Math Meth Oper Res (2010) 72:217–247
DOI 10.1007/s00186-010-0318-1

ORIGINAL ARTICLE

Continuous review inventory models for perishable items ordered in batches

Opher Baron · Oded Berman · David Perry

Received: 12 June 2009 / Accepted: 20 June 2010 / Published online: 14 July 2010

Abstract This paper is an in-depth treatment of an inventory control problem with perishable items. We focus on two prototypes of perishability for items that have a common shelflife and that arrive in batches with zero lead time: (i) sudden deaths due to disasters (e.g., spoilage because of extreme weather conditions or a malfunction of the storage place) and (ii) outdating due to expirations (e.g., medicine or food items that have an expiry date). By using known mathematical tools we generalize the stochastic analysis of continuous review (s, S) policies to our problems. This is achieved by integrating with each inventory cycle stopping times that are independent of the inventory level. We introduce special cases of compound Poisson demand processes with negative jumps and consider demands (jumps) that are exponentially distributed or of a unit (i.e., Poisson) demand. For these special cases we derive a closed form expression of the total cost, including that of perishable items, given any order up to level. Since the stochastic analysis leads to tractable expressions only under specific assumptions, as an added benefit we use a fluid approximation of the inventory level to develop efficient heuristics that can be used in general settings. Numerical results comparing the solution of the heuristics with exact or simulated optimal solutions show that the approximation is accurate.

O. Baron · O. Berman (✉)
Joseph L. Rotman School of Management, University of Toronto, 105 St. George Street, Toronto, Ontario, M5S 3E6, Canada
e-mail: berman@rotman.utoronto.ca

O. Baron
e-mail: OBaron@rotman.utoronto.ca

D. Perry
Department of Statistics, University of Haifa, 31905 Haifa, Israel
e-mail: dperry@stat.haifa.ac.il

Keywords Perishable items · Continuous review · Stochastic (s, S) · Inventory system · Kella-Whitt martingale · Fluid approximation

1 Introduction

The importance of inventory management in modern industries is well known. The classical inventory theory did not take into account that items are perishable. However, many items are subject to perishability, due to excessive storage time or because they become obsolete. Examples for perishables items include certain foods, seasonal products, chemical, medicines and so on. Managing the inventory of perishable items requires paying special attention for the time to perishability, different reasons for perishability and cost of salvage items. Therefore, special algorithms for managing inventory of perishable items are required.

A recent comprehensive review paper focusing on the management of items with finite shelf life is Karaesmen et al. (2009). They classify continuous review models into three categories: without fixed ordering cost and lead times, without fixed ordering cost having positive lead time and with fixed ordering cost (typically with zero lead time). The first category was originated by Graves (1982) who assumed that items are continuously produced, perish after a deterministic time, and that the demand follows a compound Poisson process with either a single-unit or an exponential demand at each arrival. The second category was originated by Pal (1989) who investigated the performance of an $(S - 1, S)$ control policy. For the first two categories Karaesmen et al. (2009) survey several works (e.g., Nahmias et al. (2004) and references therein), in which it was assumed that the input and the output are random processes. These papers typically analyzed the stochastic models focusing more on performance analysis than on optimization. The third category, originated by Weiss (1980), is most relevant for our paper. Lian and Liu (2001) and Liu and Shi (1999) made significant contributions to models in this category. Lian et al. (2005) consider discrete demand for items and time to perishability that is either fixed-and-known or that follows a Phase-Type distribution. The discussion of heuristics for continuous review of perishables in Lian and Liu (2001) was extended by Gürler and Özkaya (2003) to allow for a general renewal arrival process with general interarrival times. Recently, Berk and Gürler (2008) analyzed the (Q, r) periodic review inventory model for perishable items with lead time using the effective shelf life distribution.

There are two alternative families of inventory models that are relevant to our work. The first assumes that the number of items is slowly decreasing with time and the second is of models with obsolescence. For an early review of work in the first family of models see Sect. 2.2 of Nahmias (1982). This family, originated by Ghare and Scharder (1963), assume exponential decaying inventory. The tractability of such models leads to many works with similar assumptions, even though, only few practical examples accurately fit the case of an exponential decay of inventory. Kalpakam and Arivarignan (1988) consider a model where items are removed from the shelf one at a time due to either depletion (as a result of demand) or perishability. This models cases when only the exhibited item is prone to perishability. David and Mehrez (1995) present two models: one with fixed perishability times (that—unlike in our models—are independent of the items' age) and another with exponentially decaying inventory. Rajan et al. (1992) consider perishable items with order set up cost. They assume that the quantity of good items deteriorates in an exponential rate and solve the problem of finding the joint optimal ordering and pricing strategy. Recently, Varghese and Krishnamoorthy (2004) present a discrete demand model where a disaster hits a certain fraction of the existing items. There are two main differences between this family of models and our models: (i) we assume that when a disaster occurs all items perish, while they assume that during a disaster only a proportion of the items perish; (ii) these models typically result in more tractable formulations and thus their methodology is substantially different from the one we use.

The family of models with obsolescence assumes that the items perish at each period with some probability (which is increasing with time). This models items that become obsolete due to the introduction of a new product that replaces them in the market. Thus once the items become obsolete the items will no longer be ordered. This is in contrast to our infinite horizon model, where items are ordered even after they perished in some cycle. One of the first papers to discuss a model with obsolescence is Pierskalla (1969). Cobbaert and Van Oudheusden (1996) discuss this model in an EOQ like setting, Song and Zipkin (1996) discuss the difficulties in managing the inventory of such items, and more recently, Song and Lau (2004) consider a periodic version of this model.

The closest work to our paper is Gürler and Özkaya (2008) who analyzed the (s, S) policy for perishables with zero lead time and backlog allowing $s < 0$. They express the expected cost function for their model using integrations and sums of relevant distribution functions which can be evaluated only using a computer numerical method and are therefore very difficult to use in optimization. In contrast, we derive closed form expressions for the relevant cost in our model.

In this paper, we consider batch arrival of items where in each batch the items have a common shelflife, i.e. they all perish together at some deterministic or stochastic time. We also assume a compound Poisson demand process. Note that our methodology allows to consider demand that is discrete, continuous, or both.

We assume that management aims at stocking policies that minimize the inventory cost required to satisfy the demand (or some fraction of it). For perishable items, the inventory control costs include holding cost, order set up cost and cost of the perished items. We analyze in depth two cases of the time to perishability, $T(\xi)$. The first case is $T(\xi) \backsim \exp(1/\xi)$, modeling items that are subject to a disaster e.g., spoilage because of extreme weather conditions or a malfunction of a refrigerator that stores them. Alternatively, this can reflect situations where management may have a little control or information on the state of the batch received by the supplier. The second case is when $T(\xi) \equiv t_0 = 1/\xi$, modeling perishability of items with an expiry date, i.e., when dealing with medicines or food items that are ordered in batch and have an expiry date, e.g., milk.

We focus on special cases of compound Poisson demand processes and consider exponential (μ) and unit (i.e., Poisson) demand sizes. In principle, our approach can be extended to demand sizes of other phase-type distributions. However, complicated phase-type distributions lead to cumbersome expressions of the relevant functionals which are needed for optimization. As will be seen in the sequel, the models presented here are quite intricate even for the simple cases considered.

Thus, we provide an in depth analysis of four cases according to two types of time to perishability (deterministic and exponential) and two types of compound Poisson demands (exponential size and unit steps). Additional treatment explaining how to extend our results to phase type demands and time to perishability can be found in Baron et al. (2010).

We assume that the lead time is zero and the inventory level is raised to S immediately once it reaches 0. While this assumption is quite restrictive, relaxing it for models with perishable items would require $s > 0$, coupling consecutive review periods and would make the model significantly more difficult to solve and is thus left for future research. For continuous demand, this assumption is equivalent to allowing no backlog once a cycle ends due to depletion. When demand is intense and backlog is allowed, Weiss (1980) shows that it is never optimal to order before the inventory level is -1 or below.

For the models considered we develop a methodology to find the optimal inventory order quantity, S, to minimize the inventory cost of perishable items. Under the assumptions above, inventory cycles are i.i.d. where the inventory level starts at level S and drops down to 0, allowing to apply renewal theory to analyze the inventory process. Due to the compound Poisson arrival assumption the inventory process is a stopped Levy process. Thus, we use the special cases of the *Kella–Whitt* martingale (Kella and Whitt 1992) to express the joint Laplace Transform (LT) of the inventory level and cycle length. For similar usages of the Kella-Whitt martingale see Boxma et al. (2001) and references therein. Their paper studies the M/PH/1 queue and is relevant for our work because like in our paper it expresses relevant quantities for the minimum of two independent stopping times. A main difference between our paper and this one is that we consider batch arrivals and a discrete arrival process that are more plausible for inventory models with perishability.

The main contribution of our paper is from a methodological point of view. By using known mathematical tools we generalize the stochastic analysis of the traditional (s, S) models to our two perishability models and obtain closed form expressions for the main functionals. Additionally, our methodology allows considering demand that is either discrete, continuous, or both. From an application point of view, this generalization models perishable items arriving in batches, i.e., with order set up cost where all the items in a batch have a common shelflife. That is, fresh items arrive together, age together and those who perish (if any), perish together. Moreover, since the stochastic analysis leads to tractable expressions only under specific assumptions, as an additional benefit we use a fluid approximation for the inventory level to develop heuristic solutions. Numerical results comparing the solution of the heuristics with exact or simulated optimal solutions show that the approximation is accurate.

We note that all proofs that are not in the text are included in the Appendix. The paper is organized as follows. Section 2 introduces the model considered. As a building block for the analysis of the perishable items, we first consider a traditional $(0, S)$ model in which cycles are terminated only due to depletion (i.e., items are not perishable). Thus, we analyze in Sect. 3 the equilibrium law of the regenerative control level process whose cycles end only due to depletion. The proofs of this simpler model are provided in the body of the paper. We see the derivation in Sect. 3 as the best way to expose our technique to the readers. In Sect. 4 we compute the relevant functionals for the cases with exponential time to perishability and with a deterministic expiry date. In Sect. 5 we discuss the optimization and sensitivity analysis of the models developed. In Sect. 6 we present a heuristic based on the fluid approximation of the inventory level and demonstrate its effectiveness. Section 7 includes a summary and suggestions for future research.

2 The demand and perishability model

A rich subset of inventory models, both under continuous or periodic review apply (s, S) control policies. In these policies, the inventory is controlled such that an order is placed to bring the inventory level up to level S as soon as it reaches or drops below level s $(s < S < \infty)$. The decision variables (s, S) are chosen to answer two interrelated questions: "when to order" and "how much to order". For a comprehensive literature review of (s, S) inventory models see, e.g., Arrow et al. (1958), Nahmias (1997), Sahin (1989), and Scarf (1960) for periodic review variants, and Porteus (2002) and Zipkin (2000) for continuous review ones. For continuous review inventory models these decisions assume that the sample path of the inventory level $\mathbf{V} = \{V(t) : t \geq 0\}$ is a regenerative process with cycles τ of type

$$\tau = T_{S-s} := \inf\{t : V(0) = S, V(t) \leq s\}. \quad (1)$$

The cycle (1) simply says that the beginning of the cycle is shifted to the origin such that $V(0) = S$, $V(\tau) \leq s$ and $V(\tau+) = S$ (the inventory level at the beginning of the next cycle).

Traditionally, (s, S) inventory models are analyzed with the assumption that the shelflife of the items stored is infinite. We in contrast consider inventory model for perishable items where the cycle of type (1) is not applicable. Thus, when the shelflife of the items stored is finite the (s, S) control policy as introduced by the stopping time (1) should be generalized.

We assume that the lead time is 0 and do not allow back orders. It is also assumed that the inventory level is raised to S immediately once it reaches 0. In cases where a demand is larger than the inventory level the next order includes also the remaining portion of unsatisfied demand. With these assumptions the sample path of $\mathbf{V}$ within a cycle is a non-increasing step function where the (negative) jumps are the demand sizes. That is

$$V(t) = S - (X_1 + ... + X_{N(t)}), \quad 0 < t \leq \tau$$

where $\mathbf{N} = \{N(t) : t \geq 0\}$ is a compound renewal process with interarrival distribution $A(\cdot)$, LT $A^*(\cdot)$, and demands at arrivals, $X_1, X_2, \ldots$ are i.i.d. with distribution $B(\cdot)$. Also, the distribution of the time to perishability $T(\xi)$ is independent of $X_1, X_2, ...$ and $\mathbf{N}$. Note that, unless the regeneration time τ is smaller than the first interarrival time, τ is a point of discontinuity with $V(\tau-) \leq V(\tau) < V(\tau+) = S$, because at the end of the cycle the entire inventory is refilled.

We focus on two prototypes of perishability; sudden deaths due to disasters and outdatings due to expirations. We express these two prototype cycles in terms of the stopping time T_S, which is a special case of the cycle (1)

$$\tau = T_S := \inf\{t : V(0) = S, V(t) \leq 0\}. \tag{2}$$

The first prototype cycle is

$$\tau = \tau_S(\xi) := \min\{T_S, T(\xi)\}, \tag{3}$$

where $T(\xi)$ is an exponentially distributed(ξ) random variable that is independent of $\{V(t) : 0 < t \leq T_S\}$ ($T(\xi)$ and T_S are also independent, since T_S is a stopping time with respect to $\mathbf{V}$). Cycles of type (3) indicate that the items stored are subject to disasters which arrive according to a Poisson process with rate ξ. That is, whenever $\{T(\xi) < T_S\}$ the disaster destroys all the items present. Note that whenever $T_S \leq T(\xi)$, all items are used to satisfy the demand.

The second prototype cycle is

$$\tau = \tau_S(t_0) := \min\{T_S, t_0\} \tag{4}$$

where t_0 is a given constant. Cycles of type (4) represent the case in which the items stored have a fixed predetermined shelflife t_0, e.g., the items stored have an expiry date t_0.

For the generalized control policies with cycles of types (3) and (4) we use the notation (τ, S) where τ is the type of the cycle and S indicates that $V(0) = V(\tau+) = S$ (each cycle starts at level S). Note that the (τ, S) control policy where τ is the stopping time (2) is equivalent to the traditional (s, S) control policy (with $s = 0$) for non perishable items.

From the stochastic modelling point of view both types of cycles (3) and (4), separately, comprise a large collection of inventory model variants due to the specifications of the i.i.d. negative jumps and the i.i.d. interarrival times. As mention earlier, we focus on cases of compound Poisson processes with negative jumps (corresponding to demands). In the sequel we restrict the attention to exponential (μ) and unit (i.e., Poisson) demand sizes.

The objective is to minimize the total cost required to satisfy the demand. This cost is composed of holding cost, setup cost, and cost of the perished items. The objective is to find the inventory order up to level, S, that minimizes this cost.

Let K be the order set-up cost, h be the holding cost per unit of inventory per unit of time, π be the penalty for the perishability of one unit of inventory, $V(\tau-)$ be the inventory level at the end of cycles, and E denotes the expected value operator (both $E(V)$ and $E(V(\tau))$ are well defined because $\mathbf{V}$ is a regenerative process).

Then, the long-run average cost function $C(S)$ is defined by:

$$C(S) = \frac{K + \pi E(V(\tau)I\{\tau < T_S\})}{E(\tau)} + hE(V), \tag{5}$$

where $I\{\cdot\}$ is the indicator function with a value 1 if $\{\cdot\}$ occur and 0, otherwise.

For the discounted cost criterion recall that $\mathbf{V}$ is a regenerative process. Thus, it is enough to compute the expected discounted profit for one cycle $Z(S)$. The total expected discounted cost $H(S)$, is:

$$H(S) = \frac{Z(S)}{1 - E\left(e^{-\beta\tau}\right)}$$

where $\beta > 0$ is the discount factor. Then

$$Z(S) = K + hE\left(\int_0^\tau e^{-\beta t} V(t)\,\mathrm{d}t\right) + \pi E\left(e^{-\beta\tau} V(\tau) I\{\tau < T_S\}\right).$$

Note that while our numerical work is focused on the average cost criteria (5), our stochastic analysis also develops the LT required for expressing the discounted cost.

For convenience, we define the shortfall process $\mathbf{W} = \{W(t) : t \geq 0\}$ where $W(t) = S - V(t)$. That is,

$$W(t) = X_1 + \cdots + X_{N(t)}, \quad 0 < t \leq \tau.$$

Let $R(\cdot)$ and $R^*(\alpha)$ denote the equilibrium distribution and LT of $\mathbf{W}$, respectively. That is,

$$R(w) = \lim_{t\to\infty} \frac{1}{t} \int_0^t I\{W(s) \leq w\}\,\mathrm{d}s = \frac{E\left(\int_0^\tau I\{W(t) \leq w\}\,\mathrm{d}t\right)}{E(\tau)}$$

where the second equality holds since $\mathbf{W}$ is a regenerative process, and $R^*(\alpha) := E\left(e^{-\alpha W}\right)$.

Note that

$$E\left(e^{-\alpha V}\right) = e^{-\alpha S} R^*(-\alpha).$$

Remark 1 When the shortfall process W regenerates at the beginning of each cycle, it is mathematically identical to the clearing processes discussed in the inventory example Ross (1996) p. 118 and in Perry and Posner (2002). In these cases, Remark 2 of Perry and Posner (2002) holds in our settings as well. We restate the relevant part of that remark here using our notation: **The steady state distribution of W is independent of the exact details of the arrival process as long as it has i.i.d. demand sizes; that is, the inter-demand times need be neither independent nor identically distributed and this steady state distribution is also independent of the arrival rate.**

In our model, with perishability, the shortfall process W regenerates when an order of S items is initiated by a demand or when the arrival process is Poisson. In both cases, the times to the first arrivals following the event $W(\tau) \geq S$ are independent of $W(t)$

for $t < \tau$ and identically distributed. Therefore, W regenerates every time $W(\tau) \geq S$. This implies that for models without perishability, when every order is initiated by a demand, our analysis in Sect. 3, which is done assuming a compound Poisson demand, gives the steady state distribution of W for any compound renewal arrival process with the same demands upon arrivals. However, with perishability, $W(\tau) \geq S$ can happen at the middle of an interarrival time. Then, W does not regenerate. We overcome this difficulty by assuming that the demand process is a compound Poisson process, and that replenishment is made upon perishability. Extending the analysis to cases where the replenishment is postponed until the next arrival is simple and is discussed in Remark 2 of Sect. 5.

We consider the optimization problem

$$\min_S C(S) = \frac{K + \pi E((S - W(\tau))I\{\tau < T_S\})}{E(\tau)} + h(S - E(W)). \tag{6}$$

Since $V = S - W$, the equation above is equivalent to minimizing the cost function in (5).

In the next sections, we derive the joint LT for the shortfall process $R^*(\alpha)$ and the length of the cycle $E(e^{-\alpha W(\tau) - \beta\tau})$, for several demand and time to perishability models.

To solve (6) we derive

$$E(\tau) = -\left(\frac{d}{\mathrm{d}\beta} E\left(e^{-\alpha W(\tau) - \beta\tau}\right)|_{\alpha=0}\right)|_{\beta=0} ,$$

$$E(W) = -\frac{\mathrm{d}R^*(\alpha)}{d\alpha}|_{\alpha=0} , \quad \text{and}$$

$$E(W(\tau) I\{\tau < T_S\}) = -\left(\frac{d}{\mathrm{d}\alpha} E\left(e^{-\alpha W(\tau) - \beta\tau} I\{\tau < T_S\}\right)|_{\beta=0}\right)|_{\alpha=0} ,$$

where $E\left(e^{-\alpha W(\tau) - \beta\tau} I\{\tau < T_S\}\right)$ is developed while deriving $E\left(e^{-\alpha W(\tau) - \beta\tau}\right)$.

3 The control policy without perishability, i.e., of type $\tau = T_S$

Here we assume that items are not perishable and that the demand process is a compound renewal process with inter-renewal LT $A^*(\alpha)$, mean $1/\lambda$ and demand size distribution $B(\alpha)$. Remark 1 implies that the steady state law is insensitive to the interrenewal law between demand arrivals. Thus, the compound renewal assumption can be replaced by a compound Poisson assumption with parameter λ. According to this argument the process **W** becomes a stopped compound Poisson process with arrival rate λ. In the sequel we use special cases of the Kella–Whitt martingales for computing the LT of **W**, $R^*(\alpha)$ and the joint LT of $W(T_S)$ and T_S. We will use the LT developed here in Sect. 4 where we consider models with perishability.

3.1 Phase type demands

The only case (with continuous demands) in which $W(T_S)$ and T_S are independent occurs when the demand sizes are exponential(μ). As a result, for general demands the analysis becomes cumbersome. An exception is when the demand has a Phase-Type distribution. Then, $W(T_S)$ and T_S are conditionally independent given the exponential phase in which level S is upcrossed. Next, we discuss the solution for a general Erlang distribution and then give the exact solution for the exponential (μ) case. A more involved analysis for the case of Erlang (2, μ) case can be found in Baron et al. (2010).

3.1.1 General Erlang distributions for the demand

Assume that the demand $X \sim Er\,(n, \mu)$ for some n. Then, $W(T_S)$ and T_S are conditionally independent given the exponential phase in which level S is upcrossed. Define the event

$$G^i = \{\text{level } S \text{ is upcrossed by } \mathbf{W} \text{ with the } i\text{th phase of the demand jump}\}.$$

Lemma 1 *For the control policy of type T_S with an $Er(n, \mu)$ demand, let*

$$\phi^i(\beta) = E\left(e^{-\beta T_S} I\left\{G^i\right\}\right).$$

Then

$$E\left(e^{-\alpha W(T_S)-\beta T_S}\right) = e^{-\alpha S} \sum_{i=1}^{n} \left(\frac{\mu}{\mu+\alpha}\right)^{n-i+1} \phi^i(\beta).$$

Proof Proof. By the law of total expectation

$$E\left(e^{-\alpha W(T_S)-\beta T_S}\right) = \sum_{i=1}^{n} E\left(e^{-\alpha W(T_S)-\beta T_S} I\left\{G^i\right\}\right).$$

Given the event G^i, $W(T_S)$ and T_S are conditionally independent and by the memoryless property the law of the overflow above S is $Er(n-i+1, \mu)$. □

By using Lemma 1 characterization of the joint LT of $W(T_S)$ and T_S only requires to find $\phi^i(\beta)$. This is done below for the exponential (μ) case.

3.1.2 Exponential demand

When the demand size X is $exp\,(\mu)$ we have

Theorem 1 *For the control policy of type T_S with $exp(\mu)$ demand, the joint LT of $W\,(T_S)$ and T_S is*

$$E\left(e^{-\alpha W(T_S)-\beta T_S}\right)=\frac{\mu}{\mu+\alpha}\frac{\lambda}{\lambda+\beta}e^{-\alpha S}e^{-\frac{\mu\beta}{\lambda+\beta}S}, \tag{7}$$

$$E\left(T_S\right)=\frac{1+\mu S}{\lambda}, \tag{8}$$

and

$$R^*\left(\alpha\right)=\frac{\mu+\alpha-\mu e^{-\alpha S}}{\left(1+\mu S\right)\alpha}. \tag{9}$$

Proof We use the martingale $\mathbf{M}_1=\{M_1\left(t\right):t\geq 0\}$ where

$$M_1(t)=-\left[\lambda\left(1-\left(\frac{\mu}{\mu+\alpha}\right)\right)+\beta\right]\int_0^t e^{-\alpha W(u)-\beta u}\mathrm{d}u+1-e^{-\alpha W(t)-\beta t}. \tag{10}$$

Here, the exponent $-\lambda(1-(\frac{\mu}{\mu+\alpha}))$ corresponds to a compound Poisson process with rate λ and $exp(\mu)$ demands. For $\beta\geq 0$ we can apply the optional sampling theorem to the stopping time T_S. Thus, $E\left(M_1\left(0\right)\right)=E\left(M_1\left(T_S\right)\right)=0$ leads to the fundamental identity

$$-\left[\lambda\left(1-\frac{\mu}{\mu+\alpha}\right)+\beta\right]E\left(\int_0^{T_S} e^{-\alpha W(u)-\beta u}\mathrm{d}u\right)=-1+E\left(e^{-\alpha W(T_S)-\beta T_S}\right). \tag{11}$$

We choose α such that $\lambda(1-(\frac{\mu}{\mu+\alpha}))+\beta=0$. Then, the unique root of the left hand side is

$$\bar{\alpha}(\beta)=-\frac{\mu\beta}{\lambda+\beta}. \tag{12}$$

Substituting (12) in (11) we get

$$1=E\left(e^{-\bar{\alpha}(\beta)W(T_S)-\beta T_S}\right). \tag{13}$$

By the memoryless property, we have $W\left(T_S\right)=S+X$, where X is an $exp\left(\mu\right)$ random variable. Thus,

$$E\left(e^{-\alpha W(T_S)}\right)=\frac{\mu}{\mu+\alpha}e^{-\alpha S}. \tag{14}$$

and $W(T_S)$ and T_S are independent. Combining (13) with (14) we get

$$1=\frac{\mu}{\mu+\bar{\alpha}(\beta)}e^{-\bar{\alpha}(\beta)S}E\left(e^{-\beta T_S}\right)$$

Using (12),

$$E\left(e^{-\beta T_S}\right) = \frac{\mu - \frac{\mu\beta}{\lambda+\beta}}{\mu} e^{-\frac{\mu\beta}{\lambda+\beta}S}$$
$$= \frac{\mu\lambda}{\mu(\lambda+\beta)} e^{-\frac{\mu\beta}{\lambda+\beta}S}.$$

The independence between T_S and $W(T_S)$ gives (7). Using (7), we get

$$E(T_s) = \lim_{\beta\to\infty} \frac{d\left(E\left(e^{-\alpha W(T_s)-\beta T_s}\right)\right)}{\mathrm{d}\beta}|_{\alpha=0} = \frac{1+\mu S}{\lambda}.$$

To find $R^*(\alpha)$ we set $\beta = 0$ in (11) and divide both sides by $-[\lambda(1-(\frac{\mu}{\mu+\alpha}))]E(T_S)$ to get

$$\frac{E\left(\int_0^{T_S} e^{-\alpha W(u)}\mathrm{d}u\right)}{E(T_S)} = \frac{1 - E\left(e^{-\alpha W(T_S)}\right)}{[\lambda\left(1-\frac{\mu}{\mu+\alpha}\right)]E(T_S)}.$$

By renewal theory the left hand side above equals $R^*(\alpha)$. By (14) and (8) we get

$$R^*(\alpha) = \frac{1 - \frac{\mu}{\mu+\alpha}e^{-\alpha S}}{\lambda\left(1-\frac{\mu}{\mu+\alpha}\right)\frac{1+\mu S}{\lambda}}.$$

Then, (9) follows after some algebra. □

Observation 1 *For the control policy of type T_S with $exp(\mu)$ demand the density of T_S is given by*

$$f_{T_S}(t) = \sum_{n=1}^{\infty} \frac{e^{-\mu S}(\mu S)^{n-1}}{(n-1)!} \frac{e^{-\lambda t}(\lambda t)^{n-1}\lambda}{(n-1)!}. \quad (15)$$

Proof We condition on that the nth arrival induced an order, i.e., was at time T_S. Then, the total demand up to this arrival, i.e., in $(n-1)$ arrivals, is Erlang $(n-1, \mu)$. The probability that this total demand is less than S and the next demand (the nth one) will cause the total demand to be more than S is identical to the probability that a Poisson random variable with rate μS has $n-1$ arrivals. This gives the first ratio in the right hand side of (15). The second ratio is the probability that $(n-1)$ customers arrive until time t and $\lambda\,\mathrm{d}t$ is the probability of an arrival at $[t, t-\lambda t)$. This together with the summation releases the condition. An alternative, but less insightful proof can be obtained from setting $\alpha = 0$ in (7) and inverting this LT. □

4 Control policies with perishability

In this section we extend the analysis to consider stopping times that are the minimum between T_S and a stopping time that is independent of the demand. This models perishable items. The derivation of the Theorems below (given in the Appendix) uses relevant LT that were developed in the last Section.

4.1 The control policy of type $\tau = \min(T(\xi), T_S)$

In this section we apply the model to the policy of type $\tau_S(\xi) := \min(T(\xi), T_S)$ where $T(\xi) \backsim \exp(\xi)$ and is independent of T_S.

4.1.1 Exponential demands

Assume that $X \backsim \exp(\mu)$, then using (30) we have

Theorem 2 *For the control policy of type* $\tau_S(\xi)$ *with* $\exp(\mu)$ *demand,*

$$E\left(e^{-\alpha W(\tau_S(\xi))-\beta\tau_S(\xi)}\right) = \frac{\mu e^{-\alpha S}}{\mu+\alpha}\frac{\lambda e^{-\frac{\mu(\beta+\xi)}{\lambda+\beta+\xi}S}}{\lambda+\beta+\xi} + \frac{(\mu+\alpha)\,\xi}{\mu(\xi+\beta)+\alpha(\lambda+\beta+\xi)} - \frac{\xi\mu\lambda e^{-\left(\frac{\mu(\beta+\xi)}{\lambda+\beta+\xi}+\alpha\right)S}}{(\mu\,(\beta+\xi)+\alpha\,(\lambda+\beta+\xi))\,(\lambda+\beta+\xi)}, \tag{16}$$

$$E\left(\tau_S\left(\xi\right)\right) = \frac{1}{\xi}\left(1-\frac{\lambda}{\lambda+\xi}e^{-\frac{\mu\xi}{\lambda+\xi}S}\right) \tag{17}$$

$$R^*\left(\alpha\right) = \frac{\left(1-E\left(e^{-\alpha W(\tau_S(\xi))}\right)\right)(\mu+\alpha)}{E\left(\tau_S\left(\xi\right)\right)\lambda\alpha}. \tag{18}$$

The analysis above can be extended to cases in which $T(\xi)$ has a phase-type distribution as described in Baron et al. (2010).

4.1.2 Poisson demands

When customers' demand is for a single unit we restrict the discussion to the case where the order up to level S is an integer. We use the notation for the Gamma, Upper Incomplete Gamma, and Lower Incomplete Gamma functions, defined, respectively, as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \mathrm{d}t,$$

$$\Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} \mathrm{d}t \quad \text{and}$$

$$\gamma(a, x) = \int_0^x e^{-t} t^{a-1} \mathrm{d}t.$$

These functions are implemented in many commercial mathematical software. Thus, using them simplifies the numerical solution of the optimization problems in Sect. 5. The derivation that involves these functions here and in the next section follows from their definition and is easiest to follow using a mathematical software (e.g., Maple).

Theorem 3 *For the control policy of type $\tau_S(\xi)$ with a Poisson demand process, we have*

$$E\left(e^{-\alpha W(\tau_S(\xi))-\beta\tau_S(\xi)}\right) = e^{-\alpha S}\left(\frac{\lambda}{\lambda+\beta+\xi}\right)^S + \frac{\xi\left(1-e^{-\alpha S}\left(\frac{\lambda}{\lambda+\beta+\xi}\right)^S\right)}{\lambda\left(1-e^{-\alpha}\right)+\beta+\xi}$$

$$= \frac{e^{-\alpha S}\left(\frac{\lambda}{\lambda+\beta+\xi}\right)^S\left(\lambda\left(1-e^{-\alpha}\right)+\beta\right)+\xi}{\lambda\left(1-e^{-\alpha}\right)+\beta+\xi}, \tag{19}$$

$$E\left(\tau_S(\xi)\right) = \frac{\left(\frac{\lambda}{\lambda+\xi}\right)^S - 1}{\xi}, \tag{20}$$

$$R^*(\alpha) = \frac{1-e^{-\alpha S}\left(\frac{\lambda}{\lambda+\xi}\right)^S}{E\left(\tau_S(\xi)\right)\left(\lambda\left(1-e^{-\alpha}\right)+\xi\right)}.$$

4.2 The control policy of type $\tau = \min(t_0, T_S)$

In this section we apply the model to the policy of type $\tau_S(t_0) = \min(t_0, T_S)$ when t_0 is constant.

4.2.1 Exponential demands

Theorem 4 *For the control policy of type $\tau_S(t_0)$ with* $\exp(\mu)$ *demand, we have*

$$E\left(e^{-\alpha W(\tau_S(t_0))-\beta\tau_S(t_0)}\right) = e^{-\left(\frac{\lambda\alpha}{\mu+\alpha}+\beta\right)t_0}\sum_{n=1}^{\infty}\frac{e^{-\mu S}(\mu S)^{n-1}}{((n-1)!)^2}\Gamma\left(n,\lambda t_0\right)$$
$$+\sum_{n=1}^{\infty}\frac{e^{-\mu S}(\mu S)^{n-1}}{((n-1)!)^2}\left(\frac{\mu+\alpha}{\mu+2\alpha}\right)^n$$
$$\gamma\left(n,\lambda\left(\mu+2\alpha\right)\frac{t_0}{\mu+\alpha}\right) \quad (21)$$

and

$$R^*\left(\alpha\right) = \frac{1-E\left(e^{-\alpha W(\tau_S(t_0))}\right)}{\left(\frac{\lambda\alpha}{\mu+\alpha}\right)E\left(\tau_S\left(t_0\right)\right)}. \quad (22)$$

4.2.2 Poisson demands

A similar analysis when demand is Poisson leads to

Theorem 5 *For the control policy $\tau_S(t_0)$ with Poisson demand, we have:*

$$E\left(e^{-\alpha W(\tau_S(t_0))-\beta\tau_S(t_0)}\right) = e^{-\alpha S}\left(\frac{\lambda}{\lambda+\beta}\right)^S\frac{\gamma\left(S,\left(\lambda+\beta\right)t_0\right)}{\left(S-1\right)!}$$
$$+e^{-\left(\lambda\left(1-e^{-\alpha}\right)+\beta\right)t_0}\frac{\Gamma\left(S,\lambda t_0 e^{-\alpha}\right)}{\Gamma\left(S\right)}, \quad (23)$$
$$R^*\left(\alpha\right) = \frac{1-E\left(e^{-\alpha W(\tau_S(t_0))}\right)}{\lambda\left(1-e^{-a}\right)E\left(\tau_S\left(t_0\right)\right)}. \quad (24)$$

5 Optimization

In this section we carry out an optimization of the order up to level, S and asymptotic analysis of the control policies of type $\tau_S\left(\xi\right)$ as in (3) with exponential (μ) demands and of type $\tau_S\left(t_0\right)$ as in (4) with Poisson demand. As in (6), for both models we consider the optimization problem

$$\min_S C(S) = \frac{K+\pi E((S-W(\tau))I\{\tau<T_S\})}{E(\tau)} + h(S-E\left(W\right)).$$

We denote the optimal order up to level by S^*. When demand is discrete we also include a constraint that S is an integer.

Let $E(perish\ c) \equiv \frac{E((S-W(\tau))I\{\tau<T_S\})}{E(\tau)}$ denote the expected number of units perished per cycle.

Remark 2 In practice, when there is no lead time, ordering the batch after items are perished, can (and should) be postponed until the next customer's arrival. Such postponement would increase $E(\tau)$ and can be incorporated into our analysis by adding the expected time to an arrival to the expected cycle length whenever cycles end due to perishability. Denoting the length of cycles with such postponement by τ^p, we have $E(\tau^P) = E(\tau) + \frac{E(I\{\tau<T_S\})}{\lambda}$. A similar postponement and correction of the expected cycle length can be done in the unit demand case when demands that ends the cycle lead to $W(\tau) = 0$ (there is no downcrossing as in the continuous demand case). Because such postponement is not viable when lead time is positive we use $E(\tau)$ in this and the next sections.

5.1 The control policy of type $\tau = \tau_S(\xi)$ with exponential(μ) demand sizes

For this model $E\left(e^{-\alpha W(\tau_S(\xi))-\beta\tau_S(\xi)}\right)$ and $R^*(\alpha)$ are given in (16) and (18), respectively. ($E(\tau_S(\xi))$ is given in (17).) Thus, we obtain

$$E(W) = \frac{\lambda}{\mu\xi}\frac{\mu\xi S + (\xi+\lambda)\left(1-e^{\mu\xi\frac{S}{\lambda+\xi}}\right)}{\left(\lambda-(\lambda+\xi)\,e^{\mu\xi\frac{S}{\lambda+\xi}}\right)}, \text{ and}$$

$$E(perish\ c) = \frac{\lambda(\lambda+2\mu\xi S+\xi)\left(e^{\left(-\frac{\mu\xi S}{\lambda+\xi}\right)}\right)}{\left(\lambda e^{\left(-\frac{\mu\xi S}{\lambda+\xi}\right)}-\lambda-\xi\right)\mu} - \frac{(\lambda+\xi)(\mu\xi S+\lambda)}{\left(\lambda e^{\left(-\frac{\mu\xi S}{\lambda+\xi}\right)}-\lambda-\xi\right)\mu},$$

Using these expressions we derived the following asymptotic results (can be verified by taking the limits):

Proposition 1 *For the $\tau_S(\xi)$ Policy with compound Poisson (λ) arrival and exponential (μ) demand sizes, we have as $\lambda \to 0$ or $\mu \to \infty$,*

$$\lim_{\lambda\to 0} E(V) = \lim_{\mu\to\infty} E(V) = S$$
$$\lim_{\lambda\to 0} E(\tau_S(\xi)) = \lim_{\mu\to\infty} E(\tau_S(\xi)) = \frac{1}{\xi}$$
$$\lim_{\lambda\to 0} E(perish\ c) = \lim_{\mu\to\infty} E(perish\ c) = \xi S$$
$$\lim_{\lambda\to 0} C(S) = \lim_{\mu\to\infty} C(S) = K\xi + hS + \pi S\xi$$
$$\lim_{\lambda\to 0} S^* = \lim_{\mu\to\infty} S^* = 0,$$
$$\lim_{\lambda\to 0}(C(S^*) = K\xi,$$

as $\mu \to 0$

$$\lim_{\mu\to 0} E(V) = S$$
$$\lim_{\mu\to 0} E(\tau_S(\xi)) = \frac{1}{\lambda+\xi}$$

$$\lim_{\mu\to 0} E(perish\ c) = S\xi$$
$$\lim_{\mu\to 0} C(S) = K(\lambda+\xi) + hS + \pi\xi S$$
$$\lim_{\mu\to 0} S^* = 0,$$
$$\lim_{\mu\to 0} C(S^*) = K(\lambda+\xi),$$

and as $\xi \to 0$

$$\lim_{\xi\to 0} E(V) = \frac{S(2+S\mu)}{2(1+S\mu)}$$
$$\lim_{\xi\to 0} E(perish\ c) = 0$$
$$\lim_{\xi\to 0} E(\tau_S(\xi)) = \frac{\mu S+1}{\lambda}$$
$$\lim_{\xi\to 0} C(S) = \frac{2K\lambda + 2hS + h\mu S^2}{2(\mu S+1)}$$
$$\lim_{\xi\to 0} S^* = \frac{1}{h}\frac{\sqrt{2h\mu K\lambda - h^2} - h}{\mu}.$$

Note that $\xi \to \infty$ or $\lambda \to \infty$ imply that the cycle length tends to 0. Therefore, the cost tends to infinity, making the asymptotic analysis of these cases less relevant. It is reasonable that the cases $\lambda \to 0$ or $\mu \to \infty$ lead to identical results, because both imply diminishing demand so that $S^* = 0$. The case $\mu \to 0$ implies that at each arrival demand is huge and leads to the end of the cycle. Because demand not met from stock is immediately satisfied from the following order there is no point in holding stock between orders, implying that $S^* = 0$.

The case with $\xi \to 0$, results in the standard non-perishable model investigated in Sect. 3. Then, due to the memoryless, $W(\tau_S(\xi)) = S + 1/\mu$ and from Wald's equality we get $E(\tau_S(\xi)) = (\mu S + 1)/\lambda$ and $E(W) = \frac{1}{2}S^2\mu/(1+\mu S)$, leading to the expressions shown above. The positive solution of the first order condition of $C(S)$ is

$$S^* = \frac{\sqrt{2hK\lambda\mu - h^2} - h}{h\mu},$$

which is rational only if

$$2K\lambda\mu > h. \tag{25}$$

We note that as we can normalize both $\lambda = 1$ and $\mu = 1$, without loss of generality, then (25) is equivalent to requiring $2K > h$. The latter condition should hold for our model, otherwise the order set-up cost is smaller than the holding cost causing a 0 inventory policy to be optimal (as lead time is 0). Finally, we note that (25) is sufficient to verify the second order condition.

Table 1 Optimal S^* and functional values for Policy $\tau_S(T(\xi))$ when λ is varied

λ	S^*	$E(\tau_S(\xi))$	$E(V)$	$E(perish\ c)$	$C(S^*)$
0.01	0	4.9975	0	0	2.0010
0.5	0.9155	3.3705	0.6859	0.2290	4.1108
1.0	1.4048	2.9358	0.9602	0.3699	5.1061
1.5	1.7841	2.6496	1.1625	0.4812	5.8989
2.0	2.1062	2.4407	1.3308	0.5763	6.5805
2.5	2.3912	2.2787	1.4782	0.6608	7.1883
3.0	2.6496	2.1478	1.6109	0.7377	7.7422
3.5	2.8878	2.0389	1.7327	0.8086	8.2546
4.0	3.1099	1.9462	1.8459	0.8748	8.7336
4.5	3.3187	1.8661	1.9521	0.9371	9.1851
5.0	3.5165	1.7958	2.0524	0.9961	9.6132
10	5.1177	1.3723	2.8607	1.4749	13.0975
20	7.3897	1.02514	4.0018	2.1555	18.0677
50	11.9065	0.6805	6.2645	3.5097	27.9788
500	38.5132	0.2276	19.5727	11.4908	86.4829
500,000,000	44862.07	0.0003	22431.4	13458.55	86500.13

For the numerical sensitivity analysis we consider a base case where $K = 10$, $h = 1$, $\pi = 2$, $\lambda = 2$, $\mu = 3$, and $\xi = 0.2$.

In Tables 1, 2, and 3 we show the values of S^*, $E(\tau_S(\xi))$, $E(V)$, $E(perish\ c)$, and $C(S^*)$ when varying respectively λ, ξ and μ. The results nicely demonstrate the asymptotic behavior. As expected:

(i) When λ is increasing (i.e., more customers are arriving), S^*, $E(V)$, $E(perish\ c)$ and $C(S^*)$ are increasing while $E(\tau_S(\xi))$ is decreasing. When $\lambda \to 0$ (here ($\lambda = 0.01$)), $S^* = E(V) = E(perish\ c) = 0$, $E(\tau_S(\xi)) = \frac{1}{\xi} = 5$, and $C(S^*) = K\xi = 2$.

(ii) When μ is increasing (i.e. demand is decreasing), unless μ is very small, S^*, $E(V)$, $E(perish\ c)$ and $C(S^*)$ are decreasing while $E(\tau_S(\xi))$ is increasing. When $\mu \to 0$, $S^* = E(V) = E(perish\ c) = 0$, $E(\tau_S(\xi)) = \frac{1}{\lambda+\mu} = \frac{1}{2+.2} = .4545$, and $C(S^*) = K(\lambda + \xi) = 10(2.2) = 22$.

(iii) When ξ is increasing (time to perishability is decreasing), S^*, $E(V)$, $E(\tau_S(\xi))$ are decreasing while $C(S^*)$ and $E(perish\ c)$ are increasing. When $\xi = 0$, we obtain the results of the non-perishable model.

(iv) In Table 3 we also show the optimal cost when perishability is disregarded, denoted by $C(S^{**})$. When $\xi = 0$ (no perishability) the optimal solution is $S^{**} = 3.3$. The cost of $S^{**} = 3.3$ when perishability is not ignored ($\xi > 0$) is $C(S^{**} = 3.3)$. In the last column of Table 3 we show the relative error of ignoring perishability, $RE = \frac{C(S^{**})-C(S^*)}{C(S^*)}$. From the table it is evident that the perishability cannot be disregarded. A simple heuristic to check if perishability can be disregarded is to compare the cost of a model that ignores perishability

Table 2 Optimal S^* and functional values for Policy $\tau_S(T(\xi))$ when μ is varied

μ	S^*	$E(\tau_S(\xi))$	$E(V)$	$E(perish\ c)$	$C(S^*)$
0.000001	0	0.4545	2.8E-09	1.67E-09	22.0000
0.5	4.0380	1.2168	2.8119	1.0528	13.1361
1.0	3.2860	1.6284	2.1605	0.8823	10.0663
1.5	2.8246	1.9076	1.8177	0.7663	8.5926
2.0	2.5124	2.1213	1.6002	0.6849	7.6841
2.5	2.2835	2.2948	1.4467	0.6240	7.0524
3.0	2.1062	2.4407	1.3308	0.5763	6.5805
3.5	1.9635	2.5664	1.2392	0.5376	6.2109
4.0	1.8454	2.6765	1.1644	0.5053	5.9112
4.5	1.7454	2.7743	1.1018	0.4778	5.6620
5.0	1.6594	2.8620	1.0484	0.4541	5.4506
10	1.1696	3.4303	0.7505	0.3177	4.3012
50	0.4609	4.4406	0.3215	0.1201	2.8136
100	0.2924	4.6815	0.2130	0.0744	2.4978
500	0.0908	4.9268	0.0722	0.0219	2.1457
1,000	0.0528	4.9626	0.0432	0.0125	2.0832
10,000	0.0078	4.9962	0.0068	0.0018	2.0118

Table 3 Optimal S^*, RE and functional values for Policy $\tau_S(T(\xi))$ when ξ is varied

ξ	S^*	$E(\tau_S(\xi))$	$E(V)$	$E(perish\ c)$	$C(S^*)$	$C(S^{**}=3.3)$	RE
0.0001	3.3016	5.4506	1.8023	0.0005	3.6380	3.6380	0
0.5	1.5594	1.3723	1.0919	1.0135	10.4060	12.2207	.1744
1.0	1.1815	0.7955	0.9044	1.4587	16.3931	20.7269	.2644
1.5	0.9878	0.5597	0.7959	1.7695	22.2022	29.1526	.3130
2.0	0.8635	0.4315	0.7200	2.0139	27.9206	37.5319	.3442
2.5	0.7742	0.3511	0.6619	2.2164	33.5782	45.8852	.3665
3.0	0.7054	0.2959	0.6147	2.3886	39.1891	54.2230	.3836
3.5	0.6499	0.2557	0.5750	2.5371	44.7617	62.5510	.3974
4.0	0.6036	0.2251	0.5407	2.6661	50.3015	70.8726	.4089
4.5	0.5639	0.2010	0.5104	2.7785	55.8123	79.1896	.4188
5.0	0.5292	0.1816	0.4832	2.8763	61.2970	87.5033	.4275
10	0.3153	0.0924	0.3018	3.2881	115.0761	170.5664	.4822
50	0	0.0192	0	0	520	834.6126	.6050
100	0	0.0098	0	0	1020	1664.6116	.6319
500	0	0.0020	0	0	5020	8304.5556	.6543
100,000	0	1E-05	0	0	1,000,020	1659989.31	.6599
1,000,000	0	1E-06	0	0	10,000,020	16599851.4	.6599

with $\pi S\xi$ that is an upper bound on the cost of perishability. Then, if the difference between these costs is negligible, the model disregarding perishability is safe to use.

5.2 The control policy of type $\tau = \tau_S(t_0)$ with unit demand

For this model $E\left(e^{-\alpha W(\tau_S(t_0))-\beta\tau_S(t_0)}\right)$ and $R^*(\alpha)$ are given in (23), and (24), respectively. Thus, we obtain

$$E(\tau_S(t_0)) = \frac{S}{\lambda} + \frac{\Gamma(S, t_0\lambda)(t_0\lambda - S) - (t_0\lambda)^S e^{(-\lambda t_0)}}{\lambda(S-1)!}$$

$$\begin{aligned} E(W) = &\frac{1}{2}\frac{e^{\lambda t_0} S!(S-1)}{e^{\lambda t_0}(S! + \Gamma(S, t_0\lambda)(t_0\lambda - S)) - (t_0\lambda)^S} \\ &+\frac{1}{2}\frac{e^{\lambda t_0}\Gamma(S, t_0\lambda)\left(t_0^2\lambda^2 - S^2 + S\right)}{e^{\lambda t_0}(S! + \Gamma(S, t_0\lambda)(t_0\lambda - S)) - (t_0\lambda)^S} \\ &-\frac{1}{2}\frac{(t_0\lambda)^S(3 - t_0\lambda - S)}{e^{\lambda t_0}(S! + \Gamma(S, t_0\lambda)(t_0\lambda - S)) - (t_0\lambda)^S}, \quad \text{and} \end{aligned}$$

$$E(perish\ c) = \frac{(S - \lambda t_0)\lambda(S-1)!}{S! + \Gamma(S, t_0\lambda)(t_0\lambda - S) - (t_0\lambda)^S e^{-\lambda t_0}},$$

We can also get the following asymptotic results (no proof is provided):

Proposition 2 *For the $\tau_S(t_0)$ with Poisson(λ) demand we have as $\lambda \to 0$*

$$\lim_{\lambda\to 0} E(V) = S$$

$$\lim_{\lambda\to 0} E(\tau_S(t_0)) = t_0$$

$$\lim_{\lambda\to 0} E(perish\ c) = S/t_0$$

$$\lim_{\lambda\to 0} C(S) = \frac{K + \pi S}{t_0} + hS$$

$$\lim_{\lambda\to 0} S^* = 0$$

$$\lim_{\lambda\to 0} C(S^*) = \frac{K}{t_0}$$

and as $t_0 \to \infty$ (ignoring the requirement in the integrality of S)

$$\lim_{t_0\to\infty} E(V) = \frac{S+1}{2}$$

$$\lim_{t_0\to\infty} E(\tau_S(t_0)) = \frac{S}{\lambda}$$

$$\lim_{t_0\to\infty} E(perish\ c) = 0$$

$$\lim_{t_0\to\infty} C(S) = \frac{K\lambda}{S} + h\frac{(S+1)}{2}$$

$$\lim_{t_0\to\infty} S^* = \sqrt{\frac{2K\lambda}{h}}.$$

(Optimal integer S^* is either $\left\lfloor\sqrt{\frac{2K\lambda}{h}}\right\rfloor$ or $\left\lceil\sqrt{\frac{2K\lambda}{h}}\right\rceil$).

The case $\lambda \to 0$ is similar to that case for the $\tau_S(\xi)$ model with exponential demand per customer, discussed above (with t_0 replacing $1/\xi$). For the case $t_0 \to \infty$, we again approach the no perishable case. Thus, $W(t)$ within cycles has a $Poisson\,(\lambda t)$ distribution. Then, to find W, we consider the Markov chain where $W(t)$ has states $0, \ldots, S-1$. Moreover, as the step from each state, $i = 1, \ldots, S-2$, is only to state $i+1$, and from state $i = S-1$ is only to state 0, we have that W is uniform over $0, \ldots, S-1$. (Note that in agreement with Remark 1, this is independent of the actual arrival rate.) Thus,

$$E(W) = \sum_{i=0}^{S-1} \frac{i}{S} = \frac{S-1}{2}.$$

Moreover, using Wald's equality and that there is no overshoot $E(\tau_S(t_0)) = S/\lambda$. Then, the cost function is similar to the one of the EOQ model. Note that we initiate the next order when $V(t) = 0$ rather than when the next arrival occurs and $V(t) = -1$ as in Weiss (1980). This causes the carrying of extra unit during the cycle. Thus, the optimal order up to level here is just the order quantity of the EOQ, $S^* = \sqrt{2k\lambda/h}$ and the optimal cost is $C(S^*) = \sqrt{2K\lambda h} + h/2$.

For the numerical sensitivity analysis we consider a base case where $K = 10$, $h = 1$, $\pi = 2$, $\lambda = 2$, and $t_0 = 8$.

In Tables 4 and 5, we show the values of S^*, $E(\tau_S(t_0))$, $E(V)$, $E(perish\ c)$ and $C(S^*)$ when varying respectively λ and t_0. Again, the results nicely demonstrate the asymptotic behavior. As expected:

(i) When λ is increasing, S^*, $E(V)$, $C(S^*)$ are increasing while $E(\tau_S(t_0))$ and $E(perish\ c)$ (unless λ is very small) are decreasing. When $\lambda \to 0$, $S^* = E(perish\ c) = E(V) = 0$, $E(\tau_S(t_0)) = t_0 = 8$ and $C(S^*) = \frac{10}{8} = 1.25$.

(ii) When t_0 is increasing, (time to perishability is increasing), S^*, $E(\tau_S(t_0))$ and $E(V)$ are non-decreasing while $E(perish\ c)$ and $C(S^*)$ are non-increasing. When $t_0 \geq 5$, we obtain the standard (integer) EOQ results.

(iii) In Table 5 we also show the optimal cost when perishability is disregarded, denoted again by $C(S^{**})$. When t_0 is large (no perishability) the optimal solution is $S^{**} = 6$. The relative error RE is significant when $t_0 \leq 3$ and for such value perishability cannot be disregarded. A similar heuristic to the one discussed in point (iv) of Sect. 5.1 can be used here.

Table 4 Optimal S^* and functional values for Policy $\tau_S(t_0)$ when λ is varied

λ	S^*	$E(\tau_S(t_0))$	$E(V)$	$E(perish\ c)$	$C(S^*)$
0	0	8.0000	0.0000	0.0000	1.2500
0.5	3	5.3040	1.8619	0.0656	3.8785
1.0	4	3.9405	2.4593	0.0151	5.0272
1.5	5	3.3263	2.9907	0.0032	6.0033
2.0	6	2.9991	3.4981	0.0006	6.8337
2.5	7	2.7999	3.9996	0.0001	7.5715
3.0	8	2.6666	4.4999	2.42E-05	8.2500
3.5	8	2.2857	4.5000	1.38E-06	8.8750
4.0	9	2.2400	5.0000	2.63E-07	9.4444
4.5	10	2.2222	5.5000	5.03E-08	10.0000
5.0	10	2.0000	5.5000	2.49-E09	10.5000
10	14	1.4000	7.5000	9.11E-19	14.6429
20	20	1.0000	10.5000	3.24E-43	20.5000
50	32	0.6400	16.5000	6.7E-125	32.1250
500	100	0.2000	50.5000	0.0000	100.5000
50,000	1,000	0.0000	500.5000	0.000	1000.5000

Table 5 Optimal S^*, RE and functional values for Policy $\tau_S(t_0)$ when t_0 is varied

t_0	S^*	$E(\tau_S(t_0))$	$E(V)$	$E(perish\ c)$	$C(S^*)$	$C(S^{**} = 6)$	RE
0.5	2	0.4482	1.2948	2.4625	28.5322	45.5016	0.5947
1.0	3	0.8910	2.0000	1.3670	15.9575	23.0449	0.4441
1.5	3	1.1639	1.8717	0.5775	11.6181	15.4019	0.3257
2.0	4	1.6093	2.4734	0.4856	9.6587	11.6421	0.2053
2.5	4	1.7816	2.4236	0.2452	8.5270	9.5553	0.1206
3.0	5	2.2410	2.9768	0.2312	7.9015	8.3539	0.0572
3.5	5	2.3537	2.9533	0.1243	7.4505	7.6621	0.0284
4.0	5	2.4204	2.9545	0.0657	7.2175	7.2710	0.0074
4.5	6	2.9003	3.4708	0.0688	7.0563	7.0563	0
5.0	6	2.9450	3.4722	0.0374	6.9425	6.9425	0
10	6	3.0000	3.5000	3.09E-05	6.8333	6.8333	0
50	6	3.0000	3.5000	1.03E-34	6.8333	6.8333	0
100	6	3.0000	3.5000	2.46E-76	6.8333	6.8333	0
500	6	3.0000	3.5000	2.46E-76	6.8333	6.8333	0
100,000	6	3.0000	3.5000	2.46E-76	6.8333	6.8333	0
1,000,000	6	3.0000	3.5000	2.46E-76	6.8333	6.8333	0

6 A fluid approximation and heuristics

We observed from the asymptotic results that the optimal order up to level is close to the order quantity of the EOQ in some cases and that perishability cannot be ignored in many settings. This suggests that a deterministic approximation of the system and its cost might be useful. In this section we develop such a fluid approximation of the inventory level and heuristics to find the "optimal" S. We investigated the performance of the approximation for the cases where the time to perishability is exponential (with exponential demand). An investigation of the heuristic for a case with Poisson demand and Normal (distributed) time to perishability requires to simulate the system since we do not have its exact analysis. The results of this investigation are reported in Baron et al. (2010) where we also develop the approximation for the Poisson demand case with deterministic time to perishability ($\tau = \min\{t_0, T_S\}$). These results, as well as the ones reported below, suggest that the order quantity given by the heuristic is very close to the optimal one. A similar analysis to our heuristic can be found in David and Mehrez (1995).

6.1 The approximation

We replace the stochastic demand assumption with a fluid demand with a rate $\lambda E(X)$. Then, starting at the reorder level, S, for any time t before the end of the cycle, we have

$$V(t) = S - t\lambda E(X). \tag{26}$$

Therefore, if items do not perish within a cycle its length is $T_S = S/(\lambda E(X))$. Thus the expected cycle length is given by

$$\begin{aligned} E(\tau) &= E(\min\{\tau, T_S\}) \\ &= \int_0^{S/(\lambda E(X))} t\mathrm{d}F_\tau + \frac{S}{\lambda E(X)} P\left(\tau > \frac{S}{\lambda E(X)}\right). \end{aligned} \tag{27}$$

Similarly, using (26) for every $t \le \tau$,

$$E((S - W(\tau))\, I\{\tau < T_S\}) = \int_0^{S/(\lambda E(X))} (S - t\lambda E(X))\,\mathrm{d}F_\tau \tag{28}$$

and

$$\begin{aligned} E(V) &= E(S - W) \\ &= \int_0^{S/(\lambda E(X))} \left(S - t\lambda E(X) + \frac{t\lambda E(X)}{2}\right)\mathrm{d}F_\tau + \frac{S}{2} P\left(\tau > \frac{S}{\lambda E(X)}\right). \end{aligned} \tag{29}$$

It is interesting to note that $E(V) = E((S - W(\tau))I\{\tau < T_S\}) + \frac{\lambda}{2}E(\tau)$.

We suggest to use the fluid approximation in practice for two reasons. First, as we will demonstrate below, its performance in the tractable cases (and the simulated cases reported in Baron et al. 2010) leads to only a small increase in the cost. Second, whereas the development of the cost functionals for the stochastic system is not easy (even in the cases considered in the paper), expressing them using the fluid approximation is much simpler.

6.2 The heuristic for the control policy of type $\tau = \min\{\tau(\xi), T_S\}$

Since $T(\xi) \sim \exp(\xi)$ using (27)-(29) we get:

$$E(\tau) = \int_0^{S/(\lambda E(X))} t\xi e^{-\xi t}\mathrm{d}t + \frac{Se^{-\xi\frac{S}{\lambda E(X)}}}{\lambda E(X)}$$

$$= \frac{1 - e^{-\xi\frac{S}{\lambda E(X)}}}{\xi},$$

$$E((S - W(\tau))I\{\tau < T_S\}) = S - \frac{\lambda E(X)\left(1 - e^{-\xi\frac{S}{\lambda E(X)}}\right)}{\xi},$$

$$E(V) = S - \frac{\lambda E(X)\left(1 - e^{-\xi\frac{S}{\lambda E(X)}}\right)}{2\xi},$$

leading to the fluid approximation of the control problem:

$$\min_S C^h(S) = \frac{K + \pi\left(S - \frac{\lambda E(X)\left(1-e^{-\xi\frac{S}{\lambda E(X)}}\right)}{\xi}\right)}{\frac{1-e^{-\xi\frac{S}{\lambda E(X)}}}{\xi}} + h\left(S - \frac{\lambda E(X)\left(1 - e^{-\xi\frac{S}{\lambda E(X)}}\right)}{2\xi}\right).$$

In general there is no closed form solution to the above problem. Still, it is easily solved numerically.

Let S^h and $C\left(S^h\right)$ denote, respectively, the order up to level found using the above heuristic and its exact expected cost per time unit. In Table 6 we report $S^*, C(S^*), S^h, C\left(S^h\right)$ and the ratio $\left(C(S^h) - C(S^*)\right)/C(S^*)$ for the base case of Sect. 6.1 (i.e., with exponential demand), varying λ. We observe that $S^h > S^*$ in all cases considered. This over-estimation might be attributed to that the fluid approximation underestimates the quantity (and cost) of perished items. Still, as seen in the table

Table 6 Exact, and fluid approximation of solution, cost and relative error of Policy $\tau_S(\xi)$ when λ is varied

λ	S^*	$C(S^*)$	S^h	$C(S^h)$	$\frac{C(S^h)-C(S^*)}{C(S^*)}$
0.01	0	2.0010	0.0748	2.1845	0.0917
0.50	0.9155	4.1108	1.1167	4.1480	0.0090
1.00	1.4048	5.1061	1.6822	5.1553	0.0097
1.50	1.7841	5.8989	2.1261	5.9588	0.0102
2.00	2.1062	6.5805	2.5055	6.6501	0.0106
2.50	2.3912	7.1883	2.8428	7.2669	0.0109
3.00	2.6496	7.7422	3.1500	7.8294	0.0113
3.50	2.8878	8.2546	3.4340	8.3500	0.0116
4.00	3.1099	8.7336	3.6996	8.8369	0.0118
4.50	3.3187	9.1851	3.9499	9.2959	0.0121
5.00	3.5165	9.6132	4.1875	9.7315	0.0123
10.00	5.1177	12.0975	6.1264	13.2802	0.0139
50.00	11.9065	27.9788	14.4937	28.4868	0.0182
500	38.5132	86.4829	47.7216	88.4375	0.0226
5E+08	44862.07	86500.13	55884.05	91297.22	0.0555

the ratio $\left(C(S^h) - C(S^*)\right)/C(S^*)$ is less than 10% and ignoring the two extreme cases of λ (very small or very large) the ratio is less than 2.5%.

7 Summary and future research

In this paper we considered the stochastic analysis of (s, S) inventory models for perishable items. We developed exact solutions for several cases and suggested heuristics for finding the optimal control and its costs.

Numerical study of the heuristics show that they are accurate. An additional usage of the heuristics can be to initialize an exhaustive search, using simulation, around a good starting point (the solution of the heuristic) as we do. Therefore, we recommend to use the heuristics when the exact analysis is not possible and the lead time is relatively short.

For future research we suggest to investigate continuous review policies for perishable items when the lead time is positive.

Acknowledgements This paper was supported by NSERC grants of the first two authors. We would like to thank the Associate Editor and referees for their suggestions.

Appendix: Proofs

Proof of Theorem 2. To obtain the required functionals for the control policy $\tau_S(\xi)$ we decompose the joint LT of $W(\tau_S(\xi))$ and $\tau_S(\xi)$ according to the law of total probability:

$$E\left(e^{-\alpha W(\tau_S(\xi))-\beta\tau_S(\xi)}\right) = E\left(e^{-\alpha W(T_S)-\beta T_S} I\{T_S \le T(\xi)\}\right) + E\left(e^{-\alpha W(T(\xi))-\beta T(\xi)} I\{T_S > T(\xi)\}\right). \quad (30)$$

By applying the stopping time $\tau_S(\xi)$ to the martingale $\mathbf{M_1}$ we get the fundamental identity

$$-\left[\lambda(1-\left(\frac{\mu}{\mu+\alpha}\right)+\beta\right] E\left(\int_0^{\tau_S(\xi)} e^{-\alpha W(u)-\beta u}\mathrm{d}u\right) = -1 + E\left(e^{-\alpha W(\tau_S(\xi))-\beta\tau_S(\xi)}\right). \quad (31)$$

We will compute separately each of the functionals on the right hand side of (30).

First functional

$$E\left(e^{-\alpha W(T_S)-\beta T_S} I\{T_S \le T(\xi)\}\right) = E\left(e^{-\alpha W(T_S)-\beta T_S}\int_{T_S}^{\infty} \xi e^{-\xi s}\mathrm{d}s\right) = E\left(e^{-\alpha W(T_S)-(\beta+\xi)T_S}\right). \quad (32)$$

Replace β by $\beta+\xi$ in (11) we get:

$$-\left[\lambda\left(\frac{\alpha}{\mu+\alpha}\right)+\beta+\xi\right] E\left(\int_0^{T_S} e^{-\alpha W(u)-(\beta+\xi)u}\mathrm{d}u\right) = -1 + E\left(e^{-\alpha W(T_S)-(\beta+\xi)T_S}\right). \quad (33)$$

Thus, similar to (7) we have

$$E\left(e^{-\alpha W(T_S)-(\beta+\xi)T_S}\right) = E\left(e^{-\alpha W(T_S)}\right) E\left(e^{-(\beta+\xi)T_S}\right) = e^{-\alpha S}\frac{\mu}{\mu+\alpha} e^{-\frac{\mu(\beta+\xi)}{\lambda+\beta+\xi}S}\frac{\lambda}{\lambda+\beta+\xi}. \quad (34)$$

Second functional

Using the law of total probability

$$E\left(e^{-\alpha W(T(\xi))-\beta T(\xi)} I\{T_S > T(\xi)\}\right) = \int_0^{\infty} E\left(e^{-\alpha W(u)-\beta u} I\{T_S > u\}\right)\xi e^{-\xi u}\mathrm{d}u = \xi E\left(\int_0^{T_S} e^{-\alpha W(u)-(\beta+\xi)u}\mathrm{d}u\right).$$

Note that exchanging the order of integration and expectation above (as well as in other places we do so in the paper) is legitimate because the argument within the expectation is positive. From (33)

$$E\left(\int_0^{T_S} e^{-\alpha W(u)-(\beta+\xi)u}\mathrm{d}u\right) = \frac{1-E\left(e^{-\alpha W(T_S)-(\beta+\xi)T_S}\right)}{\lambda\left(1-\frac{\mu}{\mu+\alpha}\right)+\beta+\xi}. \qquad (35)$$

Combining the functionals
Substituting (35) and (34) into (30) we get

$$\begin{aligned} E\left(e^{-\alpha W(\tau_S(\xi))-\beta\tau_S(\xi)}\right) &= E\left(e^{-\alpha W(T_S)-(\beta+\xi)T_S}\right)+\xi\frac{1-E\left(e^{-\alpha W(T_S)-(\beta+\xi)T_S}\right)}{\lambda\left(1-\frac{\mu}{\mu+\alpha}\right)+\beta+\xi} \\ &= e^{-\alpha S}\frac{\mu}{\mu+\alpha}e^{-\frac{\mu(\beta+\xi)}{\lambda+\beta+\xi}S}\frac{\lambda}{\lambda+\beta+\xi} \\ &\quad+\xi\frac{1-e^{-\alpha S}\frac{\mu}{\mu+\alpha}e^{-\frac{\mu(\beta+\xi)}{\lambda+\beta+\xi}S}\frac{\lambda}{\lambda+\beta+\xi}}{\lambda\left(1-\frac{\mu}{\mu+\alpha}\right)+\beta+\xi}, \end{aligned}$$

which after some algebra gives (16). To get (17) observe that

$$\begin{aligned} E\left(\tau_S\left(\xi\right)\right) &= \int_0^\infty \Pr\left(T_S\geq x\right)\Pr\left(T\left(\xi\right)\geq x\right)\mathrm{d}x \\ &= \int_0^\infty \Pr\left(T_S\geq x\right)e^{-\xi x}\mathrm{d}x = \int_0^\infty \left(1-F_{T_S}\left(x\right)\right)e^{-\xi x}\mathrm{d}x \\ &= \frac{1}{\xi}\left(1-E\left(e^{-\xi T_S}\right)\right), \end{aligned}$$

where the last equality follows because $E\left(e^{-\xi T_S}\right)\equiv\int_0^\infty e^{-\xi x}dF_{T_S}\left(x\right)=\xi\int_0^\infty e^{-\xi x}$ $F_{T_S}\left(x\right)\mathrm{d}x$. Finally, $E\left(e^{-\xi T_S}\right)=\frac{\lambda}{\lambda+\xi}e^{-\frac{\mu\xi}{\lambda+\xi}S}$ is derived from (7) by setting $\alpha=0$.

To get $R^*\left(\alpha\right)$ in (18) we continue in a similar manner to that in Proposition 1. Namely, we set $\beta=0$ in (31) and divide both sides by $-\left[\lambda\left(1-\left(\frac{\mu}{\mu+\alpha}\right)\right)\right]E\left(\tau_S(\xi)\right)$ to get

$$\frac{E\left(\int_0^{\tau_S(\xi)}e^{-\alpha W(u)}\mathrm{d}u\right)}{E\left(\tau_S(\xi)\right)} = \frac{-1+E\left(e^{-\alpha W(\tau_S(\xi))}\right)}{-\left(\lambda\left(1-\frac{\mu}{\mu+\alpha}\right)\right)E\left(\tau_S(\xi)\right)}.$$

By renewal theory the left hand side above equals $R^*(\alpha)$. □

Proof of Theorem 3. We decompose $E\left(e^{-\alpha W(\tau_S(\xi))-\beta\tau_S(\xi)}\right)$ as in (30). We then compute the functionals on the right hand side of (30) in an alternative way to the one used in Theorem 2.

First functional.

Similarly to (32)

$$\begin{aligned}E\left(e^{-\alpha W(T_S)-\beta T_S}I\{T_S\leq T(\xi)\}\right)&=E\left(e^{-\alpha W(T_S)-(\beta+\xi)T_S}\right)\\&=e^{-\alpha S}\left(\frac{\lambda}{\lambda+\beta+\xi}\right)^S,\end{aligned}\tag{36}$$

because $W(T_S)=S$ and T_S are independent, $W(T_S)=S$ (there is no overshoot when demand is for a single unit), and $T_S\sim Er(S,\lambda)$.

Second functional.

To calculate this functional recall that $T(\xi)=t_0<T_S$ implies that $W(T(\xi))$ is Poisson(λt_0) with fewer than $S-1$ arrivals. Thus, by conditioning on $T(\xi)=t_0$ we get

$$\begin{aligned}E\left(e^{-\alpha W(T(\xi))-\beta T(\xi)}I\{t_0<T_S\}|T(\xi)=t_0\right)&=e^{-\beta t_0}\sum_{i=0}^{S-1}e^{-\alpha i}\frac{(\lambda t_0)^i\,e^{-\lambda t_0}}{i!}\\&=e^{-(\lambda(1-e^{-\alpha})+\beta)t_0}\frac{\Gamma\left(S,\lambda t_0e^{-\alpha}\right)}{\Gamma(S)}.\end{aligned}\tag{37}$$

By deconditioning we get:

$$\begin{aligned}E\left(e^{-\alpha W(T(\xi))-\beta T(\xi)}I\{T(\xi)<T_S\}\right)&=\int_{t_0=0}^{\infty}e^{-(\lambda(1-e^{-\alpha})+\beta)t_0}\frac{\Gamma\left(S,\lambda t_0e^{-\alpha}\right)}{\Gamma(S)}\xi e^{-\xi t_0}dt_0\\&=\frac{\xi\left(1-e^{-\alpha S}\left(\frac{\lambda}{\lambda+\beta+\xi}\right)^S\right)}{\lambda+\beta+\xi-e^{-\alpha}\lambda}.\end{aligned}\tag{38}$$

Thus, (19) is obtained by substituting (36) and (38) in (30).

To establish $R^*(\alpha)$ in (20) we continue in a similar manner to that in Proposition 1 using the martingale

$$M_1(t)=-\left[\lambda\left(1-e^{-\alpha}\right)+\beta\right]\int_0^t e^{-\alpha W(u)-\beta u}\mathrm{d}u+1-e^{-\alpha W(t)-\beta t}$$

and the fundamental identity for the unit demand case

$$-[\lambda\left(1-e^{-\alpha}\right)+\beta]E\left(\int\limits_0^{\tau_S(\xi)} e^{-\alpha W(u)-\beta u}\mathrm{d}u\right)=-1+E\left(e^{-\alpha W(\tau_S(\xi))-\beta\tau_S(\xi)}\right).$$

and continue as in Proposition 1. Specifically, we set $\beta=0$ in the above identity and divide its both sides by $-\lambda\left(1-e^{-\alpha}\right)E\left(\tau_S\left(\xi\right)\right)$ to get

$$\frac{E\left(\int\limits_0^{\tau_S(\xi)} e^{-\alpha W(u)}\mathrm{d}u\right)}{[\lambda\left(1-e^{-\alpha}\right)]E\left(\tau_S\left(\xi\right)\right)}=\frac{1-E\left(e^{-\alpha W(\tau_S(\xi))}\right)}{[\lambda\left(1-e^{-\alpha}\right)]E\left(\tau_S\left(\xi\right)\right)}.$$

Again by renewal theory the left hand side above is $R^*\left(\alpha\right)$ to get, after substituting $\beta=0$ in (19):

$$\begin{aligned}R^*\left(\alpha\right)&=\frac{1-E\left(e^{-\alpha W(\tau_S(\xi))}\right)}{\lambda\left(1-e^{-\alpha}\right)E\left(\tau_S\left(\xi\right)\right)}\\&=\frac{1-\frac{e^{-\alpha S}\left(\frac{\lambda}{\lambda+\xi}\right)^S(\lambda(1-e^{-\alpha}))+\xi}{\lambda(1-e^{-\alpha})+\xi}}{\lambda\left(1-e^{-\alpha}\right)E\left(\tau_S\left(\xi\right)\right)},\end{aligned}$$

which after some algebra gives $R^*\left(\alpha\right)$ as in the Theorem. □

Proof of Theorem 4. As in (30) we decompose the joint LT of $W(Z_S(t_0))$ and $T_S(t_0)$

$$\begin{aligned}E\left(e^{-\alpha W(\tau_S(t_0))-\beta\tau_S(t_0)}\right)&=E\left(e^{-\alpha W(t_0)-\beta t_0}I\{t_0<T_S\}\right)\\&\quad+E\left(e^{-\alpha W(T_S)-\beta T_S}I\{t_0\geq T_S\}\right).\end{aligned}\tag{39}$$

The first functional in (39) is found using (15):

$$\begin{aligned}E\left(e^{-\alpha W(t_0)-\beta t_0}I\{t_0<T_S\}\right)&=E\left(e^{-\alpha W(t_0)-\beta t_0}\int\limits_{u=t_0}^{\infty} f_{T_S}(u)\mathrm{d}u\right)\\&=e^{-\left(\frac{\lambda\alpha}{\mu+\alpha}+\beta\right)t_0}\sum_{n=1}^{\infty}\frac{e^{-\mu S}(\mu S)^{n-1}}{(n-1)!}\int\limits_{u=t_0}^{\infty}\frac{e^{-\lambda u}(\lambda u)^{n-1}\lambda}{(n-1)!}\mathrm{d}u\\&=e^{-\left(\frac{\lambda\alpha}{\mu+\alpha}+\beta\right)t_0}\sum_{n=1}^{\infty}\frac{e^{-\mu S}(\mu S)^{n-1}}{(n-1)!}\sum_{j=0}^{n-1}\frac{e^{-\lambda t_0}(\lambda t_0)^j}{j!}\\&=e^{-\left(\frac{\lambda\alpha}{\mu+\alpha}+\beta\right)t_0}\sum_{n=1}^{\infty}\frac{e^{-\mu S}(\mu S)^{n-1}}{((n-1)!)^2}\Gamma\left(n,\lambda t_0\right)\end{aligned}\tag{40}$$

where

$$E\left(e^{-\alpha W(t_0)}\right)=e^{-\lambda t_0\frac{\alpha}{\mu+\alpha}}.$$

For the second functional on the right hand side of (39) recall that by the memoryless property $W(T_S)=S+\widehat{X}$ where $\widehat{X}\sim exp\,(\mu)$. Therefore, similarly to (40), we get

$$\begin{aligned}E\left(e^{-\alpha W(T_S)-\beta T_S}I\{t_0\geq T_S\}\right)&=E\left(\int_{u=0}^{t_0}e^{-\alpha W(u)-\beta u}\mathrm{d}F_{T_S}(u)\right)\\&=\int_{u=0}^{t_0}e^{-(\frac{\lambda\alpha}{\mu+\alpha}+\beta)u}\sum_{n=1}^{\infty}\frac{e^{-\mu S}(\mu S)^{n-1}}{(n-1)!}\frac{e^{-\lambda u}(\lambda u)^{n-1}\lambda}{(n-1)!}\mathrm{d}u\\&=\sum_{n=1}^{\infty}\frac{e^{-\mu S}(\mu S)^{n-1}}{(n-1)!}\int_{u=0}^{t_0}\frac{e^{-(\lambda+\frac{\lambda\alpha}{\mu+\alpha}+\beta)u}(\lambda u)^{n-1}\lambda}{(n-1)!}\mathrm{d}u\\&=\sum_{n=1}^{\infty}\frac{e^{-\mu S}(\mu S)^{n-1}}{(n-1)!}\left(\frac{\mu+\alpha}{\mu+2\alpha}\right)^n\frac{\gamma\left(n,\lambda\left(\mu+2\alpha\right)\frac{t_0}{\mu+\alpha}\right)}{\Gamma(n)}.\end{aligned}\tag{41}$$

Adding the expressions $E\left(e^{-\alpha W(t_0)-\beta t_0}I\{t_0<T_S\}\right)$ from (40) and for $E\left(e^{-\alpha W(T_S)-\beta T_S}I\{t_0\geq T_S\}\right)$ from (41) we get (21).

Finally, (22) is obtained in a similar manner to that in Proposition 1 starting from replacing $\tau_S(\xi)$ with $\tau_S(t_0)$ in identity (31). □

Proof of Theorem 5. We decompose as in (39). Observe that $T_S\sim Er\,(S,\lambda)$. Thus, for $u\leq T_S$ we have $W(u)\sim Poisson\,(\lambda u)$, so that

$$\begin{aligned}E\left(e^{-\alpha W(T_S)-\beta T_S}I\{t_0\geq T_S\}\right)&=E\left(\int_{u=0}^{t_0}e^{-\alpha S-\beta u}\mathrm{d}F_{T_S}(u)\right)\\&=e^{-\alpha S}\int_{u=0}^{t_0}e^{-\beta u}\frac{\lambda^S u^{S-1}e^{-\lambda u}}{(S-1)!}\mathrm{d}u\\&=e^{-\alpha S}\left(\frac{\lambda}{\lambda+\beta}\right)^S\frac{\gamma\left(S,(\lambda+\beta)\,t_0\right)}{(S-1)!}.\end{aligned}\tag{42}$$

For the second functional on the right hand side of (39) we get:

$$\begin{aligned}E\left(e^{-\alpha W(t_0)-\beta t_0}I\{t_0<T_S\}\right)&=E\left(e^{-\alpha W(t_0)-\beta t_0}\int_{u=t_0}^{\infty}f_{T_S}(u)du\right)\\&=e^{-(\lambda(1-e^{-\alpha})+\beta)t_0}\frac{\Gamma\left(S,\lambda t_0e^{-\alpha}\right)}{\Gamma(S)}.\end{aligned}\tag{43}$$

Combining (42) and (43) we get (23).

To obtain (24) we use the fundamental identity for the unit demand case

$$-[\lambda\left(1-e^{-\alpha}\right)+\beta]E\left(\int_0^{\tau_S(t_0)} e^{-\alpha W(u)-\beta u}\mathrm{d}u\right)$$
$$=-1+E\left(e^{-\alpha W(\tau_S(t_0))-\beta\tau_S(t_0)}\right). \tag{44}$$

and continue as in Proposition 1. □

References

Arrow KJ, Karlin S, Scarf H (1958) Studies in the mathematical theory of inventory and production, Stanford mathematical studies in the social sciences. Stanford University Press, Stanford 340

Baron O, Berman O, Perry D (2010) Continuous review inventory models for perishable items ordered in batches. Working paper, Rotman School of Management, University of Toronto

Berk E, Gürler U (2008) Analysis of the (Q, r) inventory model for perishables with positive lead times and lost sales. Oper Res 56(5):1238–1246

Boxma OJ, Perry D, Stadje W (2001) Clearing models for M/G/1 queues. Queueing Sys 38:287–306

Cobbaert K, Van Oudheusden D (1996) Inventory models for fast moving spare parts subject to "sudden death" obsolescence. Int J Prod Econ 44(3):239–248

David I, Mehrez A (1995) An inventory model with exogenous failures. Oper Res 43(5):902–903

Ghare PM, Scharder GF (1963) A model for an exponentially decaying inventory. J Ind Eng 14:238–243

Graves S (1982) The application of queueing theory to continuous perishable inventory system. Manag Sci (28):400–406

Gürler U, Özkaya BY (2003) A note on continuous review perishable inventory systems: models and heuristics. IIE Trans 35(3):321–323

Gürler U, Özkaya BY (2008) Analysis of (s, S) policy for perishables with a random shelf life. IIE Trans 410:759–781

Kalpakam S, Arivarignan G (1988) A continuous review perishable inventory model. Statistics 19(3): 389–398

Karaesmen I, Scheller-Wolf A, Deniz B (2009) Managing perishable and aging invetories: review and future research directions. In: Kempf K, Keskinocak P, Uzsoy P (eds.) Handbook of production planning. Kluwer International Series in Operations Research and Management Science, Advancing the State-of-the-Art Subseries (in press)

Kella O, Whitt W (1992) Useful martingales for storage systems with Lévy input. J Appl Probab 29: 396–493

Lian Z, Liu L (2001) Continuous review perishable inventory systems: models and heuristics. IIE Trans 33:809–822

Lian Z, Liu L, Neuts M (2005) A discrete-time model for common lifetime inventory systems. Math Oper Res 30:718–732

Liu L, Lian Z (1999) (s; S) continuous review models for products with fixed lifetimes. Oper Res 47: 150–158

Liu L, Shi D (1999) An (s; S) model for inventory with exponential lifetimes and renewal demands. Naval Res Log 46:39–56

Nahmias S (1982) Perishable inventory theory: a review. Oper Res 30:680–708

Nahmias S (1997) Production and operations analysis, 3rd edn. Irwin/McGraw-Hill, Chicago

Nahmias S, Perry D, Stadje W (2004) Actuarial valuation of perishable inventory systems. Probab Eng Inf Sci 18:219–232

Pal M (1989) The (S−1,S) inventory model for deteriorating items with exponential lead time. Calcutta Stat Assoc Bull 38:83–91

Perry D, Posner MJM (2002) Production-inventory models with an unreliable facility operating in a two-state random environment. Probab Eng Inf Sci 16:325–338

Pierskalla W (1969) An inventory problem with obsolescence. Naval Res Log Q 16(2):217–228
Porteus IL (2002) Foundations of stochastic inventory theory. Stanford Business Books, Stanford
Ross S (1996) Stochastic processes. Wiley, New Jersey
Rajan A, Steniberg R, Steniberg R (1992) Dynamic pricing and ordering decisions by a monopolist. Manag Sci 38(2):240–262
Sahin I (1989) Regenerative inventory systems. Operating characteristics and optimization. Springer, New York
Scarf H (1960) The optimality of (s, S) policies in the dynamic inventory problem. Math Meth in Soc sci. Stanford University Press, Stanford 196–202
Song Y, Lau HC (2004) A periodic-review inventory model with application to the continuous-review obsolescence problem. Eur J Oper Res 159(1):110–120
Song J, Zipkin PH (1996) Managing inventory with the prospect of obsolescence. Oper Res 44(1):215–222
Varghese TV, Krishnamoorthy A (2004) Inventory system subject to disaster with general interarrival times. Stoc Anal Appl 22(5):1315–1326
Weiss H (1980) Optimal ordering policies for continuous review perishable inventory models. Oper Res 28:365–374
Zipkin PH (2000) Foundations of inventory management. McGraw-Hill, New York City