Shape of the Yield Curve Under CIR Single Factor Model: A Note

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Abstract

This note derives the shapes of the yield curve as a function of the current spot rate under Cox, Ingersoll, and Ross (CIR) (1985b) single factor model. Corresponding results reported in CIR are shown to be incorrect.

1 Introduction

In Cox, Ingersoll, and Ross (CIR) (1985a,b), they develop a general equilibrium model of the term structure of interest rates. The term structure of interest rates are linked directly to the specifications of preferences, technologies, and the distributions of the underlying sources of uncertainty.

As a special case, they derive a single factor model of the term structure. Under this single factor model, the interest rate dynamics can be expressed as:

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dz \tag{1}$$

where dz is a standard Weiner process, and κ, θ, and σ^2 are constants, with $\kappa\theta \geq 0$, and $\sigma^2 > 0$.

When the current spot rate is r, the yield-to-maturity of a τ-period pure discount bond is then given by:

$$R(r,\tau) = \frac{rB(\tau) - \log A(\tau)}{\tau} \tag{2}$$

where

$$A(\tau) \equiv \left[\frac{2\gamma e^{(\kappa+\lambda+\gamma)\tau/2}}{(\kappa+\lambda+\gamma)\left(e^{\gamma\tau}-1\right)+2\gamma}\right]^{2\kappa\theta/\sigma^2} \tag{3}$$

$$B(\tau) \equiv \frac{2\left(e^{\gamma\tau}-1\right)}{(\kappa+\lambda+\gamma)\left(e^{\gamma\tau}-1\right)+2\gamma} \tag{4}$$

$$\gamma = \left[(\kappa+\lambda)^2 + 2\sigma^2\right]^{1/2} \tag{5}$$

From (2), it is easy to see

$$R(r,0) = r \tag{6}$$

$$R(r,\infty) = \frac{2\kappa\theta}{\gamma+\kappa+\lambda} \tag{7}$$

In other words, when the current spot rate is r, the yield curve starts at r and approaches a limit of $\frac{2\kappa\theta}{\gamma+\kappa+\lambda}$ as maturity increases to infinity.

CIR provide a characterization of the shape of the yield curve as a function of the current spot rate. They claim that when $r \leq \frac{2\kappa\theta}{\gamma+\kappa+\lambda}$, the yield curve is uniformly rising. With $r \geq \frac{\kappa\theta}{\kappa+\lambda}$, the yield curve is falling. For $\frac{2\kappa\theta}{\gamma+\kappa+\lambda} < r < \frac{\kappa\theta}{\kappa+\lambda}$, the yield curve is humped.

It is easy to see this characterization of the yield curve is incorrect because if the yield curve is humped when $r = \frac{2\kappa\theta}{\gamma+\kappa+\lambda}$, then the yield curve cannot be uniformly increasing for all $r < \frac{2\kappa\theta}{\gamma+\kappa+\lambda}$. Otherwise, $R(r,\tau)$ will be a discontinuous function of r for some $\tau > 0$, which is contrary to (2) that $R(r,\tau)$ is a linear function of r for a fixed τ. In fact, for r that is less than but close to $\frac{2\kappa\theta}{\gamma+\kappa+\lambda}$, the yield curve must also be humped and it has to go above the long-term yield $\frac{2\kappa\theta}{\gamma+\kappa+\lambda}$ before approaching this limit. Another problem of this characterization is we need $\kappa+\lambda > 0$ in order for $\frac{2\kappa\theta}{\gamma+\kappa+\lambda} < \frac{\kappa\theta}{\kappa+\lambda}$. However, the model does not always imply this condition. In particular, when the term premium is positive, λ is negative and for κ small enough, $\kappa+\lambda$ could be negative as well.[1]

In the next section, we will provide the correct characterization of the yield curve as a function of the spot rate under CIR single factor model.

2 Characterization of the Shape of the Yield Curve

Before we derive the shape of the yield curve as a function of the current spot rate, we first note that the yield curve is in fact determined by only three parameters: $\kappa+\lambda$, $\kappa\theta$, and σ^2. Therefore,

[1] In Gibbons and Ramaswamy (1988), they estimate the CIR single factor model and find that $\hat{\kappa}+\hat{\lambda}$ is -0.7. In the next section, we will show that the yield curve can only be humped or increasing when $\kappa+\lambda \leq 0$ and $r \geq 0$.

by letting $\alpha = \kappa\theta$ and $\beta = \kappa + \lambda$, we can express the yield curve as follows:

$$R(r,\tau) = f(\tau) + g(\tau)r \tag{8}$$

where

$$\begin{aligned}
f(\tau) &\equiv \frac{-2\alpha \log\left[\frac{2\gamma e^{(\gamma+\beta)\tau/2}}{(\gamma+\beta)(e^{\gamma\tau}-1)+2\gamma}\right]}{\sigma^2\tau} \\
&= \frac{2\alpha}{\sigma^2\tau}\log\left[\frac{(\gamma+\beta)\left(e^{\gamma\tau}-1\right)+2\gamma}{2\gamma}\right] - \frac{\alpha(\gamma+\beta)}{\sigma^2} && (9)\\
g(\tau) &\equiv \frac{2\left(e^{\gamma\tau}-1\right)}{\left[(\gamma+\beta)\left(e^{\gamma\tau}-1\right)+2\gamma\right]\tau} && (10)\\
\gamma &= \left(\beta^2 + 2\sigma^2\right)^{1/2} && (11)
\end{aligned}$$

We first prove the following lemmata:

Lemma 1 *$f(\tau)$ is an increasing function of τ. It starts out at 0 with a slope of $\alpha/2$ when $\tau = 0$ and approaches a limit of $\frac{2\alpha}{\gamma+\beta}$ as $\tau \to \infty$.*

Proof:

$$\begin{aligned}
f'(\tau) &= \frac{2\alpha}{\sigma^2\tau^2}\left[\frac{(\gamma+\beta)\gamma\tau e^{\gamma\tau}}{(\gamma+\beta)\left(e^{\gamma\tau}-1\right)+2\gamma} + \log\left[\frac{2\gamma}{(\gamma+\beta)\left(e^{\gamma\tau}-1\right)+2\gamma}\right]\right] \\
&= \frac{2\alpha}{\sigma^2\tau^2}F(\gamma\tau) && (12)
\end{aligned}$$

where $F(x)$ is defined as:

$$F(x) = \frac{xe^x}{e^x - 1 + a} + \log\left(\frac{a}{e^x - 1 + a}\right) \qquad a = \frac{2\gamma}{\gamma+\beta} > 1 \tag{13}$$

With L'Hôpital's Rule, one can easily show $f(0) = 0$, $f(\infty) = \frac{2\alpha}{\gamma+\beta}$, and $f'(0) = \alpha/2$. Note that $F(0) = 0$ and $F'(x) = \frac{xe^x(a-1)}{(e^x-1+a)^2} > 0$ for $x > 0$. Therefore we have $F(x) > 0$ for $x > 0$ and $f'(\tau) > 0$ for $\tau > 0$.

Lemma 2 *$g(\tau)$ starts out at 1 with a slope of $-\beta/2$ when $\tau = 0$ and approaches a limit of 0 as $\tau \to \infty$. If $\beta \geq 0$, $g(\tau)$ is a uniformly decreasing function of τ. If $\beta < 0$, $g(\tau)$ first increases to a maximum value and then decreases to 0.*

Proof:

$$
\begin{aligned}
g'(\tau) &= \frac{-2\left[\beta\left(e^{\gamma\tau}-1\right)^2+\gamma\left(e^{2\gamma\tau}-2\gamma\tau e^{\gamma\tau}-1\right)\right]}{\left[(\gamma+\beta)\left(e^{\gamma\tau}-1\right)+2\gamma\right]^2\tau^2} \\
&= \frac{-(\gamma-\beta)}{2\sigma^2\tau^2}G(\gamma\tau) \qquad (14)
\end{aligned}
$$

where $G(x)$ is defined as:

$$
G(x) = \frac{(2-a)(e^x-1)^2+a(e^{2x}-2xe^x-1)}{(e^x-1+a)^2} \qquad a = \frac{2\gamma}{\gamma+\beta} > 1 \qquad (15)
$$

With L'Hôpital's Rule, one can easily show $g(0) = 1$, $g(\infty) = 0$, and $g'(0) = -\beta/2$. Consider the numerator of $G(x)$. By Taylor series expansion, we have

$$
\begin{aligned}
& (2-a)(e^x-1)^2+a(e^{2x}-2xe^x-1) \\
=& \sum_{i=2}^{\infty}\frac{a_i x^i}{i!} \qquad (16)
\end{aligned}
$$

where

$$
a_i = (2^i-2)(2-a)+(2^i-2i)a \qquad i \geq 2 \qquad (17)
$$

If $\beta \geq 0$, we have $a \leq 2$ and $a_i \geq 0$. Therefore $G(x) \geq 0$ for $x \geq 0$ and $g(\tau)$ is a decreasing function of τ for $\tau \geq 0$. When $\beta < 0$, we have $a_2 < 0$ and note that whenever $a_i > 0$, it implies $a_{i+1} > 0$. Therefore, a_i changes sign only once and by a simple extension of Descartes' Rule of Signs, $G(x)$ has only one positive root.[2] Hence, when $\beta < 0$, $g(\tau)$ is an increasing function when $\tau = 0$ and it reaches a maximum value before decreasing to 0 as $\tau \to \infty$.

We now describe the yield curve by the following proposition:

Proposition 1 *If $\kappa+\lambda > 0$, the yield curve is uniformly falling when $r \geq \frac{\kappa\theta}{\kappa+\lambda}$ and it is uniformly rising when $0 \leq r \leq \frac{\kappa\theta}{\gamma}$. For $\frac{\kappa\theta}{\gamma} < r < \frac{\kappa\theta}{\kappa+\lambda}$, the yield curve first increases to a maximum value and then decreases to the long-term yield $\frac{2\kappa\theta}{\gamma+\kappa+\lambda}$. If $\kappa+\lambda \leq 0$, the yield curve is uniformly rising when $0 \leq r \leq \frac{\kappa\theta}{\gamma}$. For $r > \frac{\kappa\theta}{\gamma}$, the yield curve first increases to a maximum value and then decreases to the long-term yield $\frac{2\kappa\theta}{\gamma+\kappa+\lambda}$.*

[2] Let Z be the number of positive zeros of a power series with radius of convergence $\rho = \infty$ and let the number of changes of signs in the sequence of coefficients be C. Then $Z \leq C$ and $C - Z$ is a non-negative even number. In our case, $C = 1$ and we must have $Z = 1$. See, for example, Pólya and Szegö (1976, Part V) for a proof of this result.

Proof:

$$\begin{aligned} R_2(r,\tau) &= \frac{\partial R(r,\tau)}{\partial \tau} \\ &= f'(\tau) + g'(\tau) r \\ &= \frac{2\alpha}{\sigma^2\tau^2}\left[F(\gamma\tau) - \frac{r(\gamma-\beta)}{4\alpha}G(\gamma\tau)\right] \\ &= \frac{2\alpha}{\sigma^2\tau^2} H(r,\gamma\tau) \end{aligned} \tag{18}$$

where $H(r,x)$ is defined as:

$$H(r,x) = F(x) - \frac{r(\gamma-\beta)}{4\alpha}G(x) \tag{19}$$

We first show that for a fixed r, $H(r,x)$ can have at most one positive root.

$$\begin{aligned} H_2(r,x) &= \frac{\partial H(r,x)}{\partial x} \\ &= F'(x) - \frac{r(\gamma-\beta)}{4\alpha}G'(x) \\ &= \frac{xe^x(a-1)}{(e^x-1+a)^2} - \frac{r(\gamma-\beta)}{4\alpha}\frac{2axe^x(e^x+1-a)}{(e^x-1+a)^3} \\ &= \frac{xe^x}{(e^x-1+a)^3}\sum_{i=0}^{\infty}\frac{b_i x^i}{i!} \end{aligned} \tag{20}$$

where b_i are obtained by Taylor series expansion and they are:

$$b_0 = a(a-1)\left(1 - \frac{r\beta}{\alpha}\right) \tag{21}$$

$$b_i = (a-1)\left(1 - \frac{r\gamma}{\alpha}\right) \qquad i \geq 1 \tag{22}$$

Therefore, b_i can change sign at most once and $H_2(r,x)$ can have at most one positive root. Since $H(r,0) = 0$, $H(r,x)$ can also have at most one positive root by Rolle's theorem. Hence, the yield curve can change direction at most once. If $\beta > 0$, b_i will change sign if and only if $\frac{\alpha}{\gamma} < r < \frac{\alpha}{\beta}$. If $\beta \leq 0$, b_i will change sign if and only if $r > \frac{\alpha}{\gamma}$. In other words, for $\beta > 0$, the yield curve is humped if and only if $\frac{\alpha}{\gamma} < r < \frac{\alpha}{\beta}$ and for $\beta \leq 0$, the yield curve is humped if and only if $r > \frac{\alpha}{\gamma}$.

By lemmata 1 and 2, we have $R_2(r,0) = f'(0) + g'(0)r = \frac{\alpha-\beta r}{2}$. If $\beta > 0$, $R_2(r,0) > 0$ when $r < \frac{\alpha}{\beta}$ and $R_2(r,0) \leq 0$ when $r \geq \frac{\alpha}{\beta}$. Therefore, the yield curve will be uniformly falling if $r \geq \frac{\alpha}{\beta}$ and it will be uniformly increasing if $r \leq \frac{\alpha}{\gamma}$. For $\frac{\alpha}{\gamma} < r < \frac{\alpha}{\beta}$, the yield curve is humped. If $\beta \leq 0$,

$R_2(r, 0)$ is always positive and the yield curve can only be upward sloping or humped. For $r \leq \frac{\alpha}{\gamma}$, the yield curve is uniformly increasing, and for $r > \frac{\alpha}{\gamma}$, the yield curve is humped.

3 Summary

We derive the correct characterization of the yield curve under CIR single factor model. For $0 \leq r \leq \frac{\kappa\theta}{\gamma}$, the yield curve is uniformly increasing. For $\kappa + \lambda > 0$, the yield curve is uniformly decreasing if $r \geq \frac{\kappa\theta}{\kappa+\lambda}$. For all the other cases, the yield curve is humped.

References

[1] Cox, John C., Ingersoll, Jonathan E., Jr., and Ross, Stephen A. (1985a), "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica*, 53, 363–384.

[2] Cox, John C., Ingersoll, Jonathan E., Jr., and Ross, Stephen A. (1985b), "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53, 385–407.

[3] Gibbons, Michael R., and Ramaswamy, Krishna (1988), "The Term Structure of Interest Rates: Empirical Evidence," Working Paper, Stanford University and University of Pennsylvania.

[4] Pólya, George, and Szegö, Gabor (1976), "Problems and Theorems in Analysis," Volume II, Translated by Claude Elias Billigheimer, Springer-Verlag, Berlin, Heidelberg, New York.