Technical Note No. 14*
Options, Futures, and Other Derivatives
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The Hull–White Two Factor Model

As explained in the text, Hull and White have proposed a model where the risk-neutral process for the short rate, r, is

$$df(r) = [\theta(t) + u - af(r)]\,dt + \sigma_1\,dz_1 \tag{1}$$

where u has an initial value of zero and follows the process

$$du = -bu\,dt + \sigma_2\,dz_2$$

As in the case of one-factor no-arbitrage models, the parameter $\theta(t)$ is chosen to make the model consistent with the initial term structure. The stochastic variable u is a component of the reversion level of r and itself reverts to a level of zero at rate b. The parameters a, b, σ_1, and σ_2 are constants and dz_1 and dz_2 are Wiener processes with instantaneous correlation ρ.

This model provides a richer pattern of term structure movements and a richer pattern of volatility structures than one-factor no-arbitrage models. For example, when $f(r) = r$, $a = 1$, $b = 0.1$, $\sigma_1 = 0.01$, $\sigma_2 = 0.0165$, and $\rho = 0.6$ the model exhibits, at all times, a "humped" volatility structure similar to that observed in the market. The correlation structure implied by the model is also plausible with these parameters.

When $f(r) = r$ the model is analytically tractable. The price at time t of a zero-coupon bond that provides a payoff of \$1 at time T is

$$P(t,T) = A(t,T)\exp[-B(t,T)r - C(t,T)u]$$

where

$$B(t,T) = \frac{1}{a}[1 - e^{-a(T-t)}]$$

$$C(t,T) = \frac{1}{a(a-b)}e^{-a(T-t)} - \frac{1}{b(a-b)}e^{-b(T-t)} + \frac{1}{ab}$$

and $A(t,T)$ is as given in the Appendix to this note.

The prices, c and p, at time zero of European call and put options on a zero-coupon bond are given by

$$c = LP(0,s)N(h) - KP(0,T)N(h - \sigma_P)$$

$$p = KP(0,T)N(-h + \sigma_P) - LP(0,s)N(-h)$$

where T is the maturity of the option, s is the maturity of the bond, K is the strike price, L is the bond's principal

$$h = \frac{1}{\sigma_P}\ln\frac{LP(0,s)}{P(0,T)K} + \frac{\sigma_P}{2}$$

and σ_P is as given in the Appendix. Because this is a two-factor model, an option on a coupon-bearing bond cannot be decomposed into a portfolio of options on zero-coupon bonds as described in Technical Note 15. However, we can obtain an approximate analytic valuation by calculating the first two moments of the price of the coupon-bearing bond and assuming the price is lognormal.

Constructing a Tree

To construct a tree for the model in equation (1), we simplify the notation by defining $x = f(r)$ so that

$$dx = [\theta(t) + u - ax]\, dt + \sigma_1\, dz_1$$

with

$$du = -bu\, dt + \sigma_2\, dz_2$$

Assuming $a \neq b$ we can eliminate the dependence of the first stochastic variable on the second by defining

$$y = x + \frac{u}{b-a}$$

so that

$$dy = [\theta(t) - ay]\, dt + \sigma_3\, dz_3$$

$$du = -bu\, dt + \sigma_2\, dz_2$$

where

$$\sigma_3^2 = \sigma_1^2 + \frac{\sigma_2^2}{(b-a)^2} + \frac{2\rho\sigma_1\sigma_2}{b-a}$$

and dz_3 is a Wiener process. The correlation between dz_2 and dz_3 is

$$\frac{\rho\sigma_1 + \sigma_2/(b-a)}{\sigma_3}$$

A three-dimensional tree for y and u can be constructed on the assumption that $\theta(t) = 0$ and the initial values of y and u are zero. A methodology similar to that for one-factor models can then be used to construct the final tree by increasing the values of y at time $i\Delta t$ by α_i. In the $f(r) = r$ case, an alternative approach is to use the analytic expression for $\theta(t)$, given in the Appendix to this note.

Rebonato gives some examples of how the model can be calibrated and used in practice.[2]

[2] See R. Rebonato *Interest Rate Option Models*, (2nd Ed., Chichester, England: John Wiley and Sons, 1998) pp 306-8.

APPENDIX
The Functions in the Two-Factor Hull-White Model

The $A(t,T)$ function is

$$\ln A(t,T) = \ln \frac{P(0,T)}{P(0,t)} + B(t,T)F(0,t) - \eta$$

where

$$\eta = \frac{\sigma_1^2}{4a}(1-e^{-2at})B(t,T)^2 - \rho\sigma_1\sigma_2[B(0,t)C(0,t)B(t,T) + \gamma_4 - \gamma_2]$$

$$-\frac{1}{2}\sigma_2^2[C(0,t)^2B(t,T) + \gamma_6 - \gamma_5]$$

$$\gamma_1 = \frac{e^{-(a+b)T}[e^{(a+b)t}-1]}{(a+b)(a-b)} - \frac{e^{-2aT}(e^{2at}-1)}{2a(a-b)}$$

$$\gamma_2 = \frac{1}{ab}\left[\gamma_1 + C(t,T) - C(0,T) + \frac{1}{2}B(t,T)^2 - \frac{1}{2}B(0,T)^2 + \frac{t}{a} - \frac{e^{-a(T-t)} - e^{-aT}}{a^2}\right]$$

$$\gamma_3 = -\frac{e^{-(a+b)t}-1}{(a-b)(a+b)} + \frac{e^{-2at}-1}{2a(a-b)}$$

$$\gamma_4 = \frac{1}{ab}\left[\gamma_3 - C(0,t) - \frac{1}{2}B(0,t)^2 + \frac{t}{a} + \frac{e^{-at}-1}{a^2}\right]$$

$$\gamma_5 = \frac{1}{b}\left[\frac{1}{2}C(t,T)^2 - \frac{1}{2}C(0,T)^2 + \gamma_2\right]$$

$$\gamma_6 = \frac{1}{b}\left[\gamma_4 - \frac{1}{2}C(0,t)^2\right]$$

where $F(t,T)$ is the instantaneous forward rate at time t for maturity T.

The volatility function, σ_P, is

$$\sigma_P^2 = \int_0^t \{\sigma_1^2[B(\tau,T) - B(\tau,t)]^2 + \sigma_2^2[C(\tau,T) - C(\tau,t)]^2$$

$$+2\rho\sigma_1\sigma_2[B(\tau,T) - B(\tau,t)][C(\tau,T) - C(\tau,t)]\}\,d\tau$$

This shows that σ_P^2 has three components. Define

$$U = \frac{1}{a(a-b)}[e^{-aT} - e^{-at}]$$

and

$$V = \frac{1}{b(a-b)}[e^{-bT} - e^{-bt}]$$

The first component of σ_P^2 is

$$\frac{\sigma_1^2}{2a} B(t,T)^2 (1 - e^{-2at})$$

The second is

$$\sigma_2^2 \left[\frac{U^2}{2a}(e^{2at} - 1) + \frac{V^2}{2b}(e^{2bt} - 1) - 2\frac{UV}{a+b}(e^{(a+b)t} - 1) \right]$$

The third is

$$\frac{2\rho\sigma_1\sigma_2}{a}(e^{-at} - e^{-aT}) \left[\frac{U}{2a}(e^{2at} - 1) - \frac{V}{a+b}(e^{(a+b)t} - 1) \right]$$

Finally, the $\theta(t)$ function is

$$\theta(t) = F_t(0,t) + aF(0,t) + \phi_t(0,t) + a\phi(0,t)$$

where the subscript denotes a partial derivative and

$$\phi(t,T) = \frac{1}{2}\sigma_1^2 B(t,T)^2 + \frac{1}{2}\sigma_2^2 C(t,T)^2 + \rho\sigma_1\sigma_2 B(t,T)C(t,T)$$