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A multi-factor, credit migration model for sovereign and corporate debts

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Abstract

This paper develops a multi-factor, Markov chain model for rating migrations and credit spreads that is applicable to both sovereign and corporate debts. The model's central feature is to allow transition matrices to be time-varying and driven by rating specific latent variables which encompass economic factors like the business cycle. There are three main contributions. First, the model incorporates well-documented empirical properties of transition matrices such as their dependence on business/credit cycles, and it also allows for inter-rating variations in credit quality changes. Second, instead of focusing solely on empirical modeling of rating transitions, the paper also shows how the empirical model can be implemented for actual valuations. Third, the estimation and calibration procedures are easy to follow and implement.

JEL classification: G10; G15; G20

Keywords: Credit rating; Rating migration; Credit risk; Transition matrix; Sovereign debts; Corporate debts

1. Introduction

In the past several years, credit risk modeling and credit derivatives valuation have received tremendous attention around the globe. The main impetus for the increasing attention was the high-profile debt defaults by sovereigns as well as major corporations. On the international scene, in the early 1990s, the Mexican peso crisis lead to some debt defaults; in 1997, the Asian currency crisis triggered a chain of credit

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failures; and in August 1998, the Russian ruble devaluation caused the Russian government to default on some dollar denominated debts. On the domestic scene in the US, major financial fiascos such as the collapse of the long term capital management also raised concerns about the impact of credit risks on the financial system. The ever increasing sophistication of derivative instruments, the pursuit of protection against counter-party losses in financial crises such as those listed above, and the stepping up of regulatory efforts have all spurred research on credit risk management.

The literature on credit risk management has grown along two related strands. On the one hand, many authors have modeled and empirically studied default risk and rating migrations. Examples include Altman and Kao (1992a), Lucas and Lonski (1992), Carty and Fons (1994), Fons (1994), Belkin et al. (1998), Duffee (1998), and Helwege and Turner (1999). On the other hand, some authors have proposed various credit risk/rating migration models to value credit derivatives such as default swaps and yield spread options. Examples here include Jarrow et al. (1997), Kijima and Komoribayashi (1998), Lando (1998), Bielecki and Rutkowski (2000), and Lando (2000). Recently, some studies have compared and evaluated various credit risk models currently being used in the industry. Examples include Crouhy et al. (2000), Gordy (2000), and Lopez and Saidenberg (2000). All the aforementioned studies address credit risk of corporate issuers. Very little attention has been given to the potential application of the existing models to sovereign credit risk management.

In their seminal study of credit spread, Jarrow et al. (1997) model rating transitions as a time-homogenous Markov chain. Within their model, whether a firm's rating will change in the next period is not affected by its rating history (hence, Markov), and the probability of changing from one rating (e.g., AA) to another (e.g., BBB) remains the same over time (hence, time-homogenous). Moreover, the market risk and credit risk are assumed to be independent. For valuation purposes, the observed transition matrices such as those published by Moody's and Standard and Poor's must be transformed to incorporate risk premium information embedded in the bond price data. Jarrow et al. (1997) accomplish this by relying on the time-homogeneity and Markov assumptions, and the additional assumption that the credit risk premiums are time-varying to reflect the changing credit spreads in corporate bonds. While retaining all the critical assumptions of Jarrow et al. (1997) and Kijima and Komoribayashi (1998) make a modification to the Jarrow–Lando–Turnbull framework to perfect the empirical estimation of the model.

While the study by Jarrow et al. (1997) represents a major step forward in credit risk modeling, their setup can be extended in several dimensions. First, as pointed out by the authors themselves, time-homogeneity is assumed solely for simplicity of estimation. Empirical evidence in the Moody's Special Report (Lucas and Lonski, 1992) and the Standard and Poor's Special Report (1998) indicates that transition probabilities are time-varying, especially for speculative grade bonds. Specifically, Belkin et al. (1998) and Nickell et al. (2000) have shown that probability transition matrices of bond ratings depend on business cycles. Similarly, Helwege and Kleiman (1997) and Alessandrini (1999) have shown respectively that default rates and credit spreads depend on the stage of the business cycle. Lando (1998) extends the Jarrow–Lando–Turnbull model by allowing for dependence between the market risk and credit risk, and by allowing the transition rates between ratings to depend on the state variables. Although Lando (1998) does not model business cycles per se, his framework is a major improvement over the fixed, time-homogenous setup of Jarrow et al. (1997). In addition, Bielecki and Rutkowski (2000) model the random transition probabilities among multiple ratings based on Heath et al. (1992) methodology which utilizes the credit spreads and recovery rates as inputs.

Second, a time-homogenous setup rules out not only the dependence on business cycles, but also the possibility that different ratings respond to the same credit condition change at different rates. Although there is no known empirical study that directly examines this aspect of credit risk behavior (which itself is another gap in the literature), inter-rating differences in credit quality changes are indeed a plausible conjecture. In fact, Altman and Kao (1992b) find that, over time, higher-rated bonds tend to be more stable than lower-rated bonds as far as retaining their original ratings is concerned. This can be considered as an indirect support for the conjecture.

Third, theoretically, it is not clear why credit risk premiums should change drastically year by year. Intuitively, the premium per unit of risk should remain more or less constant unless investors' risk attitude changes, and it is the varying level of risk, or credit cycle, which leads to the changes in spreads. As reported by Belkin et al. (1998), defaults are more likely in economic downturns than in economic booms.

Despite the obvious importance of recognizing the impact of business cycles on rating transitions, the literature is very scanty on this issue. There are only two known studies which explicitly link business cycles to rating transitions. Belkin et al. (1998) employed a univariate model whereby all ratings respond to business cycle shifts in the same manner, and they do not deal with estimating matrices under the equivalent martingale measure.[1] Nickell et al. (2000) propose an ordered probit model which allows a transition matrix to be conditioned on the industry, the country domicile, and the business cycle. Although they require a large quantity of data to estimate reliable parameters, their approach is conceptually very appealing. Insofar as the reference asset for most credit derivatives is company/institution/country specific, the ability to condition a transition matrix on the industry (to which the company belongs) is definitely desirable. However, since they also need to model the business cycle as a Markov chain, computing multi-period transition matrices becomes a very involved process, and as a result, it becomes quite challenging to estimate risk premiums in order to obtain the risk-neutral matrices for valuations. In addition, for estimation purposes, they need to assume cross-sectional independence in rating changes.

The objective of the current paper is to build a general, credit risk model which circumvents the aforementioned shortcomings and meanwhile retains the positive features of the existing models such as the incorporation of credit/business cycles. Specifically, I propose a multi-factor, Markov chain model for the evolution of credit

[1] Kim (1999), in a short article appearing in a special issue of Risk, proposes a model very similar to that of Belkin et al. (1998). However, he attempts to link some macroeconomic variables to the shifts of transition probabilities.

ratings for both sovereign and corporate debts. The Markov condition is employed to facilitate estimations. The multi-factor structure will allow the transition matrix to evolve according to credit cycles, and allow different ratings to respond in a correlated yet different fashion to the same change in the general economic conditions. In so doing, I will also ensure that the credit risk premiums are kept constant. The model can then be applied to value such credit derivatives as default swaps and credit spread options pertaining to sovereign or corporate debts.

The rest of the paper is organized in six sections. The next section contains a brief overview of the time-homogenous Markov chain model. Section 3 outlines the general framework and estimation procedures. Section 4 delineates the model's application to sovereign debts. Section 5 presents the model's application to corporate debts. This section has two sub-sections, the first presenting the data, and the second reporting and discussing estimation results. Section 6 contains some general discussions and caveats. The last section concludes.

2. Overview of the time-homogenous Markov chain model

Let Ω be the set of all possible credit states (including default), and i ($i = 1, 2, \ldots\ldots, K$) be the index of its elements, where K is the total number of possible states. For example, for a bond rating system consisting of AAA, AA, A, BBB, BB, B, CCC, and D (default), $i \in [1, 8]$ and $K = 8$. Furthermore, let p_{ij} denote the probability of state i transiting to state j. Then, the discrete time, time-homogenous transition matrix can be represented by

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1K} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2K} \\ \cdots & & & \cdots & \\ \cdots & & & \cdots & \\ \cdots & & & \cdots & \\ p_{K-1,1} & p_{K-1,2} & p_{K-1,3} & \cdots & p_{K-1,K} \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} \tag{1}$$

where $p_{ij} \geq 0 \ \forall i,j$ and $\Sigma_{j=1}^{K} p_{ij} = 1 \ \forall i$. The default state, K is assumed to be absorbing so that $p_{KK} = 1$. The Markovian assumption implies that the n-period transition matrix, $P_{0,\,n}$ is simply the product of the single-period matrix itself, P^n.

The matrix in Eq. (1) contains the observed or empirical transition probabilities. For valuation purposes, the empirical matrix needs to be transformed into a risk-neutral transition matrix under the equivalent martingale measure. Let Q denote such a matrix. Without further assumptions, the transition matrix under the new measure need not be Markovian, certainly not time-homogenous. To signify this, let $q_{ij}(t, t+1)$ denote the transition probability from state i to state j at time t. Then the transition matrix under the martingale measure becomes

$$Q_{t,t+1} = \begin{pmatrix} q_{11}(t,t+1) & q_{12}(t,t+1) & q_{13}(t,t+1) & \dots & q_{1K}(t,t+1) \\ q_{21}(t,t+1) & q_{22}(t,t+1) & q_{23}(t,t+1) & \dots & q_{2K}(t,t+1) \\ \dots \\ \dots \\ \dots \\ q_{K-1,1}(t,t+1) & q_{K-1,2}(t,t+1) & q_{K-1,3}(t,t+1) & \dots & q_{K-1,K}(t,t+1) \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \quad (2)$$

where conditions for Eq. (1) must also be satisfied here, together with the equivalence condition that $q_{ij}(t, t+1) > 0$ if and only if $p_{ij} > 0$. To utilize the empirical transition matrix P in estimation and to simplify the estimation itself, Jarrow et al. (1997) assume the following transformation:

$$q_{ij}(t,t+1) = \pi_i(t)P_{ij} \quad \forall i,j,\ i \neq j, \text{ and } q_{ii}(t,t+1) = 1-\sum_{j\gamma 1}\pi_i(t)p_{ij} \ \forall i \quad (3)$$

where $\pi_i(t)$ is at most a function of time, and $\pi_i(t) > 0$. Of course, a feasible set of $\pi_i(t)$ must also ensure that the entries for a particular row in the matrix represent probabilities: $q_{ij}\geq 0\ \forall j$ and $\Sigma^{K}_{j=1,\, i\neq j}q_{ij}\leq 1$. There is no guarantee though that the above conditions are met in actual estimations. The transformation in Eq. (3) together with the restrictions on $\pi_i(t)\ \forall i$ give the adjustments $\pi_i(t)\ \forall i$ an interpretation of risk premiums, and the transition matrix will be non-homogenous but the underlying process is still Markov. (If $\pi_i(t)$ is j specific and is path dependent, then the matrix Q will not be Markovian.) By necessity, $\pi_K(t) = 1$ and need not be estimated. With the above, the n-period transition matrix is now given by

$$Q_{0,n} = Q_{0,1}Q_{1,2}\cdots Q_{n-2,n-1}Q_{n-1,n}. \quad (4)$$

To estimate the risk premiums, $\pi_i(t)\ \forall i$, bond price data and assumptions on recovery rates are required. To this end, let $v_0(t, T)$ be the time-t price of a riskless unit discount bond maturing at time T, and let $v_i(t, T)$ be its risky counterpart for the rating class, i. As shown by Jarrow et al. (1997), under the assumptions that (1) the Markov process and the interest rate are independent under the equivalent martingale measure, and (2) bond holders will recover a fraction δ of the par at maturity in case default occurs any time prior to maturity, the following holds:

$$v_i(t,T) = v_0(t,T)[\delta + (1-\delta)\text{prob}_t\{\tau_i > T\}], \quad \forall i\in\Omega \quad (5)$$

where $\text{prob}_t(\tau_i > T)$ is the probability under the equivalent martingale measure that the bond with rating i will not default before time T. It is clear that

$$\text{prob}_t\{\tau_i > T\} = \sum_{j=1}^{K-1} q_{ij}(t,T) = 1-q_{iK}(t,T), \quad (6)$$

which holds for time $t\leq T$, including the current time, $t = 0$. Combining Eqs. (3), (5) and (6) leads to

$$\pi_i(0) = \frac{v_0(0,1)-v_i(0,1)}{(1-\delta)v_0(0,1)p_{iK}}. \tag{7}$$

Once $\pi_i(0)\ \forall i$ are obtained via Eq. (7), applying Eq. (3) for all entries leads to $Q_{0,1}$. With $Q_{0,1}$ on hand, Eq. (4) together with Eqs. (3), (5) and (6) can be utilized to find $\pi_i(1)\ \forall i$ and hence $Q_{1,2}$ and $Q_{0,2}$. Repeated application of the above procedures using prices of progressively longer bonds will lead to all the desired matrices, $Q_{0,t}$ for $t = 1, 2, \ldots, n$. Valuation of credit derivatives can then proceed by simply calculating risk-neutral, discounted expected payoffs, utilizing the transition probabilities.

It should be pointed out that the adjustment scheme in Eq. (3) is by no means unique. Instead of adjusting all entries other than the diagonal entry, Kijima and Komoribayashi (1998) propose to adjust all entries other than the default column entry:

$$q_{ij}(t,t+1) = \pi_i(t)p_{ij}\ \ \forall i,j,j \neq K, \text{ and } q_{iK}(t,t+1) = 1-\sum_{j\neq k}\pi_i(t)p_{ij} = 1 -\pi_i(1-p_{iK})\ \forall i \tag{8}$$

Their procedure leads to the following estimate for the risk premium:

$$\pi_i(0) = \frac{v_i(0,1)-\delta v_0(0,1)}{(1-\delta)v_0(0,1)}\frac{1}{1-p_{iK}}. \tag{9}$$

It is apparent that a zero or near-zero default probability would cause the risk premium estimate to explode in Eq. (7), but would still lead to a meaningful estimate in Eq. (9). For this reason, Kijima and Komoribayashi's approach will be used in this paper.[2]

The above framework assumes that the average, empirical transition matrix remains constant over time, and the per-period risk-neutral transition matrix varies over time to accompany the changes in bond prices. As discussed earlier, it is more plausible to assume that the empirical transition matrix adjusts over time according to business cycles while the risk premium remains constant. Moreover, different ratings may react to the same economic shock in very different intensities. In the following section, I will extend the Jarrow–Lando–Turnbull framework by incorporating the aforementioned desirable features.

3. A general multi-factor Markov chain model

As a starting point, assume that there exists an average transition matrix similar to the one in Eq. (1), whose fixed entries represent average, per-period transition

[2] Note that all entries in the default column must be strictly positive in order for Eq. (7) to be well-defined. Although not explicitly discussed by Kijima and Komoribayashi (1998), their modified procedure of estimating the risk premiums requires the same condition in order to guarantee the equivalence between the observed probability matrix and the risk-neutral matrix. To see this, notice from Eq. (9) that a risk premium is well-defined even if p_{iK} is zero. In this case, as long as the risk premium is not exactly 1.0, the corresponding risk-neutral default probability will not be zero, which violates the equivalence condition. In this paper, I replace the zero entries in the default column by the smallest non-zero entry in the transition matrix.

probabilities across all credit cycles. This matrix can be thought of as a matrix applicable to a typical, average credit condition. Depending on the condition of the economy for a particular year, the entries will deviate from the averages, and the size of deviations can be different for different rating categories. In order to facilitate modeling and estimations, I choose to work with a set of credit variables that drive the time-variations of the transition probabilities. It is therefore necessary to define a set of average credit scores which correspond to the average transition matrix, and model the movement of these credit scores or variables to reflect the period-specific transition matrices.

The first step is to devise a mapping through which the average transition probabilities can be translated into credit scores. To this end, a methodology similar to that of CreditMetrics™ will be adopted. For a sovereign bond, the link between transition probabilities and credit scores can be considered as a mapping between a country's overall macroeconomic conditions and possible ratings. For a corporate bond, the methodology can be understood as mapping a firm's future asset returns to possible ratings, assuming that higher returns correspond to higher ratings, and vice versa. The mapping may employ any meaningful statistical distribution, although ease of calculation and estimation may dictate the choice, given the absence of strong preference for a particular distribution. In this paper, I use the normal distribution. The detailed procedure is described below.

Since the row sum for any rating in a matrix is always 1.0, one could, for each rating class in the average transition matrix, construct a sequence of joint bins covering the domain of the normal variable. This is done by inverting the cumulative normal distribution function starting from the default column. To illustrate, suppose the issuer is currently rated A, and the average probabilities for A to transit to AAA, AA, A, BBB, BB, B, CCC, and D are 0.0026, 0.0159, 0.8905, 0.0740, 0.0148, 0.0013, 0.0006, and 0.0003 (the sum of which is 1.0). Since the default probability of 0.0003 corresponds to all negative values up to $N^{-1}(0.0003) = 3.432$, the first bin is $(-\infty, -3.432]$. Next, summing 0.0003 and 0.0006 gives us the total probability that the new rating is either CCC or D. Hence, $N^{-1}(0.0009) = -3.121$, and the next bin is $(-3.432, -3.121]$. By repeating the above, other bins can be calculated as $(-3.121, -2.848]$, $(-2.848, -2.120]$, $(-2.120, -1.335]$, $(-1.335, 2.086]$, $(2.086, 2.795]$, and $(2.795, +\infty)$. In other words, one could partition the domain of a standard normal variable by a series of z-scores. An average transition matrix as in Eq. (1) can then be represented as

$$Z = \begin{pmatrix} z_{12} & z_{13} & z_{14} & \cdots & z_{1K} \\ z_{22} & z_{23} & z_{24} & \cdots & z_{2K} \\ \cdots & & & & \\ \cdots & & & & \\ \cdots & & & & \\ z_{K-1,2} & z_{K-1,3} & z_{K-1,4} & \cdots & z_{K-1,K} \end{pmatrix}. \tag{10}$$

Notice that the z-score matrix is $(K-1)$ by $(K-1)$ because there is no need to convert the row for the absorbing default state, and because the upper limit of rating AA is the lower limit of the highest rating AAA. Obviously, given a z-score matrix, a corresponding transition matrix can also be obtained.

Once the average credit score matrix is obtained, the next step is to model deviations from those scores. To this end, it is assumed that the deviations are driven by K mutually independent, normally distributed factors scaled to a standard normal. Without loss of generality, let the first factor denote the common factor for all ratings, and the rest denote rating class specific factors. Formally, generalizing the framework of Belkin et al. (1998), define

$$y_{ij} = \alpha(x + x_i) + \sqrt{1-2\alpha^2}\varepsilon_{ij},\ i = 1,2,\ldots,K-1,\ j = 1,2,\ldots,K \tag{11}$$

where x is the common factor, x_i ($i = 1, 2, \ldots, K-1$) is the rating specific factor, and ε_{ij} is a non-systematic, idiosyncratic factor. By assumption, x, x_i and ε_{ij} are i.i.d. standard normal variables, and the correlation between the aggregate factors of any two rating classes is the same, viz, $\mathrm{corr}(y_{ij}, y_{ml}) = \alpha^2$ for all i, j, m, and l where $i \neq m$. For an average year, by definition, the realized deviations for all rating classes should be close to zero. For each rating or row i, ε_{ij} ($j = 1, 2, \ldots\ldots, K$) represents the idiosyncratic factor. The factors x and x_i can be considered as latent variables which encompass the impacts of all economic variables relevant to rating changes. In this sense, they can naturally be thought of as credit cycle variables.[3]

When seeking for the fitted transition matrix for each year, we are essentially implying the realized values for the latent variables x and x_i by minimizing the residual errors ε_{ij} ($j = 1, 2, \ldots\ldots, K$) as described in Procedure A below. In other words, the realized or implied deviation factors or credit cycle variables, x and x_i, are applied to Eq. (10) to shift the average z-scores, and a fitted transition matrix is then inverted from the adjusted average z-score matrix. Therefore, the key assumption is the equal magnitude of shifts in z-scores for a particular rating/row. It is easy to see that, for a given rating, a downward shift in the z-scores leads to an increase in probabilities of transiting to ratings higher than or equal to the rating in question, and a decrease in probabilities of transiting to lower ratings/states; and an upward shift in the z-scores leads to the opposite. For a given row, the deviations of probabilities from the average transition matrix need not be equal for all columns. In fact, it is almost certain that they are different, given that the shifts in z-scores are of the same size and that the density function is curved. Here, the unknown shift is subtracted from the average z-scores, so that a positive shift means an improvement in credit quality, and vice versa.

The proposed framework can now be summarized as follows. First, Eq. (11) can be fitted into the historical average z-score matrix and the realized annual z-score matrices to estimate the parameter, α, and then the annual fitted transition matrices

[3] Notice that a more general setup such as $y_{ij} = \alpha x + \beta x_i + \sqrt{1-\alpha^2-\beta^2}\varepsilon_{ij}$ is in principle the same as that in Eq. (11). Since x captures the common effect, the two setups imply the same correlation structure. The only difference is the scaling of x_i which has no qualitative consequence anyway.

can be obtained via α. Second, the constant risk premiums can be estimated using the fitted transition matrices and historical discount bond prices. Third, with the constant risk premium estimates and the current prices of discount bonds of various maturities, the future transition matrices under the equivalent martingale measure can be implied, and the valuation of credit derivative securities can then proceed. Detailed estimation procedures are outlined below.[4]

(A) *Estimating the factor realizations and fitted transition matrices*:

1. calculate the historical average transition matrix and convert it into a z-score matrix;
2. for each period t, find the shift for each row (of the z-score matrix) to minimize the sum of deviations of the fitted probabilities from the observed probabilities; this procedure will yield a time series of z-score deviations for all ratings and all periods, $\Delta z_{t,i}$ $\forall t = 1,2,\ldots,T$ and $\forall i = 1,2,\ldots,K-1$;[5]
3. calculate the average of the seven shifts for each year, denoted by $\overline{\Delta z_t}$, which represents the common/systematic shift;
4. calculate the variance of the time series obtained in Step 3, denoted by $\mathrm{Var}(\overline{\Delta z})$, and compute the quantity, $\hat{\alpha} = \sqrt{\mathrm{Var}(\overline{\Delta z})}$, which shall be the estimate of α;
5. for each period t, calculate $\overline{x_t} = \overline{\Delta z_t}/\alpha$ (since by definition x captures the common shift);
6. within the same period t, for each rating class i, calculate the rating specific deviation as (in Steps 5 and 6, use the estimated α from Step 4);
7. obtain the fitted transition matrix for each period by using the average historical matrix and the z-score adjustments or deviations estimated in Steps 5 and 6 (or simply from Step 2). (Note: Steps 5 and 6 can be omitted if the values of realized factors are not of interest.)

Notice that, in a univariate model such as that of Belkin et al. (1998), Step 2 is applied to the whole matrix for a particular year to find the common shift, and the parameter α is estimated in a similar fashion. One could follow this procedure to

[4] The constant risk premiums can also be estimated directly via the observed (as opposed to the fitted) transition matrices. In this case, estimating the parameter α will not be essential. However, the use of fitted matrices is recommended since this will be consistent with the procedure when implying transition matrices for the future. In addition, by estimating Eq. (11) and the z-score deviations, one can study the credit cycle effect, as is done in Section 5.

[5] To improve the estimation results for each row, I follow Belkin et al. (1998) to weigh the square of deviations by the inverse of the approximate sample variance of each entry's probability estimate. In my case though, the number of observations (i.e., bonds) for each row is irrelevant since it remains constant across columns. Furthermore, unlike Belkin et al. (1998), and Kim (1999), I do not scale the adjusted z-score by $\sqrt{1-2\alpha^2}$ because this scaling will lead to the unnatural result that the average z-scores are adjusted/scaled even when the shift is zero. Notice also that the above procedure will distort the meaning of the residual term in Eq. (11). Specifically, there is no guarantee that the sum of the residual is zero as it should be in a usual regression setting. However, this seems to be a reasonable price to pay, as directly minimizing the sum of squares of z-score deviations leads to very poor fit of transition matrices. The poor fit results from the negligence of the highly non-linear relation between z-scores and probabilities.

estimate α first, and then in the second pass, given the common shift, find the row-specific shifts. In the current paper, I estimate all quantities in one-pass as outlined above, in order to be consistent with the assumption of independence between x and x_i. However, as shown later, the two methods lead to very similar estimates for α.

(B) *Estimating the constant risk premiums*: Within the proposed framework, the risk premium for each rating class i is assumed to be constant. Therefore, only $(K-1)$ risk premium parameters need to be estimated. Specifically,

1. for each period t, following Kijima and Komoribayashi (1998), express the probability transition matrix under the equivalent martingale measure as the risk adjusted, fitted transition matrix obtained in Procedure A: $Q = P(\pi)$ (i.e., multiplying the entries of the fitted transition matrix by the unknown risk premiums while leaving the default column as the adjusting column to ensure row sum of 1.0);
2. estimate the risk premiums for period t via Eq. (9). Since bond prices of various maturities are typically available for each time period, a fitting procedure must be used to estimate the risk premiums.

(C) *Estimating implied future transition matrices under the equivalent martingale measure*: Since time-homogeneity is not assumed, the transition matrix for each of the future periods must be estimated or implied in order to do valuations. Similar to the procedure in Jarrow et al. (1997), the estimation is recursive: starting from one period out, and successively working out the matrices for long periods. Specifically,

1. via Eq. (5), using single-period bond prices and an assumed recovery rate to imply the default probabilities under the equivalent martingale measure for all ratings, $q_{1K}(0, 1)$, $q_{2K}(0, 1)$, …, and $q_{K-1,K}(0,1)$, as $(v_0(0,1)-v_i(0,1))/((1-\delta)v_0(0,1))$ $\forall i = 1,2,\ldots,K-1$;
2. for each row of the average historical z-score matrix, adjust the z-scores by subtracting some unknown amount: $\alpha(x + x_i) \equiv \alpha\Delta_i$ $\forall i = 1,2,\ldots,K-1$;
3. for each rating i, by construction, $1-\pi_i(1-N[(z_{iK}-\alpha\Delta_i)]) = q_{iK}(0,1)$ (where z_{iK} is defined in Eq. (10)), which leads to an estimate for the adjustment:

$$\Delta_t = \frac{z_{iK}-N^{-1}[1-(1-q_{iK}(0,1)/\pi_i)]}{\alpha}$$

   (both π_i and α are known by now);
4. repeat Step 3 for each rating/row and complete the adjustment of the z-score matrix;
5. convert the adjusted z-score matrix into a probability transition matrix, and, using the risk premium estimates, transform this matrix into a matrix that is applicable under the equivalent martingale measure, $Q_{0,\,1}$;
6. multi-period transition matrices are estimated recursively by utilizing Eq. (4) and bond prices with successively longer maturities. (Matrix inversion is necessary for the second period and beyond. For example, once $Q_{0,\,1}$ and the default column of $Q_{0,\,2}$ (calculated using the expression similar to the one in Step 1) are known, $Q_{0,\,1}$ is inverted to obtain the default column of $Q_{1,2}$.)

Once the transition matrices for all future periods are obtained under the equivalent martingale measure, valuation of credit derivatives such as default swaps can then proceed. In the next two sections, I will discuss the model's applications to sovereign and corporate debts, respectively. The estimations will be performed only for corporate bonds since they have a much longer rating history which covers a period of 18 years and larger sample size which encompasses more than 7000 issues. In contrast, the sample size for sovereign debts is much smaller (around 200 issues) and covers a much shorter period.

4. Application to sovereign debts

As mentioned in Introduction, the Asian currency crisis and the Russian default on some dollar denominated bonds have all drawn increasing attention to the modelling and management of sovereign credit risk. Sovereign ratings are gaining increasing importance as more governments with higher default risk, e.g., the emerging market economies, borrow in international bond markets. Governments themselves often seek credit ratings in order to have better access to international capital markets. For better risk assessment, investors also prefer rated securities over unrated ones, all else being equal.

Similar to corporate debts, sovereign debts are typically rated by major rating agencies such as Moody's and Standard and Poor's. Each agency rates debt issues by about 100 sovereigns. The scale of sovereign defaults is actually larger than that of corporate defaults. Figs. 1 and 2 plot the default history for sovereign and corpor-

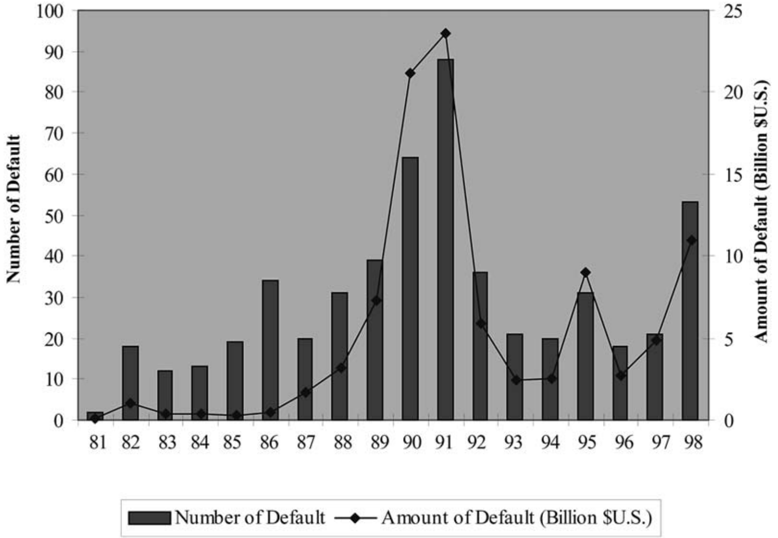

Fig. 1. Default history for corporate debts.

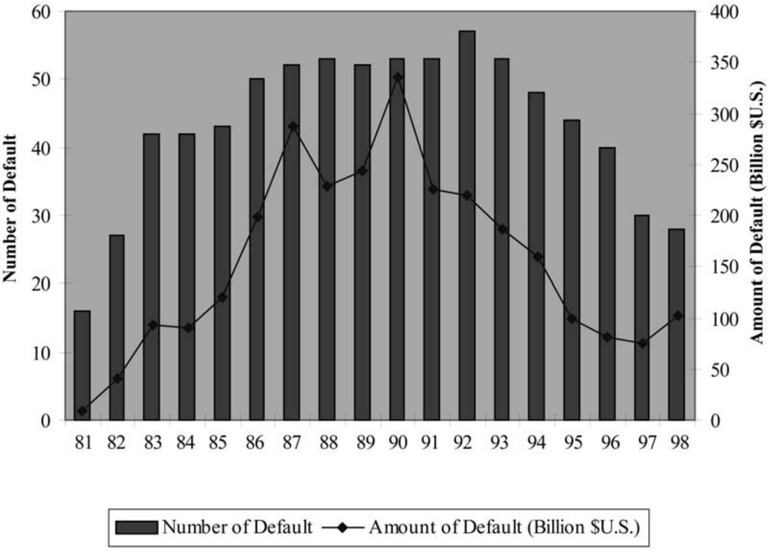

Fig. 2. Default history for sovereign debts.

ate debts. It can be seen that the dollar amount of default by sovereigns is much larger than the corporates'.

The rating of a sovereign debt is a comprehensive assessment of the issuing country's economic, social and political situation. In fact, Cantor and Packer (1996) find that rating assignments to sovereigns by Moody's and Standard and Poor's can be explained by a small number of well-defined variables such as GDP growth, inflation rate, and external debt. In addition, they found that the rankings of the sovereign debt yields broadly agree with the rating rankings: sovereign yields tend to rise as ratings decline. This correlation is also confirmed for emerging markets by Cunningham et al. (2001), and Sy (2001).

Insofar as sovereign ratings closely reflect macroeconomic fundamentals as shown by Cantor and Packer (1996) and a country's economy does go through cycles, the framework and estimation procedures outlined in Section 3 clearly apply to sovereign debts. For estimation inputs, annual transition matrices and yields for sovereign debts are required. Standard and Poor's started publishing sovereign annual transition matrices in 1998. Appendix A contains the average 1-year transition matrix for sovereign foreign currency debts based on about 200 issues. Moody's also publishes the detailed rating history for sovereign debts issued by more than 100 nations, and transition matrices can be easily constructed from the rating changes. As for bond yields, it is customary in the literature to treat the US treasury yield as default free, and use the yields on actively traded euro–dollar bonds issued by the sovereign in question to approximate the yields on sovereign debts (Cantor and Packer, 1996; Cunningham et al., 2001). Bloomberg L.P. reports on a regular basis the spread of a sovereign's bond yield over that of a comparable US treasury bond.

With the transition matrices and bond yields, estimations outlined in Section 3 can then proceed. Once the future implied transition matrices are implied, credit derivatives pertaining to sovereign debts such as credit default swaps and credit linked notes can be valued. However, as mentioned at the end of Section 3, the sovereign rating history is relatively short, and the sample is also small. I therefore use corporate bonds to demonstrate the estimations in the next section.

5. Application to corporate debts

5.1. Data

Annual transition matrices for 1981–1998 (inclusive) and the average annual transition matrix covering the same period are published by Standard and Poor's (1999) for corporate bonds. Weekly treasury and industrial corporate bond yields for various maturities (1, 5, 10, 15, 20, and 25) and ratings (AAA, AA, A, BBB, BB+, BB/BB−, and B) are obtained from the weekly publication, *Credit Week* (by Standard and Poor's). The starting date of the bond yields publication is March 1996.

For the transition matrices, several adjustments are made to smooth the transition probabilities. First of all, the raw matrices from Standard and Poor's contain a column titled "N.R."—not rated. Following Jarrow et al. (1997), I simply redistribute the "N.R." portion to other ratings on a pro rata basis. Unlike Jarrow et al. (1997), I leave the default column unchanged given that the "not rated" bonds are non-defaulting bonds (see discussions in Standard and Poor's (1999)). Second, within each row, the probability should decline monotonically on each side of the diagonal entry. Whenever there is a violation, the entry is set equal to the previous rating's entry and the difference is equally distributed among the entries between the diagonal entry and the entry in question. Third, within each column, the entries on each side of the diagonal entry should also monotonically decline. To minimize excessive arbitrary adjustments, whenever there is a violation, I simply swap the entry in question with the previous entry, and adjust the two row's diagonal entries to ensure a row sum of 1.0. In certain situations, this swapping may have to be done in several consecutive turns before the proper ranking is achieved. The default column is kept unchanged throughout the adjustments. Appendix B shows, as an illustration, the original raw matrix for the average annual transition, and the final matrix with the above adjustments. It is worth noting that the ranking adjustment is not very frequent in that the original matrices already satisfy the conditions most of the time. It should be pointed out that the monotonic smoothing is not necessary for our estimations, and there is no strong theory that dictates the monotone conditions. The smoothing is purely based on intuition. For instance, in the original raw matrix shown in Appendix B, row-wise for rating CCC, the transition probabilities to higher ratings B, BB, BBB, and A are progressively declining, which makes intuitive sense. There happened to be a transition to AAA but not AA in the sample period. Intuitively, the chance of transiting to a remote rating should be lower than that to a nearby rating, as shown in the adjusted matrix. Similarly, column-wise for rating AAA, intuitively, the probabilities for rating AA, A, BBB, and so on, to transit to AAA should be progressively lower. Presumably, when the sample is large and long enough, the observed transition matrix will automatically have the monotone feature.

For the bond yields, since annual transition matrices are of concern, they are sampled only at the beginning of the year for 1996, 1997 and 1998. To imply future matrices, I use the data of June 1999, which happen to be the end of the data set. The first three years are used to estimate the constant risk premiums, and the last year's bond yields are used to demonstrate how to imply future transition matrices. For simplicity, I will only use the one-year-maturity bond prices to estimate the constant risk premiums via minimizing the sum of squared deviations between model prices and observed prices. The bond yields are tabulated in Appendix C.

Several issues pertaining to bond yields need to be addressed. First, the corporate bond yields reported in *Credit Week* are for industrials, whereas the transition matrices are based on ratings covering a range of industries (e.g., industrials, utilities, and financial institutions) in the US and overseas. Notwithstanding the dominance of US industrials in the rating history (see Nickell et al., 2000 for statistics), the estimation results should be taken with a grain of salt. Second, *Credit Week* reports yields separately for BB+ and BB/BB−. I simply use the average of the two yields to proxy the overall yield for BB. Third, yields for rating CCC are not available. In light of the yield vs. rating profile depicted in Fig. A1 in Appendix C, I only use yields for BBB, BB and B to quadratically extrapolate the yield for rating CCC by assuming that rating classes are equally spaced. Fourth, for 1999, I use the yields of 1-, 5-, and 10-year bonds to quadratically interpolate the yields for other maturities between 1 and 10 years. Only yields with maturities up to 5 years are used to demonstrate the estimation, since most credit derivatives have a maturity less than 5 years. The extrapolated/interpolated yields are tabulated in Appendix C. Finally, when implying future transition matrices beyond one year out, yields of zero-coupon bonds should be used. Unfortunately, given the lack of information, it is impossible to infer the pure yield curves from the average yield curves. I simply assume that the reported bond yields are close approximations for discount bond yields.

5.2. Estimation results and interpretations

5.2.1. Shifts of z-scores and the fitted transition matrices

By following the estimation procedures outlined in Section 3, the parameter α in Eq. (11) is estimated to be 0.1116, which indicates that, on average, the correlation between credit migrations of any two rating classes is about 0.0125. (When the two-pass, sequential procedure is followed, the estimate for α is 0.1213, very close to the one-pass estimate.) The estimated z-score deviations (defined as $x + x_i$ in Eq. (11)) are summarized in Table 1. The sample average is −0.016 as opposed to a theoretical value of zero, and the variance of the average z-score shifts is 1.0 by design. The overall results are very similar to that of Belkin et al. (1998). For example, the 1980s saw predominantly lower than average ratings, while the 1990s saw better than average ratings. The year 1990 represents the worst year, while 1996 is the best year, similar to the findings of Belkin et al. (1998). Crouhy et al. (2000) also document that 1990 and 1991 have the most default occurrences, while 1993

Table 1
Realized z-score shifts

	AA	A	BBB	BB	B	CCC	D	Average	Goodness of fit	
									K-factor	1-factor
1981	−1.127	0.652	0.326	0.016	−5.742	0.455	1.371	−0.579	0.835	0.810
1982	−0.680	−0.386	−1.396	−2.600	−2.232	−0.76	−1.068	−1.303	0.905	0.894
1983	−5.141	0.935	2.302	0.177	−0.735	−0.273	3.207	0.067	0.916	0.861
1984	−7.013	1.418	0.869	1.150	1.098	0.273	−0.417	−0.375	0.869	0.812
1985	3.570	−2.766	−1.315	−1.326	−1.370	−1.035	5.604	0.195	0.891	0.841
1986	0.945	−0.161	−1.756	−2.317	0.021	−4.299	−1.708	−1.325	0.885	0.852
1987	1.754	1.695	−0.305	−1.116	0.016	1.171	1.416	0.662	0.923	0.910
1988	0.008	−1.543	−1.071	0.846	−0.021	0.750	−0.020	−0.15	0.93	0.924
1989	−0.188	0.296	−2.501	−0.433	1.931	0.657	−3.200	−0.491	0.936	0.913
1990	3.105	−2.407	−2.009	−0.368	−3.074	−2.584	−2.600	−1.420	0.957	0.92
1991	−1.285	−0.365	−0.940	−0.532	−1.197	−2.634	−1.384	−1.191	0.928	0.922
1992	−1.065	−0.532	0.379	0.511	1.393	0.409	−0.816	0.04	0.925	0.91
1993	1.095	0.715	0.855	−0.718	0.825	3.938	6.129	1.834	0.907	0.875
1994	−0.039	−0.497	0.765	0.510	1.318	0.410	−0.328	0.306	0.899	0.896
1995	2.465	−0.384	1.336	0.683	0.841	0.787	−1.890	0.548	0.95	0.925
1996	1.570	1.508	2.488	1.089	1.139	1.575	3.588	1.851	0.91	0.91
1997	2.230	1.204	0.554	0.204	1.498	1.016	2.288	1.285	0.944	0.931
1998	0.756	0.894	0.090	−0.657	−1.070	−0.295	−1.448	−0.247	0.894	0.883
						Average		−0.016	0.911	0.888
						Variance		1	0.001	0.001
						Quasi *R*-square			0.446	0.051

Note: Each entry under a rating represents a systematic shift in the z-score for a particular rating in a particular year. The net shift is α times a particular entry. For example, for the rating AA in 1981, the realized systematic shift factor is, $x + x_{AA} = -1.127$, and the actual shift is $\alpha(x + x_{AA}) = -0.1258$. The last column contains the average realized shift factor for a year. A negative number indicates an overall deterioration in ratings compared with the historical average, and a positive number indicates otherwise. The last two columns contains measurements of goodness of fit. Please see Section 5.2.1 for explanations of the measures. "*K*-factor" is the multi-variate model being considered, and "1-factor" is the univariate model discussed in Procedure A in Section 3.

has the least. The findings in Table 1 are further corroborated by Fig. 1 which shows the default history for corporate bonds rated by Standard and Poor's. The model's ability to capture the default behavior is rather striking. The higher default frequencies depicted in Fig. 1 for years 1982, 1986, 1990, 1991, 1995, and 1998 clearly correspond to the sizable, negative z-score deviations under the default rating in Table 1.

More striking are the inter-rating variations in rating changes for a particular year. In many cases, certain rating classes experience a credit deterioration while others enjoy an improvement. The unison in rating quality drifts is clear and strong only for the years which represent business cycle troughs (e.g., 1982, 1990, 1991) and peaks (e.g., 1996, 1997). This observation offers another way of understanding the relatively smaller average correlation estimated from the system: the average correlation is higher only when all ratings' credit quality changes are in the same direction. For most years, different rating categories experience different rating shifts, and hence the overall average correlation is low over the whole sample period. Nevertheless, the lower correlation itself is not necessarily a bad thing. In fact, it indicates that, unless the business/credit cycle is close to its peak or trough, rating specific shifts in credit quality dominate the overall change. This feature can only be accommodated by a multi-factor model such as the one considered in the current paper. As shown below, the improvement in fitting from a univariate model to a multivariate model is tremendous.

In order to assess the performance of the proposed multi-factor model, a measure of goodness of fit need to be developed. Since there is no standard goodness of fit measure for the estimation procedure here, I will develop two sensible measures here. The first measure gauges the average percentage deviation. Specifically, for each year, a statistic is calculated as one minus the L^1-norm of the matrix $(P^{\mathrm{O}}-P^{\mathrm{F}})$ divided by 7, where P^{O} and P^{F} are the observed and the fitted transition matrices, respectively. Essentially, this statistic is the (weighted) average absolute percentage deviation between the observed probabilities and the fitted probabilities. To see this, notice that for a given entry in row, i, $|P_{ij}^{\mathrm{O}}-P_{ij}^{\mathrm{F}}|/P_{ij}^{\mathrm{O}}$ represents the absolute percentage deviation. Since the row sum of a transition matrix is one, for a particular row, it is natural to use the observed probabilities as weights to calculate the row average of percentage absolute deviations. (Without weighting, small probability entries will tend to distort the true goodness of fit.) This leads to a row average of $\Sigma_{j=1}^{K}|P_{ij}^{\mathrm{O}}-P_{ij}^{\mathrm{F}}|$. Since the L^1-norm of $(P^{\mathrm{O}}-P^{\mathrm{F}})$ is simply the sum of the absolute values of its entries, and since there are seven rows, it follows that the L^1-norm divided by 7 is the average, absolute percentage deviation. One minus this quantity represents goodness of fit.

The second measure is similar to an R-square for a regression. Specifically, I calculate the following statistic,

$$\frac{\left(\sum_{i,j,t}(P_{ij,t}^{\mathrm{O}}-P_{ij}^{\mathrm{avg}})(P_{ij,t}^{\mathrm{F}}-P_{ij}^{\mathrm{avg}})\right)^2}{\sum_{i,j,t}(P_{ij,t}^{\mathrm{O}}-P_{ij}^{\mathrm{avg}})^2\sum_{i,j,t}(P_{ij,t}^{\mathrm{F}}-P_{ij}^{\mathrm{avg}})^2}$$

where $P^{O}_{ij,t}$ and $P^{F}_{ij,t}$ are defined as before, except for the time index, t. P^{avg}_{ij} represents a similar entry for the average transition matrix. Given that the mean of both $P^{O}_{ij,t}-P^{avg}_{ij}$ and $P^{F}_{ij,t}-P^{avg}_{ij}$ is zero by the nature of transition matrices, the above statistic is indeed the standard definition of R-square for a linear regression. Since the estimation procedure is slightly different from linear regressions as discussed in Footnote 5, I will call the above statistic a quasi R-square.

The last two columns of Table 1 contain the goodness of fit measures. Comparing the multi-variate model with a univariate model, although the improvement in average percentage deviations is marginal, the quasi R-square improves substantially, from 0.051 to 0.446, an almost 10-fold increase. Consistent with the magnitude of α, the large improvement in the quasi R-square indicates that, it is essential to allow inter-rating variations when modeling rating migrations. Incidentally, for the multi-factor model, the smallest entry for the first statistic is 0.835 for the year 1981, which indicates an average percentage deviation of 16.5%. The average across the 18 years is 0.911, which indicates an average deviation of 8.9%. For a fitting procedure, this is an encouraging result.

Finally, once the parameter α and the z-score shifts are estimated, a fitted transition matrix for each year can then be calculated easily. However, those fitted matrices are only useful for such purposes as estimating the risk premiums. For brevity, I only report, in Table 2, the fitted matrix for 1998, together with the actual transition matrix for the year and the average annual transition matrix for the whole sample period.

5.2.2. Risk premiums

In order to estimate the risk premiums via Eq. (9), one needs to assume a recovery rate and also calculate the bond prices. As shown by Jarrow et al. (1997) and others, the recovery rate depends on the seniority of the debt and tends to change over time. One can easily make the recovery rate in Eq. (9) time- and rating-dependent. However, for illustration purposes, I simply assume a constant recovery rate of 0.4, which is the average recovery rate for the period 1974–1991 across all ratings (see Moody's Special Report, Lucas and Lonski, 1992). Moreover, throughout the estimations, bond prices are calculated by simple discounting:

$$v_i(0,t) = \frac{1}{(1 + r_i)^t} \tag{12}$$

where r_i represents the yield for rating class i. The default probabilities are taken from the fitted transition matrices. Using fitted transition matrices for 1996, 1997 and 1998, and the one-year-maturity bond yields reported in the appendix, the risk premiums are estimated via minimizing the sum of squared deviations between bond prices based on Eqs. (9) and (12). They are reported below.

AAA	AA	A	BBB	BB	B	CCC
0.9959	0.9953	0.9941	0.9932	0.9856	1.001	1.121

Table 2
Fitted transition matrix for 1998

Average annual transition matrix, 1981–1998

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	91.93	7.46	0.48	0.08	0.04	0.00	0.00	0.00
AA	0.64	91.82	6.77	0.62	0.08	0.06	0.01	0.00
A	0.07	2.27	91.65	5.12	0.56	0.25	0.03	0.04
BBB	0.04	0.27	5.56	87.89	4.83	1.02	0.17	0.22
BB	0.04	0.10	0.61	7.76	81.55	7.90	1.11	0.92
B	0.00	0.10	0.43	0.81	7.00	82.86	3.99	4.82
CCC	0	0.04	0.28	0.47	2.57	12.69	63.56	20.39
Observed annual transition matrix for 1998								
AAA	93.12	6.56	0.31	0.00	0.00	0.00	0.00	0.00
AA	0.19	93.66	5.97	0.19	0.00	0.00	0.00	0.00
A	0.18	1.55	92.65	5.43	0.19	0.00	0.00	0.00
BBB	0.1	0.24	2.99	90.61	4.78	0.72	0.24	0.34
BB	0	0.00	0.24	5.93	83.56	6.59	3.03	0.65
B	0	0.00	0.16	1.24	6.67	81.95	5.51	4.47
CCC	0	0.00	0.00	0.67	1.24	23.60	37.82	36.67
Fitted annual transition matrix for 1998								
AAA	93.12	6.40	0.38	0.06	0.03	0.00	0.00	0.00
AA	0.84	92.94	5.64	0.47	0.06	0.04	0.01	0.00
A	0.08	2.33	91.72	5.02	0.55	0.24	0.03	0.04
BBB	0.03	0.21	4.81	87.75	5.49	1.21	0.21	0.28
BB	0.03	0.07	0.44	6.26	81.01	9.49	1.44	1.26
B	0.00	0.09	0.39	0.75	6.61	82.81	4.19	5.16
CCC	0.00	0.02	0.17	0.31	1.82	10.10	62.31	25.26

If the risk premiums were to be plotted against the ratings, a skewed, U-shaped curve would emerge, with the trough corresponding to rating BB. Interestingly, this is very similar to the results reported by Kijima and Komoribayashi (1998) who, using a different set of data, estimate the time-varying risk premiums for a specific point in time: May 16, 1997. The fact that most of the risk premiums are close to one implies that the entries for non-default ratings of a transition matrix do not change very much when the change of measure is performed. In contrast, as shown by Kijima and Komoribayashi (1998), the Jarrow–Lando–Turnbull method of changing measures can cause the probability entries to change significantly.

Eq. (8) reveals that when the risk premium is exactly unity, the default probability will remain unchanged when the change of measure is performed, i.e., $q_{iK} = p_{iK}\ \forall i$. A risk premium smaller than 1.0 means $q_{iK} > p_{iK}$, and vice versa. For higher ratings such as AAA and AA, the historical default rate is almost zero, but the observed bond prices almost always imply a non-zero default probability (in the risk-neutral world). In this case, it can be seen from Eq. (9) that, the combination of a lower p_{iK} and a bigger credit spread (or, equivalently, a smaller value of $v_i(0,1)-\delta v_0(0,1) > 0$) would lead to a smaller estimate of the risk premium. In other words, the smaller risk premium estimate compensates for the bigger discrepancy between default rates under the physical world and the risk-neutral world. The opposite holds for risk premiums larger than 1.0. The implication is that, ideally, the sample period of the bond prices should match that of the historical transition matrices to obtain more reliable estimates of risk premiums.

5.2.3. Implied transition matrices for future periods

With the risk premiums estimated above, transition matrices under the risk-neutral measure for any future year can be easily implied from the current prices of zero-coupon bonds. However, the estimation procedure is recursive for transition matrices beyond the current year. Assuming that the average bond yields tabulated in Appendix C are for zero-coupon bonds, a straightforward application of the procedures outlined in Section 3 gives us the implied matrices. Specifically, the probabilities in the default column can be computed via Eq. (5) using treasury and corporate bond yields by assuming a recovery rate of 0.4. The probabilities in other columns can be calculated based on the default column entries, the risk premiums and the average annual transition matrix. For brevity, I report in Table 3 only the cumulative transition matrices under the risk-neutral measure for 1, 2, 3, 4 and 5 years into the future. These matrices can then be used to value credit derivatives.[6]

It is seen that, for a particular rating, transitions to other ratings, especially the default state, tend to increase over time. In fact, in a Markov transition framework, all ratings eventually converge to the absorbing state, which is the default state. It is also interesting to observe the “mean-reverting” effect in rating changes: ratings A and BBB seem to be the “pulling” states toward which all other non-default states tend to move. In other words, over time, higher ratings tend to drift downward and lower ratings upward. This same effect has been observed by other authors such as Altman and Kao (1992b), and Carty and Fons (1994).

6. General discussions and caveats

Before concluding in the next section, some general discussions are in order. First, although interest rate risk is not explicitly modeled, the framework in this paper does not require the absence of interest rate risk. In fact, as in the case of Jarrow et al. (1997), as long as the interest rate process and the underlying Markov process are independent under the equivalent martingale measure, the model applies. However,

[6] See Kijima and Komoribayashi (1998) and Lando (2000) for examples of valuing credit derivatives using risk-neutral transition matrices.

Table 3
Cumulative transition matrices under the equivalent martingale measure

One-year transition matrix

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	88.06	10.46	0.82	0.15	0.08	0.00	0.00	0.43
AA	0.07	76.61	18.63	3.00	0.51	0.48	0.10	0.60
A	0.00	0.32	78.59	15.45	2.74	1.62	0.26	1.03
BBB	0.01	0.05	1.71	81.75	10.97	3.17	0.64	1.71
BB	0.01	0.03	0.23	3.96	77.18	12.76	2.21	3.62
B	0.00	0.10	0.44	0.84	7.17	82.94	3.91	4.59
CCC	0	0.09	0.58	0.88	4.37	18.45	70.98	4.65
Two-year transition matrix								
AAA	54.19	33.07	8.01	2.13	1.11	0.17	0.04	1.30
AA	0.06	48.42	34.78	10.40	2.33	1.95	0.42	1.64
A	0.01	0.37	58.93	26.80	6.69	3.98	0.72	2.50
BBB	0.01	0.08	2.62	67.04	17.96	7.00	1.43	3.87
BB	0.01	0.06	0.44	6.09	60.17	21.56	4.06	7.61
B	0.00	0.13	0.67	1.59	10.63	70.23	6.78	9.96
CCC	0	0.12	0.86	1.60	7.15	27.65	51.64	10.99
Three-year transition matrix								
AAA	28.99	38.17	19.52	6.76	2.75	1.07	0.24	2.51
AA	0.04	28.53	40.02	18.55	5.05	3.89	0.85	3.07
A	0.01	0.35	43.83	33.21	10.50	6.49	1.28	4.34
BBB	0.01	0.09	3.14	55.97	21.68	10.51	2.24	6.35
BB	0.01	0.07	0.64	7.47	48.12	26.40	5.42	11.86
B	0.00	0.12	0.79	2.24	12.15	60.14	8.69	15.86
CCC	0	0.11	0.95	2.17	8.65	31.22	38.42	18.47
Four-year transition matrix								
AAA	14.35	31.51	28.88	13.20	4.88	2.61	0.58	3.99
AA	0.02	15.72	38.83	25.36	8.06	5.87	1.31	4.83
A	0.01	0.30	33.06	36.14	13.63	8.60	1.81	6.44
BBB	0.01	0.11	3.50	47.95	23.50	12.96	2.88	9.08
BB	0.02	0.09	0.86	8.80	40.01	27.79	6.19	16.25
B	0.00	0.11	0.85	2.92	12.79	51.60	9.67	22.07
CCC	0.00	0.10	0.96	2.70	9.39	31.26	28.97	26.62
Five-year transition matrix								
AAA	8.99	19.32	33.45	20.01	7.19	4.39	1.00	5.65
AA	0.02	7.62	35.06	30.42	10.90	7.43	1.70	6.86
A	0.02	0.26	25.73	37.28	15.97	9.85	2.18	8.73
BBB	0.02	0.13	3.89	42.67	24.23	13.91	3.21	11.94
BB	0.03	0.11	1.19	10.86	34.37	26.46	6.28	20.71
B	0.01	0.09	0.94	3.87	13.02	43.99	9.71	28.37
CCC	0.01	0.08	0.98	3.40	9.70	29.09	21.74	35.00

some studies have shown that interest rate and credit risk are somewhat related. (See e.g., Longstaff and Schwartz, 1995; Duffee, 1998; Fridson et al., 1997; Alessandrini, 1999) Theoretical modeling of the direct relationship between interest rate risk and credit risk is scanty. General discussions and modeling can be found in Jarrow et al. (1997) and Jarrow and Turnbull (2000). Das and Tufano (1996) ingeniously tackle the problem by modeling a correlation between the interest rate and the stochastic recovery rate. Another important study that allows for a correlation between interest rate and credit risks is by Lando (1998). His model reduces the complex credit risk modeling to the usual modeling of term structure of interest rates. Although each paper takes a different angle and focus, as a future research avenue, the framework in this paper may be somehow combined with that of Das and Tufano (1996) or Lando (1998) in order to relax the independence assumption.

Second, the latent variables which drives the transition matrix are not identified in the current framework. As a next step in future research, efforts can be directed at explicitly linking the latent variables to business cycles as in Nickell et al. (2000) or macroeconomic variables as in Kim (1999). The benefit of this extension is the identification of the different impacts of business cycle variables on the rating shifts for each rating class. Relatedly, the current setup can not explicitly account for migration fluctuations over shorter periods of time. To explain the bond price changes over a shorter time period, one has to make some scale-based adjustments to the risk premiums. In contrast, Lando's (1998) continuous-time formulation based on intensities can easily incorporate migration fluctuations over any time interval.[7]

Third, it is known that the recovery rate depends on both the rating in question and the stage of the business cycle (see e.g., Moody's Special Report, Lucas and Lonski, 1992). The proposed framework can easily accommodate rating specific, time-varying recovery rates. For estimations, rating specific, realized historical recovery rates can be used; for implying future transition matrices, some type of forecasts would be necessary. At any rate, there is no need to make fundamental modifications to the framework.

Fourth, a normal distribution is assumed for the latent credit variables, which to a large extent describes reality quite well. Nonetheless, some empirical evidence (e.g., Carty and Lieberman, 1997) suggests that credit migration exhibits memory in its behavior in that a downgrading is more likely to be followed by another downgrading, and vice versa. Such dynamics imply autoregressive behavior and would call for ARCH or GARCH type of empirical models. Alternatively, migration memory can also be modeled by assuming finer partition of credit states as done by Arvanitis et al. (1999). The current framework does not allow memories in credit migrations.

Finally, when applied to corporate debts, the estimation procedures can be easily modified to achieve the conditioning effect similar to that in Nickell et al. (2000), despite that the usual transition matrices are based on industry-aggregate. For example, to condition on an industry, one could use the industry specific bond price data (for all ratings) to estimate the risk premiums and to subsequently imply tran-

[7] I am indebted to an anonymous referee for the insights expressed in this paragraph.

sition matrices for future periods. In this case, the conditioning is achieved through the risk premium estimations. For valuation purposes, this type of modification is meaningful and sufficient. As for business cycles, unlike that of Nickell et al. (2000), the framework here does not require an explicit modeling of the business cycle. Future dynamics of business cycles are fully captured by the observed bond prices used to imply future transition matrices. The same logic applies to sovereign debts, except that the conditioning is on a country rather than an industry.

7. Conclusions

In this paper, I propose a multi-factor Markov chain model for bond rating migrations and credit spreads that is applicable to both sovereign and corporate debts. The model takes the historical average transition matrix as the starting point, and allows the actual realized matrices to deviate from this average. The deviations are driven by a set of latent, credit cycle variables which are assumed to be normally distributed. In contrast to most existing models, the current model allows the transition probabilities to be business cycle dependent. The paper discusses the model's applications to both sovereign and corporate debts. Specifically, using historical transition matrices and bond prices for corporate debts rated by Standard and Poor's, the paper shows how to estimate the risk premiums required to convert transition matrices from the physical measure to the risk-neutral measure which can be used to value credit derivatives.

The estimation results indicate that the overall, average correlation between ratings in credit quality changes is weak. It is only the business cycle trough and peak years that saw a clear correlation in that all ratings tend to deteriorate or improve at the same time. For other years, inter-rating variations in credit quality changes are frequently present. This implies that, although incorporating the business cycle impact is important in rating migration modeling, it is crucial to allow inter-rating variations, which can only be achieved by a multi-variate model such as the one considered in this paper. It is shown that the quasi *R*-square, as a measurement of goodness of fit, improves by almost 10-fold when a univariate model is replaced by a multi-variate model.

There are several advantages of the multi-factor Markov chain model. First, it allows the rating transition probabilities to be time-varying and driven by business or economic cycles. This is desirable because the time-varying nature of transition matrices and default rates has been documented by many studies (e.g., Moody's Special Report, Lucas and Lonski, 1992; Helwege and Kleiman, 1997; Belkin et al., 1998; Standard and Poor's Special Report, 1998; Alessandrini, 1999; Nickell et al., 2000). And it is indeed intuitive to think that the ability of a firm or a sovereign nation to service their debts would depend on, among other things, the state of the economy.

Second, the model allows for different ratings to react differently to the same credit condition change. For instance, an economic downturn will increase the chance for most bonds to be downgraded. But conceivably, lower-rated bonds will be more susceptible to the overall credit deterioration. Meantime, the model also allows for the rating shifts to be cross-sectionally correlated, which is again a desirable feature.

Third, unlike most studies on credit risk or credit spreads, the framework in this paper weaves together credit risk modeling and credit derivatives valuation. It shows how the framework can be implemented for valuation purposes. This is why it is also a credit spread model. The increasing attention to debts' rating status and the growing popularity of credit derivatives in risk management have made a model like the current one ideal. The current model makes it possible to value credit derivative securities in a uniform framework where ratings and yields are jointly utilized.

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Appendix A

Table A1.

Table A1
Sovereign foreign currency average one-year transition rates (1975–2000)

	AAA	AA	A	BBB	BB	B	CCC	SD
AAA	97.32	2.68	0.00	0.00	0.00	0.00	0.00	0.00
AA	1.84	96.33	0.61	0.00	0.61	0.61	0.00	0.00
A	0.00	3.09	94.85	2.06	0.00	0.00	0.00	0.00
BBB	0.00	0.00	4.67	90.65	3.74	0.92	0.00	0.00
BB	0.00	0.00	0.00	7.02	83.33	7.02	0.00	2.63
B	0.00	0.00	0.00	0.00	13.33	80.00	3.33	3.34
CCC	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

Source: Standard and Poor's, "Rating Performance 1999", February 2000. *Note*: "SD" stands for selected default. Unlike corporate issuers, sovereigns can choose to default on only part of the total debts in foreign currency.

Appendix B

Table B1.

Table B1
The average annual transition matrix for the period of 1981–1998

Before adjustments									
	AAA	AA	A	BBB	BB	B	CCC	D	N.R.
AAA	89.48	7.26	0.47	0.08	0.04	0.00	0.00	0.00	2.67
AA	0.62	88.99	6.55	0.58	0.06	0.11	0.03	0.00	3.06
A	0.07	2.18	87.95	4.91	0.54	0.24	0.01	0.04	4.06
BBB	0.04	0.25	5.23	82.66	4.54	0.96	0.16	0.22	5.94
BB	0.04	0.09	0.55	7.04	73.98	7.17	1.01	0.92	9.20
B	0.00	0.09	0.25	0.41	6.14	73.15	3.50	4.82	11.64
CCC	0.16	0.00	0.32	0.64	2.09	10.43	52.01	20.39	13.96
After adjustments									
AAA	91.93	7.46	0.48	0.08	0.04	0.00	0.00	0.00	
AA	0.64	91.82	6.77	0.62	0.08	0.06	0.01	0.00	
A	0.07	2.27	91.65	5.12	0.56	0.25	0.03	0.04	
BBB	0.04	0.27	5.56	87.89	4.83	1.02	0.17	0.22	
BB	0.04	0.10	0.61	7.76	81.55	7.90	1.11	0.92	
B	0.00	0.10	0.43	0.81	7.00	82.86	3.99	4.82	
CCC	0.00	0.04	0.28	0.47	2.57	12.69	63.56	20.39	

Appendix C. Bond yield data

Tables C1 and C2 and Fig. A1.

Table C1
Beginning of the year bond yields for one-year maturity

Raw data								
	Treasury	AAA	AA	A	BBB	BB+	BB/BB−	B
1996	5.02	5.32	5.34	5.41	5.63	6.05	7.13	8.29
1997	5.37	5.57	5.67	5.74	5.95	6.19	7.29	7.69
1998	5.50	5.77	5.78	5.91	6.00	6.56	7.22	7.64
After combining BB sub-ratings and extrapolating for CCC								
	Treasury	AAA	AA	A	BBB	BB	B	CCC
1996	5.02	5.32	5.34	5.41	5.63	6.59	8.29	10.73
1997	5.37	5.57	5.67	5.74	5.95	6.74	7.69	8.80
1998	5.50	5.77	5.78	5.91	6.00	6.89	7.64	8.25

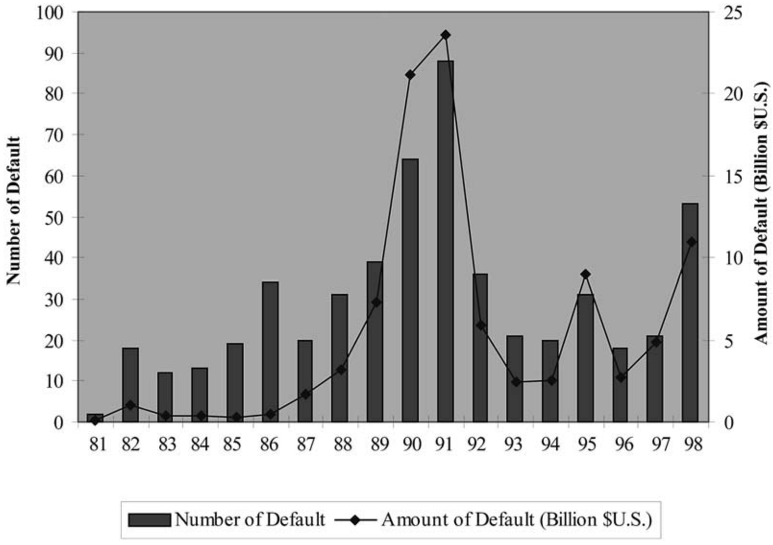

Fig. A1 Bond yields vs. rating classes.

Table C2
Bond yields at the beginning of June 1999

Raw data

Maturity (years)	Treasury	AAA	AA	A	BBB	BB+	BB/BB−	B
1	4.87	5.14	5.25	5.52	5.96	6.82	7.58	7.84
5	5.44	6.17	6.33	6.58	7.02	8.00	8.55	9.45
10	5.49	6.41	6.80	6.86	7.27	8.31	8.76	9.94

After combining BB sub-ratings and performing extrapolating for CCC and interpolation for interim maturities

Maturity (years)	Treasury	AAA	AA	A	BBB	BB	B	CCC
1	4.87	5.14	5.25	5.52	5.96	7.20	7.84	7.88
2	5.06	5.47	5.58	5.85	6.30	7.54	8.34	8.70
3	5.21	5.75	5.87	6.14	6.59	7.83	8.78	9.42
4	5.34	5.98	6.12	6.38	6.83	8.08	9.15	10.03
5	5.44	6.17	6.33	6.58	7.02	8.27	9.45	10.53
6	5.51	6.31	6.50	6.73	7.17	8.42	9.68	10.93
7	5.55	6.41	6.64	6.83	7.26	8.52	9.85	11.22
8	5.56	6.45	6.73	6.89	7.31	8.58	9.95	11.41
9	5.54	6.46	6.78	6.90	7.32	8.58	9.98	11.49
10	5.49	6.41	6.80	6.86	7.27	8.53	9.94	11.47

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