JOURNAL OF FINANCIAL AND QUANTITATIVE ANALYSIS VOL. 18, NO. 4, DECEMBER 1983

Mean-Variance Utility Functions and the Demand for Risky Assets: An Empirical Analysis Using Flexible Functional Forms

Varouj A. Aivazian, Jeffrey L. Callen, Itzhak Krinsky, and Clarence C. Y. Kwan*

I. Introduction

In a recent study, Levy and Markowitz [15] demonstrate that, at least for some utility functions, expected utility can be approximated by a judiciously chosen function defined over mean and variance. In addition to resurrecting mean-variance analysis from the limbo into which it was placed by the criticisms of Borch [10] and others, the analysis by Levy and Markowitz yields a more direct approach to portfolio analysis than that provided by the current empirical literature. The current portfolio literature is concerned with notions of efficient sets and systematic risk rather than with utility functions and mean-variance. While much has been gained from a utility-free methodology, it is ultimately predicated upon a separation theorem and, hence, an environment with zero transactions costs. But security markets are not costless and the separation theorem may not hold. In that event, a utility-dependent approach to portfolio analysis could potentially lead to more powerful results especially if such an approach could be empirically implemented.

As one step toward the implementation of a utility-dependent portfolio methodology, we empirically estimate a family of flexible functional form utility functions (defined over mean and variance). These flexible functional forms are second-order approximations to the underlying utility function, whatever that true unknown utility function may be. From this family of approximations, we choose the one specific form that best approximates the portfolio decisions of the household sector. Specifically, we select a generalized Box-Cox (flexible functional-form) utility function that takes on the generalized Leontief, generalized

* University of Toronto, Hebrew University of Jerusalem and McMaster University, McMaster University, and McMaster University, respectively. The first two authors would like to thank the Social Science and Humanities Research Council of Canada for sabbatical leave support. All the authors wish to thank Frank Denton, Ernest Oksanen, Leslie Robb, and Ralph Winter for helpful and sometimes lengthy discussions.

square root quadratic, and translog utility functions as special or limiting cases.[1] Budget share equations for risky assets are subsequently generated from the generalized Box-Cox utility function using a straightforward portfolio optimization framework. These budget share equations are then estimated from data on the household sector's holdings of risky assets and the associated market returns. A χ^2 test is employed to find which (if any) of the three specific flexible functional forms mentioned above fits the data. Finally, having determined the "optimal" flexible functional form, we check to see that it yields signs for the marginal utilities and comparative-static conditions that are consistent with the underlying theory. In this way, we can at least partially validate the mean-variance approach, although we are unable to specify the true utility function.

In what follows, Section II develops a simple portfolio model for a Lancaster-type utility function defined over portfolio mean and variance. Section III describes the generalized Box-Cox utility function and the specific flexible functional forms that can be derived from it. The associated budget share equations for the Box-Cox function are also derived in this section. Section IV describes the data and the empirical results, and Section V concludes the paper. A derivation of comparative-static conditions with respect to asset means and variances appears in the Appendix.

II. A Characteristics Model for the Demand for Risky Assets

To simplify the theory and empirical work, we adopt the common assumption of homethetic separability—that the personal sector's investment decision in specific financial assets is independent both of the overall consumption-investment decision and the amount invested in nonfinancial assets. This independence implies that the total amount of wealth to be invested in financial assets is exogenous to the model, so that the only decision of consequence is the proportion of wealth to be invested in each financial asset. The personal sector's investment preferences are assumed to be captured by a Lancaster-type utility function over portfolio characteristics of the form

$$U = U(E, V) \tag{1}$$

where E is the expected return of the portfolio of financial assets and V its variance. These two characteristics of the portfolio are defined by

[1] This technique was pioneered by Berndt and Khaled [7], Khaled [14], and Applebaum [3] but in a nonportfolio riskless framework.

$$E = \sum_{i=1}^{n} X_i \mu_i \tag{2}$$

$$V = \sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j \sigma_{ij} \tag{3}$$

$$= \sum_{i=1}^{n} X_i^2 \sigma_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j \neq i}}^{n} X_i X_j \sigma_{ij}$$

where

X_k = the proportion of the personal sector's wealth invested in financial asset k, $k = 1, \ldots, n$,

μ_k = the expected return of asset k,

σ_k^2 = the variance of the return of asset k, and

σ_{ij} = covariance of the returns between assets i and j, $i, j = 1, \ldots, n$.

In addition, the utility function is assumed to be continuous and twice differentiable with $U_E > 0$ and $U_V < 0$ where the subscript E denotes the partial derivative of U with respect to E and, similarly, the V subscript denotes the partial derivative of U with respect to V. In other words, the personal sector is assumed to be risk-averse with indifference curves in E-V space that are upward sloping and—given additional assumptions to be clarified in the Appendix—convex from below.

The personal sector is assumed to choose those proportions to invest in each financial asset—the proportions must add up to one—that maximize the utility function (1). Formally, the personal sector's financial asset choice framework is described by the program

$$\underset{\{X_1, \ldots, X_n\}}{\text{Maximize}}\ U(E, V) \tag{4}$$

subject to

$$\sum_{i=1}^{n} X_i = 1 .$$

The Appendix shows that the solution to this program yields the comparative-static conditions[2]

$$\frac{\partial X_k}{\partial \mu_r} = \left[- U_E \frac{D_{rk}}{D} \right] - X_r \left[U_{EE} \sum_{i=1}^{n} \mu_i \frac{D_{ik}}{D} + 2 U_{VE} \sum_{i=1}^{n}\sum_{j=1}^{n} X_j \sigma_{ij} \frac{D_{ik}}{D} \right] \tag{5}$$

[2] We choose to use the direct rather than indirect form of the utility function in this paper because in our portfolio framework we can easily get share equations from the direct utility function.

and

$$(6) \quad \frac{\partial X_k}{\partial \sigma_r^2} = \left[-4U_V X_r \frac{D_{rk}}{D} \right] - 2X_r^2 \left[U_{EV} \sum_{i=1}^{n} \mu_i \frac{D_{ik}}{D} + 2U_{VV} \sum_{i=1}^{n} \sum_{j=1}^{n} X_j \sigma_{ij} \frac{D_{ik}}{D} \right]$$

where D is the determinant of the bordered Hessian and D_{rk} is co-factor rk of D. The double subscripts on the utility function denote the appropriate cross-partial derivatives.

Unlike the traditional theory of demand underlying the Slutsky equation where prices appear in the budget equations, in the theory of portfolio choice expected rates of return (or variance-covariance of returns) are not in the budget equations, but in the preference function affecting the ranking of portfolios. Allingham and Morishima [2] identify effects of changes in these variables as "want-pattern" effects reflecting taste changes and distinguish them from the wealth and substitution effects of the traditional Slutsky equations. Thus, they identify the first term in (5) or (6) as a "relative want-pattern" effect and the second term as an "absolute want-pattern" effect. However, equation (5) or (6) does not represent the effect of taste changes in E-V characteristics space. Equation (5) simply represents the effects of changes in asset attributes. Thus, equation (5) does not involve a change in the investor's preference function defined over E-V characteristics, but does involve a shift in the E-V efficiency locus due to a change in the "productivity" of X_r in yielding E (i.e., μ_r). For this reason, we identify such effects as productivity effects.

Equation (5) (and similarly (6)) can be decomposed into two effects. It is shown in the Appendix that the last term in (5) is equal to $-X_r(\partial X_k/\partial T_E)$ where T_E is a "lump-sum tax" on the portfolio's (expected) return. Since a change in T_E is equivalent to a change in the average productivity of each asset in producing E, we identify the last term in (5) as the average productivity effect of a change in μ_r, while the first term on the right is the pure marginal productivity effect of a change in μ_r. Notice that for given initial quantities of the assets, a small increase in μ_r produces an increase in E of $X_r d\mu_r$, while V remains unchanged. Thus, the increase in μ_r increases the average productivity of each asset in producing E (in proportion to the change in E). We can adjust the average productivity of each asset to the original level by lump-sum taxing away the above change in E. The effect on X_k of such a compensation in the average productivities of the assets is $X_r(\partial X_k/\partial T_E)$, which is equal to the negative of the last term in (5). Hence, the first term on the right in (5) represents the effect of a pure change in the marginal productivity of asset r in producing E, netting out average productivity changes. Since we are dealing with an expected return rather than a price effect, r and k are defined to be net complements (substitutes) if this first term is positive (negative). It can be shown that the "own" expected return effect (i.e., when $r = k$) is unambiguously positive. The sign of the average pro-

Also, given the lack of analogy (discussed later in the text) between our comparative static equations and the Slutsky equation, the application of Shephard's lemma is not straightforward in a portfolio framework, and remains to be worked out in the literature. Furthermore, the number of arguments of the indirect utility function would be larger since they include the individual means and covariances among assets.

ductivity effect is ambiguous even when $r = k$. It is obvious that equation (6), the equation for risk, can be decomposed similarly into a marginal and average productivity effect in the production of portfolio risk. The "own" pure marginal productivity effect in the case of risk is unambiguously negative.[3]

III. The Choice among Flexible Functional Forms

To implement the theory developed in the previous section, we utilize a generalized Box-Cox utility function of the form

$$U(\delta) = \alpha_1 E(\lambda) + \alpha_2 V(\lambda) + 1/2\alpha_3 E(\lambda)^2 + 1/2\alpha_4 V(\lambda)^2 + \alpha_5 E(\lambda)V(\lambda) \tag{7}$$

where $U(\delta)$, $E(\lambda)$, and $V(\lambda)$ are the Box-Cox transformations

$$U(\delta) = \left(U^{2\delta} - 1\right)/2\delta \tag{8}$$

$$E(\lambda) = \left(E^{\lambda} - 1\right)/\lambda \tag{9}$$

$$V(\lambda) = \left(V^{\lambda} - 1\right)/\lambda\ . \tag{10}$$

As the parameters λ and δ take on different values, the following alternative flexible functional forms are obtained.

Case (a): $\delta, \lambda \to 0$: $U(\delta) = \mathrm{Ln}U$; $E(\lambda) = \mathrm{Ln}E$; $V(\lambda) = \mathrm{Ln}V$.
This case yields the translog utility function

$$\mathrm{Ln}U = \alpha_1 \mathrm{Ln}E + \alpha_2 \mathrm{Ln}V + 1/2\alpha_3(\mathrm{Ln}E)^2 + 1/2\alpha_4(\mathrm{Ln}V)^2 + \alpha_5(\mathrm{Ln}E)(\mathrm{Ln}V)\ . \tag{11}$$

Case (b): $\delta, \lambda = 1/2$: $U(\delta) = U - 1$; $E(\lambda) = 2(E^{1/2} - 1)$;
$V(\lambda) = 2(V^{1/2} - 1)$.
This case gives the generalized Leontief utility function

$$U = 2\alpha_3 E + 2\alpha_4 V + 4\alpha_5 E^{1/2}V^{1/2} + \left(2\alpha_1 - 4\alpha_3 - 4\alpha_5\right)E^{1/2} + \left(2\alpha_2 - 4\alpha_4 - 4\alpha_5\right)V^{1/2} + 2\alpha_3 + 2\alpha_4 + 4\alpha_5 - 2\alpha_1 - 2\alpha_2 + 1\ . \tag{12}$$

Case (c): $\delta, \lambda = 1$: $U(\delta) = (U^2 - 1)/2$; $E(\lambda) = E - 1$; $V(\lambda) = V - 1$.

[3] Aivazian [1] provides a detailed discussion of these effects.

This case results in the square root quadratic utility function

$$
\begin{aligned}
U = & \left[\alpha_3 E^2 + \alpha_4 V^2 + 2\alpha_5 EV + 2\left(\alpha_1 - \alpha_3 - \alpha_5\right)E \right. \\
& \left. + 2\left(\alpha_2 - \alpha_4 - \alpha_5\right)V + 2\alpha_5 + \alpha_3 + \alpha_4 - 2\alpha_1 - 2\alpha_2 + 1\right]^{1/2} .
\end{aligned} \tag{13}
$$

Solving the utility maximizing program (4) for the generalized Box-Cox utility function yields the budget share equations ($k = 1, \ldots, n$)

$$
X_k = \frac{\left(\alpha_1 + \alpha_3 E(\lambda) + \alpha_5 V(\lambda)\right)E^{\lambda-1}X_k\mu_k + \left(\alpha_2 + \alpha_4 V(\lambda) + \alpha_5 E(\lambda)\right)2V^{\lambda-1}\sigma_{ip}X_k}{\left(\alpha_1 + \alpha_3 E(\lambda) + \alpha_5 V(\lambda)\right)E^{\lambda} + \left(\alpha_2 + \alpha_4 V(\lambda) + \alpha_5 E(\lambda)\right)2V^{\lambda}} \tag{14}
$$

where

$$
\sigma_{ip} = \sum_{j=1}^{n} X_j \sigma_{ij} .
$$

The share equations corresponding to the translog, square root quadratic, and generalized Leontief utility functions can be obtained from equation (14) by setting λ equal to zero, one, and one-half, respectively. These share equations are homogeneous of degree zero in the α_i parameters for a given value of λ.

Appending disturbance terms to the budget share equations provides a stochastic specification for estimating the demand system. The share equations are assumed to be stochastic because of errors in optimization. The disturbance terms, denoted by ϵ_{it} where i is the asset and t the data point, are assumed to be identically and independently normally distributed,[4] with the usual properties that

$$
\begin{aligned}
\mathrm{E}\left(\epsilon_{it}\right) &= 0 \quad \text{all } i, t \\
\mathrm{E}\left(\epsilon_{it}\epsilon_{it-j}\right) &= 0 \quad \text{all } j \neq 0 \\
\mathrm{E}\left(\epsilon_{it}\epsilon_{jt}\right) &= \begin{cases} \sigma^2 & i = j \\ 0 & i \neq j \end{cases} .
\end{aligned}
$$

[4] This specification ignores the requirement that budget shares must lie between zero and one by giving positive probability to shares outside this range. The Dirichlet distribution, for example, which limits shares to the unit simplex, would have been a more appropriate stochastic specification. However, Woodland [16] provides justification for the continued use of the normal distribution specification in the estimation of share equations by showing that there are no substantial differences in empirical results using the Normal model estimator rather than the Dirichlet model. He explains, "Application of the two estimators to three different actual data sets resulted in the estimates very close to each other for each set. Moreover, the calculated standard errors were very close. These results, together with those arising from the sampling experiments, suggest that the Normal model is rather robust with respect to stochastic specification. While further evidence from alternative sampling experiments is desirable, the results of this paper suggest that, while the Normal model may not be a theoretically appropriate specification for share equations, it may, for a large number of data sets, yield valid results" (pp. 381-382).

Since the asset proportions must sum to one (implying that the sum of the error terms must be zero), only $n - 1$ share equations are independent of each other for any given value of λ. Therefore, when estimating the system, one equation is deleted. As Barten [5] and others have pointed out, it does not matter which equation is deleted.[5]

IV. The Empirical Results

A. The Data

The data that we used to estimate the demand system are from the study by Barret, Gray, and Parkin [4]. Their data base includes the financial asset holdings by quarter and the related yields of the U.K. personal sector from 1957-1967. The benefit from utilizing this data base, besides its availability, is that this ten-year period is characterized by low levels of inflation and (we presume) the absence of inflationary expectations.[6] This allows us to finesse the problem of incorporating inflationary expectations into the analysis, thereby minimizing measurement biases in the yield data.

We categorized the financial holdings of the U.K. household sector into eight asset types: five liquid assets including money; and two relatively illiquid assets, government long-term debt and equities.[7] Thus, we used all but the smallest asset categories included in the Barret et al. data base. In addition, we aggregated time deposits and ordinary savings bank deposits into one category and used the yield on time deposits for the aggregate. This was necessitated by the fact that yield on ordinary savings bank deposits was constant over the period and would have dominated money (which is assumed to have a zero return) in a portfolio framework.

Besides asset holdings, Barret et al. obtained quarterly yields for each of the asset categories from the fourth quarter of 1957 to the first quarter of 1967. We employed the first five years of yield data (20 data points) to calculate sample mean returns and variances for each asset as well as sample covariances between asset yields. These sample estimates were then used to calculate the expected return and variance of the portfolio held by the U.K. personal sector in the third

[5] Specifically, since the expenditure shares sum to unity for each t, the n disturbance terms ϵ_{it} add up to zero. Thus the variance-covariance matrix of the disturbances in each of our models is singular and nondiagonal. If the estimation procedure is to be efficient, the disturbance covariances must be taken into account. Because the estimated variance-covariance matrix of the disturbances is also singular, it is not possible to estimate the full system of n equations by traditional maximum likelihood methods (the determinant of the estimated variance-covariance matrix would be identically zero). To avoid this problem, one equation is dropped in each of our models. Barten [5] has proved that for the purpose of maximization of the likelihood function it is irrelevant which equation is dropped from the system. The same procedure of dropping one equation has been followed by Applebaum [3], Berndt and Khaled [7], Berndt, Darrough, and Diewert [8], among others.

[6] Nominal returns during this period were normally between 2 and 5 percent (the latter for more risky assets) while real returns were between -1 and 3 percent. It is unlikely that the latter range is representative of *ex ante* real returns. Given the low nominal yields during this period, we felt that the assumption of zero inflationary expectations would be less arbitrary than adjusting nominal yields by an inflationary expectations (e.g., adaptive) model.

[7] See Table 2 below for a listing of the asset categories.

quarter of 1962. The X_i's for the third quarter of 1962 were the actual proportions of each asset held by the U.K. personal sector in that quarter. The calculated sample mean and variance of portfolio yield for the third quarter of 1962 represent one data point to be utilized in estimating the budget share equations. The second (and subsequent) data point is obtained by an updating technique. Sample means, variances, and covariances were recalculated, again on five years of data, by dropping the fourth quarter data of 1957 and adding the fourth quarter data of 1962. These new estimates, together with assets proportions held by the U.K. personal sector in the fourth quarter of 1962, provide the fourth quarter portfolio mean and variance and, hence, another data point. By means of this updating procedure, a time series of eighteen data points (four and one-half years) was generated and used to estimate the utility function parameters.[8]

B. Choosing the Best Utility Function

The utility function parameters in the budget share equations were estimated using a Quasi-Newton maximum likelihood procedure. As was noted earlier, the share equations are homogeneous of degree zero in the α_i parameters. Therefore, in order to identify the parameters, a normalization on the parameters, that is not homogeneous of degree zero in the parameters, must be added. We chose to normalize the remaining parameters with respect to α_5.[9] Since only seven of the eight budget share equations are independent, one equation was dropped before estimating the system.[10] Four different budget share models were estimated; the translog ($\lambda = 0$), the generalized Leontief ($\lambda = 1/2$), the square root quadratic ($\lambda = 1$), and the unrestricted system where λ is a free parameter.

Table 1 summarizes the results for each of the estimated systems. The unrestricted system yielded a parameter estimate for λ of .5124 that is close to the value for the generalized Leontief. More rigorously, it can be shown that $-2\mathrm{Ln}L$ is asymptotically distributed $\chi^2(1)$ where L is the ratio of the value of

[8] There is, of course, potential aggregation bias in estimating a representative consumer utility function from aggregate data. It is, therefore, important that this study be replicated on panel data. However, most of the flexible functional form literature dealing with utility function estimation is based on aggregate data. See, for example, [11], [8], [12], [13], and [3].

[9] Christiensen, Jorgenson, and Lau [11], Berndt, Darrough, and Diewert [8], and Appelbaum [3] used a different normalization, namely $\Sigma\alpha_i = 1$, pointing out that the results are invariant to this normalization. It is easy to show that their normalization is identical to ours. By substituting $\alpha_5 = 1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$ into the share equation (14), we obtain

$$X_k = \frac{\left[\frac{\alpha_1}{\alpha_5} + \frac{\alpha_3}{\alpha_5} E(\lambda) + V(\lambda)\right] E^{\lambda-1} X_k \mu_k + \left[\frac{\alpha_2}{\alpha_5} + \frac{\alpha_4}{\alpha_5} V(\lambda) + E(\lambda)\right] 2V^{\lambda-1} \sigma_{ip} X_k}{\left[\frac{\alpha_1}{\alpha_5} + \frac{\alpha_3}{\alpha_5} E(\lambda) + V(\lambda)\right] E^{\lambda} + \left[\frac{\alpha_2}{\alpha_5} + \frac{\alpha_4}{\alpha_5} V(\lambda) + E(\lambda)\right] 2V^{\lambda}} .$$

$$K = 1, 2, \ldots, n ,$$

which is the system estimated.

[10] Our tests also confirmed the earlier assertion that the estimated parameters and the value of the log likelihood function are invariant to the equation omitted. This was true for all the models estimated.

the unrestricted likelihood function to the value of the restricted likelihood function.[11] In our case, the test statistic ($-2\mathrm{Ln}L$) takes on the values

Translog	24.12
Leontief	0.42
Quadratic	40.09

Since the $\chi^2(1)$ has a critical value of 6.63 at the 1 percent significance level, we cannot reject the generalized Leontief as our premier utility function, but we can reject the other two at the 1 percent significance level.[12]

TABLE 1

Parameter Estimates for Different Functional Forms

Functional Form	Value of λ	Estimated Parameters	Value of Log Likelihood
Unrestricted Maximum	–	$\lambda =$ 0.5124	709.6296
		$\alpha_1/\alpha_5 =$ 0.9313	
		$\alpha_2/\alpha_5 =$ 0.9631	
		$\alpha_3/\alpha_5 =$ −0.6414	
		$\alpha_4/\alpha_5 =$ −0.2898	
Translog	$\lambda = 0$	$\alpha_1/\alpha_5 =$ −15.5860	697.5709
		$\alpha_2/\alpha_5 =$ 3.3138	
		$\alpha_3/\alpha_5 =$ −3.9386	
		$\alpha_4/\alpha_5 =$ 0.0136	
Generalized Leontief	$\lambda = 1/2$	$\alpha_1/\alpha_5 =$ 0.8250	709.4202
		$\alpha_2/\alpha_5 =$ 1.5580	
		$\alpha_3/\alpha_5 =$ −0.7273	
		$\alpha_4/\alpha_5 =$ −0.0138	
Square Root Quadratic	$\lambda = 1$	$\alpha_1/\alpha_5 =$ 13.2030	669.5445
		$\alpha_2/\alpha_5 =$ 9.5148	
		$\alpha_3/\alpha_5 =$ 13.4630	
		$\alpha_4/\alpha_5 =$ −1.0810	

C. Validation of the Generalized Leontief Utility Function

From the theory of asset demand, we expect (i) the sign of U_E/U_V to be negative, (ii) the own marginal productivity effect with respect to expected return to be positive for all assets, (iii) the own marginal productivity effect with respect to risk to be negative for all assets, and (iv) the principal minors of the bordered Hessian to alternate in sign. Table 2 lists the signs of the cross-partial derivatives

$$\frac{\partial X_k}{\partial \mu_r} \quad (k = 1, \ldots, 8;\ r = 1, \ldots, 7)$$

[11] See [6] on this point.

[12] Note that the square root quadratic and the ordinary quadratic are empirically indistinguishable since they yield the same budget share equations. Thus, both square root quadratic and ordinary quadratic are rejected by the data.

derived from our generalized Leontief utility function for the last quarter of 1966.[13] Since in all cases the magnitudes of the marginal productivity effects outweigh the magnitudes of the average productivity effects, Table 2 also can be interpreted as summarizing the signs of the marginal productivity effects. We do not provide the signs of the variance cross-partials $\partial X_k / \partial \sigma_r^2$ since they were always of opposite sign to the expected return cross-partials. This is to be expected since U_E and U_V were of opposite sign as predicted by the theory.[14] Clearly, the own marginal productivity effects with respect to expected return (along the diagonal) are all positive, and also the own marginal productivity effects with respect to variance are all negative.[15] Therefore, our utility function yields signs that are consistent with the theory. We also checked the signs of the principal minors of the bordered Hessian but, unfortunately, they were ambiguous. This result does not contradict the theory. It just means that the sufficient conditions for a maximum did not obtain.

Although the off-diagonal cross-partials can be of any sign, theoretically, intuition suggests that (i) near-monies should be substitutes for money, (ii) long-term bonds and stocks should be complementary to money, and (iii) long-term bonds and stocks should be substitutes. While our intuition concerning long-term bonds, stocks, and money is consistent with the evidence, the results are less clear-cut for money and near-monies.[16] In particular, while time deposits plus ordinary savings bank deposits (TDSB), the most liquid of near-monies, are a substitute for money, some of the other near-monies are complementary to money. Although this result is somewhat counter-intuitive, it is consistent with what other reseachers have found using alternative methodologies.[17]

V. Conclusion

Levy and Markowitz have argued that expected utility can be adequately approximated by a function of mean and variance. Utilizing a mean-variance portfolio framework and a Box-Cox general utility function, we were able to model the demand for financial assets by the household sector. In particular, we

[13] We estimated the cross-partials for each quarter of the four and one-half year test period. The results for the last quarter of 1966 are reasonably representative and the generalizations that follow apply to the entire four and one-half year period. The only differences from one quarter to the next are that the signs of a few of the off-diagonal cross-partials do change. In these cases, intuition offered no hint as to what the signs should be.

[14] This point may require some elaboration. Since the marginal productivity effects are much larger in magnitude than the average productivity effects, the signs of $\partial X_k / \partial \mu_r$ and $\partial X_k / \partial \sigma_r^2$ are determined by the first terms on the right-hand side of equations (5) and (6). These marginal productivity terms are obviously of opposite sign if U_E and U_V are of opposite sign as indeed they were.

[15] The diagonal elements in Table 2 are in parentheses. Note that money appears only on the vertical axis. Since money has no yield or risk, the impact of changes in money's expected return or yield variance on other asset demands is not a meaningful concept. However, the impact on the demand for money as the expected returns and variances of other asset yields change can be determined.

[16] To repeat, since we are dealing with expected returns and not prices, substitutes (complements) will have a negative (positive) sign in Table 2.

[17] See Bhattacharyya [9], for example, who uses the same data base as we do. He found this same result in spite of the fact that he does not use a flexible functional form methodology.

TABLE 2

Signs of the Marginal Productivity Effects with Respect to Expected Return for the Last Quarter of 1966

	TDSB	TSB	DD	BS	NSC	BONDS	STOCKS
M	−	+	+	−	+	+	+
TDSB	(+)	+	−	+	+	−	−
TSB	+	(+)	−	−	−	−	+
DD	−	−	(+)	−	−	+	−
BS	+	−	−	(+)	−	−	−
NSC	+	−	−	−	(+)	−	+
BONDS	−	−	+	−	−	(+)	−
STOCKS	−	+	−	−	+	−	(+)

Note: M = Notes, Coin, and Demand Deposits
TDSB = Time Deposits plus Ordinary Savings Bank Deposits
TSB = Trustee Savings Bank Deposits
DD = Defense Development Bonds
BS = Building Society Deposits
NSC = National Savings Certificates
BONDS = Government Marketable and Local Authority Debt
STOCKS = Company and Overseas Securities plus Unit Trusts

were able to show that the generalized Leontief function best fits the data at least by comparison to other common flexible functional forms. Unlike Levy and Markowitz who could compare their approximation to their hypothesized real utility function, the real underlying utility function for the household sector is unknown. Instead we validated the generalized Leontief approximation by showing that, broadly speaking, it yielded signs consistent both with the theory and our intuition. However, this utility function has been estimated from aggregate data and is, therefore, a "representative consumer" utility function. The aggregation biases may be such that our results do not generalize to a heterogeneous consumer environment. Further research using our methodology on panel data would be useful in clarifying this issue.

References

[1] Aivazian, V. A. "On the Comparative-Statics of Asset Demand." Working Paper Series No. 124, McMaster University (1976).

[2] Allingham, M., and M. Morishima. "Veblen Effects and Portfolio Selection." In *Theory of Demand: Real and Monetary*, M. Morishima, et al., editors. Oxford: Clarendon Press (1973).

[3] Applebaum, E. "On the Choice of Functional Forms." *International Economic Review*, Vol. 21 (June 1979), pp. 449-458.

[4] Barrett, R. J.; M. R. Gray; and J. M. Parkin. "The Demand for Financial Assets by the Personal Sector of the U.K. Economy." In *Modelling the Economy*, G.A. Renton, ed., London: Heinemann Educational Books (1975).

[5] Barten, A. P. "Maximum Likelihood Estimation of a Complete System of Demand Equations." *European Economic Review*, Vol. 11 (Fall 1969), pp. 7-73.

[6] Berndt, E. R.; B. H. Hall; R. E. Hall; and J. A. Hausman. "Estimation and Inference in Nonlinear Structural Models." *Annals of Economic and Social Measurement*, (October 1974); pp. 653-665.

[7] Berndt, E. R., and M. S. Khaled. "Parametric Productivity Measurement and Choice among Flexible Functional Forms." *Journal of Political Economy*, Vol. 87 (December 1979), pp. 1220-1245.

[8] Berndt, E. R.; M. N. Darrough; and W. E. Diewert. "Flexible Functional Forms and Expenditure Distribution: An Application to Canadian Consumer Demand Functions." *International Economic Review*, Vol. 18 (October 1977), pp. 651-675.

[9] Bhattacharyya, D. K. *Demand for Financial Assets: An Econometric Study of the U.K. Personal Sector*. Westmead, England: Saxon House (1978).

[10] Borch, K. "A Note on Uncertainty and Indifference Curves." *Review of Economic Studies*, Vol. 36 (January 1969), pp. 1-4.

[11] Christiensen, L. R.; D. W. Jorgenson; and L.J. Lau. "Transcendental Logarithmic Utility Functions." *American Economic Review*, Vol. 65 (June 1975), pp. 367-383.

[12] Christiensen, L. R., and M. Manser. "The Translog Utility Function and the Substitution of Meats in U. S. Consumption." *Journal of Econometrics*, Vol. 5 (January 1977), pp. 37-53.

[13] Donovan, D. J. "Modeling the Demand for Liquid Assets: An Application to Canada." *International Monetary Fund Staff Papers* (December 1978), pp. 676-704.

[14] Khaled, M. "Choice among Functional Forms: A Parametric Approach Based on the Generalized Box-Cox Functional Form." Discussion Paper, University of British Columbia (1977).

[15] Levy, H., and H. M. Markowitz. "Approximating Expected Utility by a Function of Mean and Variance." *American Economic Review*, Vol. 69 (June 1979), pp. 308-317

[16] Woodland, A. D. "Stochastic Specification and the Estimation of Share Equations." *Journal of Econometrics*, Vol. 10 (July 1979), pp. 361-383.

Appendix: Comparative-Static Equations for Risky Assets

(i) Solving the utility maximizing program (4) yields the first-order conditions

$$U_E \mu_i + 2U_V \sum_{j=1}^{n} \sigma_{ij} X_j = \lambda \quad (i, j = 1, \ldots, n) \tag{A1}$$

$$1 - \sum_{i=1}^{n} X_i = 0 . \tag{A2}$$

Second-order conditions for a maximum require that the principal minors of the determinant D, obtained by differentiating the equation system (A1) and (A2) with respect to the X_i's, alternate in sign, where

$$D = \begin{bmatrix} Z_{11} & \ldots & Z_{1n} & 1 \\ \vdots & & \vdots & \vdots \\ Z_{n1} & \ldots & Z_{nn} & 1 \\ 1 & \ldots & 1 & 0 \end{bmatrix}$$

and where

$$Z_{ij} = U_{EE}\mu_i\mu_j + 2U_{EV}\mu_i\sum_{r=1}^{n}\sigma_{jr}X_r + 2U_{VE}\mu_j\sum_{r=1}^{n}X_r\sigma_{ir} + 2U_V\sigma_{ij}$$
$$+ 4U_{VV}\left(\sum_{r=1}^{n}\sigma_{ir}X_r\right)\left(\sum_{r=1}^{n}\sigma_{jr}X_r\right).$$

The comparative-static condition for the demand for asset k with respect to the expected return on asset r is obtained by differentiating (A1) and (A2) with respect to μ_r. This yields the matrix equation

$$\text{(A3)} \qquad \begin{bmatrix} Z_{11} & \cdots\cdots & Z_{1n} & 1 \\ \vdots & & \vdots & \vdots \\ Z_{r1} & \cdots\cdots & Z_{rn} & 1 \\ \vdots & & \vdots & \vdots \\ Z_{n1} & \cdots\cdots & Z_{nn} & 1 \\ 1 & \cdots\cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial X_1}{\partial \mu_r} \\ \vdots \\ \dfrac{\partial X_r}{\partial \mu_r} \\ \vdots \\ \dfrac{\partial X_n}{\partial \mu_r} \\ \dfrac{\partial \lambda}{\partial \mu_r} \end{bmatrix} = \begin{bmatrix} F_1 \\ \vdots \\ F_r \\ \vdots \\ F_n \\ 0 \end{bmatrix}$$

where

$$F_i = -U_E\delta_{ir} - X_r\left(U_{EE}\mu_i + 2U_{VE}\sum_j X_j\sigma_{ij}\right), \quad \delta_{ir} = \begin{cases} 1 & \text{for } i = r \\ 0 & \text{for } i \neq r. \end{cases}$$

Solving (A3) for $\partial X_k/\partial\mu_r$ gives

$$\text{(A4)} \qquad \begin{aligned} \frac{\partial X_k}{\partial \mu_r} &= \sum_{i=1}^{n} F_i \frac{D_{ik}}{D} \\ &= -U_E\frac{D_{rk}}{D} - X_r\left[U_{EE}\sum_{i=1}^{n}\mu_i\frac{D_{ik}}{D} + 2U_{VE}\sum_{i=1}^{n}\sum_{j=1}^{n}X_j\sigma_{ij}\frac{D_{ik}}{D}\right]. \end{aligned}$$

(ii) Suppose there is a lump-sum tax (subsidy) T_E on expected return so that

$$\text{(A5)} \qquad E = \sum_{i=1}^{n} X_i\mu_i - T_E$$

$$\text{(A6)} \qquad V = \sum_i\sum_j X_iX_j\sigma_{ij}, \quad i, j = 1, \ldots, n.$$

A change in T_E affects the average productivity of every asset in producing E (in proportion to the change in T_E). Solving the utility maximizing program with E respecified as in (A5) and then differentiating the first-order conditions with respect to T_E yields

$$\text{(A7)} \qquad \begin{bmatrix} Z_{11} & \dots & Z_{1n} & 1 \\ \vdots & & \vdots & \vdots \\ Z_{n1} & \dots & Z_{nn} & 1 \\ 1 & \dots & 1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial X_1}{\partial T_E} \\ \vdots \\ \dfrac{\partial X_n}{\partial T_E} \\ \dfrac{\partial \lambda}{\partial T_E} \end{bmatrix} = \begin{bmatrix} J_1 \\ \vdots \\ J_n \\ 0 \end{bmatrix}$$

where

$$J_i = \mu_i U_{EE} + 2U_{VE} \sum_j X_j \sigma_{ij} .$$

Solving for $\partial X_k / \partial T_E$ gives

$$\text{(A8)} \qquad \begin{aligned} \frac{\partial X_k}{\partial T_E} &= \sum_{i=1}^{n} J_i \frac{D_{ik}}{D} \\ &= U_{EE} \sum_{i=1}^{n} \mu_i \frac{D_{ik}}{D} + 2U_{VE} \sum_{i=1}^{n} \sum_{j=1}^{n} X_j \sigma_{ij} \frac{D_{ik}}{D} . \end{aligned}$$

Equation (A8) represents the effect on X_K of equi-proportional changes in the average productivities of the assets in producing E.

The comparative-static condition for risk is obtained in a similar manner. The average productivity interpretation for risk can also be motivated by setting a lump-sum tax T_V (subsidy) on risk so that

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} X_i X_j \sigma_{ij} - T_V .$$