Technical Note No. 30*
Options, Futures, and Other Derivatives
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The Return for a Security Dependent on Multiple sources of Uncertainty

In this Note we prove a result relating the excess return to market prices of risk when there are multiple sources of uncertainty.

Suppose there are n stochastic variables following Wiener processes. Consider $n+1$ traded securities whose prices depend on some or all of the n stochastic variables. Define f_j as the price of the jth security $(1 \leq j \leq n+1)$. We assume that no dividends or other income is paid by the $n+1$ traded securities.[1] It follows from Ito's lemma in Technical Note 29 that the securities follow processes of the form

$$df_j = \mu_j f_j \, dt + \sum_{i=1}^{n} \sigma_{ij} f_j dz_i \tag{1}$$

Since there are $n+1$ traded securities and n Wiener processes, it is possible to form an instantaneously riskless portfolio, Π, using the securities. Define k_j as the amount of the jth security in the portfolio, so that

$$\Pi = \sum_{j=1}^{n+1} k_j f_j \tag{2}$$

The k_j must be chosen so that the stochastic components of the returns from the securities are eliminated. From equation (1) this means that

$$\sum_{j=1}^{n+1} k_j \sigma_{ij} f_j = 0 \tag{3}$$

for $1 \leq i \leq n$. The return from the portfolio is then given by

$$d\Pi = \sum_{j=1}^{n+1} k_j \mu_j f_j \, dt$$

The cost of setting up the portfolio is

$$\sum_{j=1}^{n+1} k_j f_j$$

[1] This is not restrictive. A non-dividend-paying security can always be obtained from a dividend-paying security by reinvesting the dividends in the security.

If there are no arbitrage opportunities, the portfolio must earn the risk-free interest rate, so that

$$\sum_{j=1}^{n+1} k_j \mu_j f_j = r \sum_{j=1}^{n+1} k_j f_j \tag{4}$$

or

$$\sum_{j=1}^{n+1} k_j f_j (\mu_j - r) = 0 \tag{5}$$

Equations (3) and (5) can be regarded as $n+1$ homogeneous linear equations in the k_j's. The k_j's are not all zero. From a well-known theorem in linear algebra, equations (3) and (5) can be consistent only if for all j

$$f_j(\mu_j - r) = \sum_{i=1}^{n} \lambda_i \sigma_{ij} f_j \tag{6}$$

or

$$\mu_j - r = \sum_{i=1}^{n} \lambda_i \sigma_{ij} \tag{7}$$

for some λ_i $(1 \leq i \leq n)$ that are dependent only on the state variables and time. Dropping the j subscript, this show that for any security f dependent on the n stochastic variables

$$df = \mu f\, dt + \sum_{i=1}^{n} \sigma_i f dz_i$$

where

$$\mu - r = \sum_{i=1}^{n} \lambda_i \sigma_i$$

This proves the required result.