Properties of Ho–Lee and Hull–White Interest Rate Models

This note presents some of the math underlying the Ho–Lee and Hull–White one-factor models of the term structure. It follows the approach in Hull and White (1993).\(^1\)

In a one-factor term structure, model the process for a zero-coupon bond price in the traditional risk-neutral world must have a return equal to the short rate \(r\). Suppose that \(v(t, T)\) is the volatility. Then:

\[
dP(t, T) = rP(t, T)dt + v(t, T)P(t, T)dz
\]  

(1)

In this note, we will assume that \(v(t, T)\) is a function only of \(t\) and \(T\). Because the bond’s price volatility declines to zero at maturity \(v(t, t) = 0\).

From Ito’s lemma, for any times \(T_1\) and \(T_2\) with \(T_2 > T_1\):

\[
d \ln P(t, T_1) = \left[r - \frac{v(t, T_1)^2}{2}\right] dt + v(t, T_1)dz(t)
\]  

(2)

\[
d \ln P(t, T_2) = \left[r - \frac{v(t, T_2)^2}{2}\right] dt + v(t, T_2)dz(t)
\]  

(3)

Define \(f(t, T_1, T_2)\) as the forward rate for the period between time \(T_1\) and \(T_2\) as seen at time \(t\):

\[
f(t, T_1, T_2) = -\frac{\ln P(t, T_2) - \ln P(t, T_1)}{T_2 - T_1}
\]

From equations (2) and (3):

\[
df(t, T_1, T_2) = \left[\frac{v(t, T_2)^2 - v(t, T_1)^2}{2(T_2 - T_1)}\right] dt - \left[\frac{v(t, T_2) - v(t, T_1)}{T_2 - T_1}\right] dz(t)
\]

Define \(R(t, T)\) as the zero rate for the period between \(t\) and \(T\):

\[
R(t, T) = f(t, T) + \int_0^T df(\tau, t, T)
\]

so that:

\[
R(t, T) = f(0, t, T) + \int_0^t \left[\frac{v(\tau, T)^2 - v(\tau, t)^2}{2(T - t)}\right] d\tau - \int_0^t \left[\frac{v(\tau, T) - v(\tau, t)}{T - t}\right] dz(\tau)
\]  

(4)

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\(^1\) See J. Hull and A. White, “Bond Option Pricing Based on a Model for the Evolution of Bond Prices,” Advances in Futures and Options Research, 6 (1993), 1–13.
As $T$ approaches $t$, $R(t, T)$ becomes $r(t)$ and $f(0, t, T)$ becomes the instantaneous forward rate, $F(0, t)$ so that

$$r(t) = F(0, t) + \int_0^t \frac{\partial}{\partial t} v(\tau, t)^2 d\tau - \int_0^t \frac{\partial}{\partial t} v(\tau, t) d\tau$$

or

$$r(t) = F(0, t) + \int_0^t v(\tau, t) v_t(\tau, t) d\tau - \int_0^t v_t(\tau, t) dz(\tau)$$

(5)

where subscripts denote partial derivatives. To calculate the process for $r$ we differentiate with respect to $t$. Because $v(t, t) = 0$, this gives

$$dr = \left\{ F_t(0, t) + \int_0^t [v(\tau, t)v_{tt}(\tau, t) + v_t(\tau, t)^2]d\tau - \int_0^t v_{tt}(\tau, t) dz(\tau) \right\} dt - v_t(\tau, t)|_{\tau=t} dz(t)$$

(6)

**Case 1: Ho–Lee; $v(t, T) = \sigma(T - t)$**

In the case, equation (5) gives

$$r(t) = F(0, t) + \sigma^2 t^2/2 - \int_0^t \sigma dz(\tau)$$

(7)

and equation (6) gives

$$dr(t) = [F_t(0, t) + \sigma^2 t] dt + \sigma dz$$

This is the Ho-Lee model

$$dr = \theta(t) dt + \sigma dz$$

We have proved the equation for $\theta(t)$

$$\theta(t) = F_t(0, t) + \sigma^2 t$$

Also from equation (4)

$$R(t, T) = f(0, t, T) + \sigma^2 t T/2 - \int_0^t \sigma dz(\tau)$$

(8)

From equations (7) and (8)

$$R(t, T) = f(0, t, T) + \sigma^2 t T/2 + r(t) - F(0, t) - \sigma^2 t^2/2 = f(0, t, T) - F(0, t) + \sigma^2 t(T - t)/2 + r(t)$$

Because

$$\ln P(t, T) = -R(t, T)(T - t)$$

It follows that

$$\ln P(t, T) = -f(0, t, T)(T - t) + F(0, t)(T - t) - \sigma^2 t(T - t)^2/2 - r(t)(T - t)$$
The forward bond price \( P(0, T)/P(0, t) \) equals \( e^{-f(0,t,T)(T-t)} \) so that this becomes

\[
\ln P(t, T) = \ln \frac{P(0, T)}{P(0, t)} + F(0, t)(T - t) - \sigma^2 t(T - t)^2/2 - r(t)(T - t)
\]

This proves:

\[
P(t, T) = A(t, T)e^{-r(t)(T-t)}
\]

where

\[
\ln A(t, T) = \ln \frac{P(0, T)}{P(0, t)} + F(0, t)(T - t) - \frac{1}{2}\sigma^2 t(T - t)^2
\]

**Case 2: Hull–White;** \( v(t, T) = \sigma(1 - e^{-a(T-t)})/a \)

In this case, equation (5) gives

\[
r(t) = F(0, t) + \frac{\sigma^2}{a^2}(1 - e^{-at}) - \frac{\sigma^2}{2a^2}(1 - e^{-2at}) - \int_0^t \sigma e^{-a(t-\tau)}d\tau
\]

Equation (6) gives

\[
dr(t) = \left\{ F_t(0, t) + \frac{\sigma^2}{a}(e^{-at} - e^{-2at}) + \int_0^t \sigma ae^{-a(t-\tau)}d\tau \right\} dt - \sigma dz(t)
\]

Substituting for

\[
\int_0^t \sigma e^{-a(t-\tau)}d\tau
\]

from equation (9) into equation (10) we obtain

\[
dr(t) = \left\{ F_t(0, t) + \frac{\sigma^2}{a}(e^{-at} - e^{-2at}) - ar(t) + aF(0, t) + \frac{\sigma^2}{a}(1 - e^{-at}) - \frac{\sigma^2}{2a}(1 - e^{-2at}) \right\} dt - \sigma dz(t)
\]

or

\[
dr(t) = \left\{ F_t(0, t) + aF(0, t) - ar(t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \right\} dt - \sigma dz(t)
\]

This is the Hull–White model

\[
dr(t) = (\theta(t) - ar) dt + \sigma dz
\]

with

\[
\theta(t) = F_t(0, t) + aF(0, t) + \frac{\sigma^2}{2a}(1 - e^{-2at})
\]

From equation (4)

\[
R(t, T) = f(0, t, T) + \frac{\sigma^2[e^{-2a(T-t)} - e^{-2aT} - 1 + e^{-2at} - 4e^{-a(T-t)} + 4e^{-aT} + 4 - 4e^{-at}]}{4a^3(T - t)}
\]
\[ + \frac{\sigma(e^{-at} - e^{-at})}{a(T-t)} \int_0^t e^{a\tau} d\tau \] (11)

From equation (9)

\[
\sigma \int_0^t e^{a\tau} d\tau = -r(t)e^{at} + F(0, t)e^{at} + \frac{\sigma^2}{a^2}(e^{at} - 1) - \frac{\sigma^2}{2a^2}(e^{at} - e^{-at})
\]

so that

\[
R(t, T) = f(0, t, T) + \frac{\sigma^2[e^{-2a(T-t)} - e^{-2at} - 1 + e^{-2at} - 4e^{-a(T-t)} + 4e^{-aT} + 4 - 4e^{-at}]}{4a^3(T-t)}
\]

\[
+ \frac{(e^{-at} - e^{-at})}{a(T-t)} \left[ -r(t)e^{at} + F(0, t)e^{at} + \frac{\sigma^2}{a^2}(e^{at} - 1) - \frac{\sigma^2}{2a^2}(e^{at} - e^{-at}) \right]
\]

Now

\[
\ln P(t, T) = -R(t, T)(T-t)
\]

and the forward bond price \( P(0, T)/P(0, t) \) equals \( e^{-f(0,t,T)(T-t)} \). After some tedious algebra we get

\[
\ln P(t, T) = \ln \frac{P(0, T)}{P(0, t)} + F(0, t)B(t, T) - \frac{1}{4a^3} \sigma^2(e^{-aT} - e^{-at})^2(e^{2at} - 1) - r(t)B(t, T)
\]

where

\[
B(t, T) = \frac{1 - e^{-a(T-t)}}{a}
\]

showing that

\[
P(t, T) = A(t, T)e^{-B(t, T)r}
\]

where

\[
\ln A(t, T) = \ln \frac{P(0, T)}{P(0, t)} + F(0, t)B(t, T) - \frac{1}{4a^3} \sigma^2(e^{-aT} - e^{-at})^2(e^{2at} - 1)
\]
Bond Options

Consider a European option with strike price $K$ and maturity $T$ on a zero-coupon bond where the maturity of the bond is $s$. The forward price of the bond underlying the option as seen at time $t$, $F_B(t, T, s)$, is

$$F_B(t, T, s) = \frac{P(t, s)}{P(t, T)}$$

Using the results in equations (2) and (3) we get

$$d\ln F_B(t, T, s) = \frac{v(t, T)^2 - v(t, s)^2}{2} dt + [v(t, s) - v(t, T)] dz$$

This shows that the $P(T, s) = f_B(T, T, s)$ is lognormal when $v(t, T)$ is function only of $t$ and $T$. The variance $\ln P(T, s)$ is then

$$\sigma_P^2 = \int_0^T [v(t, s) - v(t, T)]^2 dt$$

In the case of Ho-Lee $v(t, T) = \sigma(T - t)$ and $\sigma_P^2 = \sigma^2(s - T)^2T$. In Hull-White $v(t, T) = \sigma B(t, T)$ so that

$$\sigma_P^2 = \sigma^2 \int_0^T [B(t, s) - B(t, T)]^2 dt = \frac{\sigma^2}{2a^3}[1 - e^{-a(s-T)}]^2(1 - e^{-2aT})$$

In both cases bond options can be valued using Black’s model. The average variance rate of the forward bond price is $\sigma_P^2/T$. This leads to the results for bond options in the text.