Hermite Polynomials

This note discusses various expressions of Hermite polynomials. Define the k-th order Hermite polynomial as

$$H_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{\mathrm{d}^k}{\mathrm{d}x^k} e^{-\frac{x^2}{2}}. \tag{1}$$

Note that there is another definition of Hermite polynomials that replaces the $x^2/2$ by x^2.

The first couple of Hermite polynomials are given by

$$H_0(x) = 1, \qquad H_1(x) = x, \qquad H_2(x) = x^2 - 1, \qquad H_3(x) = x^3 - 3x. \tag{2}$$

To compute the Hermite polynomials, we can use the following recurrence relation

$$H_{k+1}(x) = xH_k(x) - kH_{k-1}(x). \tag{3}$$

There are many explicit expressions of Hermite polynomials. A popular one is to write it in terms of the generalized hypergeometric function:

$$H_k(x) = x^k {}_2F_0\left(-\frac{k}{2}; -\frac{k-1}{2}; -\frac{2}{x^2}\right). \tag{4}$$

Note that this hypergeometric function terminates, and we can also write it as

$$\begin{aligned} H_k(x) &= x^k \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{\left(-\frac{k}{2}\right)_j \left(-\frac{k-1}{2}\right)_j \left(-\frac{2}{x^2}\right)^j}{j!} \\ &= x^k \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{\frac{k}{2} \times \frac{k-1}{2} \times \cdots \times \left(\frac{k-1}{2} - j + 1\right)}{j!} \left(-\frac{2}{x^2}\right)^j \\ &= \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{k!}{(k-2j)!j!} \frac{(-1)^j x^{k-2j}}{2^j}. \end{aligned} \tag{5}$$

Also note that $H_k(-x) = (-1)^k H_k(x)$.

In an earlier note, we show that the UMVU estimator of $\theta = \exp(a\mu + b\sigma)$ is $\hat{\theta}_u = \exp(a\hat{\mu})g(\hat{\sigma})$, where

$$g(\hat{\sigma}) = \sum_{r=0}^{\infty} \sum_{j=0}^{r} \frac{\Gamma\left(\frac{T-1}{2}\right) c^{r-j} d^j \hat{\sigma}^{r+j}}{j!(r-j)!\Gamma\left(\frac{T-1+r+j}{2}\right)}, \tag{6}$$

with

$$c = \sqrt{\frac{T-1}{2}} b, \qquad d = -\frac{a^2(T-1)}{4T}. \tag{7}$$

When $a \neq 0$, let $k = r + j$ and we can rewrite $g(\hat{\sigma})$ as

$$\begin{aligned} g(\hat{\sigma}) &= \sum_{k=0}^{\infty} \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{\Gamma\left(\frac{T-1}{2}\right) c^{k-2j} (-1)^j (-2d)^j \hat{\sigma}^k}{j!(r-j)!\Gamma\left(\frac{T-1+k}{2}\right) 2^j} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{T-1}{2}\right) (\sqrt{-2d}\hat{\sigma})^k}{\Gamma\left(\frac{T-1+k}{2}\right) k!} \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{k!}{j!(k-2j)!} \frac{(-1)^j \left(\frac{c}{\sqrt{-2d}}\right)^{k-2j}}{2^j} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{T-1}{2}\right) (\sqrt{-2d}\hat{\sigma})^k}{\Gamma\left(\frac{T-1+k}{2}\right) k!} H_k\left(\frac{c}{\sqrt{-2d}}\right) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{T-1}{2}\right) \left(\frac{a\tilde{\sigma}}{\sqrt{2}}\right)^k}{\Gamma\left(\frac{T-1+k}{2}\right) k!} H_k\left(\frac{\sqrt{T}b}{a}\right), \end{aligned} \tag{8}$$

where $\tilde{\sigma}^2 = (T-1)\hat{\sigma}^2/T$ is the maximum likelihood estimator of σ^2.

A direct proof of $E[g(\hat{\sigma})] = \exp(b\sigma - (a\sigma)^2/(2T))$ can be given as follows. Using the exponential generating function of Hermite polynomial, we have

$$\exp\left(xt - \frac{t^2}{2}\right) = \sum_{k=0}^{\infty} \frac{H_k(x)t^k}{k!}. \tag{9}$$

Using the fact that

$$E[\tilde{\sigma}^k] = \left(\frac{2}{T}\right)^{\frac{k}{2}} \frac{\Gamma\left(\frac{T-1+k}{2}\right)}{\Gamma\left(\frac{T-1}{2}\right)} \sigma^k, \tag{10}$$

we obtain

$$E[g(\hat{\sigma})] = \sum_{k=0}^{\infty} \frac{\left(\frac{a\sigma}{\sqrt{T}}\right)^k}{k!} H_k\left(\frac{\sqrt{T}b}{a}\right). \tag{11}$$

Putting $x = \sqrt{T}b/a$ and $t = a\sigma/\sqrt{T}$ in (9), we prove the identity.

For computational purpose, we can use the following expression of $g(\hat{\sigma})$:

$$\sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{T-1}{2}\right) \left(\frac{a\tilde{\sigma}}{\sqrt{2}}\right)^k}{\Gamma\left(\frac{T-1+k}{2}\right)} \tilde{H}_k\left(\frac{\sqrt{T}b}{a}\right), \tag{12}$$

where

$$\tilde{H}_k\left(\frac{\sqrt{T}b}{a}\right) = \frac{H_k\left(\frac{\sqrt{T}b}{a}\right)}{k!}. \tag{13}$$

$\tilde{H}_k(x)$ can be obtained using the recurrence relation

$$\tilde{H}_{k+1}(x) = \frac{x\tilde{H}_k(x) - \tilde{H}_{k-1}(x)}{k+1} \tag{14}$$

and the boundary conditions of $\tilde{H}_0(x) = 1$ and $\tilde{H}_1(x) = x$.