A Perspective-Invariant Approach to Nash Bargaining

Barry Nalebuff
Yale School of Management, barry.nalebuff@yale.edu

The Nash axioms lead to different results depending on whether the negotiation is framed in terms of gains relative to no agreement or in terms of sacrifices relative to an ideal. We look for a solution that leads to the same result from both perspectives. To do so, we restrict the application of Nash's IIA axiom to bargaining sets where all options are individually rational and none exceed either party's ideal point. If we normalize the bargaining set so that the disagreement point is $(0, 0)$ and maximal gains are $(1, 1)$, then any perspective-invariant bargaining solution must lie between the Utilitarian solution and the maximal equal-gain (minimal equal-sacrifice) solution. We show that a modified version of Nash's symmetry axiom leads to the Utilitarian solution and that a reciprocity axiom leads to the equal-gain (equal-sacrifice) solution, both of which are perspective invariant.

Key words: Nash Bargaining, Gains, Sacrifices, Perspective Invariance, Relative Utilitarianism

1. Introduction

Bargaining typically involves conflict. To resolve the conflict, the parties involved may look for general principles to help them reach a fair agreement. A normative interpretation of Nash's (1950) bargaining solution is that the parties in a negotiation would agree to the Nash axioms as being fair and consequently would be led by the axioms to a unique bargaining solution. A problem with this approach arises as the Nash axioms generally lead to different results depending on whether the negotiation is framed in terms of gains relative to no agreement (as in the standard Nash approach) or in terms of sacrifices, specifically how much each side is giving up relative to their ideal negotiation outcome. When the two perspectives lead to different outcomes, each side will pick the perspective that favors them.[1]

Just such a conflict arose in a negotiation the author was involved in. The negotiation partner was 2,000 times our size and as such had an ability to buy ingredients at a much lower price, so much so that this could save $5 million. The challenge was how to split this pie. As the larger party

[1] The issue of picking axioms that favors one's position arises more broadly as the Nash (1950) axioms lead to a different result than the axioms in Kalai and Smorodinsky (1975) or Perles and Maschler (1981). Van Damme (1986) provides a unifying perspective that shows how the Nash solution results from a procedure designed to resolve competing solutions. In our case, we are concerned that even without the complication of competing axioms, the same axioms lead to different solutions when the problem is framed differently.

correctly pointed out, the money involved would be life-changing for us, while it would not even merit a public disclosure for them. Thus they proposed an uneven dollar split that was meant to provide a more even utility split. Since money mattered much less to them, they sought 80% of the dollar pie. The $4 million to them and $1 million to us would lead each side to get roughly the same utility gain from the deal.

Our retort was that if money mattered so little to them, it would be easy for them to give up 80% of the cash. (Would anybody even notice?) Under this counterproposal, they would be giving up $4 million of the $5 million and we'd be giving up $1 million. At that point, each side would be making comparable utility sacrifices in terms of how far we were from our ideal.[2]

If one side cares much less about money, does that mean they should get more because it takes more to make them happy or does it mean they should get less since it is easier for them to make sacrifices? As an empirical matter, it seems that the sacrifice perspective wins out. Nydegger and Owen (1974) ran experiments on negotiation games where the participants had to divide up tokens with unequal value to the two of them (and no monetary transfers were allowed ex post). They found that participants were closer to the equal-sacrifice than to the equal-gain solution.[3]

As a theoretical matter, it is hard to justify why the parties should take the one perspective over the other. Why should equal gains take priority over equal sacrifices or vice versa? Any argument for one solution can be flipped to justify the other as seen above. There is, of course, one difference: the disagreement point or zero gains to each party is always a potential outcome if the negotiations break down. In contrast, the zero sacrifice for each or mutual ideal point is generally not feasible. But as each party evaluates a proposed deal, each side will consider what he or she has been offered *and* what he or she is being asked to give up. Issues of fairness can be equally-well framed using either perspective.

When the bargaining frontier is linear, the Nash (1950) solution gets around this duality by scaling the utilities so that both sides care the same amount: the maximal utility is normalized to 1 for each party. With that normalization, the bargaining set is symmetric from both the gain and the sacrifice perspective. The equal-gains solution gives each party half. The equal-sacrifice solution has each side sacrifice half, which coincides with the equal-gains solution. Thus both perspectives lead to the same 50:50 split.

The Nash solution is no longer perspective invariant once the bargaining frontier is non-linear. We look for a solution that leads to the same result no matter which perspective is taken. Before

[2] In the end, they purchased our company and created an even bigger pie that we split.

[3] In contrast, raw data from Game 3 reported in Roth and Malouf (1979) has equal gain winning out over equal sacrifice. Of the nine negotiation results reported, three chose the exact equal-gain solution and five others picked something quite close. The one exception went beyond the equal-sacrifice solution, nearly all the way to the Utilitarian solution.

suggesting such a perspective-invariant solution, we present a simple non-linear bargaining problem designed to illustrate the problem. In this example, the Nash gains solution differs from our translation of the Nash solution to the sacrifice perspective, and thus the Nash solution fails perspective invariance.

Imagine Abe and Bea are negotiating over how to divide up one serving of broccoli. Bea would like the entire serving; Abe is less hungry and only values 2/3 of the serving. For both, utility is linear in the amount received (up to the point where Abe is satiated). Absent an agreement, both get zero. As illustrated in Figure 1, the bargaining frontier is initially horizontal as Abe can give away 1/3 of the broccoli at no cost, and thereafter the frontier has a slope of $-3/2$ as Abe must sacrifice all his utility to provide Bea with an additional 2/3 of utility.

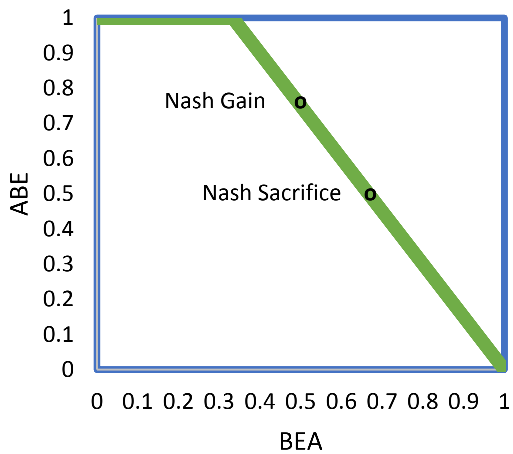

Figure 1 Abe and Bea

The Nash bargaining solution based on gains has each party receiving half a serving. That leaves Abe with seventy five percent of what he desires, while Bea receives half of her ideal result.[4] The intuition for the Nash solution is that were Abe interested in the full serving, the bargaining problem would be fully symmetric and so Abe and Bea would each get half the broccoli (with resulting utilities of $0.5 * 1.5 = 0.75$ for Abe and 0.5 for Bea). A scenario in which Abe is satiated with 2/3 has fewer opportunities to create value, and since the solution to the case where Abe wants the whole amount is still available, it should be chosen.

Consider the same bargaining problem from the sacrifice perspective.[5] As we formally show in Section 3, when we apply the Nash axioms to the sacrifice perspective, the bargaining solution is

[4] The frontier is defined by $U_{Abe} = \min[1, 1.5(1 - U_{Bea})]$. Along this frontier, $(0.5, 0.75)$ maximizes the product of the two utilities.

[5] Anbarci and Bigelow (1994), Chun (1988), Yu (1973) provide non-axiomatic bargaining solutions based on the sacrifice perspective; see further discussion in Section 8. Karagözoğlu and Tok (2018) look for a compromise point between the equal gain and equal sacrifice positions. Cao (1982) provides an objective function that is a linear combination of a player's gain and the rival's sacrifce. Our goal is to develop an approach that yields the same solution using either the gain or sacrifice perspective.

the midpoint of the Pareto frontier: the resulting utilities are 1/2 for Abe and 2/3 for Bea and the corresponding utility sacrifices are 1/2 for Abe and 1/3 for Bea.

This example shows that the Nash solution is not perspective invariant. The Nash axioms provide one answer in the gains perspective and generally a different answer in the loss perspective. To demonstrate this more formally, we start in Section 2 with a summary of the Nash axioms using the gains perspective. In Section 3, we translate the axioms into the sacrifice perspective and then apply the axioms to calculate the Nash sacrifice solution for the Abe and Bea bargaining problem. Section 4 provides a definition of perspective invariance along with examples of solutions that satisfy this condition.

Given the counterexample with the Nash solution, we know that if we want to achieve perspective invariance, we will need to relax or modify at least one of the Nash axioms. In Section 5, we employ a more limited version of Nash's Independence of Irrelevant Alternatives (IIA) axiom. Similar to a variation first proposed by Roth (1977), we exclude options where either party is getting less than its disagreement point or more than its ideal. While this weaker version of IIA does not generally lead to a unique solution, in Section 6 we show that under our weaker axioms, any efficient perspective-invariant bargaining solution must lie on the frontier between the Relative Utilitarian solution (Dhillon and Mertens (1999)) and the Raiffa (1953)/Kalai and Smorodinsky (1975) solution—the Relative Utilitarian solution maximizes the sum of utilities when each person's utility is measured on a 0 to 1 scale, and the Raiffa/Kalai-Smorodinsky solution is the equal-gain/equal-sacrifice solution under this normalization.

In section 7, we provide two axiomatic approaches that lead to a perspective invariant solution. We first show that our weaker axioms combined with a modified Nash symmetry axiom that allows for an interval solution uniquely leads to the Relative Utilitarian solution. The objective function is unique even if the maxima may not be. We then propose a reciprocity axiom that along with the weaker Nash axioms leads to a unique perspective-invariant solution that coincides with the Raiffa/Kalai-Smorodinsky solution. In Section 8, we consider examples of non-axiomatic bargaining solutions that satisfy perspective invariance. These examples provide additional insight into why the Nash solution is not perspective invariant and also lead us back to the Relative Utilitarian and the Raiffa/Kalai-Smorodinsky solutions. Section 9 offers brief conclusions.

2. The Nash Axioms

We begin with a summary of the four axioms in Nash (1950). We refer to the axiom regarding symmetry between players as Nash Symmetry so as to distinguish it from our separate interest in Perspective Invariance, which can be thought of as a symmetry between the gains and sacrifices perspectives. After presenting the axioms and the method of solution, we provide the translations of each to the sacrifice perspective.

Consider a two-person bargaining problem that is defined by a bargaining set $S \subseteq R^2$ that is convex and compact. The points $(x_1, x_2) \in S$ represent the feasible utility outcomes to the two parties. Following Nash, we use utility representations so that the disagreement point is $(0, 0)$. The set S includes the disagreement point—$(0, 0) \in S$—and some point strictly better than $(0, 0)$ for both players. Let $F(S) \in S$ represent the bargaining solution. This solution should obey four properties:

Efficiency. $F(S)$ is Pareto optimal. If $F(S) = (x_1, x_2)$ then there is no point $(v_1, v_2) \in S$ such that $v_1 \geq x_1$ and $v_2 \geq x_2$ with at least one strict inequality.

Scale Invariance. The bargaining solution to any positive linear transformation of utilities is the same linear transformation of the original solution.[6] $F(\lambda S) = \lambda F(S)$ for all $(\lambda_1, \lambda_2) > (0, 0)$, where λS is the set of points $(\lambda_1 x_1, \lambda_2 x_2)$ with $(x_1, x_2) \in S$.

Nash Symmetry. If the bargaining set is symmetric then both players must receive equal utility. If $(x_1, x_2) \in S \Rightarrow (x_2, x_1) \in S$ then $F(S) = (x, x)$ for some x.

Independence of Irrelevant Alternatives (IIA). The solution to an expanded set should also be chosen for a smaller set if it is feasible. If $S \subseteq S'$ and $F(S') \in S$, then $F(S) = F(S')$.

One way of interpreting the Nash approach is that it allows us to apply a solution from a bargaining set with a linear frontier to one with a concave frontier. Consider first a bargaining set with a linear frontier: $(0, x_2)$ to $(x_1, 0)$. Applying scale invariance, we scale the first player's utility by a factor of $1/x_1$ and the second player's by $1/x_2$, so that the frontier has slope -1. With this scaling, the bargaining set is perfectly symmetric and hence by Nash symmetry and efficiency, the bargaining solution must be the midpoint of the frontier or half to each.

To solve a bargaining problem S with a concave frontier, we apply the solution to an expanded bargaining set S' defined by a tangent line to the frontier of S. As illustrated in Figure 2, Nash picks the line tangent at the point $(x_1^*, x_2^*) \in S$ that maximizes $x_1 x_2$. This point is normalized to $(1/2, 1/2)$ via scale invariance. The resulting slope of the tangent line is $-x_2^*/x_1^* = -1$, so that S' is symmetric and thus its bargaining solution is $(1/2, 1/2)$. Since S' contains S and $F(S') = (1/2, 1/2) \in S$, by IIA it follows that $F(S) = (1/2, 1/2)$.[7] By finding the right bargaining set with a linear frontier, we determine the solution to a bargaining set with a concave frontier.

[6] Because utility representation of the disagreement point is set to $(0, 0)$, we only need consider positive linear rescaling of the utilities rather than all positive affine transformations.

[7] Kalai (1977) extends the Nash result to the case where symmetry is not required. He shows that the solution to the divide-the-dollar game determines the solution to a bargaining problem with a concave frontier. Haakea and Qin (2018) show that with a different axiom regarding how solutions to the divide-the-dollar game vary with scaling, a similar approach leads to the maximization of a generalized CES function as the bargaining solution.

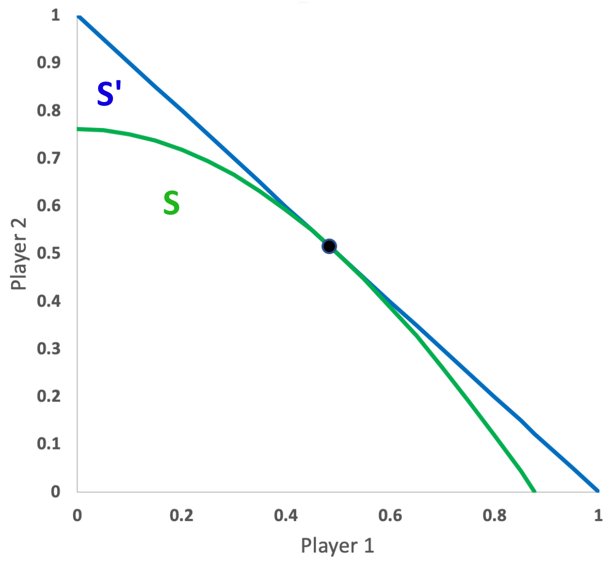

Figure 2 Nash Bargaining

3. The Nash Axioms for Sacrifices

Having presented Nash's axioms and outlined how they lead to a bargaining solution, we turn to consider how each translates into the sacrifice perspective. Set T is the compact, convex set of feasible sacrifices that corresponds to set S. Points $(y_1, y_2) \in T$ represent the feasible utility sacrifices to the two parties and include the sacrifices associated with disagreement point and some point strictly better than no agreement for both players.

Potential sacrifices are measured from a pair of ideal payoffs, $B = (B_1, B_2)$. The definition of what constitutes an ideal payoff matters as symmetry in the sacrifice perspective is measured from the zero-sacrifice point (associated with B) rather than from the sacrifices associated with no-agreement. We have to decide if the ideal payoff to each player is determined with or without regard to whether that result would ever be accepted by the other party. While Nash did not assume individual rationality, his solution would be the same if we restricted attention to the set of individually rational options. This is not the case when employing the sacrifice perspective.

We constrain the ideal payoffs to be in the set that satisfies individual rationality (as in Raiffa (1953), Kalai and Smorodinsky (1975) and Haakea and Qin (2018)). We do not think it is relevant to measure sacrifices from an ideal the other party would reject as being worse than its disagreement point. Any such sacrifice metric would be viewed as artificially high by the other party.

Player i's sacrifice when receiving utility x_i is thus $y_i = B_i - x_i$ where $B_1(S) = \{\max x_1 : (x_1, x_2) \in S, x_2 \geq 0\}$ and $B_2(S) = \{\max x_2 : (x_1, x_2) \in S, x_1 \geq 0\}$. Because we limit attention to the individually rational set, player i's ideal is defined in relationship to the other player's disagreement point. Even so, B_i is independent of the cardinal representation of player j's utility. For any strictly increasing transformation of player j's payoffs, B_i is unchanged as the rationality constraint is unchanged: $m(x_j) \geq m(0)$ is equivalent to $x_j \geq 0$ when m is strictly increasing.

While it is simpler to define the ideal payoffs $B(S)$ as a function of the gain set, we can also define the ideal payoffs $B(T)$ as a function of the sacrifice set, T.[8] As these are two representations of the same set, the ideal payoffs are the same in both perspectives.

The ideal point is different from the disagreement point in that B is generally not feasible. Even so, it serves as a beacon and a boundary. Negotiation textbooks typically frame a negotiation using the idea of a ZOPA, a zone of possible agreement (Lewicki et al. (2015)). In the textbook case, at each end of the zone one side is pushed to its reservation value while the other side obtains its ideal.[9] Negotiation experts advise participants to focus on their goals and not on their fallbacks.[10] Aiming for those goals (and measuring how close one has come) is a way of setting the negotiation in the sacrifice perspective.

Let $G(T)$ be the bargaining solution given in terms of sacrifices. The payoffs associated with these sacrifices are $B(T)-G(T)$. This solution should obey four properties:

Efficiency. $G(T)$ is Pareto optimal. Since smaller sacrifices are desirable, if $G(T)=(y_1,y_2)$ then there is no point $(v_1,v_2) \in T$ such that $v_1 \leq y_1$ and $v_2 \leq y_2$ with at least one strict inequality.

Scale Invariance. The bargaining solution to any positive linear transformation of utilities is the same linear transformation of the original solution. $G(\lambda T)=\lambda G(T)$ for all $(\lambda_1,\lambda_2)>(0,0)$, where λT is the set of points $(\lambda_1 y_1, \lambda_2 y_2)$ with $(y_1,y_2) \in T$.

Nash Symmetry. If the bargaining set of feasible sacrifices is symmetric then both players must make equal sacrifices. If $(y_1,y_2) \in T \Rightarrow (y_2,y_1) \in T$ then $G(T)=(y,y)$ for some y.

Independence of Irrelevant Alternatives (IIA). The solution to an expanded set of sacrifice options should also be chosen for a smaller set if it is feasible. If $T \subseteq T'$ and $G(T') \in T$, then $G(T)=G(T')$.

Consider this approach in the context of our original example with Abe and Bea negotiating over a serving of broccoli. We begin by observing that the ideal point is $(1,1)$ and this is associated with zero sacrifice from either party. We draw the set of feasible sacrifices in Figure 3A, reorienting

[8] Let $(B_1(T),B_2(T))$ be candidate values for the ideal payoffs. The sacrifices at the no-agreement point are the loss of the ideal payoff for each party. Individual rationality requires that no party is asked to sacrifice more their ideal payoff. Define $z_1(T)=\{\min y_1 : (y_1,y_2) \in T, y_2 \leq B_2(T)\}$ and $z_2(T)=\{\min y_2 : (y_1,y_2) \in T, y_1 \leq B_1(T)\}$—$z_i$ is the lowest feasible sacrifice for player i consistent with individual rationality for player j. We know this minimum sacrifice should be zero for each player, as the minimum arises when the player achieves his or her ideal outcome in the individually rational set. Thus the solution for $B(T)$ are the values $(B_1(T),B_2(T))$ such that $(z_1(T),z_2(T))=(0,0)$.

[9] More generally, one side's ideal may arise when the other side does better than its disagreement point.

[10] According to Malhotra and Bazerman (2008, p. 42): "Negotiators who focus on their own BATNA tend not to set high aspirations and are happy getting anything better than their reservation value. Meanwhile, those who focus on the other party's BATNA, are paying attention to the amount of value they bring to the other party. These folks tend to set higher aspirations and capture more value in the deals they negotiate."

the graph so that the zero sacrifice point is at the origin. Next, in Figure 3B, we rescale Abe's sacrifices by $2/3^{\text{rds}}$, so that the efficient frontier has slope -1 and is symmetric from the the origin. The feasible set of sacrifices is not yet fully symmetric.

To make the feasible sacrifice set symmetric, we employ IIA. In Figure 3C, we add a set of options which involve larger sacrifices for Abe and the same set for Bea. These are, of course, all inferior options. But with IIA there is free disposal of bad options. This extended set is now symmetric from the vantage point of potential sacrifices. Thus Nash symmetry requires equal sacrifice. The efficient equal-sacrifice solution $(1/3, 1/3)$ is also an option in our original set from Figure 3B. By IIA, it must also be the solution to the bargaining set with the smaller set of options.

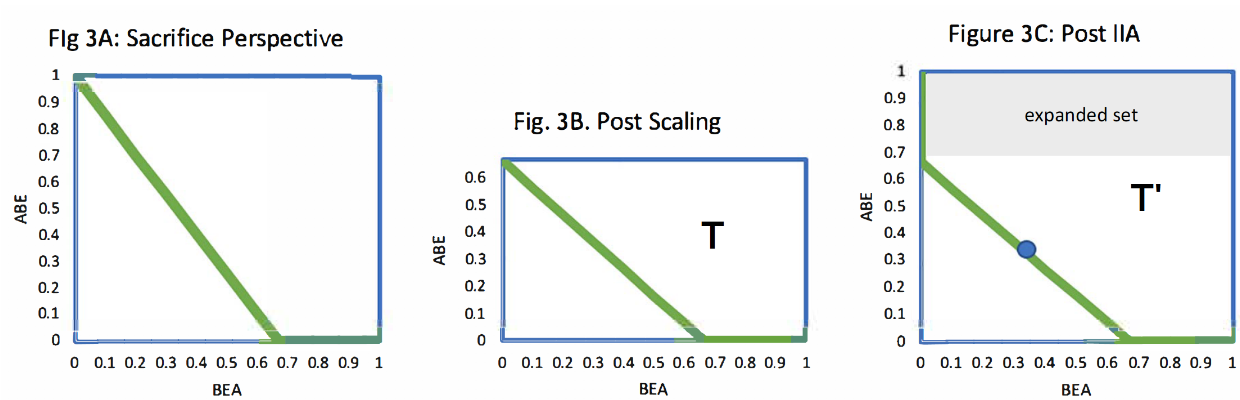

Figure 3

The Nash bargaining solution for sacrifices is the midpoint of the Pareto frontier, which is an intuitive result on its own. Each party sacrifices a utility of 1/3. In terms of percentages, each is sacrificing half relative to Abe's ideal where he gets 2/3 of a scoop and Bea gets the 1/3 Abe is happy to give away. If we translate this back into amounts of broccoli, each side gives up 1/3 of the broccoli relative to its ideal so that Abe gets 1/3 of the broccoli and Bea gets 2/3. Parallel to the Nash gains solution, the midpoint of the Pareto frontier maximizes the product of sacrifices—but only along the linear Pareto frontier.[11]

This example highlights a problem with the Nash bargaining solution. Nash's axioms lead to different choices when using the gains versus losses perspectives. This is the sense in which Nash's axioms are not invariant to perspective. There's no simple fix to this problem. For example, we can't just take the average of the gains solution and the loss solution as that average will not lie on

[11] Across the entire set, the product of sacrifices is maximized at the disagreement point. This is why we refer to the Nash sacrifice solution as being the midpoint of the Pareto frontier rather than the point that maximizes the product of the sacrifices. We can take the midpoint solution from bargaining sets with linear Pareto frontiers and use IIA to find the solution for general convex bargaining sets. For now, we don't look to find a general sacrifice solution; we only want to illustrate a simple example where the two solutions differ.

the efficient frontier when the bargaining frontier is non-linear—which is the case we care about. Before proceeding to a potential solution, let us define more precisely what we mean by perspective invariance.

4. Perspective Invariance

Under Perspective Invariance, the bargaining solution will come out the same when S and T represent the same underlying set of feasible alternatives. Our definition is presented for general solution functions $F(S)$ and $G(T)$. The larger goal is to find a common set of axioms that lead to a perspective invariant pair of solution functions.

Perspective Invariance. Let S and T be two equivalent bargaining sets, one represented in terms of feasible gains and the other represented in terms of feasible sacrifices. $F(S)$ and $G(T)$ are Perspective Invariant if for all convex bargaining sets S and their equivalent sacrifice sets T, $G(T) = B(S) - F(S)$.

Below are four examples of solutions F and G that satisfy Perspective Invariance. For simplicity, we apply scale invariance and limit attention to normalized bargaining sets with $B(S) = (1, 1)$. We discuss Example 1 in greater length below and the other examples in Sections 6, 7, and 8.

Example 1. Let $F(S)$ select the point that maximizes the minimum gain in S and $G(T)$ select the point that minimizes the maximum sacrifice in T. In both cases, the solution is the intersection of the Pareto frontier with the 45-degree line, and this intersection is the same whether one starts at (0,0) or (1,1). $F(S)$ is the Raiffa/Kalai-Smorodinsky solution, and $G(T)$, as we show below, is the result of applying the Kalai-Smorodinsky axioms to sacrifices.

Example 2. Let $F(S)$ and $G(T)$ select the midpoint of the Pareto frontier. This midpoint solution is the same whether one looks at the frontier from $(0, 0)$ or $(1, 1)$. Although $F(S)$ and $G(T)$ satisfy Perspective Invariance, Proposition 2 in Section 6 shows the solution violates the bounds implied by our weaker version of IIA combined with Nash Symmetry.

Example 3. Let $F(S)$ and $G(T)$ select the midpoint of the entire frontier; see Thomson (2010). The frontier remains the same and hence the midpoint of the frontier remains the same from either perspective. However, this solution violates IIA as well as our weaker version of IIA. In Section 8, we present a variation of this approach to find a solution that satisfies IIA along with the bounds implied by our weaker version of IIA combined with Nash Symmetry.

Example 4. Let $F(S)$ be the point or points that maximizes the sum of gains in S and $G(T)$ be the point or points that minimizes the sum of sacrifices in T. More generally, $F(S)$ is the Relative Utilitarian solution as utilities have been normalized to be on a $[0, 1]$ scale in S. $F(S)$ and $G(T)$ coincide as maximizing $x_1 + x_2$ leads to the same result as minimizing $(1 - x_1) + (1 - x_2)$. While

$F(S)$ and $G(T)$ satisfy Perspective Invariance, they fail Nash symmetry as the solution will be an interval when the Pareto frontier has a linear section with slope -1. We discuss options to resolve this issue in Section 7.

It is not enough to find a pair $F(S), G(T)$ such that $G(T) = (1,1) - F(S)$. We could always construct a $G(T)$ so that this equality holds. We are looking for axiomatic solutions $F(S)$ and $G(T)$ that satisfy Perspective Invariance where $G(T)$ is based on the same axioms as $F(S)$. The Kalai and Smorodinsky (1975) axioms provide one such example.

Kalai-Smorodinsky follow the Nash axioms except they replace IIA with a monotonicity axiom. In our variation on the monotonicity axiom, neither player is made worse off when the bargaining set provides more options while holding the disagreement point and the ideal point constant.[12]

Monotonicity If $S \subseteq S'$ and $B(S) = B(S')$ then $F(S') \geq F(S)$. For the corresponding sacrifice sets, if $T \subseteq T'$ and $B(T) = B(T')$ then $G(T') \leq G(T)$.

The Kalai-Smorodinsky result is that under Monotonicity, Efficiency, Scale Invariance, and Symmetry, $F(S')$ is the maximal point in S' along the line from $(0,0)$ to $B(S')$. To show that these axioms lead to a Perspective Invariant solution, we have to calculate the solution from both perspectives. We first provide an outline of their proof for the gains solution. For simplicity, we focus on the case where the Pareto frontier of the normalized set S' goes from $(0,1)$ to $(1,0)$.

Let (z,z) be the maximal point in S' along the 45-degree line. Consider the set S defined by the frontier from $(0,1)$ to (z,z) to $(1,0)$—this is illustrated for our Abe and Bea example by the dashed line in Figure 4. Since S is symmetric, $F(S) = (z,z)$. By construction, S also has ideal point $(1,1)$ and lies inside S'. Since the conditions for Monotonicity are satisfied, $F(S')$ must provide both players with at least z. However, as (z,z) is on the Pareto frontier of S' there are no other such points. Thus (z,z) must also be the solution to S'.

Next consider the sacrifice perspective. Let T' be the set of feasible sacrifices corresponding to S' and T be the sacrifice set that corresponds to S. Just as S is symmetric from $(0,0)$, T is symmetric from the perspective of $(1,1)$. Thus, by symmetry, $G(T) = (1,1) - (z,z)$, the equal-sacrifice point on the frontier of T. The conditions for Monotonicity hold as the maximal sacrifices in T are $(1,1)$, the same as in T', and $T \subseteq T'$. Thus $G(T') = G(T) = (1,1) - (z,z)$ as there are no other options in T' that leads to weakly lower sacrifices for both players.

Now that we know that there exists an axiomatic Perspective Invariant bargaining solution, we look to explore what other axioms lead to a Perspective Invariant solution. As the Nash bargaining

[12] The Monotonicity condition is a slight variation of the condition provided by Kalai-Smorodinsky. Their version considers monotonicity for one player at a time. In addition, in their formulation, the two bargaining sets don't have to be subsets of the other, but the Pareto frontier has to expand. This adds a step to the proof where the bargaining set is first expanded to include all the Pareto-dominated options at which point the superset relationship holds.

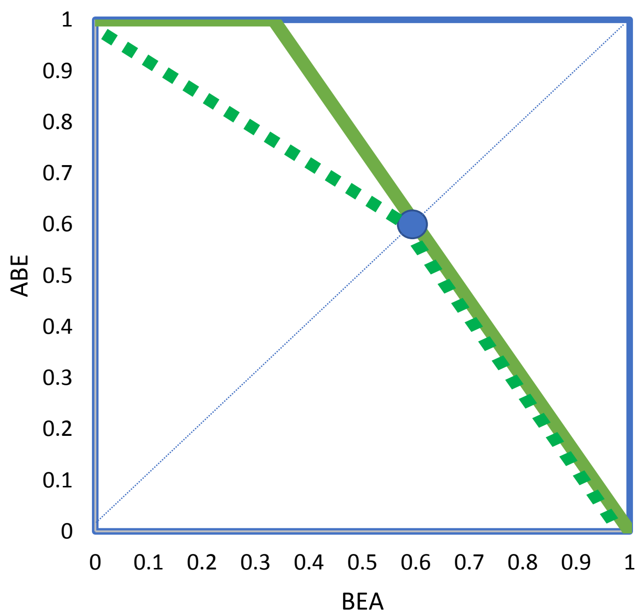

Figure 4 The Kalai-Smorodinsky Solution

solution to gains and sacrifices generally differ, we will need to relax at least one of Nash's axioms. Giving up IIA for Monotonicity is one way forward. To see what other approaches may work—while trying to stay close to the Nash framework—we start by relaxing, but not eliminating, Nash's IIA axiom.

5. Limited IIA and Individual Rationality

In our limited version of IIA, we do not consider options worse than the disagreement point or better than either party's feasible negotiation ideal. We first motivate limiting IIA to individually rational options by reevaluating the meaning of symmetry when it relies on points that fail individual rationality.

The key role of IIA in Nash's proof and in our example of the Nash sacrifice solution is that it allows us construct a larger bargaining set—in particular, one that is symmetric so that we know the solution—and then apply the larger symmetric set's solution to the original set. In a symmetric bargaining set, if player 1 makes a proposal (x_1, x_2), player 2 can make the counter-proposal (x_2, x_1) and it will always be feasible. Whatever player 1 asks for, player 2 can equally well demand the mirror image. Since every proposal can be met with its mirror image, the two parties are in a perfectly symmetric position and thus they should agree to something where they get the same amount.

In Figure 3C, we used IIA to create a larger symmetric bargaining set in the context of sacrifices. While the situation may appear to be symmetric, this isn't truly the case as seen in Figure 5. Imagine that Abe proposes a sacrifice pair of $(0.7, 0.1)$. Bea can respond with $(0.1, 0.7)$. While this option is in the feasible set, asking Abe to sacrifice 0.7 is worse than his disagreement point (in which Abe sacrifices 2/3). Abe would reject this counteroffer out of hand. In this sense, the flipped counteroffer isn't truly symmetric to the original offer and so we should not apply symmetry to argue for an equal-sacrifice solution. This suggests restricting the application of the Symmetry axiom to bargaining sets that satisfy individual rationality.

Before proposing this restriction, we first recognize that a related problem arises with IIA. In our example in Figure 3C, when we expand the bargaining set to include points that violate individual rationality, we are in effect moving the sacrifices associated with the disagreement point from $(1, 2/3)$ to $(1, 1)$. But IIA is meant to hold the disagreement point fixed. This leads us to limit IIA to options that are individually rational. Peters (1986) provides a similar concept with regard to the disagreement point through his Independence of Non-Individually Rational outcomes or INIR axiom. A bargaining solution should not depend on outcomes that are not individually rational as those options would be rejected out of hand.

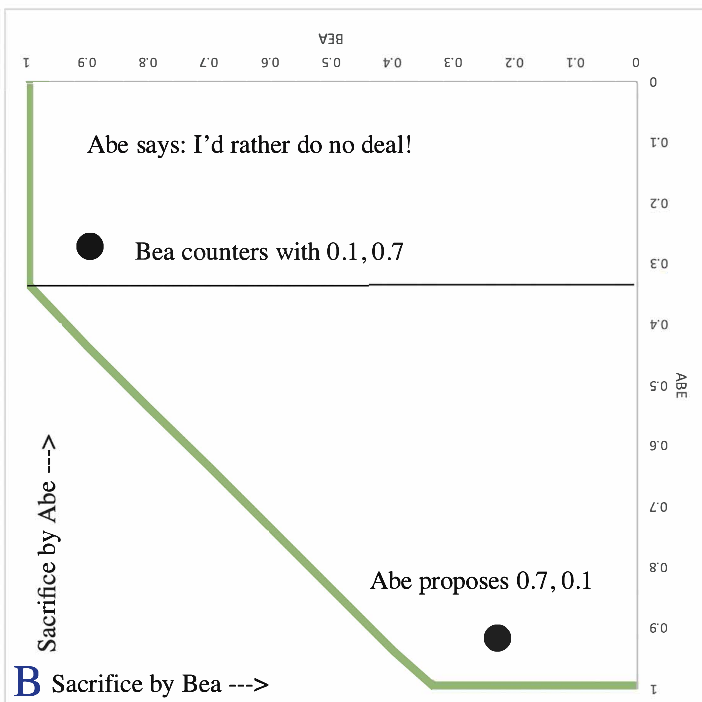

Figure 5

Nash was willing to consider bargaining sets with options worse than the disagreement point. But nothing in his analysis would be changed if bargaining sets were limited to individually rational options. In the case of gains, a bargaining set will be symmetric only if the bargaining subset which satisfies individual rationality is also symmetric. In contrast and as illustrated in Figure 5, under the sacrifice perspective it is possible that the individually rational part of the bargaining set is not symmetric, but adding options that violate individually rationality could make it so. We now rule this out. If a bargaining set satisfies individually rationality then it will be symmetric from the sacrifice perspective if and only if it is also symmetric from the gains perspective.

Henceforth, we assume that all bargainings sets satisfy Individually Rationality. We have already made this assumption when calculating B. Now we are taking it one step further and applying it directly to the bargaining sets.[13]

Individual Rationality. $(x_1, x_2) \geq (0, 0)$ for all $(x_1, x_2) \in S$. $(y_1, y_2) \leq (B_1, B_2)$ for all $(y_1, y_2) \in T$. For shorthand, we write this as $S \subseteq R^2_+$ and $B(T) - T \subseteq R^2_+$

Alongside rejecting options that are not individually rational, we propose a parallel idea with respect to the ideal outcomes for IIA. The application of IIA should also be limited to bargaining sets where the best options are no better than B.

The Nash solution relies on considering options better than B. We saw this in Figure 2 where the superset includes options that go above the maximum payoffs in the original frontier. To find the Nash solution to the bargaining set in Figure 1, we first rescale Abe's payoffs so that the range is 0 to 2/3 and the Pareto frontier has slope -1.[14] We then extend the frontier beyond Abe's maximum up to $(0, 1)$. The new set is fully symmetry, so that the bargaining solution to this extended set is $(0.5, 0.5)$; see Figure 6.

By design, the bargaining set when Abe wants the whole amount contains the set when Abe only wants 2/3. Since the $(0.5, 0.5)$ solution is available in both sets, by IIA it should also be picked in the smaller set, namely the set when Abe only wants $2/3^{\text{rds}}$.

There is something unsettling about this application of IIA. Abe is making the case that he deserves credit for making a sacrifice from 1 to 2/3, even though he only wants 2/3 of a scoop of broccoli. If Abe truly wanted the entire scoop then the bargaining problem would be symmetric, and Abe would then get 1/2. Thus Abe is asking Bea to provide him the same credit via a thought experiment: if I wanted all the broccoli, you would have agreed to give me half; don't penalize me for wanting less.

IIA is motivated by the idea that when there is a bargaining set with more alternatives, it should be easier to find a fair solution. If that solution is still available in a more restricted setting, then it should still be picked. But in our example, even though there are more bargaining alternatives

[13] In the prior literature, Individually Rationality is sometimes applied to the solution rather than the entire bargaining set; see Haakea and Qin (2018).

[14] The example is equivalent to the one provided in Luce and Raiffa (1957, p. 133)—the primary difference is that we have put a story behind the utility curves.

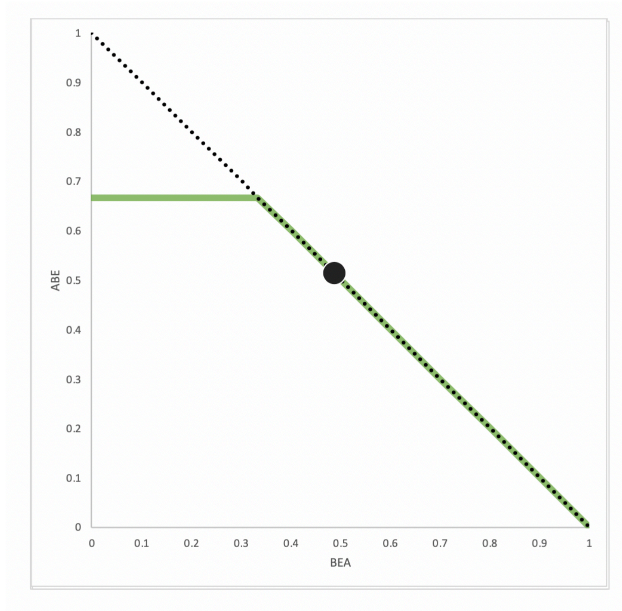

Figure 6

when Abe wants all the broccoli, this is a harder negotiation problem as the frontier is entirely zero-sum.

This critique does not suggest that we abandon IIA. Rather, it suggests that we limit the application of IIA to bargaining sets where the best option for each player remains the same. We should not give either party credit for making a sacrifice that has as its starting point something more than everything he or she has asked for. In justifying his bargaining procedure, Van Damme (1986) writes: "[W]hen a player asks for a certain amount, then he is acknowledging that he should not get more than this, so it is natural to view payoffs exceeding his demand as being irrelevant." With limited IIA, we don't use the potential to give someone more than 100 percent of what they've asked for to justify what sacrifices the other party should make.

Limited Independence of Irrelevant Alternatives (Limited IIA). A solution to an expanded set should also be chosen for a smaller set if it is feasible, the two sets have the same ideal point and satisfy Individual Rationality. For gains: if $S \subseteq S' \subseteq R^2_+$, $B(S) = B(S')$, and $F(S') \in S$, then $F(S) = F(S')$. For sacrifices: if $T \subseteq T'$, $B(T) = B(T')$, $B(T') - T' \subseteq R^2_+$, and $G(T') \in T$, then $G(T) = G(T')$.

A related version of Limited IIA was first introduced by Roth (1977) who called it "Independence of alternatives other than the disagreement point and the ideal point." Under regular IIA, the disagreement point is held fixed while under Roth's weaker form the ideal point is also held fixed. Our version and Roth's are the same except for the requirement of individual rationality.

There is a parallel between the application of Limited IIA and the monotonicity axiom in Kalai and Smorodinsky (1975). Under monotonicity, both players must do at least as well when the bargaining set expands while holding $B(S)$ constant.

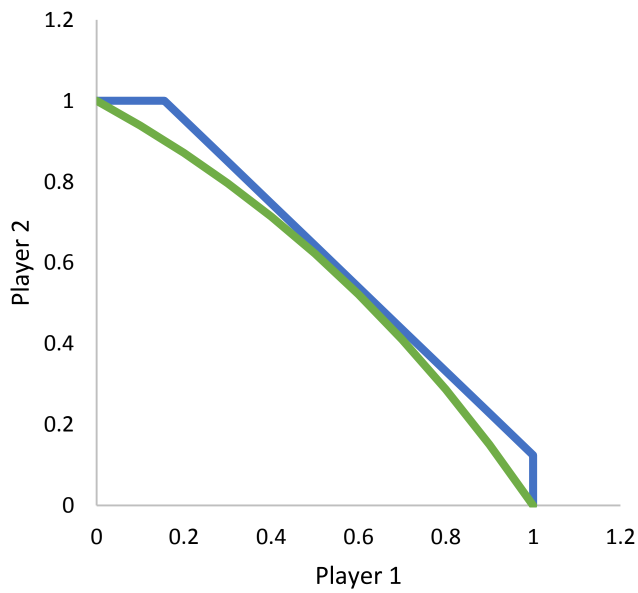

Figure 7 Limited IIA

Limited IIA is weaker than IIA as it restricts what supersets are allowed. With Limited IIA, Nash's tangent lines are truncated before they extend all the way to the two axes. The supersets bounded by the truncated tangent line will have a three-part frontier: a horizontal and a vertical segment connected by a linear sloped section that forms the Pareto frontier (as in Figure 7).

We can describe the normalized bargaining set defined by a three-part frontier by a pair (a, b): the three-part frontier starts with a horizontal segment from $(0, 1)$ to $(a, 1)$, then a linear Pareto frontier from $(a, 1)$ to $(1, b)$, followed by a vertical segment from $(b, 1)$ to $(0, 1)$. The horizontal and vertical segments are opportunities for cooperation—as one side can provide utility to the other at no cost—while the Pareto frontier is the zone of conflict.

The solution to this bargaining set can also be defined by (a, b) as these two parameters determine the frontier and hence the bargaining set. We write the gains solution as a function of the two endpoints of the Pareto frontier, $F((a, 1), (1, b))$.

If we have a continuous solution to all bargaining sets defined by a three-part frontier then we can also find a solution to a bargaining set with a concave frontier via Limited IIA. Start by employing scale invariance to rescale S so that $B = (1, 1)$. Consider all three-part frontiers that have a tangency with the normalized S (as in Figure 7). By rotating the Pareto segment of the frontier, we find there is always one bargaining set defined by a three-part frontier where the solution to that bargaining set is also on the frontier of S and hence a solution to S as well.

To demonstrate this result we first impose a continuity condition for the gains and sacrifices solution functions limited to bargaining sets defined by a three-part frontier.

Limited Continuity. $F((a,1),(1,b))$ and $G((0,1-b),(1-a,0))$ are continuous in a and b.

LEMMA 1. *Under Scale Invariance, Efficiency, and Limited Continuity, there exists an* (a,b) *such that the line segment from* $(a,1)$ *to* $(1,b)$ *is tangent to the frontier of the normalized* S *and* $F((a,1),(1,b)) \in S$.

Proof. In Appendix.

A parallel result holds for $G((0,1-b),(1-a,0))$. Under Perspective Invariance, the (a,b) will be the same for both results.

PROPOSITION 1. *Under Scale Invariance, Efficiency, Individual Rationality, Limited Continuity, and Limited IIA, there exists an* (a,b) *such that the solution to the normalized* S *coincides with* $F((a,1),(1,b))$.

Proof. Let (a,b) be a pair identified in Lemma 1. Under Limited IIA, $F((a,1),(1,b))$ must also be the solution to S since the normalized S is contained in the bargaining set defined by the three-part frontier from $(0,0)$ to $(a,1)$ to $(1,b)$ to $(1,0)$, and $F((a,1),(1,b)) \in S$. □

Again, a parallel result holds for $G((0,1-b),(1-a,0))$.

This proposition allows us to employ Limited IIA in reverse. Instead of finding the solution to a subset from a superset, we now know there is some superset—in particular one defined by a three-part frontier—that has the same solution as the subset. In the next section, we use our axioms to limit the location of any solution to a bargaining set defined by a three-part frontier and thereby also limit the location of the solutions to S and T.

6. Bounding the Solution

Propositions 2 and 3 show that the range of solutions permitted by Limited IIA and Symmetry must lie on the frontier between the Relative Utilitarian solution and the Raiffa/Kalai-Smorodinsky solution. We start by bounding the solution for bargaining sets defined by a three-part frontier. In the result below, without loss of generality, we assume $a \geq b$.

PROPOSITION 2. *Under Efficiency, Scale Invariance, Nash Symmetry, and Limited IIA,* $F((a,1),(1,b)) \in [(a,1),(z,z)]$, *where* (z,z) *is the intersection of the frontier with the 45-degree line.*

Proof. For a given (a,b) consider two bargaining set S and S': S' is defined by the three-part frontier from $(0,1)$ to $(a,1)$ to $(1,b)$ to $(1,0)$ and S is defined by the frontier $(0,1)$ to $(b,1)$ to (z,z) to $(1,b)$ to $(1,0)$, where (z,z) is the intersection of the boundary of S' with the 45-degree line.

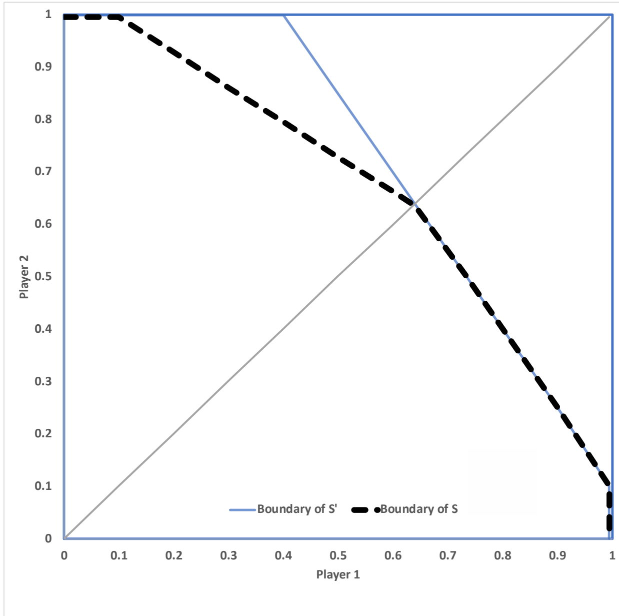

Figure 8 Bounding the Solution

This is illustrated in Figure 8 for the case $(0.4, 0.1)$. The frontier of S is the frontier of S' past the 45-degree line, combined with its symmetric flip. As $a \geq b$, S is convex and by construction the frontier segment $(0,1)$ to $(b,1)$ to (z,z) of S lies inside the frontier of S'. Past the 45-degree line, the two frontiers overlap, so that $S \subseteq S'$. We can apply Limited IIA as by construction $S' \subseteq R^2_+$.

We use Limited IIA, but in the reverse direction: we can use our knowledge about the solution to S to bound the solution of the superset S'. S is symmetric from both perspectives. By Nash Symmetry from either the gains or sacrifice perspective, the bargaining solution to S must be (z,z). It would be a contradiction if the solution to S' was in the open interval $((z,z),(1,b)]$. These points are all feasible for S and hence by Limited IIA any solution to S' in this interval would also have to be the solution to S, contradicting the fact that the solution to S is (z,z).

By efficiency, we know the solution to S' must lie in the interval $[(a,1),(1,b)]$. After eliminating the open interval $((z,z),(1,b)]$, that leaves the interval $[(a,1),(z,z)]$. A parallel argument shows that $a \leq b$ implies $F((a,1),(1,b)) \in [(z,z),(1,b)]$. $\square$

Because S is symmetric from both perspectives, an identical result and proof holds for $G((0,1-b),(1-a,0))$. In particular, if we convert the sacrifice solution back into gains, then $(1,1) - G((0,1-b),(1-a,0)) \in [(a,1),(z,z)]$.

It follows that the bargaining solution considered in Example 2—namely the midpoint of the Pareto frontier—is inconsistent with the axioms in Proposition 2. In particular, looking at the case illustrated in Figure 8, $a > b$ and the midpoint of the Pareto frontier is $\frac{1}{2}(1+a, 1+b)$ which is below the 45-degree line and thus ruled out by Proposition 2. Recall that in our initial example, the

Nash sacrifice solution was the midpoint of the Pareto frontier, a solution now ruled out. However, the construction in Figure 3 required employing bargaining supersets where the additional options violated individual rationality. Under Limited IIA, we can no longer conclude that the Nash sacrifice solution to the example with Abe and Bea is the midpoint of the Pareto Frontier.

Proposition 2 was the first step in bounding the bargaining solution. We now provide our primary bound: the solution must be between the equal-utility point and the Relative Utilitarian solution. Before presenting the result, we pause to describe and define the Relative Utilitarian solution.

Under Utilitarianism, social welfare is equal to the sum of individual utilities. A challenge here is that a different scaling of utilities leads to a different utilitarian maximum. Under Relative Utilitarianism, a specific scaling is chosen: each person's utility is measured on the same 0 to 1 scale (see Dhillon (1998) and Dhillon and Mertens (1999)). Fleurbaey and Zuber (2017) trace the history of this idea to Arrow (1951) and Luce and Raiffa (1957), both of whom are highly critical of employing Relative Utilitarianism as a social welfare function. As they observe, changing the Pareto-dominated and thus irrelevant $(0,0)$ baseline leads to a new rescaling and thus the potential for a different maximization result. But in the bargaining context, $(0,0)$ is the disagreement point and thus it is not irrelevant even though it is Pareto dominated. A bargaining result should depend on the disagreement point even if the disagreement point is never picked.

There is a second challenge. As Sobel (2001) observes, the Relative Utilitarian maxima is only a quasi-solution as it allows for an interval. We turn it to a solution by picking the point closest to 45-degree line.[15]

Relative Utilitarian solution. Rescale the bargaining set S so $B = (1,1)$. The Relative Utilitarian solution to the normalized set S is the point (x_1^u, x_2^u) that maximizes $x_1 + x_2 \in S$. If the solution is not unique, define (x_1^u, x_2^u) as the point in the maximizing interval closest to the equal-utility point (x,x) on the frontier.

For ease of exposition and without loss of generality, we assume that (x_1^u, x_2^u) is weakly to the left of (x,x) along the frontier of S. As above, we provide the proof for the gains solution. An identical result and parallel proof holds for the sacrifice solution.

PROPOSITION 3. *Under Efficiency, Scale Invariance, Limited Continuity, Nash Symmetry, Individual Rationality, and Limited IIA, any bargaining solution of the normalized set S is on the frontier of S between (x_1^u, x_2^u) and (x,x).*

Proof. Case 1: $(x_1^u, x_2^u) = (x,x)$. Since $x_1^u + x_2^u$ maximizes the sum of utilities, there is a tangent line through (x,x) with slope -1. The bargaining set defined by the resulting three-part frontier

[15] This amended Relative Utilitarian solution will not be continuous and, as Sobel (2001) shows, also fails midpoint domination

contains S and is symmetric (from both perspectives) and so has as its solution (x,x). By Limited IIA, (x,x) must also be the solution to S as $(x,x) \in S$. Proposition 3 follows directly in this case.

Case 2: (x_1^u, x_2^u) is strictly to the left of (x,x). We use Limited Continuity to apply Proposition 1. For the bargaining set S, there is an (a,b) such that $F((a,1),(1,b))$ coincides with the solution to S and the line segment $(a,1)$ to $(1,b)$ is tangent to S at $F((a,1),(1,b))$. This leads to a contradiction if $F((a,1),(1,b)) \in [(a,1),(x_1^u,x_2^u)] \cup ((x,x),(1,b)]$.

If $F((a,1),(1,b)) \in [(a,1),(x_1^u,x_2^u)]$, the tangent line must be weakly flatter than a line with slope -1 so that $a \leq b$. Applying Proposition 2, $F((a,1),(1,b)) \in [(x,x),(1,b)]$, which contradicts $F((a,1),(1,b)) \in [(a,1),(x_1^u,x_2^u)]$ as the two intervals don't overlap.

If $F((a,1),(1,b))$ is to the right of (x,x), the tangent line must be steeper than a line with slope of -1 so that $a > b$. Applying Proposition 2, $F((a,1),(1,b)) \in [(a,1),(x,x)]$ which contradicts $F((a,1),(1,b)) \in ((x,x),(1,b)]$ as the two intervals don't overlap. □

Note that (x_1^u, x_2^u) is not included in the solution except when we are in case 1 so that $(x_1^u, x_2^u) = (x,x)$. There is a weaker version of Proposition 3 that does not rely on Limited Continuity. In the appendix, an alternative version of Proposition 3 shows that the range of possible solutions is the set of points on the frontier between the equal-gains point and the point that provides the same combined utility as the equal-gains point.

Proposition 3 does not in itself require Perspective Invariance. But any perspective invariant solution satisfying the axioms in Proposition 3 must lie on the frontier interval between (x_1^u, x_2^u) and (x,x).

Cao (1982) provides a parameterized objective function where the maximization solution satisfies the conditions of Proposition 3 and the solution ranges from (x_1^u, x_2^u) to (x,x) as the parameter varies.[16] The results are complementary in that using the same set of conditions, Proposition 3 demonstrates any bargaining solution must lie in this given interval while Cao provides a family of solutions that fill out this interval. However, his family of solutions generally fail Perspective Invariance. In section 8, we look for the exceptional cases where the maximum of an objective function also satisfies Perspective Invariance. Presently, we look to see if we can find an axiomatic approach that leads to a unique solution that satisfies Perspective Invariance.

7. Two Perspective Invariant Solutions

We seek a unique perspective-invariant solution to S. The current axioms have narrowed the potential solution to a range. To generate a unique solution, we will need an additional axiom. In this section, we propose two distinct approaches. Our first approach modifies the symmetry

[16] Cao's range is slightly different as he limits the bargaining sets to be strictly convex. Consequently, the relative utilitarian solution is included in his interval, while it is excluded from ours when it does not overlap with (x,x).

axiom to give it more power. Our second approach is based on a new axiom that defines a required reciprocity when one party can provide utility to the other at no cost.

7.1. Interval Symmetry

We begin by modifying our limited symmetry to allow for an interval solution. This modification, combined with the appropriately adjusted Limited IIA axiom, leads to the interval or quasi-solution version of Relative Utilitarianism as the bargaining solution. We give up uniqueness of the maxima but get a perspective-invariant bargaining solution that uniquely maximizes the Relative Utilitarian welfare function. The result is not ideal in that the solution is not generally unique, but has the virtue that the axioms lead to a unique objective function. And when the bargaining set is strictly convex, the solution will be unique.[17]

Interval Nash Symmetry If $(x_1, x_2) \in S \Rightarrow (x_2, x_1) \in S$ then $F(S) = (x, x)$, where (x, x) is the maximal symmetric point in S unless the bargaining frontier at (x, x) has a linear interval, in which case all points in this linear interval are a solution.

If a symmetric set has a linear interval around (x, x), the slope of that interval must be -1 by symmetry. Since the set is symmetric, $B(S) = (B, B)$ and the utility scales for the two players are identical. Thus all points on the solution interval of a symmetric bargaining set maximize the Relative Utilitarian welfare function.

There is the parallel axiom from the sacrifice perspective. Note that as we restrict attention to bargaining sets that satisfy individual rationality, T will satisfy Interval Nash symmetry from the sacrifice perspective if and only if S satisfies Interval Nash symmetry from the gain perspective.

We also extend Limited IIA to allow for an interval solution. If S is a subset of S' and there is an interval solution for $F(S')$ that partially or completely overlaps with subset S, then that overlap region must be the solution for S.

Limited Interval IIA. If $S \subseteq S' \subseteq R^2_+$, $B(S) = B(S')$, and $F(S') \cap S \neq \{\emptyset\}$, then $F(S) = F(S') \cap S$.

The definition is identical to Limited IIA except that we allow for an interval solution for both S' and S. There is the parallel definition for the sacrifice perspective that we skip for brevity.

Proposition 4. *Under Efficiency, Scale Invariance, Individual Rationality, Interval Nash Symmetry, and Limited Interval IIA, any bargaining solution maximizes $x_1 + x_2$ in the normalized bargaining set S or equivalently minimizes $(1 - x_1) + (1 - x_2)$ in the normalized bargaining set T corresponding to S.*

[17] We cannot restrict attention to strictly convex bargaining sets as our proof relies on applying the solution to sets with linear boundaries.

Proof. Consider the Relative Utilitarian solution (x_1^u, x_2^u) of the normalized bargaining set S. Let S' be defined by a three-part frontier where the Pareto frontier of S' is the tangent line at (x_1^u, x_2^u) with slope -1, truncated at $x_1 = 1$ and $x_2 = 1$.

We observe that $S \subseteq S' \subseteq R_+^2$, $B(S) = B(S') = (1,1)$, and S' is symmetric from both the origin and B. With slope -1, the entire Pareto frontier of S' is a solution by Interval Nash Symmetry. Thus $(x_1^u, x_2^u) \in F(S')$ and by definition $(x_1^u, x_2^u) \in S$ so $(x_1^u, x_2^u) \in F(S') \cap S$. By Limited Interval IIA, $F(S) = F(S') \cap S$: the tangent point and any tangent interval must be the solution to the bargaining problem. The welfare function $x_1 + x_2$ is maximized at $(x_1^u, x_2^u) \in S$ and any other points on the frontier of S with slope -1, which is precisely $F(S') \cap S$.

Because S and T represent the same bargaining set, and S' is symmetric from both perspectives, the same argument shows that the sacrifices associated with (x_1^u, x_2^u) and other points on the frontier of T with slope -1 must be the solution from the sacrifice perspective. □

The Relative Utilitarian interval solution satisfies Perspective Invariance as the solutions are the same from both perspectives. The cost of allowing an interval solution is that we give up on uniqueness. This is a significant cost in that when there is an interval of solutions, the two sides will not agree about which point to pick. Thus the axioms leading to the interval version of the Relative Utilitarian bargaining solution may not help the parties reach an agreement, even if they can agree on an objective function.

To restore uniqueness and satisfy Nash symmetry, we can try to return to our Relative Utilitarian solution by picking the point closest to the 45-degree line when there is an interval solution. There are two problems with this amendment. The first is that we lose the power of Proposition 4. We can no longer claim that the Relative Utilitarian solution is the unique bargaining solution. All we can say is that it satisfies our axioms.

A second issue is that the Relative Utilitarian solution is discontinuous. The solution is $(a, 1)$ when $a > b$, $(1, b)$ when $a < b$, and $\frac{1}{2}(1 + a, 1 + a)$ when $a = b$. Picking the midpoint (rather than the entire Pareto frontier) when $a = b$ leads to a discontinuity that precludes a fixed-point solution. Specifically, the point on the bargaining frontier with slope -1 will not generally intersect with the 45-degree line.[18] Thus Relative Utilitarianism either fails Nash Symmetry due to an interval solution or is amended in a way that makes it discontinuous and consequently will not generally have a coincident tangent solution with the bargaining frontier. This helps explain why Proposition 1 no longer applies and thus why the Relative Utilitarian solution is not the unique bargaining solution that satisfies our axioms.

[18] Proposition 1 uses continuity to ensure there is a tangent line such that the solution to the bargaining set defined by this line is feasible in the original set. When the interval solution to the tangent line defined by (a, b) is upper hemi-continuous, we can apply a fixed-point argument to find such a solution. Picking the midpoint rather than the entire interval at $a = b$ leads to a violation of upper hemi-continuity for the maximizing-value correspondence, and consequently there is no general fixed-point solution.

7.2. Cooperation Reciprocity

Our second axiomatic approach starts with the idea that cooperation is valuable and should be reciprocated. Returning to the example of Abe and Bea, just because Abe can give Bea something for free doesn't mean that Abe should get all the gains from this act of cooperation. But this is what happens in the Nash solution when $b=0$ and a increases from 0 to 1/2. Nor should Bea receive all the gains. But this is what happens when the solution is the midpoint of the Pareto frontier, $b=0$ and a increases from 0 to 1/2. As we discuss below, there is a sense in which both parties are required to create the gain and thus the gain should be shared.

The Cooperation Reciprocity axiom proposes that the reward for one party's cooperation should be an equal amount of cooperation from the other side. If reciprocal gains are not possible, the other side should make some sacrifice that provides a gain to the other party, where the gain provided plus the cost of the sacrifice equals the gain received.

Cooperation Reciprocity. Consider a normalized bargaining set defined by a three-part frontier in which Player 2 can provide Player 1 with utility a for free, and Player 1 can provide Player 2 with utility b for free, where $a \geq b$. Then, in additional to b, Player 1 must provide a combination of utility u to Player 2 plus make a sacrifice r that adds up to $a-b$. This is the "reciprocal" point. On the remaining linear conflict section of the frontier, the bargaining solution is the midpoint.

To help motivate reciprocity and simplify the drawings, we illustrate the axiom for the case with $b=0$. Assume player 2 can give player 1 utility of 0.5 at no cost. The frontier from $(0,1)$ to $(a,1)$ is the cooperation component of the negotiation. The remaining frontier from $(0.5,1)$ to $(1,0)$ is the conflict component of the bargaining problem.

Player 1 does not have any ability to give any utility for free to Player 2. However, if player 1 were to make a sacrifice of 1/6, this would allow player 1 to give player 2 utility of 1/3. Added together, the gain to player 2 of 1/3 plus the 1/6 sacrifice adds up to 0.5, the amount player 2 has given to player 1. The reciprocal point is thus $(1-1/6, 0.5-1/6) = (5/6, 1/3)$.

We use these contributions to balance out the cooperative part of the bargaining. In Figure 9, the two dotted parts of the frontier balance each other out, leaving us to consider the solid line segment. This is the region of conflict that we propose to split down the middle by applying the second part of Cooperation Reciprocity. In this example, the solution is the midpoint between $(1/2, 1)$ and $(5/6, 1/3)$ which is $(2/3, 2/3)$. The fact that both parties end up equally well off (or make equal sacrifices) is true more generally.

PROPOSITION 5. *Under Efficiency, Scale Invariance, Limited IIA, Individual Rationality, and Cooperation Reciprocity, the unique solution to any convex bargaining set S is the maximal equal-gain point or equivalently the minimal equal-sacrifice point in the normalized version of S.*

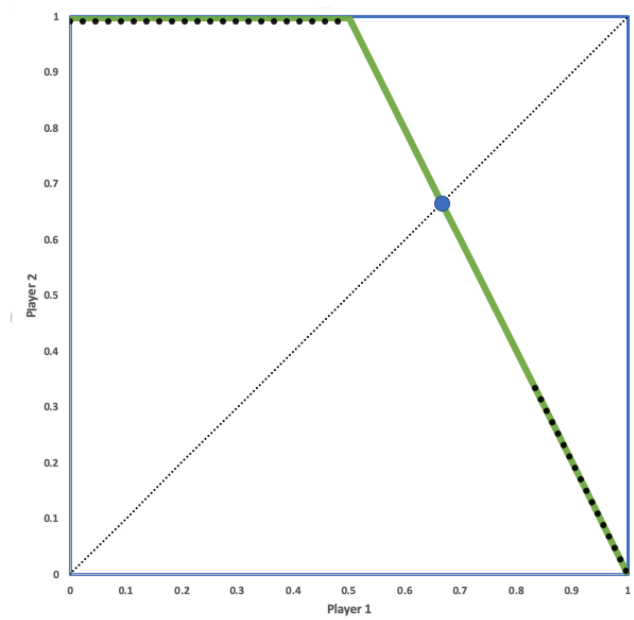

Figure 9 Post Cooperation Reciprocity

Proof. We first show that the solution to a normalized bargaining set defined by a three-part frontier is the unique intersection of the frontier with the 45-degree line. At the reciprocal point, the gain for Player 2 is $b + u = b + (a - b - r) = a - r$. Thus the reciprocal point is $(1 - r, a - r)$, where $r = \frac{(1-a)(a-b)}{(1-a+1-b)}$. This calculation of r reflects the fact that the slope from $(1 - r, a - r)$ to $(1, b)$ has the same slope as the Pareto frontier from $(a, 1)$ to $(1, b)$. Thus the overall solution under Cooperation Reciprocity is $\frac{1}{2}(1 + a - r, 1 + a - r) = \frac{1-ab}{1+a+1-b}(1, 1)$.[19]

Now consider the three-part frontier tangent at the equal-gain (equivalently, equal-sacrifice) point on the frontier of S. The solution to the bargaining set with this three-part frontier is the equal-gain (equal-sacrifice) point which by construction is an element of S. This three-part frontier also defines a superset of S, so that by Limited IIA the equal-gain (equal-sacrifice) point must be the bargaining solution of S. The result is perspective invariant as we can apply Limited IIA from either the gains or sacrifice perspective and the result is the same. □

This axiomatic solution coincides with Kalai and Smorodinsky (1975), although we have arrived at this result via a different set of axioms. We use Limited IIA and Cooperation Reciprocity rather than Monotonicity.[20] This leads us to provide additional justification for Cooperation Reciprocity.

[19] Another way to see this is that under Cooperation Reciprocity, we measure distances along the frontier using the taxicab metric: we sum the gains and the sacrifices and count them equally. The taxicab distance from $(1, 0)$ to (x_1, x_2) is $1 - x_1 + x_2$ and the taxicab distance from (x_1, x_2) to $(0, 1)$ is $x_1 + 1 - x_2$. The two distance are equal only when $x_1 = x_2$. This result is the same whether we take the gain or sacrifice perspective.

[20] While Nash is the only solution that satisfies IIA and Sobel's (1981) Midpoint Dominance, the equal-gains solution satisfies Midpoint Dominance and Limited IIA.

One reason that cooperation should be reciprocated is that creating the benefit may require something from each party. It may at first appear that our bargaining frontier in Figure 9 with $a = 0.5$, $b = 0$ suggests an asymmetric situation in that only player 2 can freely help player 1 and not vice versa. But consider the three scenarios below in which creation of this benefit requires equal coordination (or anti-coordination) from the two players. In all three cases there is a scoop of broccoli and a scoop of beets to divide.

In scenario A, neither player 1 nor player 2 likes beets—they each put a zero utility on beets (as there should be) and a value of 1 on broccoli. Here the bargaining frontier is a line with slope -1 from $(0,1)$ to $(1,0)$.

In scenario B, player 2 still does not like beets, but player 1 now likes beets more and broccoli less, so much so that this player is indifferent between beets and broccoli. Player 2's utility is unchanged while player 1 puts a utility of 1/2 on each of broccoli and beets. This is a scenario which leads to the bargaining frontier illustrated in Figure 9 with $a = 1/2$.

In scenario C, players 1 and 2 both like broccoli and beets equally; the utility of each is 1/2 for both players. As in scenario A, the bargaining frontier is a line with slope -1 from $(0,1)$ to $(1,0)$.

Scenarios A and C are both pure conflict. Scenario B expands the frontier because there is an opportunity for cooperation: Player 2 can give all the beets to player 1 with no loss in utility. To highlight the symmetry of Scenario B, observe that we can equally reach B from A or C. Going from A to B requires that player 1 likes beets more and broccoli less while going from C to B requires player 2 to like beets less and broccoli more.

Who should benefit the most in scenario B? The Nash solution gives all the benefit to player 2. But player 2 could not get this gain without the help of player 1 who now likes beets more. Nor should we give all the gains to player 1 who needs player 2 to not like beets in order to create the extra surplus. Only when we have an uncoordination—player 1 likes something relatively more and player 2 likes it relatively less—does the frontier expand. Since we need both players equally for the uncoordination, the axiom of Cooperation Reciprocity leads the players to share the gains equally.[21]

We create the potential for cooperation in moving from scenario A to B or from C to B. In moving from A to B, the gain results from a the change in Abe's preferences, while in the move from C to B, the gain results from a change in Bea's preferences. Thus it is hard to conclude that

[21] A similar interpretation can be told when there is only one good, although it is a bit more contrived. With only one good, say broccoli, Scenario A arises when Abe and Bea both want all the broccoli, and each unit is valued at 1. In scenario B, Abe is satiated with only half the broccoli (and values the half serving at 1), while Bea wants the entire amount. In scenario C, Abe is again satiated with only half the broccoli, but now Bea needs half the broccoli before she gets any benefit—the second half of the broccoli provides Bea with a value of 1.

the spoils should go to one player over the other. Under Cooperation Reciprocity, the gains are evenly split.

8. Non-Axiomatic Solutions

We end with a discussion of non-axiomatic solutions discussed in the literature. It may be that there are axioms that lead to these solutions. Even if that turns out not to be the case, these solutions provide insight into what types of solutions will satisfy perspective invariance.

A common theme is selecting the midpoint of the frontier. There are several ways of calculating what is meant by a midpoint. The natural starting point is the midpoint of distance along the bargaining frontier (Thomson (2010)).[22] As discussed in Example 3, this satisfies Perspective Invariance. The problem is that this solution violates IIA. The midpoint of the frontier is a continuous function and hence Proposition 1 applies: there is an (a, b) such that the Pareto frontier from $(a, 1)$ to $(b, 1)$ is tangent to the normalized set S and $F((a, 1), (1, b)) \in S$. By Limited IIA, the solution to S must be $F((a, 1), (1, b))$. The problem is that while $F((a, 1), (1, b))$ is the midpoint of the three-part frontier, it is not generally the midpoint of the frontier of S.

There is a workaround. We could define the midpoint-of-the-frontier solution only for bargaining sets defined by a three-part frontier and then use Proposition 1 and Limited IIA to find the solution for all convex bargaining sets.[23] While this midpoint-of-the-frontier solution satisfies our axioms and Perspective Invariance, it is not entirely satisfactory. Whatever the justification for picking the midpoint of the frontier, it is not clear why this justification applies in greater force to the boundary set defined by the three-part frontier than to the original bargaining set.

This issue does not arise when the bargaining solution is the maximization of an objective function and that objective function is increasing, symmetric, and strictly quasiconcave. In that case, Efficiency, Nash Symmetry and Limited IIA will all be satisfied. Efficiency is satisfied as maximizing an increasing strictly quasiconcave function will have a unique solution on the frontier of S. If the bargaining set is symmetric, the fact that the solution is unique implies it must be the equal-utility point on the frontier, satisfying Nash Symmetry. (Otherwise, if (x_1, x_2) maximizes

[22] Instead of the equal-length metric, Anbarci and Bigelow (1994) propose an equal-area solution: their solution is the point along the frontier where the area of S above the line from $(0, 0)$ to (z_1, z_2) equals the area of S below the line. While this solution satisfies Nash symmetry, it violates Perspective Invariance as the answer is different if (z_1, z_2) is chosen to split the sacrifice area above the frontier—from to (z_1, z_2) to $(1, 1)$—rather than the area below it. The Raiffa (1953) continuous bargaining solution provides a path from $(0, 0)$ to the frontier where each party gains in proportion to its maximal gain using the intermediate points along the path as the new baseline. This can also be considered as a way of finding a midpoint of the frontier. But, this fails Perspective Invariance as the result is different if we start at $(1, 1)$ and the two players make sacrifices in proportion to their maximal sacrifice using the intermediate points along the path as the new baseline. It is worth noting that the Raiffa solution from the sacrifice perspective picks the midpoint of the Pareto section of the three-part frontier, the same as the Nash sacrifice solution.

[23] Direct calculations show that the midpoint of the entire three-part frontier is always in the interval between the Nash gains solution and the equal-gains solution, and hence in the interval required by Proposition 3.

the objective function then so must (x_2, x_1) as the objective function is symmetric; thus, if $x_1 \neq x_2$ there would be two solutions.) Limited IIA is also satisfied as the maximum over a set must also be the maximum over a subset if the solution is available in the subset.

One intuitive objective function that satisfies the three criteria is to maximize the distance from the disagreement point. In the case of sacrifices, the parallel objective function is to minimize the distance from the ideal point (or maximize the negative distance). Here distance is measured by the power or Hölder ρ-mean. This objective function is equivalent to maximizing a CES weighted social welfare function as is discussed in Bertsimas et al. (2012) and Rachmilevitch (2015).

In the case of gains, the objective function or distance from the disagreement point is

$$\mu_\rho(x) = \left(\frac{x_1^\rho + x_2^\rho}{2}\right)^{1/\rho} \tag{1}$$

When $\rho = 1$, this is the ordinary arithmetic mean of gains. When $\rho = 0$, this converges to the geometric mean of gains. When $\rho = -1$, this is the harmonic mean of gains. When $\rho = -\infty$, this converges to the minimum gain. The Nash solution maximizes $x_1 x_2$ among $(x_1, x_2) \in S$, and so is equivalent to maximizing the geometric or 0-mean of the gains.

Following Rachmilevitch (2015), as ρ increases from $-\infty$ to 1, the point on the frontier that maximizes the ρ-mean starts at the equal-gain solution and moves monotonically towards the Relative Utilitarian solution. This provides a quick proof that the Nash gains solution ($\rho = 0$) always lies on the frontier between the equal-gain and Relative Utilitarian solutions. (Rachmilevitch goes one step further and shows the Nash solution is closer to the Relative Utilitarian solution than to the equal-gain solution.)

We focus on the case with $\rho < 1$ because $\mu_\rho(x)$ will be strictly quasiconcave. For $\rho = 1$, $\mu_\rho(x)$ is linear and there will not be a unique solution when the bargaining frontier has a linear segment with slope -1. For $\rho > 1$, $\mu_\rho(x)$ will be strictly quasiconvex and there will not be a unique solution in general.

Next we look at the same problem from the sacrifice perspective; this closely follows Yu (1973). Rather than look at gains (x_1, x_2), consider the sacrifices $(1 - x_1, 1 - x_2)$. In this case, the objective function is to minimize the distance from the ideal point or average sacrifice as measured by the ρ-mean:

$$\mu_\rho(1 - x) = \left(\frac{(1 - x_1)^\rho + (1 - x_2)^\rho}{2}\right)^{1/\rho} \tag{2}$$

When $\rho = 1$ this is the ordinary arithmetic mean of sacrifices. When $\rho = 2$ this is the Euclidean average sacrifice. When $\rho = \infty$ this converges to the maximum sacrifice. We focus on the case with $\rho > 1$, as $\mu_\rho(1 - x)$ will be strictly quasiconvex from the perspective of the origin and so there will be unique minimum along the frontier.

We have a dual set of perspectives, namely maximizing the ρ-mean of gains for $\rho \leq 1$, and minimizing the ρ-mean of sacrifices for $\rho \geq 1$. We can look to see where the two perspectives line up.

Note first that the Nash solution is not symmetric to the two perspectives. In particular, if we consider $\rho = 0$ (the Nash Solution to the gains perspective), there is no corresponding value of ρ we can apply to the sacrifice perspective that will always lead to the same solution. There are only two values of ρ that lead to consistent solutions across the two perspectives.

One solution consistent across perspectives is $\rho = 1$. Maximizing the sum of gains $(x_1 + x_2)$ leads to the same result as minimizing the sum of sacrifices $(1 - x_1 + 1 - x_2)$. This is the Relative Utilitarian interval solution discussed in the previous section.

The one other norm solution consistent between gains and sacrifices is $\rho = \{-\infty, \infty\}$. In the case of gains, we use $\rho = -\infty$, and this is the Rawlsian solution of maximizing the minimum gain. From the perspective of sacrifices, we apply $\rho = \infty$, and this is minimizing the maximum sacrifice. Both approaches lead to the maximal equal-utility solution and thus we have Perspective Invariance. This coincides with the Raiffa (1953) and Kalai and Smorodinsky (1975) solutions along with our solution using Cooperation Reciprocity.

9. Conclusions

Issues of fairness can be equally-well framed using gains or sacrifices. This leads us to look for a bargaining solution that is the same whether one frames the problem in terms of gains or sacrifices. Since the Nash Bargaining Solution is not perspective invariant, we cannot have this property and satisfy all of Nash's axioms. We relax Nash's IIA axiom so that it only applies to sets whose elements are no better than the ideal points and no worse than the disagreement point.

This limited version of IIA highlights an objection to Nash's approach that the division is based on options in which one side gets more than everything it has asked for. By limiting consideration to feasible options that are no better than either party's ideal, we are led to consider how to resolve a bargaining problem in which the frontier displays both pure cooperation as well as pure conflict.

The result of our weaker axioms alone is that there is no longer a unique solution. We demonstrate that any bargaining solution must lie along the efficient frontier between the Relative Utilitarian maximization outcome and the maximal equal-gains outcome in the normalized bargaining set. While earlier work has provided examples of parameterized bargaining solutions that fill out this range, these solutions generally fail perspective invariance.

This paper has illustrated three ways to achieve a unique perspective invariant solution. Early in the paper, we showed that the Kalai-Smorodinky axioms lead to a unique perspective invariant solution. Sticking closer to the Nash framework. we show that extending the Nash Symmetry axiom to allow for an interval solution leads to Relative Utilitarian as the unique objective function. This provides a unique solution when the bargaining set is strictly convex and will have an interval solution when the bargaining set has a linear Pareto frontier with slope -1. Our third approach adds an axiom of Cooperation Reciprocity. The way we measure reciprocity is via the end result, not the process that leads there. While the resulting perspective invariant solution coincides with Raiffa and Kalai-Smorodinsky, we arrived at the solution from a different route. Kalai-Smorodinsky replaced IIA with a monotonicity axiom. We maintained IIA in a more limited form and added a Cooperation Reciprocity axiom.

Others may apply different axioms to find other unique perspective invariant bargaining solutions. We know any such solution must be between the two solutions we have considered. Separate from the specific answer that we and others may propose, we hope this paper will help focus attention on looking at bargaining problems from a dual perspective: what does each side get and what does each side give up.

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10. Appendix

Lemma 1. *Under Scale Invariance, Efficiency, and Limited Continuity, there exists an* (a,b) *such that the line segment from* $(a,1)$ *to* $(1,b)$ *is tangent to the frontier of the normalized* S *and* $F((a,1),(1,b)) \in S$.

Proof. Define a_m as the maximal value of a such that $(a,1) \in S$. We first cover the trivial case in which $a_m = 1$. In that case, $(a,b) = (1,1)$ and $F((1,1),(1,1)) = (1,1) \in S$. Henceforth, we assume $a_m < 1$. (a_m will be zero except when there is a horizontal component to the frontier of S.)

For each $a \in (a_m, 1)$ there is a unique $b(a)$ such that the line segment from $(a,1)$ to $(1,b(a))$ has a tangency with the Pareto frontier of S. Note that $b(a)$ is continuous and weakly decreasing in a. Define $b(a_m)$ as the limit of $b(a)$ as a approach a_m from above, and $b(1)$ as the limit of $b(a)$ as a approaches 1 from below.

At $a = a_m$, either (i) $F((a_m,1),(1,b(a_m))) \in S$ in which case the result holds or (ii) $F((a_m,1),(1,b(a_m))) \in ((a_m,1),(1,b(a_m)]$ is to the right of the tangent point $(a_m,1)$. $F((a_m,1),(1,b(a_m)))$ can't be to the left of $(a_m,1)$ as $(a_m,1)$ is either the leftmost point when $a_m = 0$ or is Pareto dominated by $(a_m,1)$ when $a_m > 0$ and thus can't be a solution.

At $a = 1$, $F((1,1),(1,b(1))) = (1,1)$ as this is the unique efficient point on the three-part frontier. Having covered the trivial case where $(1,1) \in S$, we know $b(1) < 1$. It directly follows that $F((1,1),(1,b(1))) = (1,1)$ now lies to the left of (above) the tangent point $(1,b(1))$ along the line segment from $(1,1)$ to $(1,b(1))$.

By the Limited Continuity assumption combined with the continuity of $b(a)$, we have $F((a,1),(1,b(a)))$ is continuous in a over $[a_m,1]$. Thus if we are in case (ii), it follows that the solution is initially to the right of the tangent point and eventually to the left of the tangent point. By continuity, there must be some intermediate value of a so that $F((a,1),(1,b(a)))$ coincides with the tangent point (or interval) and hence is an element of S. □

Proposition 3. No Continuity *Under Efficiency, Scale Invariance, Individual Rationality, Limited IIA, and Nash Symmetry, any bargaining solution in the normalized set* S *is contained in the interval on the frontier of* S *such that* $x_1 + x_2 \geq 2x$, *where* (x,x) *is the intersection of the frontier with the 45-degree line.*

Proof. We first create a symmetric bargaining set S_A by considering the three-part frontier defined by the line through (x,x) with slope -1. (The frontier to S_A is also bounded by limiting the maximum payoff to 1 to each party.) Since S_A satisfies the hypothesis of Nash Symmetry, (x,x) is the bargaining solution to this set.

In the case where $S \subseteq S_A$, by Limited IIA it follows that (x,x) is the bargaining solution to S as it is feasible in both sets. The proposition follows immediately.

In the case where $S \nsubseteq S_A$, consider a second bargaining set $S_B = S \cap S_A$. Now we have $S_B \subseteq S_A$ and that $(x, x) \in S_B$ as it in both sets; it follows by Limited IIA that (x, x) is the solution to S_B.

By its definition, it also follows that $S_B \subseteq S$. Thus if the solution to S is feasible for S_B, the two solutions must coincide by Limited IIA. This implies that the solution to S must either be (x, x) or a point infeasible under S_B, as otherwise the implied solution to S_B would be a point other that (x, x) which contradicts the fact that (x, x) is the solution to S_B. Except for (x, x), any bargaining solution to S must lie in the frontier interval that is part of S but not S_B, which are all the points on the frontier of S for which $x_1 + x_2 > 2x$ (as S_B contains all the points in S for which $x_1 + x_2 \leq 2x$. $\square$