Bargaining with Learning[*]

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Abstract

We analyze a continuous-time bargaining game of two-sided incomplete information without time-discounting. Consistent with existing results, no trade occurs in the unique equilibrium. Next we assume that players have imperfect information about their types. We suggest a model for learning about their own types during the bargaining process. Under some conditions, there exist equilibria where trade occurs with positive probability. Moreover, there is an equilibrium that is ex-post efficient. Thus, a very simple model of learning about one's own type can circumvent the Myerson-Satterthwaite theorem. These results continue to hold when the game is extended to allow alternating offers from the buyer and the seller. We also show that the learning model can be closely approximated by a costly preference elicitation model.

JEL Classifications: C78, D83

Keywords: Bargaining, incomplete preferences, imperfect information, bounded rationality, Myerson-Satterthwaite theorem

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[*]I gratefully acknowledge Faruk Gul and Wolfgang Pesendorfer for their suggestions and advice. I have also greatly benefitted from comments by Stephen Chiu, Fahad Khalil, Eric Maskin, Gabor Virag, seminar participants at HKUST, University of Hong Kong, the 9th World Congress of the Econometric Society and, especially, Syed Nageeb Ali. I thank Hong Kong RGC for financial support. All remaining errors are mine.

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1 Introduction

Consider a buyer and a seller negotiating over the price of a used car. Suppose there is no informational asymmetry about the quality of the car and both have private valuations for it. The process of establishing a price is usually modeled as a bargaining game. In Rubinstein's (1982) seminal paper on 2-person sequential bargaining, players have complete information about all players' valuations. In equilibrium, trade occurs if and only if the buyer's valuation is above the seller's valuation. When the valuations of players are unknown to other players, bargaining ceases to be efficient. As a result, the probability of trade in bargaining with incomplete information is lower than that in the complete information case. In an important result, Ausubel and Deneckere (1992) show that no trade occurs at all in the continuous-time limit of the no-gap two-sided incomplete information bargaining game.

In practice, we see that a considerable amount of trade occurs even when offers and counter-offers are made frequently and gains from trade are not guaranteed. One potential reason for such discrepancy between theory and real life observations can be that a player's actions do not convey as much information to the other player in real life situations as standard game theoretic models suggest. Thus, in reality, players do not feel the need to "hide" their types as much. To address such issues in a formal setting, it is natural to look at models of bounded rationality. In this paper, we assume that buyers and sellers do not know their types perfectly. We suggest a learning process for players to learn about their types during the bargaining process. Interestingly, this game has an *ex-post efficient* equilibrium among many equilibria where trade occurs with positive probability.

To illustrate this particular kind of bounded rationality, suppose the buyer and the seller do not know their own reservation values for the car exactly in our used car sale example. The seller may use it as a second car if she does not sell it but is not sure how much she values having a second car. On the other hand, the buyer does not own a car currently and does not exactly know how much value having a car adds to her lifestyle. We suggest that an agent can find the answers to these questions if she thinks about her usage of the car hard enough but that mental exercise is costly. However, the negotiation process makes getting information about her own valuation in a specific manner effortless. We propose a simple model of learning where knowing her minimum possible offer in a bargaining game makes it easy for a player to learn the relation between her valuation and that price. Bargaining is not only a trading mechanism, but is also a tool for learning more about one's own type. Although a player's actions give away some information about her preferences, those also help her to learn more about her own preferences.

In many real life situations, a person may not need to know her preferences completely to participate in trade efficiently. For example, while buying milk from the supermarket, a consumer only needs to know her preferences around the asked price. In our bargaining model, the amount the buyer has to pay (or the seller has to accept) if she makes an offer, determined by a price clock, is obvious to her. That is, she knows exactly how much money is at stake. We assume that this makes it easier for the buyer to reflect on that amount and learn whether her enjoyment from the good is above that. On the other hand, comparing her valuation with a hypothetical amount has a prohibitive cognitive cost and she does not learn about the relation between her valuation and a hypothetical price. She learns about her tastes only if it is necessary.

This is reminiscent of observations that economic agents often do not realize exactly how much they like an object before they have to pay for it. For example, property purchasers frequently realize that they have agreed to way too high a price after they sign the contract and have to make payments. This psychological reaction is commonly known as buyer's remorse and is also common in purchases of other high value items such as cars, computers, jewelry, etc. (Bell, 1967). This is believed to be caused by a sense of doubt that the purchasing decision was correct. In this paper, we suggest that this kind of behavior may arise from new information about a person's preferences, not necessarily from a change in her preferences. Suppose individuals learn their preferences by contemplating about their tastes as suggested by Ergin (2003). Initially in the paper, we assume that contemplating the relation between her valuation and a price (provided by the bargaining process) is costless for an agent while contemplating the relation between her valuation and any other random number is infinitely costly. Later in the paper, we consider a model where the agent can either contemplate about her valuation at the beginning of the game or during the game, as in the main model, but both at some positive cost. Under some parameters, she expends the cost of learning only as it is necessary during the game. Equilibria where trade occurs with positive probability also exist in that model.

The goal of this paper is to show that our bargaining with learning model matches evidence from bargaining processes in real life better than do traditional models where agents know their own types. To illustrate this relatively easily, we construct a simple simultaneous bargaining model without time discounting that is rich enough to capture the essence of standard sequential bargaining games. The purpose of introducing this mechanism is not to explore a new trading mechanism, rather to explore bargaining with learning in an easy-to-analyze setting that also captures the main trade-offs present in standard bargaining models.

In this game, the buyer and the seller are bargaining over an object which they value at v_B and v_S respectively. The exact bargaining process will be clearly described in the next section. The trade price is determined by an ascending clock that both players use to make price offers to the other player. Although we refer to the "time" on the clock to distinguish the prices offered by the buyer and by the seller, this is basically a price clock. While standard bargaining games deals with two issues—the price of trade and the time it takes to reach an agreement, this paper is one-dimensional in the sense that the time needed to reach an agreement is assumed to be trivial and we only look at the occurrence and efficiency of trade. The movement of the price clock happens instantaneously in the standard sense of *time*. The bargaining process can be thought of as a process completed very fast where transactions occur at a pre-specified time in the future. No matter how long it takes to negotiate, that does not affect the final payoff—for example, a potential home buyer is negotiating with a seller before moving on to the next seller where the transaction will take place couple of days later anyway.

We first analyze the complete information game. That is, both players know that $v_B = 1$ and $v_S = 0$. In the simple bargaining model in Section 2, the game ends after any of the player makes an offer. The outcome of the unique equilibrium resembles an ultimatum game where the buyer or the seller get to be the proposer with equal probability. In a more generalized setting, in Section 4, where agents can make alternating offers, players share the pie equally. This outcome resembles that of a Rubinstein (1982) bargaining game without time discounting. When the valuations of each player are drawn from the same support and are private information, trade occurs with zero probability whether we allow only one or multiple offers. This is consistent with findings of Ausubel and Deneckere (1992).

In section 3, we assume that the players do not know their own valuations. However, if the buyer knows that the minimum amount she has to pay for the good in the remainder of the game is $\overline{p}$ then she can costlessly learn whether $v_B \geq \overline{p}$. Similarly, if the seller knows that the maximum amount she can receive is $\overline{p}$ then she can costlessly learn whether $v_S \leq \overline{p}$. As the game progresses, they get more information about their valuations. This way of gradual learning about one's own preference from a "posted" price is similar to the learning process in Hossain (2008). Theorems 3 and 5 show that, in both single-offer and alternating-offer settings, there is an equilibrium where trade occurs if and only if $v_B \geq v_S$. Section 5 shows that the learning model suggested in this paper can be rationalized by a model where offer prices or time do not play any direct informative role at all. As information acquisition is costly, players acquire more information about their own types only during the game in some equilibria where trade occurs with positive probability.

Laffont and Maskin (1979) first showed that ex-post efficiency and incentive compatibility are incompatible in the allocation of public and private goods. Myerson and Satterthwaite (1983) show that there exists no ex-post efficient incentive compatible mechanism for bargaining with common support. We will refer to these results as the *Myerson-Satterthwaite* result in the remainder of the paper. We show that this result may not hold when players' imperfect information about their own types change during the bargaining process. The change in uncertainty in our model comes from changes in the possible trade prices unlike in the model by Yildiz (2002) where uncertainty about the value of an asset is exogenously reduced over time. Our model also has a flavor of information acquisition in private-value auctions as in Compte and Jehiel (2004, 2007) and Rezende (2005).

The next section introduces the formal bargaining model and analyzes the benchmark cases of complete information and two-sided incomplete information. Section 3 analyzes the game of two-sided incomplete information where players do not know their valuations and learn about those while participating in the game. Section 4 discusses an alternating-offer bargaining model. Section 5 analyzes the situation where the cost of learning a player's type perfectly is finite and learning more about her types during the game is costly. Section 6 concludes the paper and all proofs are in the appendix.

2 The Model

A seller (S) and a buyer (B) are bargaining over the price of an object which they value at v_B and v_S respectively. The price offers in this game are determined by a stoppable ascending clock running from 0 to 1. The mark on the clock is denoted by $p \in [0, 1]$ and we will usually refer to it as *time* even though the only function of time is in determining the price.[1] At time p, if no offer has been made yet, a player may stop the clock and make an offer or let the clock continue ticking. If the buyer stops the clock, she makes an offer to buy the good at price p. If the seller stops the clock, then she makes an offer to sell the good at price $1 - p$. Suppose an offer is made and accepted at time p. Then the buyer and the seller get utilities of $v_B - \overline{p}$ and $\overline{p} - v_S$ respectively. Here $\overline{p} = p$ if the buyer makes the offer and $\overline{p} = 1 - p$ if the

[1] We do not refer to p as *price* because, for any p, the offer price depends on whether the buyer or the seller makes the offer.

seller makes the offer. If the offer is rejected, the game ends without trade and both players get zero utility. If both players stop the clock at time p then one of them is chosen to make the offer with equal probability. That is, the buyer gets to offer p or the seller gets to offer $1-p$ with probability $\frac{1}{2}$. If neither player makes an offer at any $p \in [0,1]$ then the game ends without any trade.

Any time during the game, a player has two possible actions: letting the clock tick and stopping the clock to make an offer. If the other player makes an offer, then she also has two actions: accepting or rejecting that offer. This game ends after the first offer is made irrespective of whether the offer was accepted. Hence, the game still continuing at time p implies that no offer has been made so far. Thus, a player's relevant strategy is history independent. We consider only pure strategies in this paper. A player's strategy specifies p_i^O and p_i^A, both functions of v_i, such that she makes an offer at $p = p_i^O$ if no offer has been made so far and accepts an offer made by the other player at time $p \geq p_i^A$.[2] Thus, commitment is built-in in this mechanism as once a player makes an offer, she commits not to make or accept any offer that is better for the opponent. In section 4, we analyze a bargaining protocol which is more standard in the sense that a player can make many offers. We first introduce the more stylized model in this section because it is more tractable and our main result on bargaining with learning does not depend on the commitment level of the bargaining protocol as shown in theorems 3 and 5.

The only role of time in this game is in determining the term of trade. Time does not correspond to how long it takes to reach an agreement as in the standard bargaining literature. In some sense, the clock can be thought of as merely a method for both players continuously making offers and counter-offers. Once a player stops the clock, she is basically saying (credibly) that the current offer is her final offer to the opponent and if that is rejected, she will walk away from the table without a deal. Thus, players make many non-serious offer-counteroffers and once a player makes a serious take-it or leave-it offer, the game ends.

2.1 Complete Information Bargaining Game

First we assume that there is complete information; both players know the other player's valuation and $v_B > v_S$. Without loss of generality, we assume that $v_B = 1$ and $v_S = 0$.[3] We look at subgame perfect equilibria in pure strategies. In the unique equilibrium of this game, both players stop the clock and make offers at time zero.

[2] We ignore strategies where a player accepts some offer while rejecting a better offer.

[3] Otherwise, the time on the clock can be denoted by $p \in [v_S, v_B]$ where the buyer offers to pay p at time p and the seller offers to accept $v_B - p$ at time p.

Theorem 1 *In the unique equilibrium of the complete information bargaining game, both players stop the clock at* $p = 0$*. Trade occurs at price 0 or at price 1 with equal probability.*

Given that the seller makes an offer at $p = 0$ thus offering to accept 1 and ending the game immediately, stopping her clock immediately is a best response for the buyer and vice versa. The buyer offers to pay 0 and the seller offers to accept 1. One of the offer is chosen with probability $\frac{1}{2}$ and it is accepted. The equilibrium outcome of this game resembles the outcome of an ultimatum game where either of the players get to be the offer maker with equal probability.

2.2 Standard Two-sided Incomplete Information Case

Next we analyze the game of two-sided incomplete information with fully rational players. Buyer's valuation v_B and seller's valuation v_S are independently drawn from $[0, 1]$ according to distributions F_B and F_S respectively. Following Cramton (1992), we assume that distributions F_B and F_S are symmetric. That is, $\Pr(v_B \leq y) = \Pr(v_S \geq 1 - y)$ or $F_S(y) = 1 - F_B(1 - y)$ for $y \in [0, 1]$. Let us denote F_B by F for notational convenience. Distribution F, which is common knowledge, is strictly increasing and twice continuously differentiable on $[0, 1]$. The seller and the buyer know their own valuations but do not know the other player's valuation. This allows us to compare the equilibria of our bargaining game to those resulting from a standard sequential bargaining model.

An *equilibrium* of this game is defined as a Perfect Bayesian Equilibrium in pure strategies. Since the game ends (with or without trade) once an offer is made, we only need to specify strategies along the path where no offer has been made so far. If the seller makes an offer at time p, it is optimal for the buyer to accept the offer if and only if $v_B \geq 1 - p$. Similarly, the seller accepts an offer at time p if and only if $v_S \leq p$. We show that in the unique equilibrium of this game, both players stop the clock and make offers at time zero.

Theorem 2 *With two-sided incomplete information where players know their own valuations, both players stopping the clock at* $p = 0$ *is the unique equilibrium. Trade will occur with zero probability.*

Given that the seller stops the clock at $p = 0$ thus ending the game immediately, stopping the clock at $p = 0$ is a best response for the seller and vice versa. The buyer offers to pay 0 and the seller offers to accept 1. Either offer is rejected with probability 1. Suppose there is an equilibrium that continues beyond $p = 0$.

If the buyer follows an equilibrium strategy then, her actions for high enough v_B conveys too much information about v_B to the seller. The seller can exploit the information and profit by making an earlier offer (less attractive to the buyer) than the equilibrium strategy prescribes. Hence, there is no equilibrium where the game continues beyond $p = 0$ and trade occurs with positive probability.

In some sense, this result is comparable to the "No Trade Theorem" in Ausubel and Deneckere (1992). Theorem 2 implies that when traders do not discount delayed transaction and gains from trade is not guaranteed for any $v_i \in [0, 1]$, there will be no trade at all even in a simple setting as in this paper. This is contrary to the fact that, in practice, a significant amount of trade goes on even in bargaining games where delay between offers is negligible.

3 Learning While Bargaining

In this section, we assume that a player does not know her own valuation in addition to not knowing the other player's valuation. However, she learns about her valuation during the process of negotiating the price. Thus, the nature chooses v_i and player i receives more information about v_i with time as time and the offer prices are closely related in this bargaining process. At time p, a seller learns the relation between her valuation and the price if she were to stop the clock and make an offer right at that time. That is, the buyer learns whether $v_B > p$ or $v_B \leq p$ and the seller learns whether $v_S < 1 - p$ or $v_S \geq 1 - p$. In addition, when a player makes an offer, the other player gets a signal at that price offer. If the seller makes an offer of $1 - p$, the buyer learns whether $v_B > 1-p$ and if the buyer makes an offer of p, the seller learns whether $v_S < p$. Thus, players continuously get signals about their valuations. A player only observes her own signals and not the other player's signals. In the rest of the paper, a negative signal at time p for the buyer stands for her learning that $v_B \leq p$ ($v_S \geq 1 - p$ for the seller). On the other hand, a positive signal at time p means she learns that $v_B > p$ ($v_S < 1 - p$ for the seller). Once a player gets a negative signal, all her future signals are negative. If at time p, the buyer's signal turns from positive to negative then she learns that $v_B = p$ and if the seller's signal turns from positive to negative then she learns that $v_S = 1 - p$.

At any time p, a player deciding between making an offer or not implies that no offer has been made so far. When a player receives an offer, her acceptance decision depends only on the offer and her own type as the game ends immediately. Hence, we can restrict attention to history independent strategies. Since the buyer does not know her exact type at time $p < v_B$, her action at time p can depend only on the relation between v_B and p at $p < v_B$ and on v_B only at $p \geq v_B$. Similarly, the seller's action at time p can depend only on the relation between v_S and $1-p$ at $p < 1 - v_S$ and on v_S only at $p \geq 1 - v_S$.

There is no difference in the action spaces of the players between the models in Section 2 and the model in this section. The only difference between the two sections is that player i has imperfect information about her own valuation v_i in this section. Over time, the player receives more information about v_i but their underlying preferences do not change. The players are rational apart from the fact that they have incomplete information about their own types and this is common knowledge to both players.

In this game, there exist equilibria where trade occurs with positive probability. To characterize strategies for such equilibria, we will mostly look at a particular class of strategies. As we mentioned earlier, the simple structure allows us to look at strategies that are independent of the public history. Player i's strategy at time p depends on her information about v_i which we denote by $I_{i,p}$. For example, $I_{B,p}$ is $v_B \in (p, 1]$ if the buyer has not gotten any negative signal up to time p. On the other hand, if the seller makes an offer at time p then $I_{B,p}$ tells us whether $v_B \in (1-p, 1]$ or $v_B \in [0, 1-p]$. If the buyer's signal turns from positive to negative at time p then $I_{B,p}$ is $v_B = p$.

Buyer's strategy β_τ is such that she makes an offer if she gets a negative signal or the clock reaches time $p = \tau$ and she accepts any offer by the seller if the asked price is below v_B. That is, she stops the clock at time $p < \tau$ if she learns that $v_B = p$ or at time τ if $v_B \geq \tau$. If the seller makes an offer to accept a price of $1-p$ (this may occur only if $v_B \geq p$) then the buyer accepts the offer if $v_B \geq 1-p$. Seller's strategy σ_τ is symmetric. Strategies β_τ and σ_τ can be formally expressed as

$$\beta_\tau\left(I_{B,p}\right) = \begin{cases} \text{Accept } S\text{'s offer} & \text{if } v_B \in (1-p, 1] \text{ and } S \text{ makes an offer at time } p \\ \text{Reject } S\text{'s offer} & \text{if } v_B \in [0, 1-p] \text{ and } S \text{ makes an offer at time } p \\ \text{Offer to pay } p & \text{if } v_B = p \text{ or } p = \tau \\ \text{Let the clock tick} & \text{if } v_B \in (p, 1] \text{ and time } p < \tau \end{cases}$$

and

$$\sigma_\tau\left(I_{S,p}\right) = \begin{cases} \text{Accept } B\text{'s offer} & \text{if } v_S \in [0, p) \text{ and } B \text{ makes an offer at time } p \\ \text{Reject } B\text{'s offer} & \text{if } v_S \in [p, 1] \text{ and } B \text{ makes an offer at time } p \\ \text{Offer to accept } 1-p & \text{if } v_S = 1-p \text{ or } p = \tau \\ \text{Let the clock tick} & \text{if } v_S \in [0, 1-p) \text{ and } p < \tau. \end{cases}$$

Strategy profile (β_0, σ_0) where both players make offers at $p = 0$ continues to be an equilibrium. In addition, the buyer and the seller following strategies $\beta_{\frac{1}{2}}$ and $\sigma_{\frac{1}{2}}$ respectively is an equilibrium when F is concave.

Theorem 3 *If F is concave and players learn about their own valuations during the bargaining game, then $\left(\beta_{\frac{1}{2}}, \sigma_{\frac{1}{2}}\right)$ is an equilibrium.*

The equilibrium $\left(\beta_{\frac{1}{2}}, \sigma_{\frac{1}{2}}\right)$ does not exist in the fully rational case discussed in section 2.2. When she does not know her type perfectly, player $i \in \{B, S\}$ gets more information about her type v_i as time passes. However, the offers she can make become less attractive to her and more attractive to the other player. If the buyer knows that her valuation is higher than the current price p then it is better to wait for her and make an offer more attractive to the seller in the future if p is low enough. If she learns that her valuation equals her current price offer p, she makes an offer at that price as she cannot do better otherwise. At $p = \frac{1}{2}$, the cost of waiting overtakes the benefit and she offers to pay $\frac{1}{2}$ even if $v_B > \frac{1}{2}$. Therefore, the buyer plans to make a non-zero offer with probability 1. The same is true for the seller. If a seller receives a price offer of p, she accepts the offer if $v_S \leq p$.

In this equilibrium, a player stops the clock as soon as she gets a negative signal. The buyer accepts an offer at time p if $v_B \geq 1 - p$ and the seller accepts an offer at time p if $v_S \leq p$. Trade will take place with positive probability and if trade occurs then $v_B \geq v_S$. In fact, proposition 1 shows that trade occurs if and only if $v_B \geq v_S$. Thus, equilibrium $\left(\beta_{\frac{1}{2}}, \sigma_{\frac{1}{2}}\right)$ is *ex-post efficient.*

Proposition 1 *The equilibrium $\left(\beta_{\frac{1}{2}}, \sigma_{\frac{1}{2}}\right)$ is ex-post efficient. In this equilibrium, trade occurs with probability $1 - \int_0^1 F(1-y)\, dF(y)$.*

If a player makes an offer at time $p < \frac{1}{2}$, she offers her valuation as the price. The player receiving the offer accepts it if an only if trade is efficient. Both players making offers at $p = \frac{1}{2}$ implies $v_B \geq \frac{1}{2}$ and $v_S \leq \frac{1}{2}$; that is, $v_B \geq v_S$ and trade is efficient in that case. Trade occurs with certainty when both players stop the clock at $p = \frac{1}{2}$. Therefore, this equilibrium is ex-post efficient. Thus, bounded-rationality in the form of players not knowing their own types perfectly can indeed circumvent the Myerson-Satterthwaite result.

Theorem 2 showed that there will be no trade when the players know their valuations. This implies that trade occurring with positive probability results from players learning about their valuations during the bargaining process and not just from the non-standard bargaining mechanism. When players have perfect knowledge about their own valuations, any strategy of player i where she does not make an offer at $p = 0$ would eventually give away too much information about v_i and warrant deviation from player j. When player i has imperfect knowledge of v_i, she has too little information about v_i herself for player j to learn too much about v_i from her strategies. This leads to the positive result that trade may occur in equilibrium.

In equilibrium, any transaction price $\overline{p} \in (0,1)$ can be reached. If $v_B = \overline{p} \in \left(0, \frac{1}{2}\right)$ and $v_S \leq \overline{p}$ then the transaction price is $\overline{p}$. The probability that $v_S \leq \overline{p}$ is $1 - F(1-\overline{p})$. If $v_B \geq \frac{1}{2}$ and $v_S \leq \frac{1}{2}$ then the final price is $\frac{1}{2}$. If $v_S = \overline{p} \in \left(\frac{1}{2}, 1\right)$ and $v_B \geq \overline{p}$ then the final price equals $\overline{p}$. The probability that $v_B \in [\overline{p}, 1]$ is $1 - F(\overline{p})$. Therefore, trade can occur at any price between 0 and 1.

Proposition 2 *The expected price conditional on a trade is*

$$\frac{\int_0^{\frac{1}{2}} y\,(1 - F(1-y))\,dF(y) + \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^y z f(1-z)\,dz dF(y) + \frac{1}{2}\left(1 - F\left(\frac{1}{2}\right)\right)^2}{1 - \int_0^1 F(1-y)\,dF(y)}.$$

When F is concave, there exist many equilibria that lead to trade with positive probability but are not ex-post efficient. Suppose the seller's strategy $\sigma_{a,b}$ is such that if she learns that $v_S = 1 - p$ for $p \in [0,a] \cup \left[b, \frac{1}{2}\right]$ then she stops the clock and makes an offer at time p. If $v_S \in (1-b, 1-a)$ then she does not make any offer at all. If $v_S < \frac{1}{2}$ then she stops the clock and makes an offer of $\frac{1}{2}$ at time $\frac{1}{2}$. If the buyer makes an offer at time p then she accepts the offer if $v_S \leq p$. Similarly, the buyer's strategy $\beta_{a,b}$ is such that if $v_B = p \in [0,a] \cup \left[b, \frac{1}{2}\right]$ then she stops the clock and makes an offer at time p. If $v_B \in (a,b)$ then she does not make an offer. If $v_B > \frac{1}{2}$ then she stops the clock and makes an offer of $\frac{1}{2}$ at time $\frac{1}{2}$. If the seller makes an offer at time p then she accepts the offer if $v_B \geq 1 - p$. There exists $\widetilde{b} < \frac{1}{2}$ such that $\left(\beta_{0,b}, \sigma_{0,b}\right)$ is an equilibrium for any $b \leq \widetilde{b}$. Thus, there exists infinitely many equilibria where trade occurs with positive probability when F is concave. For example, when F is uniform, $\left(\beta_{0,b}, \sigma_{0,b}\right)$ is an equilibrium for all $b \leq \frac{1}{\sqrt[3]{18}}$. If such an equilibrium is played, trade occurs if a player gets a negative signal at $p \in [0,a) \cup \left(b, \frac{1}{2}\right]$ and $v_B \geq v_S$ or if neither gets a negative signal by $p = \frac{1}{2}$ and trade is efficient. If a player gets a negative signal at $p \in [a,b]$ then trade does not occur even if $v_B \geq v_S$. There also exist asymmetric equilibria where either $\beta_{a,b}$ or $\sigma_{a,b}$ is played. If $\left(\beta_{0,b}, \sigma_{0,b}\right)$ is an equilibrium, then $\left(\beta_{\frac{1}{2}}, \sigma_{0,b}\right)$ and $\left(\beta_{0,b}, \sigma_{\frac{1}{2}}\right)$ are equilibria too.

4 An Alternating-Offer Model

The single-offer structure of the bargaining game analyzed in the previous sections is extremely tractable. As the game ends after only one offer is made, optimal strategies are essentially history free. Moreover, decision to accept or reject is belief (about opponent's type) free. This game leads to an interesting result; when players do not know their own valuations and get more information through the bargaining process, there is an ex-post efficient equilibrium. We now show that this result is not an artifact of the fact that only one offer can be made.

In this section, we assume that the buyer and the seller alternate the right to place an offer. Before any offer is made, either player can make an offer. If both players stop the clock at the same time, then one of the offers is chosen with equal probability. If player i makes the first offer then player j gets the right to make the next offer. The right to make offers alternate between the two players until an offer is accepted or the clock reaches $p = 1$. Moreover, the time gap between two offers by a player has to be at least ϕ for a small positive ϕ where $1/\phi$ is finite. Suppose the buyer makes the first offer at time p_1. If the seller does not accept the offer, she gets the right to make the next offer. If she stops the clock and makes an offer at time $p_2 > p_1$, then if the buyer rejects that offer, she (the buyer) can make the next offer. This offer by the buyer can be made earliest at time $p_3 \geq p_1 + \phi$. Similarly, a counter-offer to buyer's offer at p_3 can be made by the seller earliest at time $p_2 + \phi$. The sole function of ϕ is to restrict the possibility of any player making infinitely many offers. It does not play a role in getting any of the results presented in this section. The history at time p, when no offers have been accepted yet, lists the timing of all the rejected offers and the players who made those offers. The public history at $p = \tau$, h_τ is a list $\{(i, p_1), (j, p_2), (i, p_3), \ldots, (k, p_n)\}$ where $p_n \leq \tau$, $p_l > p_{l-1}$, $p_l \geq p_{l-2} + \phi$, $i, j \in \{B, S\}$ with $i \neq j$ and $k = i$ if n is odd and $k = j$ if n is even. After an offer is accepted, the history is irrelevant for player strategies.

Once again, we start with the complete information game where both players know that $v_S = 0$ and $v_B = 1$. There is, essentially, a unique subgame perfect equilibrium of this game. In that equilibrium, one or both of the players stop the clock to make an offer of $\frac{1}{2}$ at $p = \frac{1}{2}$ and the offer is accepted.

Theorem 4 *In the unique equilibrium outcome of the complete information bargaining game with alternating offers, trade occurs at price* $\frac{1}{2}$.

In the equilibrium, the buyer (and the seller) follows the strategy that she rejects any offer at $p < \frac{1}{2}$, makes her first offer at $p = \frac{1}{2}$ and accepts any offer made by the opponent at $p \geq \frac{1}{2}$. This equilibrium is analogous to the equilibrium suggested by Rubinstein (1982) at the limit when the common time discounting factor goes to one. On the other hand, the single-offer game leads to an equilibrium that is analogous to an ultimatum game where the offer maker is chosen randomly. Thus, the equilibrium outcome with complete information is different between the single-offer and alternating-offer games.

There is no equilibrium where an offer is accepted at $p \neq \frac{1}{2}$. If player i's offer at $p = \tau < \frac{1}{2}$ is accepted in an equilibrium, then player j can be better off by rejecting that offer and making her own offer at $p \in (\tau, 1 - \tau)$. If player i's offer at $p > \frac{1}{2}$ is accepted, then she can be better off by making a slightly earlier offer.

The alternating-offer and single-offer games lead to the same outcome with two-sided incomplete information where agents know their own types. In both games, no trade occurs in equilibrium. The structure of the alternating-offer game is very similar to that of an alternating-offer sequential bargaining game at the limit where time between two offers is infinitesimal. This result is not surprising as the Ausubel-Deneckere result of no trade can easily be extended to the case where both players can make offers. As such, we do not present a formal statement of the result.

In the remainder of this section, agents are boundedly rational like the agents in Section 3. At any time p, an agent can costlessly learn the relation between her valuation and her corresponding offer at that time. If the other player makes an offer, she also learns the relation between her valuation and that offer. Again, we look at perfect Bayesian equilibria in pure strategies.

The buyer's strategy $\widetilde{\beta}_\tau$ is such that if she has the right to make an offer and she believes $\Pr(v_S \leq p) > 0$ then she makes an offer at $p < \tau$ if $v_B = p$ and at time $p = \tau$ if $v_B \geq \tau$. If the seller makes an offer at time p, she accepts the offer if $v_B \geq 1 - p$. That is $\widetilde{\beta}_\tau$ is just like strategy β_τ with the addition that the buyer makes an offer only if she believes that the offer is going to be accepted with positive probability. Similarly, the seller's strategy $\widetilde{\sigma}_p$ is such that if she has the right to make an offer and she believes $\Pr(v_B \geq 1 - p) > 0$ then she makes an offer at time $p < \tau$ if $v_S = 1 - p$ and at time τ if $v_S \leq 1 - \tau$. She accepts the buyer's offer made at time p if $v_S \leq p$. The buyer and the seller following strategies $\widetilde{\beta}_{\frac{1}{2}}$ and $\widetilde{\sigma}_{\frac{1}{2}}$ respectively is an equilibrium of this alternating-offer game.

Theorem 5 *If F is concave and players learn about their types while bargaining, then $\left(\widetilde{\beta}_{\frac{1}{2}}, \widetilde{\sigma}_{\frac{1}{2}}\right)$ is an equilibrium of the alternating-offer bargaining game.*

When equilibrium $\left(\widetilde{\beta}_{\frac{1}{2}}, \widetilde{\sigma}_{\frac{1}{2}}\right)$ is played, both players make at most one offer. A player makes an offer at time $p < \frac{1}{2}$ upon a negative signal. The other player accepts it if it is profitable for her. If neither player has made an offer at all by $p = \frac{1}{2}$, both make an offer at time $\frac{1}{2}$. This offer is accepted. The equilibrium path is the same as that with profile $\left(\beta_{\frac{1}{2}}, \sigma_{\frac{1}{2}}\right)$ in the single-offer game. Thus, this equilibrium is ex-post efficient. Trade occurs if and only when there is benefit from trade. If players do not know their own valuations and learn about those during the bargaining process, an efficient equilibrium exists even when we extend the model to allow many offers.

5 A Costly Contemplation Model of Learning

In this section, we consider a model where a player has to pay some contemplation cost to get more information about her valuation. She can either contemplate about her valuation at the beginning of the game or during the game, both at some cost. If she contemplates before the start of the game, she can learn about her valuation at any level of precision by paying a cost according to the precision level. If she contemplates during the game, she gets less precise information at any given point of time; but she can decide whether to get more information many times throughout the game. We are interested in knowing if there is any condition under which $\left(\beta_{\frac{1}{2}}, \sigma_{\frac{1}{2}}\right)$ remains an equilibrium of the single-offer game in this costly contemplation model. If $\left(\beta_{\frac{1}{2}}, \sigma_{\frac{1}{2}}\right)$ is not an equilibrium, then we want to find out whether there is any equilibrium where trade occurs with positive probability. As we want to characterize what happens to this equilibrium at different levels of contemplation costs, we only analyze the case where the valuations are drawn from uniform distributions to provide concrete examples. Nevertheless, the qualitative result will go through for any concave F.

We assume that, before the game begins, a player can learn whether $v_i \in [v_0, v_1]$ at a cost of $c\,(v_1 - v_0)$ where she learns v_i exactly if $v_i \in [v_0, v_1]$ and learns whether $v_i < v_0$ or $v_i > v_1$ otherwise for any $v_1 \geq v_0 + \Delta$. After the game starts, the buyer decides whether to contemplate v_B around the interval $[p, p + \Delta]$ at a cost of $c\Delta$ at any time p. That is, she learns the exact v_B if $v_B \in [p, p + \Delta]$ and the fact that $v_B > p + \Delta$ otherwise if she chooses to contemplate. Similarly, the seller can learn v_s exactly if $v_S \in [1 - p - \Delta, 1 - p]$ and $v_S < 1 - p - \Delta$ otherwise at a cost of $c\Delta$. Moreover, once a player receives an offer, she can contemplate around the offer at a cost of $c\Delta$. It is relatively easy to generalize to the situation where, for example, the buyer can contemplate about any interval $[p, v_1]$ with $v_1 \geq p + \Delta$ during the game. We restrict attention to tiny intervals of contemplation around the current offer price for the player to keep the analysis algebraically simpler. Moreover, we will take the limit as Δ approaches zero.

There is no equilibrium of this game that is ex-post efficient. As time approaches $\frac{1}{2}$, the difference between expected payoffs from waiting to make an offer till $p = \frac{1}{2}$

and making an immediate offer approaches zero even when contemplation is costless. As a result, for any positive contemplation cost, the ex-post efficient equilibrium suggested in Section 3 cease to exist. However, there is an equilibrium where neither player contemplates about their valuation prior to the game. In that equilibrium, trade occurs with positive probability as both contemplate about their valuations during the game for a positive measure of time. Thus, players incur positive contemplation cost over the game.

If a player contemplates about her valuation before the game, then both will have the incentive to learn their valuations with high precision at the beginning of the game. In that case, the game will become a standard bargaining game with two-sided incomplete information as discussed in Theorem 2. Trade will not occur at all in that case. This will happen when c is small.

For appropriate values of c there will be an equilibrium where trade occurs with positive probability. Let us define the strategy profile $\left(\widehat{\beta}_\rho, \widehat{\sigma}_\rho\right)$ such that neither players contemplate before the beginning of the game. During the game, the buyer contemplates whether $v_B \in [p, p+\Delta]$ when the clock reads p for all $p \in \{0, \Delta, 2\Delta, \ldots, \rho - \Delta\}$ if $v_B > p$ paying a cost of $c\Delta$. If she learns $v_B \in [p, p+\Delta]$ (thus learning the exact v_B), she offers to buy the good at price p. If $v_B > \rho$ and the seller has not made an offer yet, she offers to buy the good at price ρ. If the seller makes an offer to accept $1-p$ at some time $p \leq \rho$, the buyer contemplates v_B around $1-p$ at cost $c\Delta$. The seller's strategy $\widehat{\sigma}_\rho$ can be similarly defined. She contemplates whether $v_S \in [1-p-\Delta, 1-p]$ at time p for all $p \in \{0, \Delta, 2\Delta, \ldots, \rho - \Delta\}$ if $v_S < 1-p$. She stops the clock to make an offer to sell the good at price $1-p$ if she learns $v_S \in [1-p-\Delta, 1-p]$. If $v_S < 1-\rho$ then she stops the clock at time ρ and offers the good at price $1-\rho$. If $\left(\widehat{\beta}_\rho, \widehat{\sigma}_\rho\right)$ is an equilibrium with $\rho = \frac{1}{2}$ then the equilibrium is ex-post efficient. We show in Proposition 3 that $\left(\widehat{\beta}_\rho, \widehat{\sigma}_\rho\right)$ is an equilibrium for a positive ρ smaller than $\frac{1}{2}$.

Proposition 3 *If $c \in (0.0286583, 0.5)$ then $\left(\widehat{\beta}_\rho, \widehat{\sigma}_\rho\right)$ is an equilibrium of the bargaining game with costly contemplation at the limit as Δ approaches zero where ρ solves $\frac{(1-2\rho)^2}{2(1-\rho)^2} = c$.*

If $c < 0.0286583$ then the buyer will contemplate whether $v_B \in \left[0, \min\left\{\frac{1+\rho}{2}, 2\rho\right\}\right]$ at the beginning of the game if the seller follows $\widehat{\sigma}_\rho$. As a result, learning during the game will not occur in equilibrium. On the other hand, if $c \geq 0.5$ then neither of the players will contemplate at the beginning of or during the game and trade will not occur in that case either as both have the same (expected) valuation. When

$c \in (0.0286583, 0.5)$ and $\left(\widehat{\beta}_{\rho}, \widehat{\sigma}_{\rho}\right)$ is played, trade occurs with probability ρ and the player with the higher valuation gets the object with probability $\frac{1}{2}+2\rho-2\rho^2$. Notice that even though the cost of each contemplation during the game is infinitesimal, the overall contemplation cost to a player over the game is not zero as she expends many such small contemplation costs during the game. The expected contemplation cost for each player in this equilibrium is $\left(\rho-\rho^2+\frac{1}{3}\rho^3\right)c$.

Thus, learning from the price during the game can be closely be replicated by using a costly contemplation model. As information is costly, they opt to gather information only when it is necessary. This reduces signaling value of their actions and facilitates trade.

6 Conclusion

Introducing the notion of imperfect information about own type, this paper suggests that buyers and sellers may use bargaining as a tool for learning their valuations. As well as getting to know about the opponents, people get to know about themselves during the process of bargaining. Bargaining occurs in continuous time and learning while bargaining is frictionless. This leads to an equilibrium that is ex-post efficient.

With a simple model of bounded rationality, we show that bargaining models may actually perform better in assigning a good to the person who values it the most than what the standard models predict if agents cannot perfectly compute their own type. Players perfectly surmising opponent's types from their observable actions leads to surprising equilibrium characteristics in many game theoretic models. The very stylized model introduced in this paper provides an example of how imperfect information about one's preferences may allow us to get around that problem. This model also provides a first step in analyzing more general and standard bargaining games when there is learning.

A Appendix

Proof of Theorem 1

Proof. Given that player i stops the clock at $p=0$, stopping the clock immediately when the game starts is the unique best response for player j. Stopping the clock at $p=0$, player j gets an expected payoff of $\frac{1}{2}$ and otherwise she gets zero payoff. As a result, both the buyer and the seller stopping the clock at $p=0$ and making price offers of 0 and 1 respectively is an equilibrium.

Now suppose there exists an equilibrium where none of the bidders make an offer at $p = 0$. Without loss of generality, suppose the buyer makes an offer at $p = \tau \geq 0$ while the seller does not make an offer before $p = \tau$. Given individual rationality, the seller accepts any offer at a strictly positive time in any equilibrium. Therefore, making an offer at $p = \tau - \epsilon$ for $\tau > \epsilon > 0$ is a profitable deviation for the buyer. Thus, there does not exist any equilibrium where the first offer is made at $p > 0$. Also, none of the players making an offer is not an equilibrium because by making an offer at $p > 0$ a player can get a payoff of $1 - p$. Hence, both players stopping the clock at $p = 0$ is the unique equilibrium. ■

Proof of Theorem 2

Proof. Given that the other player stops the clock at $p = 0$, stopping the clock immediately when the game starts is a best response irrespective of the player's type. As a result, both the buyer and the seller stopping the clock at $p = 0$ and making price offers of 0 and 1 respectively is an equilibrium.

To prove that no other equilibrium exists, in case (i), we assume that for some $i \in \{B, S\}$, player i plans to make an offer at some point during the bargaining game for all $v_i \in [0, 1]$. We discuss offer schedules where, for both i, there exist intervals $[a_i, b_i]$ such that player i makes no offer if $v_i \in [a_i, b_i]$ in case (ii).

Since only one offer can be made in the game, in an equilibrium player i accepts player j's offer at time p if $v_B \geq 1 - p$ when $i = B$ or $v_S \leq p$ when $i = S$. Suppose there is an equilibrium where player i's equilibrium strategy can be characterized by $p_i : [0, 1] \to \{N\} \cup [0, 1]$ that determines at what time player i makes an offer if player j has made no offer so far. If player i does not make any offer if $v_i = v$ then $p_i(v) = N$ and if she plans to make an offer at time $\widetilde{p} \in [0, 1]$ then $p_i(v) = \widetilde{p}$. For trade to occur with positive probability, there must a v and i such that $p_i(v) \in (0, 1]$.

Therefore, if the game is still continuing at time $p = \gamma$, the expected payoff for B from making an offer at time $\theta \geq \gamma$ is

$$\begin{aligned}
& (v_B - \theta) \Pr(S \text{ makes no offer at } p \in [\gamma, \theta] \text{ and } v_S \leq \theta | S \text{ made no offer at } p \in [0, \gamma]) \\
& + (v_B - \mathbf{E}[1 - p | p \in [\gamma_a, \theta]]) \Pr(S \text{ makes an offer at } p \in [\gamma_a, \theta] \, | S \text{ made no offer in } p \in [0, \gamma]) \\
= \quad & (v_B - \theta) \frac{1 - G(\theta)}{1 - G(\gamma)} H(\theta) + \frac{1}{1 - G(\gamma)} \int_{\gamma_a}^{\theta} (v_B - 1 + p) \, dG(p)
\end{aligned}$$

where $\gamma_a = \max[\gamma, 1 - v_B]$, $G(y)$ is the probability that the seller makes an offer at $p \in [0, y]$ with $G(1) \leq 1$ and $H(\theta)$ is the probability that $v_S \leq \theta$ given that the seller made no offer at $p \in [0, \theta]$. Therefore, $\mathbf{E}[1 - p | p \in [\gamma_a, \theta]]$ is the buyer's expected payment if the seller makes an offer at $p \in [\gamma_a, \theta]$ according to the equilibrium offer schedule. Taking derivatives with respect to v_B and θ, we get

$$(1 - G(\theta)) H'(\theta) + (1 - H(\theta)) G'(\theta)$$

which is positive.[4] If $\theta \leq 1 - v_B$ then expected payoff of B is

$$(v_B - \theta) \Pr (S \text{ makes no offer at } p \in [\gamma, \theta] \text{ and } v_S \leq \theta | S \text{ made no offer at } p \in [0, \gamma)) .$$

At $p = \theta$, her expected payoff is $(v_B - \theta) \Pr (v_S \leq \theta | S$ made no offer at $p \in [0, \theta])$. If the buyer makes no offer then her expected payoff at $p = \gamma$ is

$$\Pr (S \text{ makes an offer at } p \in [\gamma_a, 1] | S \text{ made no offer in } p \in [0, \gamma)) (v_B - \mathbf{E} [1 - p | p \in [\gamma_a, 1]]) .$$

The seller's expected payoff can be found similarly. Notice that the probability of getting an offer at some time interval is independent of v_B. Given these expected payoffs, the players' preferences satisfy the single-crossing property. Therefore, if schedules p_B and p_S exist then p_B is non-decreasing in v_B and p_S is non-increasing in v_S. That is, if $p_i(a_i), p_i(b_i) \in [0, 1]$ and $a_i < b_i$, then $p_B(b_B) \geq p_B(a_B)$ and $p_S(a_S) \geq p_S(b_S)$. Moreover, if it is optimal for the buyer not to make an offer when $v_B = a_B$, that is, $p_B(a_B) = N$, then $p_B(v_B) = N$ for all $v_B \geq a_B$. Similarly, if it is optimal for the seller not to make an offer when $v_S = b_S$ then it is optimal for her not to make an offer for all $v_S \leq a_B$.

Case (i): Suppose τ_i is the maximal stopping time for player i in an equilibrium where trade occurs with positive probability and $\tau = \min[\tau_B, \tau_S] > 0$. Lemma 1 shows that $\tau \leq \frac{1}{2}$. Next we will show that there is also no equilibrium where $\tau \in \left(0, \frac{1}{2}\right]$. Without any loss of generality, we assume the seller plans to make an offer for all $v_S \in [0, 1]$. That is, $p_S(v) \in [0, 1]$ for all $v \in [0, 1]$.

First we assume that there is an $\eta < \tau_S$ such that for $v \in [\eta, 1]$, $p_S(v) \leq \eta$ as Figure 1 illustrates.[5] Suppose $v_B = \widetilde{v}$ where $p_B(\widetilde{v}) > \eta$. At time η, if the seller has not made an offer yet, B deduces that $v_S \leq \eta$. Hence, if B makes an offer, it will be accepted with probability 1. That is, the buyer's expected payoff from making an offer at $p = \eta$ is $v_B - \eta$. If she makes an offer at $p = p_B(\widetilde{v})$ or accepts an offer by the seller between time η and $p_B(\widetilde{v})$, she gets a lower payoff. Hence it is optimal for the buyer to deviate from her equilibrium stopping time $p_B(\widetilde{v})$ and making an offer at $p = \eta$. Then p_B cannot be an equilibrium offer schedule for the buyer. Therefore,

[4] We implicitly assume G and H are differenriable on $(0, 1)$. The results would go through even when they are piecewise differentiable.

[5] This happens, for example, if t_S is strictly decreasing.

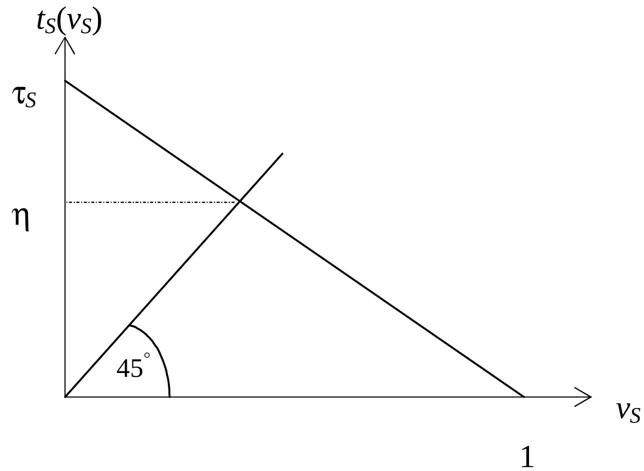

Figure 1: Seller's stopping time as a function of v_S

there is no equilibrium with a stopping schedule such that there is an $\eta < \tau_S$ where $p_S(v) \leq \eta$ for all $v_S \leq \eta$.

Now suppose, as Figure 2 illustrates, there exists no $\eta < \tau_S$ such that for $v \in [0, \eta]$, $p_S(v) \geq \eta$. Suppose for $v_S \leq \nu$ for $p_S(v_S) = \tau_S$. First suppose $\tau_S \geq \frac{1}{2}$ which implies $\nu \geq \tau_S \geq \frac{1}{2}$ (If $\nu < \tau_S$ then there exists $\eta < \tau_S$ such that for $v \leq \eta$, $p_S(v) \geq \eta$). If $v_S = \nu$, the seller gets negative or zero payoff from making an offer at $p = \tau_S \geq \frac{1}{2}$. She can be better off by making an earlier offer (such that her offer exceeds $1 - p > \nu$). Therefore, $\tau_S < \frac{1}{2}$.

First we assume that $\tau_S = \tau_B = \tau$. If $v_S = \nu$, then expected payoff of S from stopping the clock at $p = \tau$ if B has not stopped the clock yet is given by

$$\frac{1}{2}(1 - \tau - \nu)\Pr(v_B \geq 1 - \tau).$$

If S stops the clock at $\tau - \epsilon$ for a small ϵ, her expected payoff is

$$(1 - \tau + \epsilon - \nu)\Pr(v_B \geq 1 - \tau + \epsilon).$$

For small ϵ and $\tau > 0$, the seller will be better off by deviating.

Now we assume that $\tau_S \neq \tau_B$. Suppose (p_B, p_S) is an equilibrium and without loss of generality, assume $\tau_B > \tau_S = \tau$. That means that if v_B is that $p_B(v_B) \geq \tau$ then the probability that B will make an offer or her offer will be accepted (if $p_B(v_B) = \tau$) is zero.[6] Therefore, $(\widetilde{p}_B, p_S)$ where $\widetilde{p}_B(v) = \max[p_B(v), \tau]$ is also

[6] If the probability that B's offer at τ is accepted is positive then (t_B, t_S) is not an equilibrium because then for B with high v_B, it is better to choose $t_B(v_B)$ to be τ instead of higher than τ.

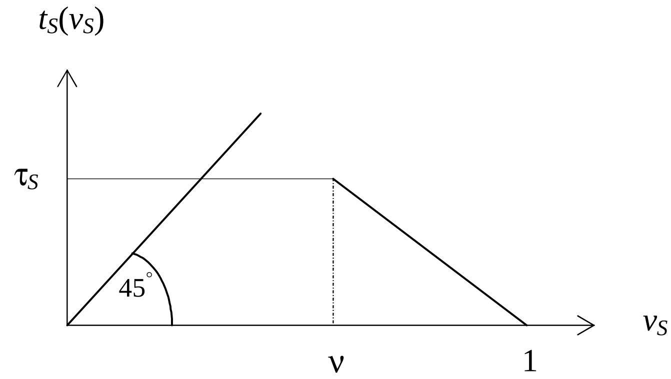

Figure 2: Seller's stopping time as a function of v_S

an equilibrium schedule profile because that leads to the same outcome as (p_B, p_S). Then $\max_v \widetilde{p}_B(v) = \max_v p_S(1-v)$. Using the argument in the previous paragraph, we can show that $(\widetilde{p}_B, p_S)$ cannot be an equilibrium. That implies that (p_B, p_S) is not an equilibrium either.

Case (ii): We show that there is no equilibrium where both players make no offer with some probability. Let there be an equilibrium where for both i, there exist intervals $[a_i, b_i]$ such that $p_i(v_i) = N$ for $v_i \in [a_i, b_i]$ with $b_i > a_i$.[7] This implies that $a_B > 1 - \tau_S$ and, similarly, $b_S < \tau_B$. Otherwise, the buyer gets zero payoff from not not making an offer at any p and gets positive payoff if she makes an offer at some time p if $v_B = a_B$. Using the single-crossing property, $b_B = 1$ and $a_S = 0$. Without loss of generality, we assume $\tau_S \leq \tau_B$. Suppose at $p = \tau_S$ the seller has made no offer. Then for some $v^* \in (a_B, 1)$, $v^* > \tau_S$ and an offer at that time will be accepted by the seller with positive probability. Therefore, the buyer will make an offer right after $p = \tau_S$ if the seller has made no offer even if $v_B = v^*$. Therefore, there is no equilibrium where both bidders plan to make no offer with positive probability.

Hence, both players stopping the clock at $p = 0$ is the unique equilibrium. ■

Lemma 1 *For at least one of the players, the maximal stopping time is below or equal to* $\frac{1}{2}$.

Proof. Let $\tau = \min[\tau_B, \tau_S] > 0$ where τ_i is the maximal stopping time of player i and $\tau > \frac{1}{2}$. That is, both players make an offer after time $\frac{1}{2}$ with positive probability.

[7]The arguments in this section hold when the intervals are open.

Suppose the seller makes an offer for all possible values of v_S. If none of the players have made an offer at $p = \frac{1}{2}$, player B knows that player S will make an offer at some $p \in \left(\frac{1}{2}, 1\right]$. Then, it is better for B not to make any offer at time $p \in \left(\frac{1}{2}, 1\right]$. Instead she lets S make an offer of $1-p$ at $p > \frac{1}{2}$. Hence, $\tau \in \left(0, \frac{1}{2}\right]$.[8] ■

Proof of Theorem 3

Proof. Suppose the seller follows the strategy $\sigma_{\frac{1}{2}}$. Knowing that $v_B > p$, the buyer's expected utility from staying following $\beta_{\frac{1}{2}}$ is

$$\frac{\left(1 - F\left(\frac{1}{2}\right)\right) \int_{\frac{1}{2}}^{1} \left(z - \frac{1}{2}\right) dF(z) + \int_{\frac{1}{2}}^{1-p} \int_{y}^{1} (z-y) f(z) f(1-y)\, dz dy}{(1 - F(p))^2}. \tag{I}$$

Her expected utility from stopping the clock at time $p \leq \frac{1}{2}$ is

$$\frac{(1 - F(1-p)) \int_{p}^{1} (z-p)\, dF(z)}{(1 - F(p))^2}. \tag{II}$$

Knowing that $v_S < p$, the seller's expected payoffs at time p from $\sigma_{\frac{1}{2}}$ and from stopping the clock at time p (σ_p) are given by I and II respectively. At $p = \frac{1}{2}$ the buyer's (and the seller's) expected utility from either B or S stopping the clock are the same. Knowing that seller follows $\sigma_{\frac{1}{2}}$, at $p = \frac{1}{2}$, stopping the clock is a best response for the buyer even if $v_B > \frac{1}{2}$.

Now we need to ensure that the buyer does not prefer β_p over $\beta_{\frac{1}{2}}$ at some $p < \frac{1}{2}$. It is enough to show that for all $p \in \left[0, \frac{1}{2}\right]$,

$$\left(1 - F\left(\frac{1}{2}\right)\right) \int_{\frac{1}{2}}^{1} \left(z - \frac{1}{2}\right) dF(z) + \int_{\frac{1}{2}}^{1-p} \int_{y}^{1} (z-y) f(z) f(1-y)\, dz dy$$
$$- (1 - F(1-p)) \int_{p}^{1} (z-p)\, dF(z) \geq 0.$$

The left hand side is strictly positive if $p = 0$ and equals zero if $p = \frac{1}{2}$. Differentiating the left hand side with respect to p, we get

$$-f(p) \int_{1-p}^{1} (z - 1 + p)\, dF(z) - f(1-p) \int_{p}^{1} (z-p)\, dF(z) + (1 - F(1-p))(1 - F(p)).$$

Integrating by parts, this equals

$$-f(p) \int_{1-p}^{1} (1 - F(z))\, dz - f(1-p) \int_{p}^{1} (1 - F(z))\, dz + (1 - F(1-p))(1 - F(p)).$$

[8] If $\tau = 0$ then trade occurs with zero probability.

Thus, showing that

$$\begin{aligned}
-\frac{f(p)\int_{1-p}^{1}(1-F(z))\,dz+f(1-p)\int_{p}^{1}(1-F(z))\,dz}{(1-F(p))(1-F(1-p))}+1 &\leq 0\\
\Rightarrow \int_{1-p}^{1}\frac{f(p)(1-F(z))}{(1-F(p))(1-F(1-p))}dz+\int_{p}^{1}\frac{f(1-p)(1-F(z))}{(1-F(p))(1-F(1-p))}dz &\geq 1.
\end{aligned}$$

is sufficient to prove that I≥II.

Using lemma 2,

$$\begin{aligned}
&\int_{1-p}^{1}\frac{f(p)(1-F(z))}{(1-F(p))(1-F(1-p))}dz+\int_{p}^{1}\frac{f(1-p)(1-F(z))}{(1-F(p))(1-F(1-p))}dz\\
\geq\ &\int_{1-p}^{1}\frac{1-z}{(1-p)p}dz+\int_{p}^{1}\frac{1-z}{(1-p)p}dz=\frac{1}{(1-p)p}\left(\frac{p^2}{2}+\frac{(1-p)^2}{2}\right)\geq 1.
\end{aligned}$$

Therefore, $\beta_{\frac{1}{2}}$ is a best response to $\sigma_{\frac{1}{2}}$ and vice versa and $\left(\beta_{\frac{1}{2}},\sigma_{\frac{1}{2}}\right)$ is an equilibrium when F is concave. ■

Lemma 2 *When F is concave,*

$$\frac{f(p)(1-F(z))}{(1-F(p))(1-F(1-p))}\geq\frac{1-z}{(1-p)p}$$

for $z\in[p,1]$ where $p\leq\frac{1}{2}$

Proof. Notice that if $z=p$ then, given concavity of F,

$$f(p)\geq f(1-p)\geq\frac{1-F(1-p)}{p}$$

and when $z=1-p$,

$$\frac{f(p)}{(1-F(p))}\geq\frac{1}{(1-p)}.$$

When, $z=1$, the left hand side equals the right hand side. Notice that the derivative of the left hand side is $\frac{-f(p)f(z)}{(1-F(p))(1-F(1-p))}$ and the derivative of the right hand side is $\frac{-1}{(1-p)p}$. That is, if the right hand side derivative is larger for some z then it is larger for all $z'\geq z$ when F is concave. This implies that if

$$\frac{f(p)(1-F(z))}{(1-F(p))(1-F(1-p))}<\frac{1-z}{(1-p)p}$$

for some z then it will hold true for all $z'\geq z$. Since

$$\frac{f(p)(1-F(z))}{(1-F(p))(1-F(1-p))}\geq\frac{1-z}{(1-p)p}\qquad\text{(III)}$$

for $z=p$ and $z=1$, equation III holds for all $z\in[p,1]$. ■

Proof of Proposition 1

Proof. Suppose $v_B \leq \frac{1}{2}$ or $v_S \geq \frac{1}{2}$. Then, if $v_B = \tau \leq 1 - v_S$ then the buyer gets a negative signal first and makes an offer of τ at time τ which the seller accepts only if $v_S < \tau$. If $v_S = 1 - \tau > 1 - v_B$ then the seller gets a negative signal first and makes an offer of $1 - \tau$ at time τ which the seller accepts only if $v_B > 1 - \tau$. If $v_B > \frac{1}{2}$ and $v_S < \frac{1}{2}$. then $v_B > v_S$. In that case, both stop the clock at $p = \frac{1}{2}$ and accept the offer of $\frac{1}{2}$. Therefore, the equilibrium $\left(\beta_{\frac{1}{2}}, \sigma_{\frac{1}{2}}\right)$ is ex-post efficient. Moreover, the probability of trade is $\int_0^1 \Pr(v_S \leq y)\, dF(y)$ equaling $1 - \int_0^1 F(1-y)\, dF(y)$. ■

Proof of Proposition 2

Proof. Trade occurs if and only if $v_B \geq v_s$. Conditional on trade, the price is $\frac{1}{2}$ if $v_B \geq \frac{1}{2}$ and $v_S \leq \frac{1}{2}$. Otherwise, conditional on trade, the price equals v_B if $v_B < \frac{1}{2}$ and v_S if $v_B \geq \frac{1}{2}$. Hence, the expected price conditional on trade is

$$\frac{\int_0^{\frac{1}{2}} \int_0^y y f(1-z)\, dz dF(y) + \int_{\frac{1}{2}}^1 \left(\int_{\frac{1}{2}}^y z f(1-z)\, dz + \int_0^{\frac{1}{2}} \frac{1}{2} f(1-z)\, dz \right) dF(y)}{1 - \int_0^1 F(1-y)\, dF(y)}$$
$$= \frac{\int_0^{\frac{1}{2}} y(1 - F(1-y))\, dF(y) + \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^y z f(1-z)\, dz dF(y) + \frac{1}{2}\left(1 - F\left(\frac{1}{2}\right)\right)^2}{1 - \int_0^1 F(1-y)\, dF(y)}.$$

■

Proof of Theorem 4

Proof. In this equilibrium, the buyer (and the seller) follows the strategy that she rejects any offer at $p < \frac{1}{2}$, makes her first offer at $p = \frac{1}{2}$ and accepts any offer made by the other player at $p \geq \frac{1}{2}$. Given the opponent's strategy, it is a best response for a player to make an offer at $p = \frac{1}{2}$, reject any offer at $p < \frac{1}{2}$ and accept the opponent's offer if it is made at $p \geq \frac{1}{2}$.

There is no equilibrium where an offer is accepted at $p \neq \frac{1}{2}$. Suppose there exists an equilibrium where none of the bidders make an offer. If a player deviates and makes an offer at $p \in (0,1)$ then that offer (or some other future offer) has to be accepted in any equilibrium strategy as the payoff from following the proposed equilibrium of no offers is zero for both players. Hence, one of the players will deviate and make an offer. Similarly, there cannot exist an equilibrium where offers are made but none are accepted.

Now, without loss of generality, suppose there is an equilibrium where the buyer makes an offer at $p = \tau$ and the seller accepts the offer. If $\tau < \frac{1}{2}$, then the buyer's strategy must include that she rejects any offer by the seller made at $p \in (\tau, 1-\tau)$. Suppose the seller deviates by rejecting the buyer's offer at $p = \tau$ and then makes a counter-offer at $p = \tau_S \in (\tau, 1-\tau)$.[9] In this subgame at time τ_S, if the buyer rejects the seller's offer then she must make a future offer at $p \geq \tau + \phi$ in any equilibrium strategy because otherwise she gets zero payoff. Then, the seller gets higher payoff by rejecting the buyer's offer at $p = \tau$. Thus, the seller accepting the buyer's offer at $p = \tau < \frac{1}{2}$ cannot be an equilibrium.

If $\tau > \frac{1}{2}$ then, instead of making an offer at $p = \tau$, if the buyer makes an offer at $p = \tau - \epsilon$ for a positive $\epsilon \leq \frac{1}{2} - \tau$, the seller will either accept the offer or make a counter-offer. Either way, the buyer will be better off than making an offer at $p = \tau$ that is accepted.[10] Thus, player i accepting player j's offer at $p = \tau$ can be an equilibrium only if $\tau = \frac{1}{2}$ and in the unique equilibrium an offer is made and accepted at $p = \frac{1}{2}$. ■

Proof of Theorem 5

Proof. We show that there is a belief structure such that a player makes at most one offer and accepts the other player's offer if trading at that price gives her positive payoff when $\left(\widetilde{\beta}_{\frac{1}{2}}, \widetilde{\sigma}_{\frac{1}{2}}\right)$ is played.

The beliefs that arise from the equilibrium path is enough to ensure that $\left(\widetilde{\beta}_{\frac{1}{2}}, \widetilde{\sigma}_{\frac{1}{2}}\right)$ is an equilibrium. If the seller makes an offer at time p, B believes that $v_S = 1-p$ if $p < \frac{1}{2}$ and $v_S \leq \frac{1}{2}$ if $p = \frac{1}{2}$. If the seller rejects the buyer's offer at time p, B believes that $v_S > p$. Similarly, if B makes an offer at time p, S believes that $v_B = p$ if $p < \frac{1}{2}$ and $v_B \geq \frac{1}{2}$ if $p = \frac{1}{2}$. If the buyer rejects the seller's offer at time p, S believes that $v_B < 1-p$.

If B makes an offer at time $p < \frac{1}{2}$, given her beliefs, it is optimal for the seller to accept p if and only if $v_S \leq p$. If $v_S > p$, then not making any offer after the buyer's offer is optimal for S as the buyer rejects any price offer above p according to S's belief. Similarly, if S learns that $v_S = 1-p$ at time p, then it is optimal for her to make an offer immediately. After her offer is rejected, she will not make any more offer as she gets negative payoff if the offer is accepted. If S makes an offer at $p = \gamma < \frac{1}{2}$, B does not make any offer after rejecting S's offer (if $v_B < 1-\gamma$) because S rejects any offer below $1-\gamma$. At $p = \frac{1}{2}$, it is optimal for S to make an offer given

[9] Here we assume that the seller's prior offers, if any, were made before time $1 - \tau - \phi$. If this does not hold, she can always change her bid schedule to make sure that this holds true and the logic used in the contradiction will go through.

[10] Here we assume that the buyer's prior offers, if any, were made before time $\tau - \phi$.

the buyer's strategy. As a player does not make an offer at $p > \frac{1}{2}$, rejecting an offer at $p = \frac{1}{2}$ if $v_S < \frac{1}{2}$ ($v_B > \frac{1}{2}$) is not optimal.

Suppose the seller deviates and makes an offer at time $\gamma < \frac{1}{2}$ even though $v_S < 1 - \gamma$, the buyer accepts if and only if $v_B \geq 1 - \gamma$. If the buyer rejects that offer, she (B) does not make any more offers. As a result, the seller does not get to make any more offers. Thus, if B follows strategy $\widetilde{\beta}_{\frac{1}{2}}$ then S can make at most one offer and if S follows strategy $\widetilde{\sigma}_{\frac{1}{2}}$ then B can make at most one offer. Therefore, S's expected payoff from making an offer at time p is same as it would be in the one offer case. As theorem 3 shows, at time $p < \frac{1}{2}$, it is better for S to let the clock tick if $v_S < p$. Hence, $\widetilde{\sigma}_{\frac{1}{2}}$ is a best response to $\widetilde{\beta}_{\frac{1}{2}}$. The buyer's problem is symmetric to the seller's problem. Using similar arguments, we can show that $\widetilde{\beta}_{\frac{1}{2}}$ is a best response to $\widetilde{\sigma}_{\frac{1}{2}}$. Therefore, $\left(\widetilde{\beta}_{\frac{1}{2}}, \widetilde{\sigma}_{\frac{1}{2}}\right)$ is an equilibrium even when players can make alternating offers as she wants before an offer is accepted. ■

Proof of Proposition 3

Proof. First we show that the buyer will make tiny contemplations at intervals of Δ until time ρ or until he gets a negative signal where $\frac{(1-2\rho)^2}{2(1-\rho)^2} = c$ instead of making an offer immediately at any time $p < \rho$. If $v_B \geq p$, if the buyer does not contemplate whether $v_B \in [p, p + \Delta]$ at a cost of $c\Delta$, then she will make an immediate offer earning an expected payoff of $\frac{p}{2}$. If she contemplates and learns that $v_B \in [p, p + \Delta]$, she immediately stops the clock making an expected payoff of $\frac{p\int_p^{p+\Delta}(y-p)dy}{(1-p)^2}$. If $v_B > p$ then she lets the clock tick till $p + \Delta$ and then takes a similar decision. If she let's the clock tick then the seller makes an offer of $1 - p$ if $v_S \in [1 - p - \Delta, 1 - p]$.[11] Then the buyer's expected payoff is

$$\begin{aligned}
& \frac{p\int_p^{p+\Delta}(y-p)\,dy}{(1-p)^2} + \frac{(p+\Delta)\int_{p+\Delta}^{1}(y-p-\Delta)\,dy}{(1-p)^2} + \frac{\Delta\left(\int_{1-p}^{1}(y-1+p)\,dy - c\Delta\right)}{(1-p)^2} - c\Delta \\
= \; & \frac{p}{2} + \Delta\left(\frac{(1-2p)^2 + \Delta(-2+4p-c) + \Delta^2}{2(1-p)^2} - c\right).
\end{aligned}$$

When Δ is very small, the buyer will contemplate in the interval $[p, p + \Delta]$ if $\frac{(1-2p)^2}{2(1-p)^2} \geq c$. At any time before ρ, it is more profitable for the buyer to contemplate at least one more time than not contemplating and making an offer. By symmetry, we can show that the seller will also contemplate until time ρ unless she gets a

[11] We assume that each player assumes that the other player moves $\psi \to 0$ time later than she does so she herself has the opportunity of making the offer first to avoid ties that arise from the discreteness of contemplation even when Δ is arbitrarily small.

negative signal. If players follow $\left(\widehat{\beta}_\rho, \widehat{\sigma}_\rho\right)$, then a player's expected payoff at the beginning of the game at the limit as $\Delta \to 0$ is:

$$\begin{aligned} & -c\int_0^\rho y\left(1-\frac{y}{2}\right)dy + \rho\int_\rho^1 \left((y-\rho) - c\left(1-\frac{\rho}{2}\right)\right)dy + \int_{1-\rho}^1\int_y^1 (z-y)\,dzdy \\ = \; & \frac{1}{6}\left(3\rho(1-\rho)^2 + \rho^3 - 2\rho\left(3-3\rho+\rho^2\right)c\right). \end{aligned}$$

On the other hand, if the seller follows $\widehat{\sigma}_\rho$ then if the buyer contemplates at the beginning of the auction, she will contemplate about the interval $\left[0, \min\left\{\frac{1+\rho}{2}, 2\rho\right\}\right]$. She learns v_B exactly if $v_B \leq \min\left\{\frac{1+\rho}{2}, 2\rho\right\}$ and learns that $v_B > \min\left\{\frac{1+\rho}{2}, 2\rho\right\}$ otherwise. Suppose $\rho \geq \frac{1}{3}$. If $v_B \leq \frac{2}{3}$, the buyer will make an offer at $\widetilde{p}$ that maximizes $(v_B - \widetilde{p})\,\widetilde{p}$ and if $v_B > \frac{2}{3}$ then she will make an offer at $\min\left\{\widetilde{p}, \frac{1}{2}\right\}$ where $\widetilde{p}$ maximizes $(v_B - \widetilde{p})\,\widetilde{p} + \int_{1-\widetilde{p}}^{v_B}(v_B - y)\,dy$. The buyer's optimal offer schedule is:

$$\widetilde{p}(v_B) = \begin{cases} \frac{v_B}{2} & \text{if } v_B \leq \frac{2}{3} \\ 2v_B - 1 & \text{if } \frac{2}{3} \leq v_B \leq \frac{1+\rho}{2} \\ \rho & \text{otherwise.} \end{cases}$$

The buyer's expected payoff from the above strategy is:

$$\begin{aligned} & \int_0^{\frac{2}{3}} \frac{y}{2}\left(y - \frac{y}{2}\right)dy + \int_{\frac{2}{3}}^{\frac{1+\rho}{2}}\left((y-2y+1)(2y-1) + \int_{2-2y}^{y}(y-z)\,dz\right)dy \\ & + \int_{\frac{1+\rho}{2}}^1\left(\rho(y-\rho) + \int_{1-\rho}^y (y-z)\,dz\right)dy - \frac{1+\rho}{2}c \\ = \; & \frac{1}{12}\left(\frac{1}{3} + 3\rho - 3\rho^2 + \rho^3 - 6(1+\rho)c\right). \end{aligned}$$

If $\rho < \frac{1}{3}$ then the buyer makes an offer at time $\frac{v_B}{2}$ if $v_B \leq 2\rho$ and at time ρ otherwise. Her expected payoff from that strategy is:

$$\begin{aligned} & \int_0^{\frac{1+\rho}{2}} \frac{y}{2}\left(y - \frac{y}{2}\right)dy + \int_{\frac{1+\rho}{2}}^1\left(\rho(y-\rho) + \int_{1-\rho}^y (y-z)\,dz\right)dy - 2\rho c \\ = \; & \frac{1}{32} + \frac{7}{32}\rho - \frac{5}{32}\rho^2 - \frac{1}{96}\rho^3 - 2\rho c. \end{aligned}$$

Now $\frac{(1-2\rho)^2}{2(1-\rho)^2} = c$ implies $\rho = \frac{2-2c-\sqrt{2c}}{4-2c}$ and we can show that following $\left(\widehat{\beta}_\rho, \widehat{\sigma}_\rho\right)$ is better than contemplating at the beginning of the game if $c \geq 0.0286583$. However, the players will not contemplate at all, even during the game, if $c \geq 0.5$. Therefore, $\left(\widehat{\beta}_\rho, \widehat{\sigma}_\rho\right)$ is an equilibrium where trade occurs with positive probability if $c \in (0.0286583, 0.5)$. The probability of trade occurring is

$$\int_0^\rho y\,dy + \int_{1-\rho}^1 (1-y)\,dy + (1-\rho)\,\rho = \rho.$$

Moreover, the outcome will be efficient with probability

$$\int_0^{\rho} dy + \int_{\rho}^{1-\rho} (\rho + 1 - y)dy + \int_{1-\rho}^{1} dy = \frac{1}{2} + 2\rho - 2\rho^2.$$

■

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