of a visit depends on some sort of aggregate participation, there will be a positive externality in visits, and transportation subsidies will add to welfare.

This analysis also has broader implications for urban transportation policy. It has long been recognized that negative externalities associated with transportation can be corrected through the imposition of congestion or pollution taxes (i.e., Vickrey (1963)). Our analysis has identified another margin on which transportation may be mis-priced. However, in contrast to models of transportation congestion, in our model the private cost of transportation is too high. In the presence of the sort of interaction externality present in our analysis, welfare may be improved by subsidizing the cost of urban travel. This is not an argument for the abandonment of congestion pricing. Rather, it is an appeal for a more complete accounting of the costs and benefits of urban transportation, where the latter include the interaction benefits that inexpensive transportation may allow.

IV. Urban spatial structure and land use policy

A. Constrained efficiency

The previous section examined a first-best program where the planner could directly control interactions. This section considers a second-best program where the planner cannot directly control the visit choices of consumers. The resulting constrained efficient allocation corresponds to the development pattern that would be chosen by a developer who controls all of the land in the city and chooses density to maximize profit, anticipating the visit choices of consumers. In the next section, we will contrast this allocation to the equilibrium with a limited developer who controls some, but not all, of the land in the city.

Formally, the constrained efficient allocation consists of a function n(x) and values for K and x^b that maximize aggregate land rent,

$$\int_0^{x^b} r(K,x)n(x) - c(n(x))dx \tag{29}$$

subject to

$$K = \int_0^{x^b} F\big(v(K,x)\big)n(x)dx. \tag{30}$$

Interaction quality in the constrained efficient allocation is a fixed point of equation (30). The first-order conditions for problem are:

$$n(x):\ \ r(K,x) - c'(n(x)) + \lambda F(v(K,x)) = 0, \text{ for all } x \in [0,x^b], \tag{31}$$

$$K:\ \ \int_0^{x^b} \frac{\partial r}{\partial K} n(x) + \lambda F'\big(v(K,x)\big)n(x)\frac{\partial v}{\partial K}dx - \lambda = 0, \tag{32}$$

$$x^b:\ \ r(K,x^b)n(x^b) - c(n(x^b)) + \lambda F(v(K,x^b))n(x^b) = 0. \tag{33}$$

See the Mathematical Appendix for sufficient conditions that this solution be unique.

Equation (31) is the Euler equation that governs the optimal density function n(x), where λ is again the shadow price of interaction quality. From equation (32),

$$\lambda = \frac{\int_0^{x^b} \frac{\partial r}{\partial K} n(x)dx}{1 - \int_0^{x^b} F'\big(v(K,x)\big)n(x)\frac{\partial v}{\partial K}dx} > 0. \tag{34}$$

The denominator in equation (34) is positive by the contraction condition that guarantees that (30) has a unique fixed point. Then, the fact that r(K,x) is increasing in K (equation (8)) implies that $\lambda > 0$. Equation (31) then implies that at the optimal density,

$$r(K,x) + \lambda F(v(K,x)) - c'(n(x)) = 0. \tag{35}$$

Since c(n) is strictly convex, comparing equations (35) and (11) shows that constrained optimum densities are larger than equilibrium densities, *ceteris paribus*. Increasing density increases visits, which in turn increases interaction quality and aggregate land rent. Individual builders ignore the impact of their density choices on the number of visits that consumers make to the center, the quality of interactions, and the rent on space at other locations. This externality leads to inefficient private development decisions.

Equation (33) implies that

$$r(K,x^b)n(x^b) - c((x^b)) = -\lambda F(v(K,x^b))n(x^b) < 0. \tag{36}$$

Thus, the bid rent on land is negative at the border of the city in the constrained efficient allocation. Comparing equations (36) and (17) implies that the border of the city is larger in the constrained efficient allocation than in the atomistic equilibrium, *ceteris paribus*. As before, the intuition is that internalizing the externality in density choices leads to a higher value of K, which increases the rent on space at every location, making it optimal to bring additional land into the urban use.

B. Construction subsidies

The second term on the left of equation (35) is the external benefit of increasing density at location x. This also represents the Pigovian subsidy to the rent on space that would lead builders to make efficient density choices. The required subsidy is

$$s(x) = \lambda F(v(K,x)), \tag{37}$$

where

$$s'(x) = \lambda F'(\cdot)\frac{\partial v}{\partial x} < 0. \tag{38}$$

Since consumers near the center choose to visit more, subsidizing density at central locations has a large value because the contribution to interaction quality is greater.

There is a long history of redevelopment projects that seem to be designed with interactions in mind. One example is the development of the World Trade Center in New York City. The project's context was the gradual erosion of Lower Manhattan's pre-eminence as a center of business in New York City as important companies moved uptown. With less activity downtown, there was less interaction among businesses. This, in turn, made downtown less attractive relative to midtown, completing the vicious cycle. The essence of the project was the redevelopment of a large part of Lower Manhattan, adding more than 10,000,000 square feet of office space. The World Trade Center project had an essential public component. Without the ability to condemn properties and without the Port Authority's ability to finance development, it is difficult to believe that the project would have overcome its many political obstacles. The key argument in favor of public involvement is the project's public benefit: the support of "trade" in lower Manhattan. Ensuing projects around the World Trade Center, including most notably Battery Park City, suggest that the project did make Lower Manhattan a better place to do business and therefore also to engage in commercial real estate development.

The phenomenon of supporting development in order to encourage interaction continues to the present day. Of the ten tallest buildings in the world, eight have been built since 1990. This skyscraper boom has been particularly pronounced in Asia. The Petronas Towers in Kuala Lumpur, Malaysia, were the world's tallest buildings from 1998-2004. It has recently been replaced as the world's tallest by Taipei 101, in Taipei, Taiwan. It has 2,336,000 square feet of office space, 796,000 square feet of retail space, and 893,000 square feet of parking space for more than 1,800 vehicles. The magnitude of these figures is made clear by comparing them to the average for the total central Taipei office market for the previous three years, 2,099,000 square feet.

All of these buildings have benefited from public assistance in their construction. It is common to attribute such assistance to government objectives or the peculiarities of local political systems. This paper's analysis suggests a different explanation. Urban interactions are both valuable and under-produced. Increasing the central density both directly adds to interaction there and indirectly increases interaction by encouraging visits from consumers at other locations. Thus, it is possible that subsidies to high-density construction may be a second-best policy toward inefficiencies in urban interactions.

This analysis also has broader implications for land use policy. A properly designed zoning code could implement the constrained-efficient allocation directly. However, the political process that creates zoning laws is complicated and frequently appears to be at odds with efficiency. It is common for zoning regulations to discourage the sort of high-density development that we have shown to be necessary for efficient interaction. Our analysis argues that zoning should not be designed only with reference to direct land use externalities but also with consideration given to the indirect effect of density on urban interactions.

V. Urban interactions with a limited developer

The previous section argued that the allocation of resources with atomistic builders is inefficient. Individual builders ignore the impact of their density choices on visits to the center and thus on the value of interactions. If there were a large agent who controlled a discrete fraction of the city's land, then it would be possible to internalize some of the externalities that lead to inefficiency. In fact, such agents do exist: they are the developers of large buildings and large scale development projects in cities. This section will consider the implications of such a developer for urban spatial structure, interactions, and efficiency.

There is a long tradition in urban economics that focuses on the ability of large agents to correct externalities in land markets. In early treatments (Henderson (1974)), developers were endowed with unlimited abilities, and so were able to internalize all externalities. However, there are strong reasons to believe that the scope of developers, while substantial, is limited. Credit constraints, difficulties with land assembly, and political pressures all constrain the scale of actual development activities.[8] Thus, limited developers cannot internalize all externalities.

To illustrate the impact of limits to developer power in this context, we suppose that a limited developer controls all of the land in the interval $[0,x^d]$, where $x^d < x^b(K)$, evaluated at the equilibrium interaction quality. The remainder of the city operates as in the atomistic equilibrium described above. We refer to the area outside of the developer’s control as the competitive fringe. In this case, the developer’s profit equals

$$\int_0^{x^d} r(K,x)n^D(x) - c(n^D(x))dx\,, \tag{39}$$

[8] See Helsley and Strange (1997) for a formal model of land assembly problems as a limit to developer control. See also Strange (1994) for a model of land assembly or Stiglitz and Weiss (1981) for a model of credit constraints.

where $n^D(x)$ is the density chosen by the developer on the interval of land that it controls. n(K,x) will continue to represent the density chosen by an individual atomistic builder at x.

The developer chooses $n^D(x)$ to maximize profit taking as given the density levels chosen in the competitive fringe, and anticipating how these density choices will impact the quality of interactions at the center. We could have considered a sequential game in which the developer anticipates the effects of its choices on fringe builders. We have not because the comparisons between equilibrium, optimum and developer allocations are more transparent in a model with simultaneous choices. In this case, the definition of interaction quality becomes

$$K = \int_0^{x^d} F\big(v(K,x)\big)n^D(x)dx + \int_{x^d}^{x^b(K)} F\big(v(K,x)\big)n(K,x)dx\,. \tag{40}$$

The first-order conditions for the developer's profit maximization problem are:

$$n^D(x):\;\; r(K,x) - c'(n^D(x)) + \mu F(v(K,x)) = 0,\text{ for all } x \in [0,x^d], \tag{41}$$

$$K:\;\; \int_0^{x^d} \frac{\partial r}{\partial K} n^{dD}(x) + \mu F'\big(v(K,x)\big)n^D(x)\frac{\partial v}{\partial K}dx - \mu\int_{x^d}^{x^b(K)} F'\big(v(K,x)\big)n(K,x)\frac{\partial v}{\partial K}dx - \mu = 0\,, \tag{42}$$

where μ is the multiplier on the definition of interaction quality. See the Mathematical Appendix for details. Comparing equations (41) and (42) with equations (31) and (32) shows that the key difference between the developer and constrained efficient allocations is that the developer only considers the impact of externalities on rents for the land that it controls. As noted earlier, if the developer were unlimited in the sense that it controlled all of the land in the city, then the developer and constrained-efficient allocations would coincide.

Equation (42) implies

$$\mu = \frac{\int_0^{x^d} \frac{\partial r}{\partial K} n^D(x)dx}{1 - \int_0^{x^d} F'\big(v(K,x)\big)n^D(x)\frac{\partial v}{\partial K}dx + \int_{x^d}^{x^{b(K)}} F'\big(v(K,x)\big)n(K,x)\frac{\partial v}{\partial K}dx} > 0 . \tag{43}$$

We assume that the denominator in equation (43) is positive: this is another instance of the contraction condition (20), and it ensures that equation (40) has unique solution. Equation (41) implies that the developer sets $n^d(x)$ so that

$$r(K,x) + \mu F(v(K,x)) - c'(n^D(x)) = 0. \tag{44}$$

Since $\mu > 0$ and c(n) is strictly convex, comparing equations (44) and (11) shows that the developer chooses a higher level of density on the land that she controls than individual builders do in the atomistic equilibrium. However, since the developer only partially internalizes the externality in density and interaction quality, she does not set $n^d(x)$ at its constrained efficient level.

To see this, suppose that the developer did set $n^D(x) = n^{CE}(x)$, where $n^{CE}(x)$ is the constrained efficient density at x, the value that solves equation (31). Then the numerator in (43) would be smaller than the numerator in (34), while the denominator would be larger, implying, $\mu < \lambda$. Then (41) and (31) could not give the same solution for density. Since $\mu < \lambda$ when evaluated at $n^{CE}(x)$, and c(n) is strictly convex, (44) implies that the developer would choose a lower level of density, compared to the constrained efficient allocation, at all of the locations that it controls. Thus, a limited developer would choose a density level that is between the atomistic equilibrium and constrained efficient levels. The intuition is that, since the developer only controls some of the land in the city, it only partially internalizes the externality in interaction quality. A higher level of density implies a higher interaction quality compared to the atomistic equilibrium. This in turn implies a larger number of visits at all locations, a greater level of fringe density, and a more distant city border.

This analysis also has implications for the differences between edge cities and downtowns. Land ownership is fragmented in downtowns. In edge cities, as Garreau (1991) reports, large plots of land can be assembled more easily. This implies that land assembly is a more formidable barrier to developer control in a downtown than in an edge city. This suggests that one advantage of an edge city may be that they provide an environment conducive to large scale development. This, in turn, permits greater internalization of externalities in density and interactions.

VI. Conclusion

This paper has considered urban interactions in a monocentric model. The inability to control these interactions means that equilibrium in unlikely to be efficient.

The equilibrium can be improved upon in several ways. By increasing density, construction subsidies encourage greater interaction. A subsidy to transportation would do this directly. If a land developer were to try to profit from correcting inefficiencies, the developer could implement part of the second-best efficient construction subsidy. The developer's limited control, however, would leave some inefficiency uncorrected.

These results have implications for both downtowns and edge cities. Our analysis suggests that there are external benefits associated with projects designed to achieve a critical mass of interaction. In the paper, we mentioned the World Trade Center and the current boom in skyscraper construction. The currently considered large scale redevelopment of Manhattan's Westside is similar in spirit. It is important to recognize, however, that this sort of large-scale high-density redevelopment can also be beneficial if it takes place at the urban periphery. The many instances of edge cities documented by Garreau (1991), where the density is considerably higher than in surrounding traditional suburbs, clearly illustrate this.

This is not to say that the paper's results are neutral regarding the relative abilities of developers to control interactions in downtowns and in edge cities. The ability of developers to address density depends on the ability to assemble land and the ability to finance large-scale redevelopment. Both of these favor developers of edge cities relative to those who would redevelop downtowns. In addition, the result that a transportation subsidy improves resource allocation clearly favors edge cities, with their heavily subsidized interstate highways.

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Mathematical Appendix.

This appendix provides formal proofs of the paper's key results.

Equilibrium

Assume:

A0. (a) $y > u^*$; (b) $c(0) = 0$ and $c'(0) = 0$.

Remark: Condition (a) is sufficient to obtain positive rent when visits are positive. Condition (b) is sufficient to obtain positive density when rents are positive. If these conditions are not satisfied, there is no urban population, and hence no interaction. This seems uninteresting.

Assume:

A1. For all K, $\lim_{v \to 0+} \partial u/\partial v \to \infty$

Lemma 1. For all $K \geq 0$, (a) $x^b > 0$ and (b) $v(K,x) > 0$, $n(K,x) > 0$, and $r(K,x) > 0$ for all $x < x^b$.

Proof: For any $K \geq 0$, (A1) implies that there exists an x_0 such that $v(K,x_0) > 0$. (7) and (A0) then imply that $r(K,x_0) > 0$. (A0) also implies that $n(K,x_0) > 0$. $x^b = \text{argmax}_x$ s.t. $r(K,x) \geq 0$. It must be positive. For $x < x^b$, $v(K,x) > 0$, $n(K,x) > 0$, and $r(K,x) > 0$ because rent is decreasing in x.

Lemma 2. $\Phi(0) > 0$.

Proof: Lemma 1 and (19).

Remark: Lemma 1 implies that there will be positive urban population densities and visits to the center even when interaction quality is zero. Thus (19) establishes the result.

A2. Assume that:

$$1-\int_0^{x^b(K)} F'(\cdot)\frac{\partial v}{\partial K}n(\cdot)+F(\cdot)\frac{\partial n}{\partial K}dx - F(v(K,x^b(K))n(K,x^b(K))x^{b\prime}(K)>0$$

Remark: $\Phi'(K) < 1$ means that an increase in interaction does not increase interaction quality by so much that it would induce a subsequent unbounded increase in interaction.

Proposition 1. If (A1) and (A2) hold, there exists a unique, globally stable equilibrium.

Proof: (A1) implies $\Phi(0) > 0$ by Lemma 1. (A2) means that $\Phi'(K) < 1$. Together, these imply existence, uniqueness, and global stability by Banach's Theorem (Kolmogorov and Fomin (1970)).

Efficient allocation (rent-maximization)

Let the solutions for v(x), n(x), and x^b from (23), (24), and (26) -- with λ given by (25) -- be given by $v^{FB}(x)$, $n^{FB}(x)$, and x^{bFB}. Let $\Gamma(K)$ define the mapping produced by substituting $v^{FB}(x)$, $n^{FB}(x)$, and x^{bFB} into the definition of interaction quality, (19).

Lemma 3. In the efficient allocation, (a) $x^{bFB} > 0$ and (b) $v^{FB}(x) > 0$, $n^{FB}(x) > 0$ for all $x < x^b$.

Proof: Let K^{FB} represent the level of interaction quality in the efficient allocation. Comparing (23) to (4), $v^{FB}(x) > v(K^{FB},x)$, the visit level arising from individual maximization. Comparing (24) to (11), $n^{FB}(x) > n(K^{FB},x)$, the density level arising from individual maximization. Comparing (26) to (17), $x^{bFB} > x^b$, the border arising from individual maximization. $v(K^{FB},x)$, $n(K^{FB},x)$, and x^b are all positive by Lemma 1.

Lemma 4. $\Gamma(0) > 0$.

Proof: Lemma 3 and (19).

A3. Assume that:

$$1 - \int_0^{x^{b^{FB}}(K)} F'(\cdot)\frac{\partial v^{FB}}{\partial K} n^{FB}(\cdot) + F(\cdot)\frac{\partial n^{FB}}{\partial K} dx - F(v^{FB}(K, x^{b^{FB}}(K))n^{FB}(K, x^{b^{FB}}(K))x^{b^{FB}\prime}(K) > 0$$

Proposition 2. If (A1) and (A3) hold, there exists a unique, globally stable solution to the efficiency (rent-maximization) problem.

Proof: (A1) implies $\Gamma(0) > 0$ by Lemma 3. (A3) means that $\Gamma'(K) < 1$. Together, these imply existence, uniqueness, and global stability by Banach's Theorem.

Remark: Lemma 4 and (A3) are parallel to Lemma 2 and (A2) above.

Constrained efficient

Let the solutions for n(x), and x^b from (31), and (33) -- with λ given by (32) -- be given by $n^{CE}(x)$, and x^{bCE}. Let v(K,x) denote the visit level arising from individual maximization. Let K^{CE} denote the level of interaction quality at the constrained efficient allocation. Let $\Omega(K)$ define the mapping obtained by substituting v(K,x), $n^{CE}(x)$, and x^{bCE} into (19).

Lemma 5. In constrained efficient allocation, (a) $x^{bCE} > 0$ and (b) $v(K^{CE},x) > 0$, $n^{CE}(x) > 0$ for all $x < x^{bCE}$.

Proof: The result for $v(K^{CE},x)$ is proved in Lemma 1. Comparing (31) to (11), $n^{CE}(x) > n(x)$, the density level arising from individual maximization. Comparing (33) to (17), $x^{bCE} > x^b$, the border arising from individual maximization. $v(K^{CE},x)$, n(x), and x^b are all positive by Lemma 1.

Lemma 6. $\Omega(0) > 0$.

Proof: Lemma 5 and (19).

A4. Assume that:

$$1 - \int_0^{x^b} F'\left(v^{FB}(K,x)\right) n^{FB}(x) \frac{\partial v^{FB}}{\partial K} dx > 0$$

Proposition 3. If (A1) and (A4) hold, there exists a unique, globally stable solution to the constrained efficiency problem.

Proof: (A1) implies $\Omega(0) > 0$ by Lemma 5. (A4) means that $\Omega'(K) < 1$. Together, these imply existence, uniqueness, and global stability by Banach's Theorem.

Developer

Let $n^D(x)$ denote the density chosen by the developer on the interval $[0, x^d]$, given by (41), with λ given by (42). n(K,x) denotes the density arising from individual construction decisions on $[x^d, x^b]$. v(K,x) denotes the visit level arising from individual maximization. Let K^D denote the level of interaction quality at the developer solution. Let $\Delta(K)$ denote the mapping defined in (40), obtained by substituting v(K,x), $n^D(x)$, n(K,x), x^d and x^b into (19).

A5. $x^d < x^b$

Remark: This is a sufficient condition that ensures that the developer solution is different from the equilibrium.

Lemma 7. In the developer solution, $x^b > 0$ and $v(K^D,x) > 0$ for all $x < x^b$. $n^D(x) > 0$ for all $x < x^d$. $n(K^D,x) > 0$ for all x on $[x^d, x^b]$.

Proof: The results for $v(K^D,x)$, $n(K^D,x)$, and x^b are proved in Lemma 1. Comparing (41) to (11), $n^D(x) > n(x)$. $v(x)$, $n(x)$, and x^b are all positive by Lemma 1.

Lemma 8. $\Delta(0) > 0$.

Proof: Lemma 7 and (19).

A6. Assume that:

$$1 - \int_0^{x^d} F'\big(v(K,x)\big) n^D(x) \frac{\partial v}{\partial K} dx + \int_{x^d}^{x^{b(K)}} F'\big(v(K,x)\big) n(K,x) \frac{\partial v}{\partial K} dx > 0$$

Proposition 4. If (A1), (A5) and (A6) hold, there exists a unique, globally stable solution to the developer's problem.

Proof: (A1) implies $\Delta(0) > 0$ by Lemma 8. (A6) means that $\Delta'(K) < 1$. Together, these imply existence, uniqueness, and global stability by Banach's Theorem.