Technical Note No. 22*
Options, Futures, and Other Derivatives
John Hull

Valuation of a Variance Swap

This note proves the result in the text for the valuation of a variance swap.

Suppose that the stock price follows process

$$\frac{dS}{S} = (r - q)\,dt + \sigma\,dz$$

in a risk-neutral world where σ is itself stochastic. From Ito's lemma

$$d\ln S = (r - q - \sigma^2/2)\,dt + \sigma\,dz$$

By subtracting these two equations we obtain

$$\frac{\sigma^2}{2}dt = \frac{dS}{S} - d\ln S$$

Integrating between time 0 and time T, the realized average variance rate, $\overline{V}$, between time 0 and time T is given by

$$\frac{1}{2}\overline{V}T = \int_0^T \frac{dS}{S} - \ln\frac{S_T}{S_0}$$

or

$$\overline{V} = \frac{2}{T}\int_0^T \frac{dS}{S} - \frac{2}{T}\ln\frac{S_T}{S_0} \tag{1}$$

Taking expectations in a risk-neutral world

$$\hat{E}(\overline{V}) = \frac{2}{T}(r-q)T - \frac{2}{T}\hat{E}\left(\ln\frac{S_T}{S_0}\right)$$

or

$$\hat{E}(\overline{V}) = \frac{2}{T}\ln\frac{F_0}{S_0} - \frac{2}{T}\hat{E}\left(\ln\frac{S_T}{S_0}\right) \tag{2}$$

where F_0 is the forward price of the asset for a contract maturing at time T.

Consider

$$\int_{K=0}^{S^*} \frac{1}{K^2}\max(K - S_T,\, 0)dK$$

for some value S^* of S. When $S^* < S_T$ this integral is zero. When $S^* > S_T$ it is

$$\int_{K=S_T}^{S^*} \frac{1}{K^2}(K - S_T)dK = \ln\frac{S^*}{S_T} + \frac{S_T}{S^*} - 1$$

Consider next

$$\int_{K=S^*}^{\infty} \frac{1}{K^2} \max(S_T - K,\, 0) dK$$

When $S^* > S_T$ this is zero. When $S^* < S_T$ it is

$$\int_{K=S^*}^{S_T} \frac{1}{K^2}(S_T - K) dK = \ln \frac{S^*}{S_T} + \frac{S_T}{S^*} - 1$$

From these results it follows that

$$\int_{K=0}^{S^*} \frac{1}{K^2} \max(K - S_T,\, 0) dK + \int_{K=S^*}^{\infty} \frac{1}{K^2} \max(S_T - K,\, 0) dK = \ln \frac{S^*}{S_T} + \frac{S_T}{S^*} - 1$$

for all values of S^* so that

$$\ln \frac{S_T}{S^*} = \frac{S_T}{S^*} - 1 - \int_{K=0}^{S^*} \frac{1}{K^2} \max(K - S_T,\, 0) dK - \int_{K=S^*}^{\infty} \frac{1}{K^2} \max(S_T - K,\, 0) dK \quad (3)$$

This shows that a variable that pays off $\ln S_T$ can be replicated using options. This result can be used in conjunction with equation (1) to provide a replicating portfolio for $\overline{V}$. Taking expectations in a risk-neutral world in equation (3)

$$\hat{E}\left(\ln \frac{S_T}{S^*}\right) = \frac{F_0}{S^*} - 1 - \int_{K=0}^{S^*} \frac{1}{K^2} e^{RT} p(K) dK - \int_{K=S^*}^{\infty} \frac{1}{K^2} e^{RT} c(K) dK \quad (4)$$

where $c(K)$ and $p(K)$ are the prices of European call and put options with strike price K and maturity T and R is the risk-free interest rate for a maturity of T.

Combining equations (2) and (4) and noting that

$$\hat{E}\left(\ln \frac{S_T}{S_0}\right) = \ln \frac{S^*}{S_0} + \hat{E}\left(\ln \frac{S_T}{S^*}\right)$$

$$\hat{E}(\overline{V}) = \frac{2}{T} \ln \frac{F_0}{S_0} - \frac{2}{T} \ln \frac{S^*}{S_0}$$

$$-\frac{2}{T}\left[\frac{F_0}{S^*} - 1\right] + \frac{2}{T}\left[\int_{K=0}^{S^*} \frac{1}{K^2} e^{RT} p(K) dK + \int_{K=S^*}^{\infty} \frac{1}{K^2} e^{RT} c(K) dK\right]$$

which reduces to

$$\hat{E}(\overline{V}) = \frac{2}{T} \ln \frac{F_0}{S^*} - \frac{2}{T}\left[\frac{F_0}{S^*} - 1\right] + \frac{2}{T}\left[\int_{K=0}^{S^*} \frac{1}{K^2} e^{RT} p(K) dK + \int_{K=S^*}^{\infty} \frac{1}{K^2} e^{RT} c(K) dK\right]$$

This is the result in the text.