A Markovian continuous review inventory model with leadtimes and disasters

Opher Baron, Oded Berman, David Perry

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Abstract

We consider a continuous review (s, S) model with lost sales in the presence of unexpected events. Such unexpected events include external disasters (e.g. theft or earthquakes) and internal disasters (e.g. malfunctioning of a storage equipment or fire). Once a disaster occurs the inventory drops instantaneously to zero. The total cost includes the cost of: ordering, unsatisfied demand, units destroyed, holding and fixed cost of disasters. Both the time between disasters and leadtimes are assumed to be exponentially distributed while two cases of demand distribution are considered: Poisson and compound Poisson with general demand sizes. We study the average cost criterion and discuss an extension for the discounted cost criterion. We also provide asymptotic expressions for some of the cost functionals and computational results on the problem of finding the optimal re-order, s, and order up to, S, levels. None of the known work on the subject is as general as the model presented here. We compared the cost resulting from the exact analysis to the one of a heuristic that ignores disasters (and is also analyzed here). This comparison shows that ignoring disasters is more costly when demand variability is high and, not surprisingly, when disasters are frequent.

Subject Classification: Inventory/Production: Perishable/aging items, Uncertainty: Stochastic

Area of Review: Stochastic Processes

1 Introduction

Recently, there has been growing attention in the literature on the importance of planning a resilient supply chain. Sheffi (2005) discusses how should businesses best prepare for confronting unexpected events. However, much of the recent discussion in this literature is on a high level and there is little guidance on how should the firm take its daily operational decisions, such as inventory and capacity decisions, in order to improve its resiliency. In contrast, there is much literature on stochastic inventory theory, starting with the seminal paper of Arrow, Harris and Marschak (1951), presenting a multi-period stochastic inventory problem. Since then, there have been several books on inventory management, including Whitin (1957), Zipkin (2000) and Porteus (2002). These books and many papers on inventory control typically assume that items have an infinite lifetime. That is, a vast majority of inventory theory ignores unexpected events. In this paper we aim at filling some of this gap. We consider a continuous review inventory model in the presence of unexpected events that cause the inventory level to instantaneously drop down to zero.

Unexpected events include: external disasters such as thefts, earthquakes, weather related (e.g., a hurricane, a tsunami or a super storm) and terrorism; internal disasters such as fires, malfunctioning of storage equipment and strikes; and emergency orders from other vendors that are willing to generously compensate the supplier for the order. We believe that even though an individual unexpected event is not likely to occur frequently, the occurrence of any unexpected event may not be that rare and better be taken into account when making inventory control decisions. We model the time between consecutive unexpected events as exponentially distributed and once they occur all items (if any) are removed from the shelf.

Specifically, we consider a continuous review (s, S) inventory model with lost sales and aim at cost minimizing. Other common controls such as (Q, r) and models with backlogging are also possible, but are more complicated for analysis and we leave their investigation for future work. The total cost includes the costs of: ordering, unsatisfied demand, units destroyed, holding, and fixed cost of disasters. The stochasticity of the system is due to the occurrence of unexpected events (which will be referred to from now on also as *disasters*), the leadtime, and the demand. As mentioned earlier, both the leadtime and the time between disasters are exponentially distributed and the demand follows a compound Poisson with general distributed demand sizes. We focus on the average cost criterion where we obtain exact closed form expressions for all the cost functionals required. We also discuss approximations required for the discounted cost criterion.

The (s, S) inventory model we consider in this paper is comprehensive as it includes the following components together: compound Poisson demand, stochastic leadtime and unexpected events. To the best of our knowledge, no other article is that general. As a matter of fact even the special case of our model with general demand size without disasters (which is easily derived by taking the rate of disasters to be zero), as far as we know, has never been investigated in the literature.

In our derivation, we use three methodologies: renewal theory, level crossing theory, and Markov chains. Utilizing intricate derivations enables us to provide exact, closed form, analysis for this comprehensive model. The closed form results allow an immediate optimization of the values of the reorder point, s, and order up to level, S. We provide numerical results (and asymptotic results in the on-line supplement) for the optimal cost and controls of systems with disasters. We further compare these results to the ones of a heuristic that ignores disasters. We show that for the Poisson demand case, the heuristic of ignoring disasters is quite effective (with a typical error of less than 1%) if disasters are not too frequent. In contrast, in the compound Poisson demand case, the heuristic performed quite poorly (errors between 7% to 113%). Therefore, we conclude that considering disasters in inventory control is important as the variability of the demand size increases. Of course such consideration is also important if disasters are quite frequent.

Our plan for the paper is as follows. We next provide a brief literature review, Then, the model is presented in Section 2. The cost functionals are derived in Section 3. Examples of Poisson and compound Poisson with exponential demand sizes are discussed in Section 4. In Section 5 we focus on optimization and computational results. In Section 6 we discuss an extension of the model with the discounted cost criterion. We conclude the paper in Section 7 We provide asymptotic analysis of both models in the on-line Appendix.

1.1 Literature Review

The continuous review (s, S) policy with lost sales is presented in Archibald (1981) using the average cost criterion, assuming constant leadtime, and a discrete compound Poisson demand. The continuous review (s, S) policy with compound Poisson demand, exponential demand sizes, and with zero leadtimes is discussed in Presman and Sethi (2006) for both the average and discounted cost criteria. Continuous

(s, S) policies with discrete demand and backlogging for the average cost criterion is discussed in Federgruen and Zipkin (1985). In the above models it is assumed that there are no disasters and the rate at which the inventory is depleted is the demand rate. There is much literature were inventories are perishable in the sense that they are subject to *decay*, *obsolescence*, *outdating* or *disasters*.

Decay means that as time progresses a fixed fraction of the inventory is vanished. It is either lost continuously over time (under models of continuous review, see e.g., Rajan, Steinberg, and Steinberg (1992), which also consider pricing) or after every planning period (under models of periodic review). More recently, Li et al. (2014) and Xue et al. (2014) studied managing perishable items whose value deteriorates with time by increasing demand using sales. *Obsolescence* means that an item is superseded by an improved version and is typical for high-tech products, software, and maps. In these examples the items themselves do not change, but the environment around them changes and, as a result, their utility declines. Usually the demand for the items is then gradually decreasing and their sale prices are changed accordingly (see e.g., Song and Lau (2004) and references therein). Both *Outdatings* and *Disasters* consider the perishability of individual items where a perishable unit has a binary (0 -1) utility; full utility before perishability and zero utility after. Several surveys of inventory models with perishable items are those of Nahmias (1982), Karaesmen, Scheller-Wolf, and Deniz (2009), and Baron (2011). Outdating means that every item has a specific expiration date and when time exceeds this expiration date the item's utility changed to 0. Most of the literature in the surveys above is devoted to models with Outdatings. We note that such models are similar to models of single server queue with abandonment (see Boxma, Perry, and Stadje (2011) and Perry and Asmussen (1995)). In models with disasters all items stored are subject to external unexpected events that instantaneously bring the utility of all items on the shelf to 0. In practice, disasters are followed by a random set-up recovery time. That is after a random leadtime a new batch of items arrives at the shelf.

Several of the recent continuous (s, S) models with perishability are: Liu and Shi (1999) who considered the average cost case with exponential lifetime of inventory and general renewal demand through a Markov process with zero leadtime. A continuous (s, S) model with random shelf life with a general distribution, zero leadtime and renewal arrivals is discussed in Gürler and Özkayab (2008). A continuous (s, S) model with exponential lifetime, exponential leadtime, Poisson demand, and with backlogging is presented in Liu and Yang (1999) for the average cost case. A similar model, but with lost sales with the restriction that the number of outstanding replenishment orders is at most one, is introduced in Kalpakam and Sapna (1994). Recently, Baron, Berman, and Perry (2010) developed a continuous review $(0, S)$ model of perishable items under compound Poisson demand with either unit or exponentially distributed demand sizes, exponential time to perishability, and zero leadtimes. They discussed perishability due to either disasters or outdating. We note that with zero leadtime there is no reason to order when inventory is positive. Thus, the inclusion of leadtime in the model substantially changes the derivation due to that $s > 0$.

We note that inventory models with *clearings* (such as the ones in Kella, Perry and Stadje (2003), Perry and Stadje (2001), Perry, Stadje, and Zacks (2005), Serfozo and Stidham (1978), and Stidham (1986), are different than those with disasters, because the timing of the clearing is determined by the controller and is therefore dependent on the inventory level. There are several queueing models with *negative customers*, see Gelenbe (1991), Jain and Sigman (1996), and Boucherie, Boxma, and Sigman (1997), that are similar to disasters (e.g., their arrival rate is independent of the inventory level). However, such models focus on performance analysis without considering control of the system.

2 Preliminaries and Problem Formulation

The inventory model we consider assumes an (s, S) control without backlogging. We let $\mathbf{W} = \{W(t) : t \geq 0\}$ denote the content level process. This process, $\mathbf{W}$, is a regenerative process with a step function where the negative jump sizes represent the satisfied demands and the positive jumps are order arrivals. Let T be the generic cycle that is the time between two order arrivals into the warehouse. We assume that $0 \leq \mathbf{W} \leq S$ with $W(0) = S$.

Two important features of our model are the random leadtime L_ξ that is exponentially distributed with a rate ξ and the random time between disasters D_η that is exponentially distributed with a rate η. The leadtime L_ξ is the time it takes from the instant that an order is placed until it arrives at the system and the end of the leadtime is also the end of the cycle for W (i.e., at the end of the leadtime $W = S$). At an instant of a disaster all items (if any) are removed from the shelf. However, some of the disasters are not effective (if $W(t) = 0$). As a result, the times between *effective disasters* (times at which items are removed from the shelf) are not exponentially distributed, but as will be seen in the sequel, are independent and identically distributed (iid) random variables (RV), so that the effective disaster process forms a renewal process.

We consider a compound Poisson demand process, with interarrival times $V_i \sim \exp(\lambda)$, and demand sizes that are iid random variables denoted by Y_i that follow a general distribution, $F_Y(\cdot)$. We demonstrate our derivation and provide numerical results on the two most common (and easy) demand size examples—the Poisson demand, where demand sizes $Y_i = 1$, and—the compound Poisson demand with demand sizes $Y_i \sim \exp(\mu)$ (with Y being the generic RV).

To describe the dynamic of $\mathbf{W}$ we define first the compound Poisson process (with arrival rate λ) $\bar{W}= \{\bar{W}(t) : t \geq 0\}$ where

$$\bar{W}(t) = S - [Y_1 + \cdots + Y_{N(t)}] \tag{1}$$

and $\mathbf{N} = \{N(t) : t \geq 0\}$ is the Poisson arrival process with rate λ. We assume that the three random quantities, time between disasters, D_η, leadtime, L_ξ, and the compound Poisson process, $\bar{\mathbf{W}}$, are independent of each other.

The content level $\mathbf{W}$ is regulated by the process $\bar{W}$ and 4 stopping times. Before presenting the stopping times for $\mathbf{W}$, we introduce several relevant stopping times for the process $\bar{\mathbf{W}}$. Recall that $\bar{W}(0) = S$ and define the stopping times $\tilde{\tau}_{S-s} = \inf\{t > 0 : \bar{W}(t) \leq s\}$, denoting the first time when $\bar{W}$ drops to or below level s, and $\tilde{\tau}_S = \inf\{t > 0 : \bar{W}(t) \leq 0\}$ denoting the first time when $\bar{W}$ drops to or below level 0. Also, let $\tilde{\tau}_{s-Y} = \tilde{\tau}_S - \tilde{\tau}_{S-s} \geq 0$, denote the duration of time from the instant when $\bar{W}$ drops to or below level s and until $\bar{\mathbf{W}}$ drops at or below level 0.

Note that in the Poisson demand case $\bar{W}(\tilde{\tau}_{S-s}) = s$, and when demand sizes are exponential, it follows by the memoryless property that $\bar{W}(\tilde{\tau}_{S-s}) = s - Y$. Moreover, for any demand size, it follows, by the strong Markov property, that $\tilde{\tau}_{S-s}$ and $\tilde{\tau}_{s-Y}$ are conditionally independent. Also, the distribution of $\tilde{\tau}_{s-Y}$ may have an atom at 0. The Laplace Stieltjes Transform (LST) of the stopping times $\tilde{\tau}_{S-s}$, $\tilde{\tau}_{s-Y}$ (where for the Poisson demand case $s - Y = s$), and $\tilde{\tau}_S$ are known before for several demand sizes such as a unit demand or phase type, see e.g., Baron, Berman and Perry (2010).

We next define the 4 stopping times that are associated with $\mathbf{W}$: The first stopping time is the time of a disaster D_η. The second stopping time is

$$\tau_{S-s} = \min\{\tilde{\tau}_{S-s}, D_\eta\}. \tag{2}$$

The stopping time τ_{S-s} can be interpreted as the first time the content level drops to or below level s, either due to demand, when $\tau_{S-s} = \tilde{\tau}_{S-s}$ or due to a disaster, when $\tau_{S-s} = D_\eta$. We assume that orders are initiated whenever $W(t) \leq s$, whether the cause for the decrease is a demand or a disaster, so that after a leadtime L_ξ, an order arrives, bringing the content level back to S; then, the cycle ends. We define the third stopping time as the end of a cycle

$$T = \tau_{S-s} + L_\xi. \tag{3}$$

Finally, the fourth stopping time is

$$\tau_S = \min\{\tilde{\tau}_S, D_\eta\}. \tag{4}$$

The stopping time τ_S can be interpreted as the first time, after τ_{S-s}, when the content level drops to or below level 0. Note that in the compound Poisson case $\tau_S = \tilde{\tau}_{S-s}$ with a positive probability. This occurs whenever $\tilde{\tau}_{s-Y} = 0$, then τ_S may be longer than T.

Formally, we define the content level process $\mathbf{W}$ during any one cycle by the compound Poisson process $\bar{W}$ and the four stopping times as follows (see Figure 1):

$$W(t) = \begin{cases} \bar{W}(t) & 0 \le t \le D_\eta & if \;\; \tau_{S-s} = D_\eta, \\ 0 & D_\eta \le t < T, & (cycle\ 1,\ in\ Figure\ 1), \\ \bar{W}(t) & 0 \le t \le D_\eta & if \;\; \tau_{S-s} < D_\eta \;\; and \;\; D_\eta \le \min(T, \tilde{\tau}_S) \\ 0 & D_\eta \le t < T, & (cycle\ 2,\ in\ Figure\ 1), \\ \bar{W}(t) & 0 \le t \le \tau_S & if \;\; \tau_S < D_\eta \;\; and \;\; \tilde{\tau}_{s-Y} < L_\xi \\ 0 & \tau_S \le t < T, & (cycle\ 3,\ in\ Figure\ 1), \\ \bar{W}(t) & 0 \le t \le T, & if \;\; \tau_S < D_\eta \;\; and \;\; \tilde{\tau}_{s-Y} > L_\xi \\ & & (cycle\ 4,\ in\ Figure\ 1). \end{cases}$$

2.1 Objective Function

We will focus on deriving the average cost per time unit and will discuss extensions of our analysis to the discounted cost in Section 6. Both costs are composed of 4 components: ordering cost (both variable and fixed), holding costs, cost of unsatisfied demand, and cost of disasters. We denote by K_o the order set-up cost, c the cost per item, K_u the cost per unit of an unsatisfied demand, K_d be the fixed penalty for effective disasters (disasters that occur when the shelf is not empty), and h the holding cost for one unit of inventory per time unit. We further denote the time between unsatisfied demands by U and the time between effective disasters by Z.

With these definitions and using the key renewal theorem the average cost can be expressed as follows: First, in the beginning of each cycle the controller pays the set-up ordering cost K_o; resulting

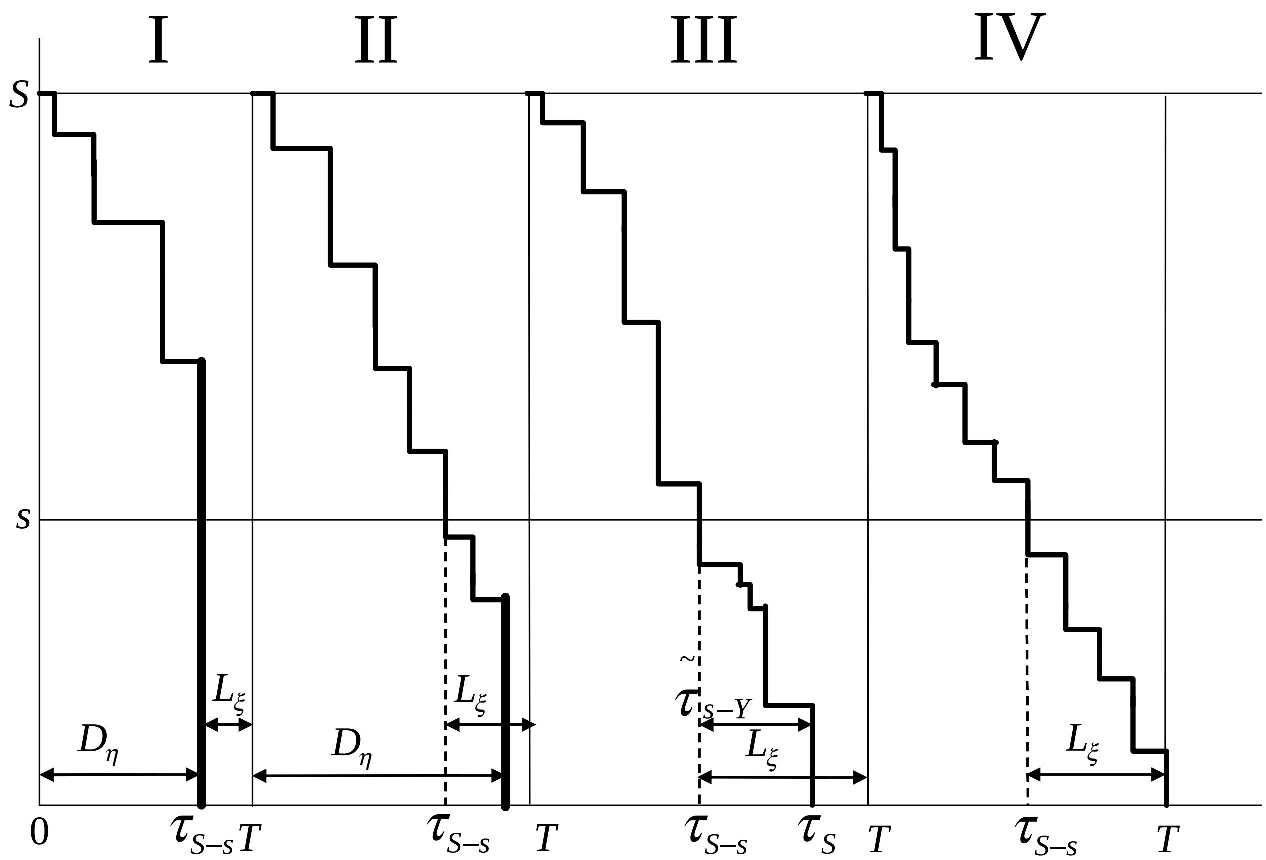

Figure 1: Sample paths of the inventory process for the 4 different possible cycles. The bold line of downwards jumps denote disasters. (This exmaple shows a compound Poisson demand, as can be seen from the different sizes of the orthogonal jumps.)

in an average cost rate of $K_0/E(T)$. Second, because each demand is for $1/\mu$ units on average, demand arrives at a rate λ and is unsatisfied at a rate $1/E(U)$, the variable cost rate of ordering is $\frac{c}{\mu}\left(\lambda - \frac{1}{E(U)}\right)$. In general there are two types of unsatisfied demands: arriving at an empty system, i.e., when $W(t) = 0$, with an average size of $1/\mu$ units, and arriving when there is some but not enough inventory in the system, i.e., $0 < W(t) < Y$, with an average size of $E(Y|Y \geq W(t))$ units. In the Poisson demand case, where $Y = 1$, every unsatisfied demand arrives when $W(t) = 0$ and is for 1 unit. Similarly, in the compound Poisson case with $Y \sim \exp(\mu)$ demands, due to the memoryless property, we have $E(Y) = E(Y - W(t)|Y \geq W(t)) = 1/\mu$. Nevertheless, we characterize the exact rate of unsatisfied demand below. Third, by Poisson Arrivals See Time Averages, each disaster destroys $E(W)$ items on average and these are destroyed at a rate η and a cost c_1. For simplicity, we do not consider any additional cost per items destroyed as this cost is captured via their replenishment cost. Thus, we use $c_1 = c$. Fourth, the holding cost rate is $hE(W)$ and, as in the calculation of the variable ordering cost rate, the expected cost of unsatisfied demand is $K_u/(\mu E(U))$. Finally, the average fixed cost of disasters is $K_d/E(Z)$. Thus, letting R denote the average cost we have:

$$\begin{aligned} R &= \frac{K_o}{E(T)} + c\left(\frac{1}{\mu}\left(\lambda - \frac{1}{E(U)}\right) + \eta E(W)\right) + hE(W) + \frac{K_u}{\mu E(U)} + \frac{K_d}{E(Z)} \\ &= \frac{K_o}{E(T)} + c\frac{\lambda}{\mu} + c\eta E(W) + hE(W) + \frac{K_u - c}{\mu E(U)} + \frac{K_d}{E(Z)}. \end{aligned} \tag{5}$$

Note that the cost of unsatisfied demand (lost sales) is offset by the cost savings from not having to meet this demand, so that the actual cost of a unit of un-met demand is $K_u - c$. From a modeling perspective it is reasonable that $K_u - c > 0$. That is, a unit of lost demand is more costly than a unit of a satisfied one.

In the next sections we analyze the cost components one at a time. To express these costs, let $\mathbf{\Lambda_o} = \{\Lambda_o(t) : t \geq 0\}$ be the counting process of order arrivals, $\mathbf{\Lambda_u} = \{\Lambda_u(t) : t \geq 0\}$ be the counting process of the unsatisfied demand, $\mathbf{\Lambda_d} = \{\Lambda_d(t) : t \geq 0\}$ be the counting process of the effective disasters. Let $F(\cdot)$ denote the steady state probability distribution of the inventory level (this exists as the system is regenerative).

We use three different methodologies to express the different costs. First, we express the LST of T, U, and Z by considering the different stopping times related to $\mathbf{W}$. We derive these LSTs using the laws of the stopping times $\tilde{\tau}_{S-s}$, $\tilde{\tau}_{s-Y}$, and $\tilde{\tau}_S$ for $\bar{\mathbf{W}}$. A critical building block in this derivation is the memoryless property of $\mathbf{W}$; a property that follows because the arrival process is a compound

Poisson and both the leadtime and time to disaster are exponentially distributed. Second, to express the holding cost, we derive $F(\cdot)$, which allows us to calculate $E(W)$ using level crossing theory. This derivation requires solving the functional equation for $F(\cdot)$. For the Poisson demand case we also derive the holding costs rate from the Markov chain representation of the content level process.

Our general derivation, in the next section, holds for any general demand size distribution Y_i. As mentioned above, we focus on the two easiest discrete and continuous demand cases: Poisson demand where $Y_i = 1$ at each arrival and the compound Poisson case with $Y_i \sim \exp(\mu)$. For these cases we provide the exact expressions for the different LSTs and the solution for $F(\cdot)$. We note that while in both cases the solutions results in closed form expressions for the cost function in (5), these expressions are cumbersome and, therefore, some of them are not explicitly written below. Still, the availability of closed form expressions allow us to easily optimize the inventory control parameters, the base stock level, S, and the reorder point, s, and derive some asymptotic results.

For the discounted cost criterion with a discount rate β, we derive the LSTs for most of the expressions and provide approximations for the others, in Section 6.

3 General Derivation of the Cost Functionals

Here, we derive the cost functionals for any compound Poisson demand case in terms of the LST of the different stopping times defined for the well studied $\bar{\mathbf{W}}$ process: $\tilde{\tau}_{S-s}$, $\tilde{\tau}_{s-Y}$, and $\tilde{\tau}_S$. In the next sections we express the LST for the cases of Poisson and a compound Poisson (with exponential demand sizes) processes to derive the corresponding costs. We remind that the expected value of a RV X, denoted by $E(X)$ can be obtained from the LST of X, $E\left(e^{-\beta X}\right)$ as

$$E(X) = -\lim_{\beta \to 0} \frac{d\left(E\left(e^{-\beta X}\right)\right)}{d\beta}.$$

3.1 Ordering Cost Functional

We consider the time between two order arrivals, T, as a generic cycle. We have:

Proposition 1 *The LST of the length of cycles is*

$$E\left(e^{-\beta T}\right) = \frac{\xi}{\xi + \beta}\left(\frac{\eta}{\beta + \eta} + \frac{\beta E(e^{-(\beta+\eta)\tilde{\tau}_{S-s}})}{\beta + \eta}\right).$$

As expected, the length of the cycle is a convolution of the length of the time to order with the exponentially distributed leadtime (with rate ξ). The time until an order is placed is either the time until a disaster or the time until the content level downcrosses s for the process without disasters, $\tilde{\tau}_{S-s}$.

3.2 Unsatisfied Demand Cost Functional

It follows by the strong Markov property that the counting process of the unsatisfied demands is a renewal process. And let U be generic interarrival time between unsatisfied demands. To express $E\left(e^{-\beta U}\right)$ define the random variable X as the time it takes from the instant of order arrival until the next unsatisfied demand. We then have:

Criterion 1

$$X = \begin{cases} D_\eta + \hat{U}, & D_\eta \leq \min\left\{T, \tau_S\right\}, \\ \tau_S + \hat{U}\mathbf{1}_{\{W(\tilde{\tau}_S)=0\}}, & \tau_S < \min\left\{D_\eta, T\right\}, \\ T + \hat{X}, & T \leq \tau_S, \end{cases} \tag{6}$$

where $\hat{U}$ is stochastically equal to U, $\hat{X}$ is stochastically equal to X and all the random variables on the right hand side of (6) are independent.

To understand the idea behind (6) we consider the three stopping times D_η, τ_S, and T. Suppose first that the event $\{D_\eta \leq \min\{T, \tau_S\}\}$ occurred; this event means that the system becomes empty due to disaster and not due to an unsatisfied demand, so that the system is empty before the cycle ends. From that instant the time to the next unsatisfied demand is a probabilistic replication of the time between two unsatisfied demands. Thus, when $\{D_\eta \leq \min\{T, \tau_S\}\}$, X is stochastically equal to $D_\eta + U$. Next, suppose that $\{\tau_S < \min\{D_\eta, T\}\}$; this event means that level 0 is downcrossed before a disaster and also before the cycle ends, so that X equals to τ_S. Note that the indicator function is required for the cases where level 0 is not strictly downcrossed. In these cases, the time to the next unsatisfied demand is a probabilistic replication of the time between two unsatisfied demands. (Such cases can only occur if demand size is discrete) Finally, suppose that $\{T \leq \tau_S\}$; this event means that the cycle ends without an unsatisfied demand, the content level is filled up to level S and the time to the next unsatisfied demand is stochastically equal to X.

Proposition 2 *The LST's of X and U satisfy the two equations:*

$$E\left(e^{-\beta U}\right)=\frac{\lambda}{\lambda+\beta+\xi}+\frac{\xi}{\lambda+\beta+\xi}E\left(e^{-\beta X}\right). \tag{7}$$

$$\begin{aligned} E\left(e^{-\beta X}\right) &= \frac{\frac{E\left(e^{-\beta U}\right)\eta}{\eta+\beta}\left(1-E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\left(\frac{\xi+(\beta+\eta)E(e^{-(\beta+\eta+\xi)\tilde{\tau}_{s-Y}})}{\beta+\eta+\xi}\right)\right)}{\left(1-E(e^{-(\eta+\beta)\tilde{\tau}_{S-s}})\frac{\xi\left(1-E(e^{-(\xi+\beta+\eta)\tilde{\tau}_{s-Y}})\right)}{\xi+\eta+\beta}\right)} \\ &+\frac{E(e^{-(\eta+\beta)\tilde{\tau}_{S-s}})E(e^{-(\xi+\beta+\eta)\tilde{\tau}_{s-Y}})\left(P\left(W\left(\tilde{\tau}_S<0\right)\right)+E\left(e^{-\beta U}\right)P\left(W\left(\tilde{\tau}_S=0\right)\right)\right)}{\left(1-E(e^{-(\eta+\beta)\tilde{\tau}_{S-s}})\frac{\xi\left(1-E(e^{-(\xi+\beta+\eta)\tilde{\tau}_{s-Y}})\right)}{\xi+\eta+\beta}\right)}. \end{aligned} \tag{8}$$

The intuition behind (7) is that after an unsatisfied demand there are two competing independent exponential RVs to the next possible event: the next demand arrival, with rate λ, and the leadtime, with rate ξ. If (i) the next demand arrives first, with probability $\lambda/\left(\lambda+\xi\right)$, then the next unsatisfied demand occurs after an exponential time with rate $(\lambda+\xi)$. Then, the partial LST of U (together with the event that it comes before the lead time) is $\lambda/\left(\lambda+\xi\right)*(\lambda+\xi)/\left(\lambda+\xi+\beta\right)$). If (ii) the next event is the leadtime, with probability $\xi/\left(\lambda+\xi\right)$, then the time until the next un-met demand is a convolution of the time for the leadtime (with LST $\xi/\left(\lambda+\xi\right)*(\lambda+\xi)/\left(\lambda+\xi+\beta\right)$) and the RV X as defined above.

For example, for a continuous demand case, when $P\left(W\left(\tau_S=0\right)\right)=0$, the solution of Proposition 2 gives

$$\begin{aligned} E\left(e^{-\beta U}\right) &= \frac{\lambda}{\lambda+\beta+\xi} \\ &+\frac{\frac{\xi}{\lambda+\beta+\xi}\frac{\lambda}{\lambda+\beta+\xi}\frac{\eta}{\eta+\beta}\left(1-\frac{E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\left(\xi+(\beta+\eta)E(e^{-(\beta+\eta+\xi)\tilde{\tau}_{s-Y}})\right)}{\beta+\eta+\xi}\right)}{1-\frac{E(e^{-(\eta+\beta)\tilde{\tau}_{S-s}})\left(\xi-\xi E(e^{-(\xi+\beta+\eta)\tilde{\tau}_{s-Y}})\right)}{\xi+\eta+\beta}-\frac{\xi}{\lambda+\beta+\xi}\frac{\eta}{\eta+\beta}\left(1-\frac{E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\left(\xi+(\beta+\eta)E(e^{-(\beta+\eta+\xi)\tilde{\tau}_{s-Y}})\right)}{\beta+\eta+\xi}\right)} \\ &+\frac{\frac{\xi}{\lambda+\beta+\xi}E(e^{-(\eta+\beta)\tilde{\tau}_{S-s}})E(e^{-(\xi+\beta+\eta)\tilde{\tau}_{s-Y}})}{1-\frac{E(e^{-(\eta+\beta)\tilde{\tau}_{S-s}})\left(\xi-\xi E(e^{-(\xi+\beta+\eta)\tilde{\tau}_{s-Y}})\right)}{\xi+\eta+\beta}-\frac{\xi}{\lambda+\beta+\xi}\frac{\eta}{\eta+\beta}\left(1-\frac{E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\left(\xi+(\beta+\eta)E(e^{-(\beta+\eta+\xi)\tilde{\tau}_{s-Y}})\right)}{\beta+\eta+\xi}\right)}. \end{aligned} \tag{9}$$

3.3 Effective Disasters Cost Functional

Disasters arrive according to a Poisson process with rate η, but effective disasters occur only when the system is not empty. It is clear that the effective disaster process is a renewal process. Recalling that Z denotes the time between effective disaster:

Proposition 3 *The LST of Z is given by*

$$E\left(e^{-\beta Z}\right)=\frac{\xi}{\xi+\beta}\frac{\eta}{\eta+\beta}\frac{\left(1-E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\frac{\xi+(\beta+\eta)E\left(e^{-(\beta+\xi+\eta)\tilde{\tau}_{s-Y}}\right)}{\beta+\xi+\eta}\right)}{1-\xi E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\left(\frac{\eta E\left(e^{-(\beta+\eta+\xi)\tilde{\tau}_{s-Y}}\right)}{(\xi+\beta)(\xi+\beta+\eta)}+\frac{1}{\xi+\beta+\eta}\right)}. \tag{10}$$

It is seen from (10) that the time between effective disasters is the convolution of (i) the leadtime, exponential with rate ξ, in which no effective disasters can occur, with (ii) the time to disaster, exponential with rate η, and with (iii) a RV that corrects for the probability that upon a disaster the inventory level is 0 again. In this third case the disaster is not effective and the time to the next effective disaster has the same distribution as before. (It can be verified that the third fraction represents the LST of a RV by taking its limit as $\beta\to 0$, which is indeed 1.)

3.4 Holding Cost Functional

By the dominated convergence theorem $\lim_{t\to\infty}E\left(W(t)\right)=E\left(W\right)$, since the content level is bounded in $[0,S]$. To calculate $E\left(W\right)$ we first evaluate the steady state distribution $F(\cdot)$ or in terms of the steady state density $dF(\cdot)$ and then take expectation. It should be noted that $F(\cdot)$ if the distribution of the demand size, $F_Y\left(\cdot\right)$, is an absolutely continuous distribution, so is $F\left(\cdot\right)$ for all $0<x<S$ but, for any demand size distribution, $F\left(\cdot\right)$ has an atom π_S at S and an atom π_0 at 0, so that $dF\left(S\right)=\pi_S$ and $dF\left(0\right)=\pi_0$. In the next Proposition we introduce the balance equation generated by level crossing theory that can be used to express $dF\left(\cdot\right)$. We first remind that $F_Y\left(\cdot\right)$ denotes the distribution of the demand size Y.

Proposition 4 *We have*

$$\eta\int_x^S dF\left(w\right)+\lambda\int_x^S\left(1-F_Y\left(w-x\right)\right)dF\left(w\right)=\begin{cases}\xi\int_0^x dF\left(w\right), & x\le s,\\ \xi\int_0^s dF\left(w\right), & s<x\le S.\end{cases} \tag{11}$$

alternatively, letting $f\left(\cdot\right)$ be the density of $F\left(\cdot\right)$ in $(0,S)$, we have

$$\eta\int_x^S f(w)dw+\eta\pi_S+\lambda\int_x^S\left(1-F_Y\left(w-x\right)\right)f(w)dw+\lambda\pi_S e^{-\mu(S-x)}$$
$$=\begin{cases}\xi\pi_0+\xi\int_0^x f(w)dw, & x\le s,\\ \xi\pi_0+\xi\int_0^s f(w)dw, & s<x\le S.\end{cases}$$

The boundary conditions can be found from the normalizing condition, and by taking $x = S$, $x = s$, and $x = 0$, in (11):

$$\begin{aligned}
\int_0^S f(w)\,dw &= 1 - \pi_S - \pi_0, \\
(\lambda + \eta)\pi_S &= \xi\pi_0 + \xi \int_0^s f(w)\,dw, \\
\eta \int_s^S f(w)\,dw + \eta\pi_S + \lambda \int_s^S (1 - F_Y(w-s)) f(w)\,dw + \lambda\pi_S F_Y(S-s) &= \xi\pi_0 + \xi \int_0^s f(w)\,dw, \\
\eta(1-\pi_0) + \lambda \int_0^S (1 - F_Y(w)) f(w)\,dw + \lambda\pi_S(1 - F_Y(S)) &= \xi\pi_0.
\end{aligned}$$

Now, $E(W) = \int_0^S wf(w)\,dw + S\pi_S$.

4 Examples of Exact Cost Analysis

Here we consider two examples for the demand size at each arrival: a Poisson demand, with unit demand sizes, and a compound Poisson demand, with exponential demand sizes. The case of Poisson demand can be directly solved from an analysis of a Markov chain. However, it is brought here to demonstrate the generality of our approach.

4.1 Poisson Demand Case

We assume that the demand follows a Poisson process, i.e., that the size of each demand is one unit. The stopping times $\tilde{\tau}_{S-s}$, $\tilde{\tau}_s$, (here $s - Y = s$), and $\tilde{\tau}_S$ that are required for substituting in the results of Section 3 are all $Erlang(n, \lambda)$ where n is the number of demands realized:

$$E\left(e^{-\beta\tilde{\tau}_n}\right) = \left(\frac{\lambda}{\lambda+\beta}\right)^n,$$

where $n = S - s$, $n = s$, and $n = S$ for the LST of $\tilde{\tau}_{S-s}$, $\tilde{\tau}_s$, and $\tilde{\tau}_S$, respectively. The results below follow by substitution of the relevant LST in the results derived in Section 3, noting that in this case $P(W(\tau_S) = 0) = 1$, and some (tedious) algebra, no further proof is provided. Because expressing the steady state distribution for the inventory level (and the implied holding cost) for this case can be also done directly from the Markov chain representation of the inventory level process, we provide this direct derivation in the next subsection.

Proposition 5 *The LST of the cycles is*

$$E\left(e^{-\beta T}\right)=\frac{\xi}{\xi+\beta}\frac{\left(\frac{\lambda}{\lambda+\beta+\eta}\right)^{S-s}\beta+\eta}{\beta+\eta} \tag{12}$$

so that

$$E\left(T\right)=\frac{\eta+\xi-\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s}\xi}{\xi\eta}=\frac{1}{\eta}+\frac{1}{\xi}-\frac{\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s}}{\eta}. \tag{13}$$

The LST for the time between unsatisfied demand is

$$E(e^{-\beta U})=\frac{\lambda(\eta+\beta)(\eta+\beta+\xi+\xi(\frac{\lambda}{\lambda+\beta+\eta})^{S-s}((\frac{\lambda}{\lambda+\beta+\eta+\xi})^{s}-1))}{(\beta+\eta+\xi-\xi(\frac{\lambda}{\lambda+\beta+\eta})^{S-s})(\beta(\beta+\xi+\eta)+(\eta+\beta)\lambda)+\xi(\eta+\beta)\lambda(\frac{\lambda}{\lambda+\beta+\eta})^{S-s}(\frac{\lambda}{\lambda+\beta+\eta+\xi})^{s}}. \tag{14}$$

So that

$$E(U)=\left(\lambda\left(\frac{\eta\xi}{((\eta+\xi)-\xi(\frac{\lambda}{\lambda+\eta})^{S-s})(\lambda+\eta)}\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s-1}\left(\frac{\lambda}{\lambda+\eta+\xi}\right)^{s}\frac{\lambda}{\xi+\eta}+\frac{\eta}{\xi+\eta}\right)\right)^{-1} \tag{15}$$

The LST of the time between effective disasters is

$$E\left(e^{-\beta Z}\right)=\frac{\xi}{\xi+\beta}\frac{\eta}{\eta+\beta}\frac{\left(1-\left(\frac{\lambda}{\lambda+\beta+\eta}\right)^{S-s}\frac{\xi+(\beta+\eta)\left(\frac{\lambda}{\lambda+\beta+\xi+\eta}\right)^{s}}{\beta+\xi+\eta}\right)}{1-\xi\left(\frac{\lambda}{\lambda+\beta+\eta}\right)^{S-s}\left(\frac{\eta\left(\frac{\lambda}{\lambda+\beta+\xi+\eta}\right)^{s}}{(\xi+\beta)(\xi+\beta+\eta)}+\frac{1}{\xi+\beta+\eta}\right)}. \tag{16}$$

so that

$$E\left(Z\right)=\frac{\xi+\eta}{\xi\eta}\frac{\xi+\eta-\xi\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s}}{\xi+\eta-\xi\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s}-\eta\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s}\left(\frac{\lambda}{\lambda+\xi+\eta}\right)^{s}} \tag{17}$$

Recalling our discussion after Proposition 1, the cycle, T, is a convolution of the length of time to order with the leadtime. For this demand case the LST of the time to order for a model with no leadtime was derived in equation (19) of Baron, Berman, and Perry (2010). (It agrees with (12), after substituting $\alpha\to 0$, $\xi\to\eta$ and $S\to S-s$ in their result).

4.1.1 Holding Cost Functional

To calculate $E\left(W\right)$ we first evaluate the steady state distribution $F(i)=\sum_{j=0}^{i}P_j$, where P_j is the steady state probability that the inventory level is j, and then take expectation. To this end, we consider the Markov chain representation of the inventory level to get:

$$\begin{aligned}
(\lambda+\eta) P_S &= \xi F(s) \\
(\lambda+\eta) P_i &= \lambda P_{i+1} \quad i=s+1,...,S-1 \\
(\lambda+\eta+\xi) P_i &= \lambda P_{i+1} \quad i=1,...,s \\
(\xi+\eta) P_0 &= \eta+\lambda P_1.
\end{aligned} \tag{18}$$

Solving (18) we get

$$P_i = \begin{cases} P_S\left(\frac{\lambda}{\lambda+\eta}\right)^{S-i} & i=s+1,...,S \\ P_S\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s-1}\left(\frac{\lambda}{\lambda+\eta+\xi}\right)^{s+1-i} & i=1,...,s \\ P_S\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s-1}\left(\frac{\lambda}{\lambda+\eta+\xi}\right)^{s}\frac{\lambda}{\xi+\eta}+\frac{\eta}{\xi+\eta} & i=0, \end{cases}$$

where P_S is given from the solution of (after some tedious algebra):

$$\begin{aligned}
\frac{1-\frac{\eta}{\xi+\eta}}{P_S} &= \sum_{i=s+1}^{S}\left(\frac{\lambda}{\lambda+\eta}\right)^{S-i}+\sum_{i=1}^{s}\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s-1}\left(\frac{\lambda}{\lambda+\eta+\xi}\right)^{s+1-i} \\
&\quad+\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s-1}\left(\frac{\lambda}{\lambda+\eta+\xi}\right)^{s}\frac{\lambda}{\xi+\eta} \\
P_S &= \frac{\eta\xi}{\left((\eta+\xi)-\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s}\xi\right)(\lambda+\eta)}.
\end{aligned}$$

It can be verified that $(\lambda+\eta) P_S = \xi F(s)$.

Now

$$\begin{aligned}
E(W) &= P_S\left(\sum_{i=s+1}^{S} i\left(\frac{\lambda}{\lambda+\eta}\right)^{S-i}+\sum_{i=1}^{s} i\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s-1}\left(\frac{\lambda}{\lambda+\eta+\xi}\right)^{s+1-i}\right) \\
&= P_S\left(\frac{\eta S-\lambda}{\eta^2}+\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s}\left(\frac{s(\xi+\eta)+\lambda\left(\left(\frac{\lambda}{\lambda+\eta+\xi}\right)^{s}-1\right)}{(\xi+\eta)^2}-\frac{(\eta s-\lambda)}{\eta^2}\right)\right).
\end{aligned} \tag{19}$$

By Poisson arrivals see time averages and since all unsatisfied demand arrives while there is no inventory, i.e., during P_0 of the time (recall with Poisson demand there is no unsatisfied demand if inventory is positive), the rate of unsatisfied demand $1/E(U)$ equals the rate of arrivals that see an empty system, i.e.,

$$\lambda P_0 = \frac{1}{E(U)},$$

which, of course, agrees with (15).

4.2 Compound Poisson with Exponential Demand

Here we assume that the demand follows a compound Poisson process where the size of each demand is exponentially(μ) distributed. The stopping times $\tilde{\tau}_{S-s}$, $\tilde{\tau}_{s-Y}$, and $\tilde{\tau}_S$ that are required for substituting in the results of Section 3 are known (see, e.g., Baron, Berman, and Perry (2010)). Specifically, setting $\alpha = 0$ and substituting $S-s$ rather than S in equation (7) of Baron, Berman, and Perry (2010) we get:

$$\begin{aligned} E\left(e^{-\beta\tilde{\tau}_S}\right) &= \frac{\lambda}{\lambda+\beta} e^{-\left(\frac{\mu\beta S}{\lambda+\beta}\right)}, \quad \textit{and} \\ E\left(e^{-\beta\tilde{\tau}_{S-s}}\right) &= \frac{\lambda}{\lambda+\beta} e^{-\left(\frac{\mu\beta(S-s)}{\lambda+\beta}\right)}. \end{aligned} \tag{20}$$

Using that $\bar{W}\left(\tilde{\tau}_{S-s}\right) = s - Y$ and (20) we get

$$\begin{aligned} E\left(e^{-\beta\tilde{\tau}_{s-Y}}\right) &= \int_0^s \frac{\lambda}{\lambda+\beta} e^{-\left(\frac{\mu\beta(s-y)}{\lambda+\beta}\right)} \mu e^{-\mu y} dy + \int_s^\infty \mu e^{-\mu y} dy \\ &= e^{-\mu\frac{\beta}{\lambda+\beta}s}. \end{aligned} \tag{21}$$

Note that the independence of $\tilde{\tau}_S$ and $\tilde{\tau}_{S-s}$ implies that $E(e^{-\beta\tilde{\tau}_{s-Y}}) = E\left(e^{-\beta\tilde{\tau}_S}\right)/E(e^{-\beta\tilde{\tau}_{S-s}})$, which leads to (21) as well. Substituting (20) and (21) into the derivation in Section 3 gives (no further proof is provided).

Proposition 6 *The LST of the cycles is*

$$E\left(e^{-\beta T}\right) = \frac{\xi}{\xi+\beta} \frac{\eta + \beta\frac{\lambda}{\lambda+\beta+\eta} e^{-\frac{\mu(\beta+\eta)}{\lambda+\beta+\eta}(S-s)}}{\beta+\eta} \tag{22}$$

so that

$$E(T) = \frac{1}{\eta} + \frac{1}{\xi} - \frac{\lambda e^{\left(-\frac{\eta\mu(S-s)}{\eta+\lambda}\right)}}{\eta(\lambda+\eta)}. \tag{23}$$

The LST for the time between unsatisfied demand is:

$$E\left(e^{-\beta U}\right) = \frac{\lambda}{\lambda+\beta+\xi} + \frac{\xi}{\lambda+\beta+\xi}$$

$$\times \frac{\frac{\lambda}{\lambda+\beta+\xi}\frac{\eta}{\eta+\beta}\left(1 - \frac{\lambda e^{-\left(\frac{\mu(\beta+\eta)(S-s)}{\lambda+\beta+\eta}\right)}}{\lambda+\beta+\eta}\frac{\xi+(\beta+\eta)e^{-\mu\frac{\beta+\eta+\xi}{\lambda+\beta+\eta+\xi}s}}{\beta+\eta+\xi}\right) + \frac{\lambda e^{-\frac{\mu(\eta+\beta)(S-s)}{\lambda+\eta+\beta}}}{\lambda+\eta+\beta} e^{-\frac{\mu(\xi+\beta+\eta)s}{\lambda+\xi+\beta+\eta}}}{\left(1 - \frac{\lambda e^{-\left(\frac{\mu(\eta+\beta)(S-s)}{\lambda+\eta+\beta}\right)}}{\lambda+\eta+\beta}\frac{\xi\left(1-e^{-\frac{\mu(\xi+\beta+\eta)s}{\lambda+\xi+\beta+\eta}}\right)}{\xi+\eta+\beta} - \frac{\eta}{\eta+\beta}\frac{\xi}{\lambda+\beta+\xi}\left(1 - \frac{\lambda e^{-\left(\frac{\mu(\beta+\eta)(S-s)}{\lambda+\beta+\eta}\right)}}{\lambda+\beta+\eta}\frac{\xi+(\beta+\eta)e^{-\mu\frac{\beta+\eta+\xi}{\lambda+\beta+\eta+\xi}s}}{\beta+\eta+\xi}\right)\right)}. \tag{24}$$

The LST of the time between effective disasters is:

$$E\left(e^{-\beta Z}\right)=\frac{\xi}{\xi+\beta}\frac{\eta}{\eta+\beta}\frac{\left(1-\frac{\lambda e^{-\left(\frac{\mu(\beta+\eta)(S-s)}{\lambda+\beta+\eta}\right)}}{\lambda+\beta+\eta}\frac{\xi+(\beta+\eta)e^{-\mu\frac{\beta+\eta+\xi}{\lambda+\beta+\eta+\xi}s}}{\beta+\eta+\xi}\right)}{1-\xi\frac{\lambda e^{-\left(\frac{\mu(\beta+\eta)(S-s)}{\lambda+\beta+\eta}\right)}}{\lambda+\beta+\eta}\left(\frac{\eta e^{-\mu\frac{\beta+\eta+\xi}{\lambda+\beta+\eta+\xi}s}}{(\xi+\beta)(\xi+\beta+\eta)}+\frac{1}{\xi+\beta+\eta}\right)}. \tag{25}$$

Recall that the cycle, T, is a convolution of the time to order with the leadtime. In a model with disasters but no leadtime equation (16) of Baron, Berman, and Perry (2010) gives the LST of the time to order for this demand case. Indeed, their (16) (with the required corrections as in the Poisson case) agrees with (22).

4.2.1 Holding cost functional

Following our discussion in Section 3.4 we calculate $E(W)$. The next Proposition that follows from Proposition 4 (no detailed proof is provided) and substitution of the demand distribution, we introduce the balance equation generated by level crossing theory that can be used to express $dF(\cdot)$ in closed form as explained below.

Proposition 7 *We have*

$$\eta\int_x^S dF(w)+\lambda\int_x^S e^{-\mu(w-x)}dF(w)=\begin{cases}\xi\int_0^x dF(w), & x\le s,\\ \\ \xi\int_0^s dF(w), & s<x\le S.\end{cases} \tag{26}$$

or

$$\eta\int_x^{S^-}f(w)dw+\eta\pi_S+\lambda\int_x^{S^-}e^{-\mu(w-x)}f(w)dw+\lambda\pi_S e^{-\mu(S-x)}=\begin{cases}\xi\pi_0+\xi\int_{0^+}^x f(w)dw, & x\le s,\\ \\ \xi\pi_0+\xi\int_{0^+}^s f(w)dw, & s<x\le S,\end{cases}$$

where $S^-=\sup\{x|x<S\}$ *and* $0^+=\inf\{x|x>0\}$.

Solving for $dF(\cdot)$ in (26) we get

$$dF(x)=\begin{cases}k_0e^{ax} & 0<x\le s\\ k_1e^{bx} & s<x<S\end{cases} \tag{27}$$

where $a=\frac{\mu(\xi+\eta)}{\lambda+\xi+\eta}$ and $b=\frac{\mu\eta}{\lambda+\eta}$. We now have 4 unknowns: π_0, π_S, k_0, and k_1. To find these unknowns we use the following 4 equations:

$$\begin{aligned}
\int_{0^+}^{s} k_0 e^{aw} dw + \int_{s}^{S^-} k_1 e^{bw} dw &= 1-\pi_S-\pi_0 \\
(\lambda+\eta)\pi_S &= \xi\pi_0+\xi\int_{0^+}^{s} k_0 e^{aw} dw \\
\eta\int_{s}^{S^-} k_1 e^{bw} dw + \eta\pi_S + \lambda\int_{s}^{S^-} e^{-\mu(w-s)} k_1 e^{bw} dw + \lambda\pi_S e^{-\mu(S-s)} &= \xi\pi_0+\xi\int_{0^+}^{s} k_0 e^{aw} dw \\
\eta\left(1-\pi_0\right)+\lambda k_0\int_{0^+}^{s} e^{-(\mu-a)w} dw + \lambda k_1\int_{s}^{S^-} e^{-(\mu-b)w} dw + \lambda\pi_S e^{-\mu S} &= \xi\pi_0
\end{aligned}$$

The first equation is the normalizing condition, the second, third, and fourth are obtained by taking $x=S$, $x=s$, and $x=0$, respectively, in (26). These equations are equivalent to:

$$\begin{aligned}
k_0\frac{e^{as}-1}{a}+k_1\frac{e^{bS}-e^{bs}}{b} &= 1-\pi_S-\pi_0 \\
(\lambda+\eta)\pi_S &= \xi\pi_0+\xi k_0\frac{e^{as}-1}{a} \\
\eta k_1\frac{e^{bS}-e^{bs}}{b}+\eta\pi_S+\lambda e^{\mu s}k_1\frac{e^{(b-\mu)S}-e^{(b-\mu)s}}{b-\mu}+\lambda\pi_S e^{-\mu(S-s)} &= \xi\pi_0+\xi k_0\frac{e^{as}-1}{a} \\
\eta\left(1-\pi_0\right)+\lambda k_0\frac{e^{(-\mu+a)s}-1}{-\mu+a}+\lambda k_1\frac{e^{(-\mu+b)S}-e^{(-\mu+b)s}}{-\mu+b}+\lambda\pi_S e^{-\mu S} &= \xi\pi_0.
\end{aligned}$$

These 4 linear equations in 4 unknowns can be solved in closed form (which is too cumbersome to include here). We then have

$$\begin{aligned}
E\left(W\right) &= k_0\int_0^s x e^{ax} dx + k_1\int_s^S x e^{bx} dx + S\pi_S \\
&= k_0\frac{e^{as}\left(as-1\right)+1}{a^2}+k_1\frac{e^{bS}\left(bS-1\right)-e^{bs}\left(bs-1\right)}{b^2}+S\pi_S.
\end{aligned} \tag{28}$$

5 Optimization

In this section we discuss the optimization of the order up to level, S, and the re-order level, s, for the Poisson and compound Poisson demand distributions. For both cases we consider the problem (see (5)):

$$\min_{S,s} R=\frac{K_0}{E(T)}+c\frac{\lambda}{\mu}+c\eta E(W)+hE(W)+\frac{k_u-c}{\mu E(U)}+\frac{K_d}{E(Z)}$$

where for the Poisson demand case $E(T)$ and $E(W)$ are given respectively in (13), (19) and $E(U)$ and $E(Z)$ are derived from (14) and (17) and for the compound Poisson demand case $E(T)$ and $E(W)$ are given in (23) and (28) and $E(U)$ and $E(Z)$ are derived from (24) and (25). We provide numerical results on the sensitivity of parameters for the optimal s^* and S^*.

We present asymptotic expressions of $E(W)$, $E(T)$, $E(U)$, $E(Z)$ and R in the on-line Appendix. Two important asymptotics, leading to relevant special cases, are the one when there is no lead time, i.e., $\xi \to \infty$, analyzed in Baron, Berman and Perry (2010), and the one when there are no disasters, i.e., $\eta \to 0$. To capture the value of considering disasters, we compare the value of the exact analysis of the system with disasters to the one that ignores disasters. We refer to the system without disasters as the heuristic.

5.1 Poisson Demand Case

Our exact, closed form analysis allowed us to use Maple to numerically find the optimal policy (controls) in "no time" (few seconds for 10 runs, the same is true also for the numerical results in Section 5.2 for the compound Poisson demand). For our numerical results we considered a base case with $K_0 = 50$, $h = 1$, $K_u = 10$, $K_d = 50$, $c = 5$, $\xi = 0.2$, $\eta = .05$, and $\lambda = 50$ and derived the optimal policy. We then varied each of the parameters ξ, η, and λ one at a time under the optimal policy.

We note that when demand is Poisson, there is no value in ordering a non integer number of units, so after finding the optimal continuous controls, if needed, we run a quick search comparing the costs for any round up and round down combination of S and s. The same procedure was used with the heuristic (where, of course, we used the heuristic cost function in this comparison). We note that the optimal S^* and s^* were the round up results of the continuous solution in all our numerical experiments (when the continuos $s = 0$ it is the optimal level without the need to round up). As is common, we let S^*, s^*, and R^*, denote the optimal order up to level, reorder point, and cost, respectively. We let R^H denote the cost of the heuristic and report the $Loss\ \% = \left(R^H - R^*\right)/R^*$.

From Table 1 we observe that when ξ increases, as expected, R^*, S^*, s^*, $E(T)$, and $E(Z)$ decrease while $E(W)$ and $E(U)$ increase. The no-disaster heuristic performs very well ($Loss\%$ of less than 0.72%). This is true despite that its controls may be quite far from optimal (e.g., for the base case, $S^* = 145$, $s^* = 81$, whereas the heuristic chooses $S = 183$ and $s = 108$). This implies that the cost function is quite flat.

From Table 2 we observe that when varying η: As expected, when η increases, because there are more disasters, S^* and s^*, decrease, R^* increases, and $E(T)$, $E(Z)$, $E(W)$, and $E(U)$ decrease. As

ξ	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
S^*	169	159	151	145	139	134	130	126	122	119
s^*	106	96	88	81	76	71	67	63	59	56
R^*	483	469	458	449	440	434	427	422	417	413
ET	21.22	11.22	7.89	6.24	5.22	4.55	4.08	3.72	3.44	3.22
EZ	140.87	80.58	60.41	50.19	44.17	40.15	37.21	35.09	33.52	32.16
EW	12.63	21.04	26.92	31.35	34.39	36.65	38.53	39.73	40.34	41.09
EU	0.02	0.03	0.03	0.03	0.04	0.04	0.04	0.05	0.05	0.05
R^H	485	472	461	452	444	437	430	425	420	416
Loss %	0.40	0.58	0.66	0.71	0.69	0.72	0.70	0.67	0.65	0.66

Table 1: Results when varying ξ– Poisson demand

η	.05	.1	.15	.2	.25	.3	.35	.4	.45	.5
S^*	145	120	102	88	78	70	63	57	52	48
s^*	81	62	49	39	32	26	22	18	15	12
R^*	449	449	467	473	477	481	484	487	489	491
ET	6.24	6.09	5.98	5.89	5.82	5.77	5.71	5.67	5.63	5.6
EZ	50.19	30.26	23.66	20.48	18.47	17.15	16.27	15.68	15.22	14.83
EW	31.35	21.57	15.67	11.73	9.25	7.46	6.08	5	4.19	3.58
EU	0.03	0.03	0.03	0.03	0.03	0.02	.02	.02	.02	.02
R^H	452	469	485	500	512	524	535	545	554	562
Loss %	0.71	2.14	3.83	5.57	7.2	8.9	10.42	11.85	13.17	14.39

Table 2: Results when varying η– Poisson demand

λ	10	20	30	40	50	60	70	80	90	100
S^*	32	61	89	117	145	172	200	228	256	284
s^*	4	21	40	60	81	103	126	148	171	194
R^*	96.45	184.77	272.80	360.71	448.57	536.40	624.21	712.00	799.79	887.56
ET	7.61	6.90	6.57	6.37	6.24	6.12	6.03	5.98	5.92	5.88
EZ	51.85	50.52	50.35	50.26	50.19	50.29	50.20	50.16	50.12	50.08
EW	6.56	12.90	19.02	25.16	31.35	37.23	43.50	49.70	55.94	62.18
EU	0.16	0.08	0.06	0.04	0.03	0.03	0.02	0.02	0.02	0.02
R^H	97.36	186.26	275.14	363.30	451.75	541.29	629.64	717.83	806.17	893.45
Loss %	0.94	0.81	0.86	0.72	0.71	0.91	0.87	0.82	0.80	0.66

Table 3: Results when varying λ– Poisson demand

expected the no-disaster case heuristic is only close to optimal for small η. Consider the second column in the table where $\eta = .1$ and recall that here $\xi = 0.2$. That is in this column a disaster occurs on average every 2 lead times. The results in columns 3-10 are for even more frequent disasters. Therefore, ignoring disasters, as the heuristic does, is costly in these settings. However, we believe that such short times between disasters is not often the case in practice.

From Table 3 we observe that when varying λ: As expected, when λ increases, S^*, s^*, R^*, and $E(W)$ increase, and $E(T)$, and $E(U)$ decrease. The value of $E(Z)$ does not change much and it may increase or decrease with λ (see the case where $\lambda = 60$). The gap between the optimal and heuristic costs is less than 1%; again this is despite that the heuristic may be considerably off in calculating its controls.

Overall from the above tables we conclude that in the Poisson demand case, where there is no demand size variability, the heuristic of ignoring disasters performs well when disasters are not too frequent.

5.2 Compound Poisson with Exponential Demand

Here we consider the same base case with $K_0 = 50$, $h = 1$, $K_u = 10$, $K_d = 50$, $c = 5$, $\xi = 0.2$, $\eta = .05$, $\lambda = 50$ and $\mu = 1$. We again vary each of the parameters ξ, η and λ one at a time and compare the values of the exact analysis with disasters to the system with no disaster which we refer to again as the heuristic. As for the Poisson demand case we let S^*, s^*, R^*, and R^H denote, respectively, the optimal

ξ	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
S^*	196.5	142.4	113.5	95.7	83.6	75.1	68.7	63.9	59.2	54.9
s^*	134	79.8	50.9	33.0	21.0	12.5	6.1	1.3	0	0
R^*	459	445	438	435	433	432	431	431	431	432
ET	21.23	11.23	7.90	6.23	5.23	4.56	4.09	3.73	3.39	3.09
EZ	123.73	87.76	75.49	69.22	65.39	62.79	60.91	59.46	58.07	56.80
EW	16.87	17.17	15.81	14.43	13.27	12.34	11.61	11.03	10.47	9.93
EU	0.02	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03
R^H	491.7	514.8	538.6	558.9	575.6	589.3	600.8	610.3	618.4	625.4
Loss %	7.07	15.70	22.92	28.59	33.03	36.53	39.31	41.55	43.37	44.84

Table 4: Results when varying ξ–compound Poisson arrival with $\exp(\mu)$ demand

η	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
S^*	95.65	52.59	34.79	25.86	20.52	16.98	14.46	12.58	11.12	9.95
s^*	33.04	0	0	0	0	0	0	0	0	0
R^*	435	467	481	488	492	495	497	499	500	501
ET	6.23	6.01	5.68	5.51	5.41	5.34	5.29	5.26	5.23	5.21
EZ	69.22	59.28	55.91	54.27	53.25	52.51	51.93	51.44	51.02	50.64
EW	14.43	4.60	2.17	1.26	0.82	0.58	0.43	0.33	0.27	0.22
EU	0.03	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
R^H	559	742	864	945	997	1029	1049	1060	1064	1065
Loss %	28.59	58.84	79.85	93.69	102.53	107.91	110.95	112.39	112.76	112.42

Table 5: Results when varying η–compound Poisson arrival with $\exp(\mu)$ demand

S, s, R and the cost of the heuristic R^H.

The results when varying ξ, η, and λ are depicted in Tables 4, 5 and 6, respectively. The direction of changes of s^*, S^*, R^*, $E(T)$, $E(W)$, and $E(U)$ in these tables is similar to the direction of changes in Tables 1-3. Similarly, $E(Z)$ in Tables 4 and 5 decreases, as in Tables 1 and 2. In Table 6 $E(Z)$ increases with λ whereas at Table 3 $E(Z)$ is not monotone; but this increase is quite mild. We further observe that the heuristic performance is deteriorating when the lead time, time between disasters and the demand rate increase.

Overall, the most interesting results from these tables is that, unlike in the case of Poisson demand, the heuristic performs quite poorly in comparison with the optimal policy; its best performance are in

λ	10	20	30	40	50	60	70	80	90	100
S^*	22.07	41.54	59.67	77.69	95.65	113.59	131.50	149.39	167.28	185.15
s^*	0	2.14	11.29	21.74	33.04	44.93	57.30	70.03	83.06	96.35
R^*	93.37	178.89	264.22	349.43	434.59	519.71	604.82	689.90	774.98	860.05
ET	7.17	6.92	6.58	6.37	6.23	6.13	6.05	5.98	5.92	5.88
EZ	66.10	68.79	69.03	69.15	69.22	69.27	69.31	69.33	69.35	69.37
EW	3.55	6.30	8.99	11.71	14.43	17.16	19.89	22.63	25.37	28.11
EU	0.14	0.07	0.05	0.04	0.03	0.02	0.02	0.02	0.02	0.01
R^H	118	229	339	449	559	669	779	889	999	1109
Loss %	26.96	27.76	28.2	28.44	28.59	28.7	28.78	28.83	28.88	28.92

Table 6: Results when varying λ–compound Poisson arrival with $\exp(\mu)$ demand

the case where the lead time is quite long, $\xi = 0.05$. In all other cases the heuristic is much worse with a maximum error of about 45%, 113%, and 29% in tables, 1, 2 and 3, respectively. Therefore, our main conclusion from the numerical results is that considering disasters in inventory control is important as the variability in the demand size increases. Of course, such consideration is also important if disasters are quite frequent.

6 Extension

Here we consider the discounted cost criteria. For this let $0 < \beta < 1$ be the discount factor. To express the costs, we use $\mathbf{\Lambda_o} = \{\Lambda_o(t) : t \geq 0\}$ as the counting process of order arrivals, $\mathbf{\Lambda_u} = \{\Lambda_u(t) : t \geq 0\}$ as the counting process of the unsatisfied demand, and $\mathbf{\Lambda_d} = \{\Lambda_d(t) : t \geq 0\}$ as the counting process of the effective disasters.

Let $W(t^-)$ denote the level of inventory just before time t. Then, at time t the number of units paid for is $(S - W(t^-))\,d\Lambda_o(t)$, the fixed order cost paid is $K_o d\Lambda_o(t)$, the cost of unsatisfied demand is $K_u/\mu d\Lambda_u(t)$ (we again assume for simplicity that $E(1/Y d\Lambda_u(t)) = E(1/\mu d\Lambda_u(t))$, the fixed cost of disaster is $K_d d\Lambda_d(t)$, and the holding cost is $hW(t)dt$. Thus, the total expected discounted costs is given by

$$\begin{aligned} R^\beta \;&=\; cE\left(\int_0^\infty e^{-\beta t}\left(S - W\left(t^-\right)\right) d\Lambda_o(t)\right) + K_o E\left(\int_0^\infty e^{-\beta t} d\Lambda_o(t)\right) \\ &\quad + \frac{K_u}{\mu} E\left(\int_0^\infty e^{-\beta t} d\Lambda_u(t)\right) + K_d E\left(\int_0^\infty e^{-\beta t} d\Lambda_d(t)\right) + hE\left(\int_0^\infty e^{-\beta t} W(t)dt\right). \end{aligned} \tag{29}$$

The cost component pertaining to the arrival processes can be expressed in closed form but the exact analysis of the holding cost functional, $hE\left(\int_0^\infty e^{-\beta t}W(t)dt\right)$, and the discounted items' cost $cE\left(\int_0^\infty e^{-\beta t}W\left(t^-\right)d\Lambda_o(t)\right)$ are too complicated and we suggest the approximations below. Let Λ_g, represent the counting process for arrivals of either the order, or the unsatisfied demand, or the effective disasters.

Proposition 8 *For the discounted cost criteria we have:*

$$
\begin{aligned}
E\left(\int_0^\infty e^{-\beta t}d\Lambda_g\left(t\right)\right) &= \frac{E\left(e^{-\beta G}\right)}{1-E\left(e^{-\beta G}\right)}, && (30)\\
E\left(\int_0^\infty e^{-\beta t}W\left(t^-\right)d\Lambda_o(t)\right) &\approx E\left(W|W<s\right)\frac{E\left(e^{-\beta T}\right)}{1-E\left(e^{-\beta T}\right)}, && (31)\\
E\left(\int_0^\infty e^{-\beta t}W\left(t^-\right)d\Lambda_o(t)\right) &\approx E\left(W|W<s\right)F_W\left(s\right)\frac{\xi}{\beta}, \quad \text{and} && (32)\\
E\left(\int_0^\infty e^{-\beta t}W\left(t\right)dt\right) &\approx \frac{E\left(W\right)}{\beta}. && (33)
\end{aligned}
$$

So for example, in the compound Poisson demand case we have from (30) and $E\left(e^{-\beta T}\right)$ and $E\left(e^{-\beta U}\right)$, which are given in (12) and (24), respectively, that

$$
\begin{aligned}
K_oE\left(\int_0^\infty e^{-\beta t}d\Lambda_o(t)\right) &= \frac{K_o\left(\frac{\xi}{\xi+\beta}\frac{\eta+\beta e^{-\frac{\lambda(\beta+\eta)}{\mu+\beta+\eta}(S-s)}\frac{\lambda}{\lambda+\beta+\eta}}{\beta+\eta}\right)}{1-\frac{\xi}{\xi+\beta}\frac{\eta+\beta e^{-\frac{\lambda(\beta+\eta)}{\mu+\beta+\eta}(S-s)}\frac{\lambda}{\lambda+\beta+\eta}}{\beta+\eta}}, \quad \text{and} && (34)\\
\frac{K_u}{\mu}E\left(\int_0^\infty e^{-\beta t}d\Lambda_u(t)\right) &= \frac{K_u}{\mu}\frac{E\left(e^{-\beta U}\right)}{1-E\left(e^{-\beta U}\right)}.
\end{aligned}
$$

As for the approximations, due to the stationarity of W we have for the Poisson demand case (using $\left(\lambda+\eta\right)P_S=\xi F\left(s\right)$ and after some algebra)

$$
\begin{aligned}
E\left(W\mid W\leq s\right) &= \frac{P_S\left(\sum_{i=1}^{s}i\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s-1}\left(\frac{\lambda}{\lambda+\eta+\xi}\right)^{s+1-i}\right)}{F\left(s\right)}\\
&= \xi\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s}\frac{\left(\eta s+\xi s-\lambda\right)+\left(\frac{\lambda}{\lambda+\eta+\xi}\right)^{s+1}\left(\lambda+\eta+\xi\right)}{\left(\eta+\xi\right)^2}
\end{aligned}
$$

and for the compound Poisson case we have

$$
\Pr\left(W\leq x|W\leq s\right)=\begin{cases} 0 & x<0\\ \frac{\pi_0+\int_0^x k_0e^{ay}dy}{\pi_0+\frac{k_0\left(\exp^{as}-1\right)}{a}} & x\in[0,s]\\ 1 & x\geq s\end{cases},
$$

whereπ_0, k_0, and a are derived in Section 4.2.1 and where the expression for $x \in [0, s]$ follows from normalization of the CDF and

$$F(s) = \pi_0 + \int_0^s k_0 e^{ax} dx = \pi_0 + \frac{k_0 (\exp^{as} - 1)}{a}$$

From these we can approximate

$$E(W \mid W \le s) = \frac{\int_0^s k_0 y e^{ay} dy}{\pi_0 + \frac{k_0(\exp^{as} - 1)}{a}} = \frac{k_0 \frac{e^{as}(as-1)+1}{a^2}}{\pi_0 + \frac{k_0(\exp^{as} - 1)}{a}} = \frac{k_0 e^{2as} (as - 1 + e^{-as})}{a (\pi_0 a e^{as} + k_0)}. \tag{35}$$

7 Conclusions

In this paper we presented a continuous review (s, S) model with lost sales in the presence of unexpected events (disasters). Both the time between disasters and leadtimes are assumed to be exponentially distributed while two types of demand distribution were studied: Poisson and compound Poisson with general demand sizes.

The following are our contributions:

- We provided exact closed form expressions for all the cost functionals required to evaluate the system under the average cost criterion.

- Using the closed form expressions we managed to minimize the total cost and obtained optimal re-order and order on up-to levels. We compared the cost resulting from the exact model to the one of a heuristic that ignores disasters. This comparison shows that ignoring disasters is costly (i) if demand size variability is high (i.e., in the compound Poisson case with exponential demand size), and, not surprisingly, (ii) if disasters are frequent.

- Using the closed form expressions we obtained asymptotic expressions (for the compound Poisson we assume exponential demand sizes) for some of the cost functionals. For few cases these expressions provided a useful model for a less general system. For example, the heuristic when the time between disasters approaches 0 results in a model with all the features of our model except for the occurrence of disasters. As far as we know, this model has not been studied for the compound Poisson case.

- We performed sensitivity analysis for both the Poisson and the compound Poisson with exponential leadtimes cases. For both models we compared our results with those of the heuristic ignoring disasters. Our exact optimization in all cases totally dominated the heuristic.

- We extended our model to the discounted cost criterion. For some of the functionals we had to provide approximations.

A Proofs

Proof. of Proposition 1: To calculate the functional $E\left(e^{-\beta T}\right)$, we notice that L_ξ and τ_{S-s} in (3) are independent. Thus,

$$E\left(e^{-\beta T}\right) = \frac{\xi}{\xi+\beta} E\left(e^{-\beta \tau_{S-s}}\right)$$

where $\frac{\xi}{\xi+\beta}$ is the LST of L_ξ. So we only need to calculate the LST of τ_{S-s}, $E\left(e^{-\beta \tau_{S-s}}\right) = E\left(e^{-\beta \min\{\tilde{\tau}_{S-s}, D_\eta\}}\right)$. By definition

$$E\left(e^{-\beta \tau_{S-s}}\right) = \beta \int_0^\infty e^{-\beta t} d \Pr(\tau_{S-s} \leq t).$$

Thus,

$$\begin{aligned} E\left(e^{-\beta \tau_{S-s}}\right) &= E\left(e^{-\beta \min\{\tilde{\tau}_{S-s}, D_\eta\}}\right) \\ &= \beta \int_0^\infty e^{-\beta t} \left(1 - \Pr(\min\{\tilde{\tau}_{S-s}, D_\eta\} > t)\right) dt \\ &= \beta \int_0^\infty e^{-\beta t} \left(1 - e^{-\eta t} \Pr(\tilde{\tau}_{S-s} > t)\right) dt \end{aligned} \tag{36}$$

so that

$$\begin{aligned} E\left(e^{-\beta \tau_{S-s}}\right) &= 1 - \frac{\beta}{\beta+\eta} \int_0^\infty (\beta+\eta) e^{-(\eta+\beta)t} \left(1 - \Pr(\tilde{\tau}_{S-s} \leq t)\right) dt \\ &= 1 - \frac{\beta}{\beta+\eta} + \frac{\beta E(e^{-(\beta+\eta)\tilde{\tau}_{S-s}})}{\beta+\eta} = \frac{\eta + \beta E(e^{-(\beta+\eta)\tilde{\tau}_{S-s}})}{\beta+\eta}. \end{aligned}$$

■

Proof. of Proposition 2: Let V denote the time between two consecutive demands. Using the memoryless of the leadtime we have

$$E\left(e^{-\beta U}\right) = E\left(e^{-\beta V} \mathbf{1}_{\{V<L_\xi\}}\right) + E\left(e^{-\beta L_\xi} \mathbf{1}_{\{L_\xi \leq V\}}\right) E\left(e^{-\beta X}\right),$$

where $\mathbf{1}_{\{V<L_\xi\}}$ is a competition between two exponentials, so on this event V the time between consecutive demands is $\exp(\lambda+\xi)$ and

$$E\left(e^{-\beta V} \mathbf{1}_{\{V<L_\xi\}}\right) = \frac{\lambda}{\lambda+\beta+\xi}.$$

Similarly, we have:

$$E\left(e^{-\beta L_\xi}\mathbf{1}_{\{L_\xi\le V\}}\right)=\frac{\xi}{\lambda+\beta+\xi}.$$

Establishing (7).

By (6)

$$\begin{aligned} E\left(e^{-\beta X}\right) &= E\left(e^{-\beta D_\eta}e^{-\beta U}\mathbf{1}_{\{D_\eta\le\min\{T,\tau_S\}\}}\right) \qquad (37)\\ &\quad +E\left(\left(e^{-\beta\tau_S}\left(\mathbf{1}_{\{W(\tilde\tau_S)<0\}}+e^{-\beta U}\mathbf{1}_{\{W(\tilde\tau_S)=0\}}\right)\right)\mathbf{1}_{\{\tau_S<\min\{D_\eta,T\}\}}\right)+E\left(e^{-\beta(T+X)}\mathbf{1}_{\{T\le\tau_S\}}\right). \end{aligned}$$

The functional $E\left(e^{-\beta D_\eta}e^{-\beta U}\mathbf{1}_{\{D_\eta\le\min\{T,\tau_S\}\}}\right)$. We first note that due to the definitions in (2-4) the following events are equivalent

$$\begin{aligned} \{D_\eta\le\min\{T,\tau_S\}\} &= \{D_\eta\le\min\{\tau_{S-s}+L_\xi,\tilde\tau_S\}\}=\{D_\eta\le\min\{\tilde\tau_{S-s}+L_\xi,\tilde\tau_{S-s}+\tilde\tau_{s-Y}\}\}\\ &= \{D_\eta\le\tilde\tau_{S-s}+\min\{L_\xi,\tilde\tau_{s-Y}\}\}. \end{aligned}$$

So that,

$$\begin{aligned} E\left(e^{-\beta D_\eta}e^{-\beta U}\mathbf{1}_{\{D_\eta\le\min\{T,\tau_S\}\}}\right) &= E\left(e^{-\beta U}\right)E\left(\int_0^{\tilde\tau_{S-s}+\min\left\{L_\xi,\tilde\tau_{s-Y}\right\}}e^{-\beta z}\eta e^{-\eta z}dz\right)\\ &= \frac{E\left(e^{-\beta U}\right)\eta}{\eta+\beta}\left(1-E\left(e^{-(\eta+\beta)\left(\tilde\tau_{S-s}+\min\left\{L_\xi,\tilde\tau_{s-Y}\right\}\right)}\right)\right). \end{aligned}$$

We remind that $\tilde\tau_{S-s}$, L_ξ, and $\tilde\tau_{s-Y}$ are all independent, so that $E\left(e^{-(\eta+\beta)\left(\tilde\tau_{S-s}+\min\left\{L_\xi,\tilde\tau_{s-Y}\right\}\right)}\right)=E\left(e^{-(\eta+\beta)\tilde\tau_{S-s}}\right)E\left(e^{-(\eta+\beta)\min\left\{L_\xi,\tilde\tau_{s-Y}\right\}}\right)$. To calculate $E\left(e^{-\beta\min\left\{L_\xi,\tilde\tau_{s-Y}\right\}}\right)$, we use a similar argument to the one in (36), replacing $\tilde\tau_{S-s}$ with $\tilde\tau_{s-Y}$ and η with ξ, to get:

$$E\left(e^{-\beta\min\left\{L_\xi,\tilde\tau_{s-Y}\right\}}\right)=\frac{\xi+\beta E\left(e^{-(\beta+\xi)\tilde\tau_{s-Y}}\right)}{\beta+\xi}. \qquad (38)$$

The functional $E\left(\left(e^{-\beta\tau_S}\left(\mathbf{1}_{\{W(\tilde\tau_S)<0\}}+e^{-\beta U}\mathbf{1}_{\{W(\tilde\tau_S)=0\}}\right)\right)\mathbf{1}_{\{\tau_S<\min\{D_\eta,T\}\}}\right)$

We notice that the following events are equivalent

$$\{\tau_S<\min\{D_\eta,T\}\}=\{\{\tilde\tau_{s-Y}\le L_\xi\}\cup\{\tilde\tau_S\le D_\eta\}\}=\{\{\tilde\tau_{s-Y}\le L_\xi\}\cup\{\tilde\tau_{s-Y}+\tilde\tau_{S-s}\le D_\eta\}$$

and that on $\{\tau_S<\min\{D_\eta,T\}\}$ we have

$$\tau_S=\tilde\tau_{s-Y}+\tilde\tau_{S-s}.$$

So that by the independence of U, $\tilde{\tau}_{s-Y}$ and $\tilde{\tau}_{S-s}$ we have

$$\begin{aligned}
E\left(\left(e^{-\beta\tau_S}\left(\mathbf{1}_{\{W(\tilde{\tau}_S)<0\}}+e^{-\beta U}\mathbf{1}_{\{W(\tilde{\tau}_S)=0\}}\right)\right)\mathbf{1}_{\{\tau_S<\min\{D_\eta,T\}\}}\right) &= E\left(e^{-\beta\tau_S}\mathbf{1}_{\{\tau_S<\min\{D_\eta,T\}\}}\right)\\
&\quad *E\left(\mathbf{1}_{\{W(\tilde{\tau}_S)<0\}}+e^{-\beta U}\mathbf{1}_{\{W(\tilde{\tau}_S)=0\}}\right)\\
&= E\left(e^{-\beta\tau_S}\mathbf{1}_{\{\tau_S<\min\{D_\eta,T\}\}}\right)\\
&\quad *\left(P\left(W\left(\tilde{\tau}_S\right)<0\right)+E\left(e^{-\beta U}\right)P\left(W\left(\tilde{\tau}_S\right)=0\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
E\left(e^{-\beta\tau_S}\mathbf{1}_{\{\tau_S<\min\{D_\eta,T\}\}}\right) &= E\left(e^{-\beta(\tilde{\tau}_{s-Y}+\tilde{\tau}_{S-s})}\mathbf{1}_{\left\{\tilde{\tau}_{s-Y}\le L_\xi\right\}}\mathbf{1}_{\{\tilde{\tau}_{s-Y}+\tilde{\tau}_{S-s}\le D_\eta\}}\right)\\
&= E\left(e^{-\beta(\tilde{\tau}_{s-Y}+\tilde{\tau}_{S-s})}\int_{\tilde{\tau}_{s-Y}}^{\infty}\xi e^{-\xi y}dy\int_{\tilde{\tau}_{s-Y}+\tilde{\tau}_{S-s}}^{\infty}\eta e^{-\eta x}dx\right)\\
&= E\left(e^{-\beta(\tilde{\tau}_{s-Y}+\tilde{\tau}_{S-s})}e^{-\xi\tilde{\tau}_{s-Y}}e^{-\eta(\tilde{\tau}_{s-Y}+\tilde{\tau}_{S-s})}\right)\\
&= E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)E\left(e^{-(\xi+\beta+\eta)\tilde{\tau}_{s-Y}}\right).
\end{aligned}$$

The Functional $E\left(e^{-\beta(T+X)}\mathbf{1}_{\{T\le\tau_S\}}\right)$

We notice that the following events are equivalent

$$\begin{aligned}
\{T\le\tau_S\} &= \{\tau_{S-s}+L_\xi\le\min\{\tilde{\tau}_S,D_\eta\}\}=\{\min\{\tilde{\tau}_{S-s},D_\eta\}+L_\xi\le\min\{\tilde{\tau}_S,D_\eta\}\}\\
&= \{\tilde{\tau}_{S-s}+L_\xi\le\min\{\tilde{\tau}_S,D_\eta\}\}=\{\tilde{\tau}_{S-s}+L_\xi\le D_\eta\}\cup\{L_\xi\le\tilde{\tau}_{s-Y}\}
\end{aligned}$$

so that

$$E\left(e^{-\beta(T+X)}\mathbf{1}_{\{T\le\tau_S\}}\right)=E\left(e^{-\beta X}\right)E\left(e^{-\beta T}\mathbf{1}_{\left\{\tilde{\tau}_{S-s}+L_\xi\le D_\eta\right\}}\mathbf{1}_{\left\{L_\xi\le\tilde{\tau}_{s-Y}\right\}}\right).$$

Next, since on $\{T\le\tau_S\}$ we have $T=\tau_{S-s}+L_\xi=\tilde{\tau}_{S-s}+L_\xi$ and due to the independence of D_η, $\tilde{\tau}_{S-s}$, and L_ξ we have:

$$\begin{aligned}
E\left(e^{-\beta T}\mathbf{1}_{\left\{\tilde{\tau}_{S-s}+L_\xi\le D_\eta\right\}}\mathbf{1}_{\left\{L_\xi\le\tilde{\tau}_{s-Y}\right\}}\right) &= E\left(e^{-\beta\tilde{\tau}_{S-s}}\int_0^{\tilde{\tau}_{s-Y}}e^{-\beta L_\xi}\left(\int_{\tilde{\tau}_{S-s}+L_\xi}^{\infty}\eta e^{-\eta x}dx\right)\xi e^{-\xi L_\xi}dL_\xi\right)\\
&= E\left(e^{-\beta\tilde{\tau}_{S-s}}\int_0^{\tilde{\tau}_{s-Y}}e^{-\beta L_\xi}e^{-\eta\left(\tilde{\tau}_{S-s}+L_\xi\right)}\xi e^{-\xi L_\xi}dL_\xi\right)\\
&= \xi E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)E\left(\int_0^{\tilde{\tau}_{s-Y}}e^{-(\xi+\beta+\eta)L_\xi}dL_\xi\right)\\
&= \frac{\xi E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\left(1-E\left(e^{-(\xi+\beta+\eta)\tilde{\tau}_{s-Y}}\right)\right)}{\xi+\beta+\eta}. \qquad (39)
\end{aligned}$$

Substituting these functionals into (37), establishes (8). ■

Proof. of Proposition 3: To compute the LST of Z we shift the origin to the time of an outdating and recall that

$$E\left(e^{-\beta Z}\right) = \frac{\xi}{\xi+\beta} E\left(e^{-\beta B}\right) \tag{40}$$

where the random variable B is the time it takes from the instant the content level is S until the next disaster. The explanation for (40) is simple. Since outdatings never occur when the system is empty, we should wait until the end of the leadtime which is exponentially distributed(ξ). From that instant the time to the next outdating is B. We now have:

$$\begin{aligned} E\left(e^{-\beta B}\right) &= E\left(e^{-\beta D_\eta}\mathbf{1}_{\{D_\eta<\tilde{\tau}_{S-s}+\min(\tilde{\tau}_{s-Y},L_\xi)\}}\right) \\ &+E\left(e^{-\beta\tilde{\tau}_S}\mathbf{1}_{\{\tilde{\tau}_S\le D_\eta\}}\mathbf{1}_{\{\tilde{\tau}_{s-Y}\le L_\xi)\}}\right)\tfrac{\xi}{\xi+\beta}E\left(e^{-\beta B}\right) \\ &+E\left(e^{-\beta(\tilde{\tau}_{S-s}+L_\xi)}\mathbf{1}_{\{\tilde{\tau}_{S-s}+L_\xi\le D_\eta\}}\mathbf{1}_{\{L_\xi\le\tilde{\tau}_{S-Y})\}}\right)E\left(e^{-\beta B}\right) \end{aligned} \tag{41}$$

The first term on the right of (41) indicates the partial LST of the time to the next disaster if it arrives before $\tilde{\tau}_{S-s}+\min(\tilde{\tau}_{S-Y},L_\xi)$. The second term indicates that an unsatisfied demand occurs before a disaster and before the end of the leadtime. Then, we have to wait until the end of the leadtime and from that instant the time to the next outdating starts afresh. The third term on the right of (41) is for the case when the time until the content level equals S is $\tilde{\tau}_{S-s}+L_\xi$, where $\tilde{\tau}_{S-s}+L_\xi$ occurs before a disaster and also the leadtime comes before $\tilde{\tau}_{S-Y}$. Again, since the content level is refilled up to level S, the time to the next outdating starts afresh.

For the first component of (41) we obtain (using the independence between $\tilde{\tau}_{S-s}$ and $\min(\tilde{\tau}_{s-Y},L_\xi)$)

$$\begin{aligned} E\left(e^{-\beta D_\eta}\mathbf{1}_{\{D_\eta<\tilde{\tau}_{S-s}+\min(\tilde{\tau}_{s-Y},L_\xi)\}}\right) &= E\left(\int_0^{\tilde{\tau}_{S-s}+\min(\tilde{\tau}_{s-Y},L_\xi)} e^{-\beta t}\eta e^{-\eta t}dt\right) \\ &= \frac{\eta}{\eta+\beta}\left(1-E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)E\left(e^{-(\beta+\eta)\min(\tilde{\tau}_{s-Y},L_\xi)}\right)\right) \\ &= \frac{\eta}{\eta+\beta}\left(1-E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\frac{\xi+(\beta+\eta)E\left(e^{-(\beta+\xi+\eta)\tilde{\tau}_{s-Y}}\right)}{\beta+\xi+\eta}\right), \end{aligned}$$

where $E\left(e^{-(\beta+\eta)\min(\tilde{\tau}_{s-Y},L_\xi)}\right)$ is given in (38).

For the second component, using that $\tilde{\tau}_S=\tilde{\tau}_{S-s}+\tilde{\tau}_{s-Y}$ and that the last two RVs are independent we get:

$$
\begin{aligned}
E\left(e^{-\beta\tilde{\tau}_S}\mathbf{1}_{\{\tilde{\tau}_S\le D_\eta\}}\mathbf{1}_{\{\tilde{\tau}_{s-Y}\le L_\xi\}}\right)\frac{\xi}{\xi+\beta}E\left(e^{-\beta B}\right) &= \frac{\xi}{\xi+\beta}E\left(e^{-\beta B}\right) \\
&\quad *E\left(e^{-\beta\tilde{\tau}_S}\int_{\tilde{\tau}_S}^{\infty}\eta e^{-\eta x}dx\int_{\tilde{\tau}_{s-Y}}^{\infty}\xi e^{-\xi z}dz\right) \\
&= \frac{\xi E\left(e^{-\beta B}\right)}{\xi+\beta}E\left(e^{-\beta\tilde{\tau}_S}e^{-\eta\tilde{\tau}_S}e^{-\xi\tilde{\tau}_{s-Y}}\right) \\
&= \frac{\xi E\left(e^{-\beta B}\right)}{\xi+\beta}E\left(e^{-(\beta+\eta+\xi)\tilde{\tau}_{s-Y}}\right)E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right).
\end{aligned}
$$

For the third component, because on these events $T=(\tilde{\tau}_{S-s}+L_\xi)$ we can use (39)

$$
E\left(e^{-\beta(\tilde{\tau}_{S-s}+L_\xi)}\mathbf{1}_{\{\tilde{\tau}_{S-s}+L_\xi\le D_\eta\}}\mathbf{1}_{\{L_\xi\le\tilde{\tau}_{s-Y})\}}\right)=\frac{\xi E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\left(1-E\left(e^{-(\xi+\beta+\eta)\tilde{\tau}_{s-Y}}\right)\right)}{\xi+\beta+\eta}.
$$

Solving for $E\left(e^{-\beta B}\right)$ we get

$$
E\left(e^{-\beta B}\right)=\frac{\frac{\eta}{\eta+\beta}\left(1-E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\frac{\xi+(\beta+\eta)E\left(e^{-(\beta+\xi+\eta)\tilde{\tau}_{s-Y}}\right)}{\beta+\xi+\eta}\right)}{1-\xi E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\left(\frac{E\left(e^{-(\beta+\eta+\xi)\tilde{\tau}_{s-Y}}\right)\eta}{(\xi+\beta)(\xi+\beta+\eta)}+\frac{1}{\xi+\beta+\eta}\right)}.
$$

Using (40), we obtain (10). ■

Proof. of Proposition 4: The left hand side of (11) is the long run average number of downcrossings of level x. There are two types of downcrossings. The first occurs at times of disasters, so that when a disaster arrives (at rate η) and the content level is above level x, level x is downcrossed. Therefore the long run average number of downcrossings that are generated by disasters is $\eta\int_x^S dF(w)$. The second type is due to the demands that arrive at rate λ. Every demand starting at inventory level w downcrosses x if the demand size is greater than $w-x$, i.e., with probability $1-F_Y(w-x)$. Therefore, the second type of downcrossings of level x has a rate $\lambda\int_x^S(1-F_Y(w-x))\,dF(w)$. The right hand side of (11) is the long run average upcrossing of level x. When $x\le s$, the upcrossings are generated by the replenishments (that arrive at rate ξ); every replenishment is an upcrossing if the content level is less than x. However, if $s<x\le S$ level x is upcrossed only by jumps that start below level s, since the leadtime starts only after downcrossings of level $S-s$. Finally, the density $dF(\cdot)$ is the same in both sides of (11) by PASTA. ■

Proof. of Proposition 8: For the exact analysis, let G_n be the time of the n^{th} arrival in the Λ_g arrival process, so that e.g., T_n is the time between the $n-1^{th}$ and n^{th} order arrival, i.e., the length of the n^{th}

cycle. Then, because all of these processes are renewal processes, G_n are iid and for (30) the discounted expected costs we have

$$\begin{aligned}
E\left(\int_0^\infty e^{-\beta t} d\Lambda_g(t)\right) &= E\left(\sum_{n=1}^\infty e^{-\beta \sum_{j=1}^n G_j}\right) = \sum_{n=1}^\infty E\left(e^{-\beta \sum_{j=1}^n G_j}\right) \\
&= \sum_{n=1}^\infty \left(E\left(e^{-\beta G}\right)\right)^n = \frac{E\left(e^{-\beta G}\right)}{1-E\left(e^{-\beta G}\right)}.
\end{aligned} \tag{42}$$

To establish the approximation in (31) let $S_n = \sum_{i=1}^n T_n$ and construct a modified process $\hat{\mathbf{W}} = \{\hat{W}(t) : t > 0\}$ by deleting from $\mathbf{W}$ the time periods in which $\mathbf{W}$ is above level s and gluing together the time periods in which $\mathbf{W}$ is below s. Clearly, $\hat{\mathbf{W}}$ is a regenerative process, such that the end of its cycles occur at instants of upward Poisson jumps (with rate ξ). (These upward jumps are the only upward jumps for $\hat{\mathbf{W}}$). Thus, by PASTA, these jumps see time average; moreover, these jumps occurs at times S_n in the original process. Thus, $W(S_n^-)$ in the original process is stochastically equal to $\hat{W}$ - the steady state random variable of $\hat{\mathbf{W}}$. Finally, by the construction of $\hat{\mathbf{W}}$ from $\mathbf{W}$ we have

$$E\left(W(S_n^-)\right) = E\left(\hat{W}\right) = E\left(W \mid W \leq s\right). \tag{43}$$

Our first approximation assumes that $W\left(S_n^-\right)$ is independent of S_n^-. To see that this is an approximation it is enough to consider the first cycle where $W(S_1^-)$ and $S_1 = T_1 = \tau_{S-s} + L_\xi$, from (3), are dependent- a longer cycle implies a longer lead time and therefore a lower $W\left(S_1^-\right)$. With this assumption, (43), and the exact derivations above we have

$$\begin{aligned}
E\left(\int_0^\infty e^{-\beta t} W\left(t^-\right) d\Lambda_o\left(t\right)\right) &= E\left(\sum_{n=1}^\infty W\left(S_n^-\right) e^{-\beta S_n}\right) \approx E\left(\sum_{n=1}^\infty E\left(W | W \leq s\right) e^{-\beta S_n}\right) \\
&= E\left(W \mid W \leq s\right) \frac{E\left(e^{-\beta T}\right)}{1-E\left(e^{-\beta T}\right)},
\end{aligned}$$

as in (31).

To establish the approximation in (32) we use Poisson thinning of the leadtime process, $\Lambda_o(t)$. This process is a Poisson process with rate ξ and $d\Lambda_o(t) = \xi dt$. However, when the inventory level is above s, an arrival of this process has no effect on costs. If we assume that given $W\left(t\right) \leq s$, we have that $W\left(t\right)$ and $d\Lambda_o(t)$ are independent, we can approximate the discounted set-up and ordering cost functionals by

$$
\begin{aligned}
E\left(\int_0^{\infty} e^{-\beta t} W\left(t^{-}\right) I\left\{W\left(t^{-}\right) \leq s\right\} d\Lambda_o(t)\right) &\approx E\left(\int_0^{\infty} e^{-\beta t} \xi * \left(W\left(t^{-}\right) I\left\{W\left(t^{-}\right) \leq s\right\}\right) dt\right) \\
&= \int_0^{\infty} e^{-\beta t} \xi * E\left(W\left(t^{-}\right) I\left\{W\left(t^{-}\right) \leq s\right\}\right) dt \\
&= \frac{\xi}{\beta} E\left(W I\left\{W \leq s\right\}\right) = \frac{\xi}{\beta} E\left(W | W \leq s\right) F_W\left(s\right).
\end{aligned}
$$

where the first equality follows because the integrand is positive so that we can change the order of expectation and integration.

As for the approximation in (33), it is based on replacing the transient content level process $\mathbf{W}$ by its steady state random variable. In other words, in (33) we replace the pre-determined constants $W\left(t^{-}\right)$ for the unit purchase cost and $W(0) = S$, for the holding cost, by the random variable W_e such that $\lim_{t\to\infty} \Pr(W_e(t) \leq x) = F(x)$ at all points of continuity of $F(\cdot)$, i.e., $W_e(t)$ designates that the content level process is in equilibrium. It should be noted that this approximation improves as: (i) the speed of convergence to steady state increases, and (ii) the discounted rate β decreases, thus costs in the far future still impact the total expected discounted costs. Formally, if the real content process is replaced by its steady state version we have for the holding cost:

$$
E\left(\int_0^{\infty} e^{-\beta t} W(t) dt\right) \approx E\left(\int_0^{\infty} e^{-\beta t} W_e(t) dt\right) = \int_0^{\infty} e^{-\beta t} E(W_e(t)) dt = \int_0^{\infty} e^{-\beta t} E\left(W\right) dt = \frac{E\left(W\right)}{\beta}. \tag{44}
$$

■

B Asymptotic Results

B.1 Poisson Demand

Taking the limit of $E(W)$, $E(T)$, $E(Z)$ and $E(U)$ with respect to ξ, $\lim\limits_{\xi\to\infty}$, using Maple we obtain:

$$
\begin{aligned}
\lim_{\xi\to\infty} E\left(W\right) &= \frac{\lambda^{S-s}\left(s\eta - \lambda\right) - \left(\lambda + \eta\right)^{S-s}\left(S\eta - \lambda\right)}{\eta\left(\lambda^{S-s} - \left(\lambda + \eta\right)^{S-s}\right)} \\
\lim_{\xi\to\infty} E\left(T\right) &= \frac{1 - \left(\frac{\lambda}{\lambda+\eta}\right)^{S-s}}{\eta} \\
\lim_{\xi\to\infty} E\left(Z\right) &= \frac{1}{\eta} \\
\lim_{\xi\to\infty} E\left(U\right) &= \infty.
\end{aligned}
$$

This results are not surprising and, of course, agree with the ones in Baron, Berman and Perry (2010).

For $\lim_{\eta\to 0}$, i.e., the heuristic with no disasters, we obtain

$$\begin{aligned}
\lim_{\eta\to 0} E(W) &= \frac{\xi\left(\left(S+S^2\right)-\left(s+s^2\right)\right)}{2\left(\lambda+\xi\left(S-s\right)\right)} \\
\lim_{\eta\to 0} E(T) &= \frac{\lambda+\xi\left(S-s\right)}{\lambda\xi} \\
\lim_{\eta\to 0} E(U) &= \left(\frac{\lambda+\xi}{\lambda}\right)^s \frac{\lambda+\xi\left(S-s\right)}{\lambda^2}.
\end{aligned}$$

We get no exact expression for $\lim_{\eta\to 0} E(Z)$ (its dependency on the controls is not trivial). For $\lim_{\xi\to\infty}\lim_{\eta\to 0}$ we obtain (note that the order of taking the limits does not matter)

$$\begin{aligned}
\lim_{\xi\to\infty}\lim_{\eta\to 0} E(W) &= \frac{1}{2}S+\frac{1}{2}s+\frac{1}{2} \\
\lim_{\xi\to\infty}\lim_{\eta\to 0} E(T) &= \frac{S-s}{\lambda} \\
\lim_{\xi\to\infty}\lim_{\eta\to 0} E(Z) &= \infty \\
\lim_{\xi\to\infty}\lim_{\eta\to 0} E(U) &= \infty
\end{aligned}$$

When $\xi\to\infty$ and $\eta\to 0$, i.e., for our model with no disasters and no leadtime, the cost function becomes (recall that $\mu=1$ in the Poisson case):

$$R=\frac{K_o}{E(T)}+c\lambda+hE(W)+\frac{K_u-c}{E(U)}+\frac{K_d}{E(Z)}.$$

$$R=\frac{\lambda K_o}{S-s}+c\lambda+\frac{h}{2}\left(S+s+1\right).$$

It is easy to verify that all the elements in R are increasing with $s>0$, so that $s=0$ is optimal. Therefore,

$$R=\frac{\lambda K_o}{S}+c\lambda+\frac{h}{2}\left(S+1\right),$$

which agrees with the expression in Proposition 2 for the case $t_0\to\infty$ from Baron, Berman, and Perry (2010) for this case (where the fixed cost $c\lambda$ is not included and t_0 is the time to perishability).

B.2 Compound Poisson with Exponential Demand

In this demand size case exact asymptotics are not always available. Taking the limit of the expression $E(W)$, $E(T)$, $E(Z)$ and $E(U)$ we obtain the following:

$$\begin{aligned}
\lim_{\eta\to 0}\lim_{\xi\to\infty} E(W) &= \frac{\mu\left(S^2-s^2\right)+2S}{2\mu(S-s)+2} \\
\lim_{\xi\to\infty} E(T) &= -\frac{\left(\lambda\left(e^{\frac{\mu\eta(s-S)}{\lambda+\eta}}-1\right)-\eta\right)}{\eta(\lambda+\eta)} \\
\lim_{\eta\to 0}\lim_{\xi\to\infty} E(T) &= \frac{\mu(S-s)+1}{\lambda} \\
\lim_{\xi\to\infty} E(Z) &= \frac{1}{\eta} \qquad \left(\lim_{\xi\to\infty\eta\to 0} E(Z)=\infty\right)
\end{aligned}$$

$$\begin{aligned}
\lim_{\xi\to\infty} E(U) &= \frac{\lambda\left(e^{\frac{\mu(s\lambda+S\eta)}{\lambda+\eta}}-e^{\mu s}\right)+\eta e^{\frac{\mu(s\lambda+S\eta)}{\lambda+\eta}}}{\lambda\eta} \\
\lim_{\eta\to 0}\lim_{\xi\to\infty} E(U) &= \frac{e^{\mu s}\left(\mu(S-s)+1\right)}{\lambda}
\end{aligned}$$

In accordance with this, as $\xi\to\infty$ and $\eta\to 0$, i.e., for our model with no disasters and no leadtime with a compound Poisson demand with rate λ where each demand size is $\exp(\mu)$, the cost function becomes

$$R=\frac{K_o}{E(T)}+c\frac{\lambda}{\mu}+hE(W)+\frac{K_u-c}{\mu E(U)}=\frac{\lambda K_o}{\mu(S-s)+1}+c\frac{\lambda}{\mu}+h\left(\frac{\mu\left(S^2-s^2\right)+2S}{2(\mu(S-s)+1)}\right)+\frac{\lambda(K_u-c)}{\mu e^{\mu s}(\mu(S-s)+1)}.$$

It is easy to verify that other than the cost of lost sales (the last element above) the other elements are all increasing with $s>0$.

When there are no lost sales

$$R=\frac{\lambda K_o}{\mu(S-s)+1}+c\frac{\lambda}{\mu}+h\left(\frac{\mu\left(S^2-s^2\right)+2S}{2\left(\mu(S-s)+1\right)}\right).$$

Now,

$$\frac{d\left(\frac{\lambda K_o}{\mu(S-s)+1}+c\frac{\lambda}{\mu}+h\left(\frac{\mu\left(S^2-s^2\right)+2S}{2(\mu(S-s)+1)}\right)\right)}{ds}=\frac{\mu}{2}\frac{2\lambda K_o+\mu h(S-s)^2+2h(S-s)}{\left(\mu(S-s)+1\right)^2}>0$$

so that $s=0$ is optimal and

$$R=\frac{2\lambda K_o+2hS+h\mu S^2}{2(\mu S+1)}+c\frac{\lambda}{\mu},$$

which agrees with the expression from Baron, Berman, and Perry (2010) for this case (where the fixed cost $c\frac{\lambda}{\mu}$ is not included).

For the model $\xi\to\infty$ we still get a closed form solution for $E(W)$, but it is quite cumbersome and we do not report it here. Still, this implies that we can write the closed form expression for this model, i.e., for the lost sales model with disasters but with no leadtime of an (S, s) policy with a compound Poisson demand with rate λ where each demand size is $\exp(\mu)$. This model is similar to Baron, Berman, and Perry (2010) and we get

$$\begin{aligned}
\lim_{\xi\to\infty} E(W)\,|_{s=0} &= \frac{e^{\mu S}\left(e^{\mu S}-1\right)\left(\eta\lambda-\mu\eta^2 S-\lambda\mu\eta S+\lambda^2\right)+\left(e^{\frac{S\mu\lambda}{\lambda+\eta}}-e^{\frac{\mu S(2\lambda+\eta)}{\lambda+\eta}}\right)\left(\eta\lambda+\lambda^2\right)}{\left(\lambda e^{\frac{\mu S(2\lambda+\eta)}{\lambda+\eta}}+(\eta+\lambda)\,e^{\mu S}\left(1-e^{\mu S}\right)\right)\eta\mu} \\
&= \frac{-e^{\mu S}\left(e^{\mu S}-1\right)(\lambda+\eta)(\mu S\eta-\lambda)+\left(e^{\frac{S\mu\lambda}{\lambda+\eta}}-e^{\frac{\mu S(2\lambda+\eta)}{\lambda+\eta}}\right)(\eta+\lambda)\,\lambda}{\left(\lambda e^{\frac{\mu S(2\lambda+\eta)}{\lambda+\eta}}+(\eta+\lambda)\,e^{\mu S}\left(1-e^{\mu S}\right)\right)\eta\mu} \\
&= \frac{(\lambda+\eta)}{\eta\mu}\frac{e^{\mu S}\left(e^{\mu S}-1\right)(\mu S\eta-\lambda)-\lambda\left(e^{\frac{\mu S(2\lambda+\eta)}{\lambda+\eta}}-e^{\frac{S\mu\lambda}{\lambda+\eta}}\right)}{(\eta+\lambda)\,e^{\mu S}\left(1-e^{\mu S}\right)+\lambda\left(e^{\frac{\mu S(2\lambda+\eta)}{\lambda+\eta}}-e^{\frac{S\mu\lambda}{\lambda+\eta}}\right)}.
\end{aligned}$$

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