Algorithm for Index Updating of Partitions that are Less than a Given Partition with the Sum of the Partition Smaller than a Positive Integer

Given a vector $\kappa = (k_1, \cdots, k_r)$ and a constant n, $\nu = (\nu_1, \cdots, \nu_r)$ are vectors of nonnegative integers that satisfy $\nu \le \kappa$ and $|\nu| \le n$. We assume $k_i > 0$. If $k_i = 0$, then $\nu_i = 0$ and we only need to consider the subvector of ν that can take positive value. Denote $s = (s_1, \cdots, s_r)$ as the cumulative sum of ν, i.e., $s_i = \sum_{j=1}^{i} \nu_j$. Suppose ν's are ordered first on an ascending order on s_r, second on an ascending order on s_{r-1}, and then on s_{r-2} and so on, this note tries to provide an algorithm to update the index for a ν from $s_r = k$ to $s_r = k+1$. There are, at most, r different ways for this updating. That is, we can add 1 to ν_j for $j = 1, \ldots, r$. However, due to the constraint of $\nu \le \kappa$, it is possible that the number of ways of updating is less than r.

Suppose that we are given a matrix $F(i,j)$, for $i = 0, \cdots, n$ and $j = 1, \cdots, r$, where $F(i,j)$ is the number of different partitions that satisfy $\nu_{(j)} \le \kappa_{(j)}$ and $0 \le |\nu_{(j)}| \le i$, where $\nu_{(j)} = (\nu_1, \cdots, \nu_j)$ and $\kappa_{(j)} = (k_1, \cdots, k_j)$. It can be shown that

$$F(i,j) = \begin{cases} F(i-1,j) + F(i,j-1) - F(i-k_j-1,j-1) & \text{if} \quad k_j + 1 \le i, \\ \\ F(i-1,j) + F(i,j-1) & \text{if} \quad k_j + 1 > i, \end{cases} \tag{1}$$

using the convention that $F(i,j) = 0$ for $i < 0$.

Let p be the list of ν's with $\nu \le \kappa$ and $0 \le |\nu| \le n$, so the length of p is $F(n,r)$. We partition p into $n+1$ blocks of ν's, differ in terms of their $s_r = |\nu|$. We label these blocks as block 0 to block n. Let $\eta = (\eta_1, \ldots, \eta_r)$ be a given partition with partial sums $s_i(\eta)$. We show that the position of η within the list of p is given by

$$I(\eta) = F(s_r - 1, r) + \sum_{i=1}^{r-1} [F(s_i - 1, i) - F(b_i - 1, i)] + 1, \tag{2}$$

where $b_i = \max[0, s_{i+1} - k_{i+1}]$. To understand this expression, we note that the first term of $I(\eta)$ is $F(s_r - 1, r)$, which is the number of ν's with $\nu \le \kappa$ and $|\nu| < |\eta| = s_r(\eta)$. For ν's with $s_r(\nu) = s_r(\eta)$, those that are ahead of η can take value of $b_{r-1}(\eta) \le s_{r-1}(\nu) < s_{r-1}(\eta)$. This is because of the constraint of $\nu_r \le k_r$, so the minimum value of $s_{r-1}(\nu)$ can take is $\max[0, s_r(\nu) - k_r] = \max[0, s_r(\eta) - k_r] = b_{r-1}(\eta)$. The number of ν's with $b_{r-1}(\eta) \le s_{r-1}(\nu) < s_{r-1}(\eta)$ is given by $F(s_{r-1}(\eta) - 1, r-1) - F(b_{r-1}(\eta) - 1, r-1)$. The same logic applies to other s_i's for $1 \le i \le r-1$.

For a given η with $|\eta| = g$, its position within block g is

$$I_g(\eta) = \sum_{i=1}^{r-1} [F(s_i - 1, i) - F(b_i - 1, i)] + 1, \tag{3}$$

Suppose $\eta^{(j)} = (\eta_1, \cdots, \eta_j + 1, \eta_{j+1}, \cdots, \eta_r)$ satisfies the constraint $\eta^{(j)} \le \kappa$ (i.e., $\eta_j < \kappa_j$). Then $|\eta^{(j)}| = g + 1$ and its position within block $g + 1$ is

$$I_{g+1}\left(\eta^{(j)}\right) = \sum_{i=1}^{j-1} F(s_i - 1, i) + \sum_{i=j}^{r-1} F(s_i, i) - \sum_{i=1}^{r-1} F(c_i - 1, i) + 1, \tag{4}$$

where $c_i = \max[0, s_{i+1}(\eta^{(j)}) - k_{i+1}]$. Since $s_i(\eta^{(j)}) = s_i(\eta)$ for $i < j$ and $s_i(\eta^{(j)}) = s_i(\eta) + 1$ for $i \ge j$, we have $c_i = b_i$ for $i < j - 1$ and $c_i = \max[0, s_{i+1}(\eta) + 1 - \kappa_{i+1}]$ for $i \ge j - 1$.

Therefore, for $2 \le j \le r$, we have

$$I_{g+1}\left(\eta^{(j)}\right) - I_g(\eta) = \sum_{i=j}^{r-1} [F(s_i, i) - F(s_i - 1, i)] + \sum_{i=j-1}^{r-1} [F(b_i - 1, i) - F(c_i - 1, i)]\,. \tag{5}$$

For $j = 1$, we have

$$I_{g+1}\left(\eta^{(1)}\right) - I_g(\eta) = \sum_{i=1}^{r-1} [F(s_i, i) - F(s_i - 1, i)] + \sum_{i=1}^{r-1} [F(b_i - 1, i) - F(c_i - 1, i)]\,. \tag{6}$$

For the special case that $\min_{1 \le i \le r} k_i \ge n$, the constraint $\nu \le \kappa$ is not binding and we have

$$F(i, j) = \begin{pmatrix} i + j \\ j \end{pmatrix} \tag{7}$$

and $b_i = c_i = 0$. It follows that

$$I_{g+1}\left(\eta^{(j)}\right) - I_g(\eta) = \sum_{i=j}^{r-1} [F(s_i, i) - F(s_i - 1, i)] = \sum_{i=j}^{r-1} F(s_i, i - 1). \tag{8}$$

Note that the displacement only depends on s_i for $i = 1, \ldots, r - 1$. Within each block, the elements are sorted first on ascending order of s_{r-1}, then on ascending order of s_{r-2} and so on. Between block to block, the displacement matrix is largely the same. The only difference between the displacement matrix of updating block $g + 1$ to block $g + 2$ vs. the one for updating block g to $g+1$ is that the former block has elements that have $s_{r-2} = g+1$ (which come after the elements that have $s_{r-2} = g$). Therefore, once we establish the displacement matrix from block $n - 1$ to block n, we can use its submatrix for updating all the other blocks.

For block $n-1$, we let D be an $F(n-1, r-1) \times (r-1)$ matrix with its ith column containing the elements of $F(s_i, i-1)$. The last column of D is straightforward to create. It is because for $s_{r-1} = k$, there are $F(k, r - 1) - F(k - 1, r - 1) = F(k, r - 2)$ elements, and for each one of them, we assign a value of $F(k, r - 2)$ to its last element in the corresponding row of the D matrix. For the block with $s_{r-1} = k \le n - 1$, we know the first $r - 2$ columns is the same as the previous block whenever $s_{r-2} \leq k-1$, so we can just copy it from the previous block. The rest of the block is for the case that $s_{r-2} = s_{r-1} = k$. We need to assign a value of $F(k, r-3)$ to the $(r-2)$th column within this block. The length of this subblock is of length $F(k, r-2) - F(k-1, r-2) = F(k, r-3)$. Within this subblock, we need to set its first $(r-3)$th columns. The part that $s_{r-3} \leq k-1$ is the same as the previous block which has $s_{r-3} \leq s_{r-2} = k-1$, so we just simply copy it. The rest of the block are for the case that $s_{r-2} = k$. We need to assign a value of $F(k, r-4)$ to the $(r-3)$th column within this subblock and there are again $F(k, r-4)$ elements. We repeat the same process for the other columns. In general, the last $F(k, r-i-1)$ elements of the $(r-i)$th column of the D matrix should take a value of $F(k, r-i-1)$.