On the distribution of the sample autocorrelation coefficients

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Abstract

Sample autocorrelation coefficients are widely used to test the randomness of a time series. Despite its unsatisfactory performance, the asymptotic normal distribution is often used to approximate the distribution of the sample autocorrelation coefficients. This is mainly due to the lack of an efficient approach in obtaining the exact distribution of sample autocorrelation coefficients. In this paper, we provide an efficient algorithm for evaluating the exact distribution of the sample autocorrelation coefficients. Under the multivariate elliptical distribution assumption, the exact distribution as well as exact moments and joint moments of sample autocorrelation coefficients are presented. In addition, the exact mean and variance of various autocorrelation-based tests are provided. Actual size properties of the Box-Pierce and Ljung-Box tests are investigated, and they are shown to be poor when the number of lags is moderately large relative to the sample size. Using the exact mean and variance of the Box-Pierce test statistic, we propose an adjusted Box-Pierce test that has a far superior size property than the traditional Box-Pierce and Ljung-Box tests.

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1. Introduction

Sample autocorrelation coefficients are among the most commonly used test statistics for examining the randomness of economic and financial time series. Despite the widespread popularity, tests based on sample autocorrelation coefficients typically use critical values based on their asymptotic normal distribution. However, it is found that this approximation is, in general, unsatisfactory, and tests based on it are usually undersized. Therefore, some alternative approximations are proposed in the literature. Dufour and Roy (1985) suggest using a normal approximation based on the exact mean and variance of the sample autocorrelation coefficients. Ali (1984) suggests using a four-parameter Pearson distribution approximation based on the first four moments of the sample autocorrelation coefficients. For some popular significance levels, simulation results show that these alternative approximations provide a far better size property than the asymptotic normal approximation.

Ideally, one would like to evaluate the exact distribution of the sample autocorrelation coefficients so that exact inferences can be made. Although this exact distribution is well known (see Ali (1984) and Provost and Rudiuk (1995)), its practical use has been very limited. The main obstacle in its computation is obtaining the eigenvalues of a symmetric $n \times n$ matrix, where n is the length of the time series. As the computation time required to calculate the eigenvalues for a general symmetric $n \times n$ matrix is of $O(n^3)$, it is impractical to directly compute the eigenvalues of the matrix concerned when the sample size is large. In this paper, we overcome this problem by exploiting the structure of the symmetric matrix that characterizes the sample autocorrelation coefficient and derive an efficient algorithm to compute its eigenvalues. With the eigenvalues available, the exact distribution as well as the exact moments and joint moments of the sample autocorrelation coefficients can be easily computed under the assumption that the time series is multivariate elliptically distributed.

In addition, we evaluate several popular autocorrelation-based tests of randomness. Explicit expressions of the exact mean and variance are obtained for Knoke's test, the variance-ratio test, the long-horizon regression test, the Box-Pierce test, and the Ljung-Box test. It is found that the exact mean and variance of these tests differ significantly from the approximate mean and variance based on their asymptotic distributions. Based on the exact mean and variance of the Box-Pierce test statistic, a simple adjusted Box-Pierce test is proposed and it is shown to have much more satisfactory size properties than the traditional Box-Pierce and Ljung-Box tests.

The remainder of the paper is organized as follows. Section 2 presents the eigenvalues that determine the distribution of the sample autocorrelation coefficients. Section 3 discusses the exact distribution and moments of the sample autocorrelation coefficients under the multivariate elliptical distribution assumption. Section 4 evaluates some popular tests based on sample autocorrelations. Section 5 concludes the paper and the Appendix contains proofs of all propositions.

2. Sample autocorrelation coefficients

Several definitions of the sample autocorrelation coefficients have been proposed in the literature. We consider the most standard one: given n observations of a time series $x_1, \ldots, x_n$, the sample autocorrelation coefficient at lag k is given by

$$\hat{\rho}(k) = \frac{\sum_{i=1}^{n-k}(x_i - \bar{x})(x_{i+k} - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}, \tag{1}$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the sample mean, and $1 \leq k \leq n-1$. In matrix notation,

$$\hat{\rho}(k) = \frac{x' M_n A_k M_n x}{x' M_n x}, \tag{2}$$

where $x = [x_1,\ \ldots\ , x_n]'$, A_k is an $n \times n$ symmetric Toeplitz matrix with $[0_k',\ \frac{1}{2},\ 0_{n-k-1}']$ as its first row, where 0_k stands for a k-vector of zeros, and $M_n = I_n - \frac{1}{n}1_n 1_n'$, where I_n is an identity matrix with dimension n and 1_n is an n-vector of ones.

2.1. The eigenvalue problem

To further simplify the expression for $\hat{\rho}(k)$, we note that M_n is an idempotent matrix with rank $n-1$ and it is orthogonal to 1_n, so we can write $M_n = P_n P_n'$ where P_n is an $n \times (n-1)$ orthonormal matrix such that $P_n' 1_n = 0_{n-1}$ and $P_n' P_n = I_{n-1}$. Letting $y = P_n' x$ and denoting $B_k = P_n' A_k P_n$, we can write

$$\hat{\rho}(k) = \frac{x' P_n P_n' A_k P_n P_n' x}{x' P_n P_n' x} = \frac{y' P_n' A_k P_n y}{y' y} = \frac{y' B_k y}{y' y}. \tag{3}$$

Suppose $B_k = H'\Xi H$, where $\Xi = \mathrm{Diag}(\xi_1, \ldots, \xi_{n-1})$ is a diagonal matrix of the eigenvalues of B_k and H is a matrix with its columns equal to the corresponding eigenvectors of B_k. Using the transformation $z = Hy$, we can then simplify $\hat{\rho}(k)$ to

$$\hat{\rho}(k) = \frac{y'H'\Xi Hy}{y'H'Hy} = \frac{z'\Xi z}{z'z} = \sum_{i=1}^{n-1} \xi_i u_i^2, \tag{4}$$

where $u_i = z_i/(z'z)^{\frac{1}{2}}$. From this expression, we can see that the distribution of $\hat{\rho}(k)$ depends only on the distribution of $u = z/(z'z)^{\frac{1}{2}}$ and the eigenvalues of B_k, so we need to obtain the eigenvalues of B_k to study the distribution of $\hat{\rho}(k)$. In addition, these eigenvalues also give us a bound on $\hat{\rho}(k)$. This is because by the Rayleigh-Ritz theorem, we have

$$\min_{1\le i\le n-1} \xi_i \le \hat{\rho}(k) \le \max_{1\le i\le n-1} \xi_i, \tag{5}$$

and these bounds only depend on n and k but not on the distribution of u.

As it turns out, the eigenvalues of B_k are closely related to the eigenvalues of A_k. The eigenvalues and eigenvectors of A_k were obtained by Provost and Rudiuk (1995). We summarize their results in the following Proposition.

Proposition 1. *Suppose $n = mk + l$, where $m = \lfloor n/k \rfloor$ denotes the integral part of n/k. Let $\phi = [\phi_1, \ldots \phi_{m+1}]'$ and $\theta = [\theta_1, \ldots, \theta_m]'$, where $\phi_s = s\pi/(m+2)$ and $\theta_s = s\pi/(m+1)$. Define Q_a and Q_b as discrete sine transform matrices of order $m+1$ and m:*

$$Q_a = \left(\left(\frac{2}{m+2}\right)^{\frac{1}{2}} \sin(r\phi_s)\right)_{r,s=1}^{m+1}, \qquad Q_b = \left(\left(\frac{2}{m+1}\right)^{\frac{1}{2}} \sin(r\theta_s)\right)_{r,s=1}^{m}. \tag{6}$$

The eigenvalues of A_k are given by

$$\lambda = \begin{bmatrix} \cos(\phi) \otimes 1_l \\ \cos(\theta) \otimes 1_{k-l} \end{bmatrix}, \tag{7}$$

and the corresponding matrix of the eigenvectors of A_k is given by

$$Q = [I_n,\ 0_{n\times(k-l)}] \left[Q_a \otimes E_L,\ \begin{bmatrix} Q_b \\ 0'_m \end{bmatrix} \otimes E_R \right], \tag{8}$$

where E_L equals the first l columns of I_k and E_R equals the last $k-l$ columns of I_k.

In Proposition 1, we adopt the convention that when $l = 0$, 1_l and E_L are null matrices which implies $\lambda = \cos(\theta) \otimes 1_k$ and $Q = Q_b \otimes I_k$. This allows us to present the eigenvalues and eigenvectors of A_k more compactly than the expressions in Provost and Rudiuk (1995). In addition, the eigenvectors that we present are normalized such that $QQ' = I_n$ and $Q'Q = I_n$. Finally, our proof of Proposition 1 is significantly shorter than the one in Provost and Rudiuk (1995).

With the explicit solutions to the eigenvalues and eigenvectors of A_k available, our next lemma presents the relation between the eigenvalues of A_k and B_k, which is the first step in our effort to obtain the eigenvalues of B_k.

Lemma 1. *Let $\Lambda = \mathrm{Diag}(\lambda)$ and $\tilde{q} = (I_n + \Lambda)^{\frac{1}{2}} Q' 1_n$, where λ and Q are defined in (7) and (8). Suppose $\xi_1 \geq \xi_2 \cdots \geq \xi_{n-1} > \xi_n = -1$ are the eigenvalues of $\Lambda - \frac{1}{n}\tilde{q}\tilde{q}'$. Then, ξ_1 to ξ_{n-1} are the eigenvalues of B_k.*

Lemma 1 suggests that the eigenvalues of B_k can be obtained from the eigenvalues of the matrix $\Lambda - \frac{1}{n}\tilde{q}\tilde{q}'$, which differs from Λ by a matrix of rank one. In numerical matrix algebra, this problem is known as the rank-one update of a diagonal matrix, and there is a fast and stable algorithm with computation time of $O(n^2)$ (see Li (1993) and Gu and Eisenstat (1994) and references therein).[1] Although Lemma 1 allows us to significantly reduce the computation time for obtaining the eigenvalues of B_k, the rank-one update problem in Lemma 1 can be further deflated because λ has many repeated eigenvalues and many of the elements of $\tilde{q}$ are zero. The following Proposition shows that the eigenvalues of B_k can be divided into two sets: one set consists of the eigenvalues for A_k, and the other set is obtained from solving a potentially very small rank-one update problem.

Proposition 2. *Suppose $n = mk + l$, where $m = \lfloor n/k \rfloor$ denotes the integral part of n/k. Denote ϕ^o and ϕ^e as the vectors of odd and even elements of ϕ, and θ^o and θ^e as the vectors of odd and even elements of θ. When $l > 0$, the eigenvalues of B_k are given by $\xi = [\xi_a', \ \xi_b']'$, where*

$$\xi_a = [\cos(\phi^o)' \otimes 1_{l-1}', \ \cos(\phi^e)' \otimes 1_l', \ \cos(\theta^o)' \otimes 1_{k-l-1}', \ \cos(\theta^e)' \otimes 1_{k-l}']' \tag{9}$$

[1]In addition to the reduction of computation time, the requirement of memory storage is also significantly reduced because there is no need to create and store the $(n-1) \times (n-1)$ matrix B_k in order to obtain its eigenvalues.

and ξ_b are the eigenvalues of

$$D - \frac{1}{n}\tilde{r}\tilde{r}', \tag{10}$$

that are not equal to -1*, where* $D = \mathrm{Diag}([\cos(\phi^o)', \ \cos(\theta^o)'])$ *and*

$$\tilde{r} = \left[\left(\frac{4l}{m+2}\right)^{\frac{1}{2}}\left[\csc\left(\frac{\phi^o}{2}\right) - \sin\left(\frac{\phi^o}{2}\right)\right]', \ \left(\frac{4(k-l)}{m+1}\right)^{\frac{1}{2}}\left[\csc\left(\frac{\theta^o}{2}\right) - \sin\left(\frac{\theta^o}{2}\right)\right]'\right]'. \tag{11}$$

When $l = 0$*, the terms that involve* ϕ^o *and* ϕ^e *drop out.*

Proposition 2 suggests that most of the eigenvalues of B_k are the same as the eigenvalues of A_k and they are readily available. The rest of the eigenvalues can be obtained by performing a rank-one update on D. In the next subsection, we provide an explicit equation for solving this rank-one update problem.

2.2. Finding the distinct eigenvalues

Let p be the dimension of D in Proposition 2. We have $p = \lfloor (m+1)/2 \rfloor$ when $l = 0$ and $p = m+1$ when $l > 0$, and p can be significantly smaller than n. The eigenvalues of $D - \frac{1}{n}\tilde{r}\tilde{r}'$ are the solutions to the equation $|D - \frac{1}{n}\tilde{r}\tilde{r}' - xI_p| = 0$. Since D is diagonal, the eigenvalues can be written as the solutions to the following equation

$$f(x) = \frac{1}{n}\sum_{i=1}^{p}\frac{\tilde{r}_i^2}{d_i - x} - 1 = 0, \tag{12}$$

where d_i is the ith diagonal element of D and $\tilde{r}_i$ is the ith element of $\tilde{r}$. Since we know the smallest root of $f(x) = 0$ is -1 and it is not an eigenvalue of B_k, we can solve for ξ_b from the roots of the equation $g(x) = f(x)/(1+x) = 0$. Our objective is to obtain an explicit expression for $g(x)$.

Using the fact that

$$[\csc(\varphi/2) - \sin(\varphi/2)]^2 = \frac{[1+\cos(\varphi)]^2}{2[1-\cos(\varphi)]}, \tag{13}$$

we can write $f(x)$ as

$$f(x) \quad = \quad \frac{2l}{n(m+2)}\sum_{i=1}^{\lfloor (m+2)/2 \rfloor}\frac{[1+\cos(\phi_{2i-1})]^2}{[1-\cos(\phi_{2i-1})][\cos(\phi_{2i-1}) - x]}$$

$$+\frac{2(k-l)}{n(m+1)}\sum_{i=1}^{\lfloor (m+1)/2\rfloor}\frac{[1+\cos(\theta_{2i-1})]^2}{[1-\cos(\theta_{2i-1})][\cos(\theta_{2i-1})-x]}-1. \tag{14}$$

Since $f(-1)=0$, we have

$$0=\frac{2l}{n(m+2)}\sum_{i=1}^{\lfloor (m+2)/2\rfloor}\frac{1+\cos(\phi_{2i-1})}{1-\cos(\phi_{2i-1})}+\frac{2(k-l)}{n(m+1)}\sum_{i=1}^{\lfloor (m+1)/2\rfloor}\frac{1+\cos(\theta_{2i-1})}{1-\cos(\theta_{2i-1})}-1. \tag{15}$$

Subtracting (15) from (14), we obtain an alternative expression of $f(x)$

$$\begin{aligned} f(x) &= \frac{2l}{n(m+2)}\sum_{i=1}^{\lfloor (m+2)/2\rfloor}\frac{[1+\cos(\phi_{2i-1})](1+x)}{[1-\cos(\phi_{2i-1})][\cos(\phi_{2i-1})-x]} \\ &\quad +\frac{2(k-l)}{n(m+1)}\sum_{i=1}^{\lfloor (m+1)/2\rfloor}\frac{[1+\cos(\theta_{2i-1})](1+x)}{[1-\cos(\theta_{2i-1})][\cos(\theta_{2i-1})-x]}. \end{aligned} \tag{16}$$

Denoting $\delta=k/n$, we can use the fact that $[1+\cos(\varphi)]/[1-\cos(\varphi)]=\cot^2(\varphi/2)$ to write $g(x)=f(x)/(1+x)$ as

$$g(x)=\frac{2(1-m\delta)}{m+2}\sum_{i=1}^{\lfloor (m+2)/2\rfloor}\frac{\cot^2(\phi_{2i-1}/2)}{\cos(\phi_{2i-1})-x}+\frac{2[(m+1)\delta-1]}{m+1}\sum_{i=1}^{\lfloor (m+1)/2\rfloor}\frac{\cot^2(\theta_{2i-1}/2)}{\cos(\theta_{2i-1})-x}. \tag{17}$$

Since $g(x)$ only depends on m and δ, the roots of $g(x)=0$ (i.e., ξ_b) are completely determined by m and δ. When $l=0$ (i.e., $\delta=1/m$), the first term in $g(x)$ vanishes and the second term is only a function of m, so ξ_b is only a function of m when $l=0$. From (17), it is easy to see that when $l>0$, $g(x)$ has $m+1$ poles at $\cos(\phi^o)$ and $\cos(\theta^o)$ and it strictly increases between these poles, so the roots of $g(x)=0$ must be between the poles. As a result, we have the following bounds on the elements of ξ_b when $l>0$:

$$\cos\left(\frac{i\pi}{m+2}\right)>\ \xi_{bi}\ >\cos\left(\frac{i\pi}{m+1}\right) \qquad \text{if } i \text{ is odd}, \tag{18}$$

$$\cos\left(\frac{(i-1)\pi}{m+1}\right)>\ \xi_{bi}\ >\cos\left(\frac{(i+1)\pi}{m+2}\right) \qquad \text{if } i \text{ is even}. \tag{19}$$

Similarly, we have the following bounds on the elements of ξ_b when $l=0$:

$$\cos\left(\frac{(2i-1)\pi}{m+1}\right)>\xi_{bi}>\cos\left(\frac{(2i+1)\pi}{m+1}\right). \tag{20}$$

These bounds are useful because they allow us to efficiently locate the roots of $g(x)=0$. With some additional work, we can further tighten the upper bounds in (19) and (20). The results are summarized in the following lemma.

Lemma 2. *When $l > 0$, the elements of ξ_b are within the following bounds:*

$$\cos\left(\frac{i\pi}{m+2}\right) > \ \xi_{bi} \ > \cos\left(\frac{i\pi}{m+1}\right) \qquad \text{if } i \text{ is odd,} \tag{21}$$

$$\cos\left(\frac{i\pi}{m+1}\right) > \ \xi_{bi} \ > \cos\left(\frac{(i+1)\pi}{m+2}\right) \quad \text{if } i \text{ is even.} \tag{22}$$

When $l = 0$, the elements of ξ_b are within the following bounds:

$$\cos\left(\frac{2i\pi}{m+1}\right) > \xi_{bi} > \cos\left(\frac{(2i+1)\pi}{m+1}\right), \qquad i = 1, \ldots, \lfloor (m-1)/2 \rfloor. \tag{23}$$

For m up to a certain order, the roots of $g(x) = 0$ can be solved analytically. In Table 1, we use Mathematica to further simplify the equation $g(x) = 0$ and report the solutions of ξ_b for $3 \leq m \leq 14$ when $l = 0$, and for $1 \leq m \leq 10$ when $l > 0$. From Table 1, we can see that when $l = 0$, analytical solutions for ξ_b are available for m up to 10, and when $l > 0$, analytical solutions for ξ_b are available for m up to 4. With higher m, one needs to numerically solve for the roots of a polynomial. With the help of Table 1, computing ξ for $k \geq n/10$ takes almost no time even for very large n.

Table 1 about here

As m goes up (i.e., k goes down), the computation time for solving the rank-one update problem increases. In terms of computation time, the worst case scenario is for $k = 1$ which requires one to solve a rank-one update problem of $\lfloor (n+1)/2 \rfloor$ dimension. Nevertheless, the rank-one update method is still much faster than the standard way of computing the eigenvalues of B_1. To provide an idea of how fast our method is, we perform an experiment using the Windows version of Matlab running on an Opteron 165 processor. For $n = 2400$, it takes only 0.054 second for our program to compute the eigenvalues of B_1 and 0.438 second to compute all the eigenvalues of B_1 to B_{2399}. In contrast, using the standard `eig` function of Matlab, it takes 31.36 seconds to compute the eigenvalues of B_1, and if one needs to compute all the eigenvalues of B_1 to B_{2399}, it would take 21 hours.[2] Therefore, our method provides a phenomenal speedup of the computation of the eigenvalues of B_k relative to the traditional method, which in turn gives us a practical way to numerically compute the exact moments and distribution of $\hat{\rho}(k)$ even for very large n.

[2]For a fair comparison, we call `eig` to only compute the eigenvalues (but not the eigenvectors) of B_k.

2.3. Range of sample autocorrelation coefficients

From (5), we know that $\hat{\rho}(k)$ is bounded by the smallest and largest eigenvalue of B_k. Using the results in Proposition 2 and Lemma 2, the following lemma provides an explicit characterization of the range of $\hat{\rho}(k)$.

Lemma 3. *Suppose $n = mk + l$ where $m = \lfloor n/k \rfloor$. Denote ξ_b^* and ξ_b^{**} as the smallest and largest element of ξ_b, respectively. When $k = 1$, we have*

$$\cos\left(\frac{2\pi}{n+1}\right) \geq \hat{\rho}(1) \geq -\cos\left(\frac{\pi}{n+1}\right) \qquad \text{if } n \text{ is even,} \tag{24}$$

$$\cos\left(\frac{2\pi}{n+1}\right) \geq \hat{\rho}(1) \geq \xi_b^* \qquad \text{if } n \text{ is odd.} \tag{25}$$

When $k > 1$ and $l = 0$, we have

$$\cos\left(\frac{\pi}{m+1}\right) \geq \hat{\rho}(k) \geq -\cos\left(\frac{\pi}{m+1}\right). \tag{26}$$

When $k > 1$ and $l = 1$, we have

$$\xi_b^{**} \geq \hat{\rho}(k) \geq \xi_b^* \qquad \text{if } m \text{ is even,} \tag{27}$$

$$\xi_b^{**} \geq \hat{\rho}(k) \geq -\cos\left(\frac{\pi}{m+2}\right) \qquad \text{if } m \text{ is odd.} \tag{28}$$

When $k > 1$ and $l > 1$, we have

$$\cos\left(\frac{\pi}{m+2}\right) \geq \hat{\rho}(k) \geq -\cos\left(\frac{\pi}{m+2}\right). \tag{29}$$

Lemma 3 suggests that while we can have $-1 < \rho(k) < 1$ for the population autocorrelation coefficient, the range that the sample autocorrelation coefficient $\hat{\rho}(k)$ can take is more limited, especially when k is large relative to n. For example, when $n/2 \leq k \leq n - 1$, we have $|\hat{\rho}(k)| \leq 1/2$ and for $n/3 \leq k < (n-1)/2$, we have $|\hat{\rho}(k)| \leq 1/\sqrt{2}$. In Figure 1, we plot the range of $\hat{\rho}(k)$ for $k = 1, \ldots, n-1$ under four different assumptions on the sample size. It clearly shows that when k goes up, the range of $\hat{\rho}(k)$ becomes narrower.[3] In addition, the

[3]Some simple degrees of freedom correction (like multiplying $\hat{\rho}(k)$ by $(n-1)/(n-k-1)$) can help to restore the range of $\hat{\rho}(k)$ to be closer to $(-1, 1)$, especially for $k < n/2$. We thank an anonymous referee for pointing this out to us.

range of $\hat{\rho}(k)$ is a step function, it shrinks whenever n/k goes through an integer value. As our bounds on $\hat{\rho}(k)$ work for any time series, this implies that we should expect to observe a decaying $\hat{\rho}(k)$ as k increases, regardless of whether the time series is stationary or not. In addition, the fact that $\hat{\rho}(k)$ is bounded suggests that a test that is based on the normal or Pearson distribution may have a size quite different from its level.[4]

Figure 1 about here

3. Exact distribution and moments of $\hat{\rho}(k)$

For the sake of deriving the exact distribution and moments of $\hat{\rho}(k)$, we need to make an assumption on the joint distribution of x. We assume x has a multivariate elliptical distribution with mean $\mu 1_n$ and variance-covariance matrix $\sigma^2 I_n$, so that the time series is uncorrelated. Under this assumption, the exact distribution as well as the exact moments and joint moments of $\hat{\rho}(k)$ can be obtained, and they are provided in the following subsections.

3.1. Exact distribution of $\hat{\rho}(k)$

From (4), we know that $\hat{\rho}(k)$ can be written as

$$\hat{\rho}(k) = \frac{y' B_k y}{y'y} = \frac{z'\Xi z}{z'z} = u'\Xi u, \tag{30}$$

where Ξ is the diagonal matrix of the eigenvalues of B_k. Under the assumption that $x = [x_1, \ldots, x_n]'$ is multivariate elliptically distributed with mean $\mu 1_n$ and variance-covariance matrix $\sigma^2 I_n$, z has a spherical distribution. Since the distribution of $u = z/(z'z)^{\frac{1}{2}}$ is the same for all spherical distributions z (see, for example, Theorem 1.5.6 of Muirhead (1982)), we can assume without loss of generality that $z \sim N(0_{n-1}, I_{n-1})$ or $y \sim N(0_{n-1}, I_{n-1})$. Therefore, in order to compute the cumulative density function of $\hat{\rho}(k)$, we only need to evaluate the following probability

$$P[\hat{\rho}(k) < c] = P\left[z'\Xi z < cz'z\right] = P\left[\sum_{i=1}^{n-1} (\xi_i - c) z_i^2 < 0\right], \tag{31}$$

[4]We find that when $k > 3n/4$, the Pearson distribution that matches the first four moments of $\hat{\rho}(k)$ is Type IV, which is a distribution with unbounded support.

where ξ_i's are the eigenvalues of B_k. This amounts to computing the probability for a linear combination of $n-1$ independent χ_1^2 random variables to be less than zero. This problem has been well studied in the statistics and econometrics literature. Using the results from Gil-Pelaez (1951) and Imhof (1961), we can express

$$P[\hat{\rho}(k) < c] = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\sin\tau(t)}{t\eta(t)}\mathrm{d}t, \tag{32}$$

where

$$\tau(t) = \frac{1}{2}\sum_{i=1}^{n-1}\arctan(2t(\xi_i - c)), \tag{33}$$

$$\eta(t) = \prod_{i=1}^{n-1}[1 + 4t^2(\xi_i - c)^2]^{\frac{1}{4}}. \tag{34}$$

Numerical evaluation of this integral was studied by Imhof (1961), Davies (1980), Ansley, Kohn, and Shively (1992), and Lu and King (2002). The only time-consuming part for evaluating this integral is to obtain the ξ_i's. With our fast algorithm for the computation of ξ_i's, we can numerically evaluate the exact distribution of $\hat{\rho}(k)$ with high accuracy and efficiency.[5]

With our numerical method of computing the exact distribution of $\hat{\rho}(k)$, we plot the lower and upper fifth percentiles of the exact distribution of $\hat{\rho}(k)$ as a function of k in Figure 2 using the solid lines, for $n = 60$, 240, 600, and 2400. The lower and upper fifth percentiles of the asymptotic normal distribution are also plotted using the dotted lines for comparison. It can be seen from the plot that the asymptotic normal distribution does a very poor job in approximating the true distribution, especially when k is large. This suggests that we could have a serious size problem when using the asymptotic normal distribution to test the null hypothesis.

Figure 2 about here

In Figures 3a and 3b, we evaluate the actual size properties of three approximation tests of $H_0 : \rho(k) = 0$ as a function of k under four different choices of n. The first approximation

[5] From Proposition 2, we know ξ has many repeated elements, this allows us to further speed up the computation of $\sin\tau(t)$ and $\eta(t)$.

test is based on the asymptotic normal distribution and assumes $\hat{\rho}(k) \sim N(0, 1/n)$. The second approximation test is also a normal approximation, and uses the exact mean and variance of $\hat{\rho}(k)$. This approximation test is suggested by Dufour and Roy (1985). The third test is based on Pearson's approximation, and it is suggested by Ali (1984). All tests are two-sided tests and are assumed to have a nominal size of 10% in Figure 3a and 5% in Figure 3b. If the tests have a good size property, the actual probabilities of rejection should be close to the nominal size. We use the solid lines to represent the actual size of the normal approximation test based on the asymptotic mean and variance, the dotted lines to represent the normal approximation test based on the exact mean and variance, and the dashed lines to represent the test based on Pearson's approximation. We can see that the normal approximation test based on the asymptotic distribution behaves very poorly even for moderately large k. From Lemma 3, we know that the range of $\hat{\rho}(k)$ shrinks when k goes up. But the critical values based on the asymptotic normal distribution do not change with k. Therefore, we expect the asymptotic test to be too conservative as k increases. Compared to the test based on the asymptotic distribution, the normal approximation test based on the exact mean and variance and the test that is based on Pearson's approximation behave far better. Except when k is close to n, the actual probabilities of rejection are close to the assumed nominal size. This is consistent with the simulation results of Dufour and Roy (1985) and Ali (1984). However, given that $\hat{\rho}(k)$ is bounded, there can be situations that the normal distribution (which is unbounded) could not approximate the exact distribution of $\hat{\rho}(k)$ well, especially in the two tails. Similarly, there are situations under which the Pearson's distribution does not provide a good approximation of the exact distribution of $\hat{\rho}(k)$. With the development of our efficient numerical method of computing the exact distribution of $\hat{\rho}(k)$, we can overcome this problem by evaluating the exact distribution directly.

Figure 3a about here

Figure 3b about here

3.2. Exact moments of $\hat{\rho}(k)$

We now turn our attention to the problem of computing the exact moments of $\hat{\rho}(k)$. The following Proposition provides a fast method for computing all the moments of $\hat{\rho}(k)$.

Proposition 3. *Suppose $x = [x_1, \ldots, x_n]'$ is multivariate elliptically distributed with mean $\mu 1_n$ and variance-covariance matrix $\sigma^2 I_n$. The sth moment of $\hat{\rho}(k)$ is given by*

$$E[\hat{\rho}(k)^s] = \frac{E[(y'B_k y)^s]}{(n-1)(n+1)\cdots(n-3+2s)}, \tag{35}$$

where $y \sim N(0_{n-1}, I_{n-1})$, and $E[(y'B_k y)^s]$ is obtained by using the following recursive relation[6]

$$E[(y'B_k y)^s] = (s-1)! \sum_{j=1}^{s} \frac{2^{j-1}\mathrm{tr}(B_k^j) E[(y'B_k y)^{s-j}]}{(s-j)!}. \tag{36}$$

In order to obtain the moments of $\hat{\rho}(k)$ using Proposition 3, we need to be able to compute $\mathrm{tr}(B_k^j)$. Since

$$\mathrm{tr}(B_k^j) = \sum_{i=1}^{n-1} \xi_i^j, \tag{37}$$

where ξ_i are the eigenvalues of B_k, it is very easy to compute $\mathrm{tr}(B_k^j)$ once we obtain ξ_i using our efficient algorithm. As a result, evaluating the moments of $\hat{\rho}(k)$ up to any order can be easily accomplished. In addition, with the noncentral moments of $\hat{\rho}(k)$ available, the central moments of $\hat{\rho}(k)$ can also be easily obtained.

For small s, it is possible to derive $E[\hat{\rho}(k)^s]$ as an explicit polynomial of n and k and these expressions are in fact available in the literature. The first four moments of $\hat{\rho}(k)$ are

$$\begin{aligned}
E[\hat{\rho}(k)] &= -\frac{n-k}{n(n-1)}, & (38)\\
E[\hat{\rho}(k)^2] &= \frac{(n-k)(n^2+n-3k)}{n^2(n^2-1)} - \frac{2(n-2k)^+}{n(n^2-1)}, & (39)\\
E[\hat{\rho}(k)^3] &= -\frac{3(n-k)[n^2(n+1)-n(n+4)k+5k^2]}{n^3(n^2-1)(n+3)} + \frac{6(n-3k)(n-2k)^+}{n^2(n^2-1)(n+3)}
\end{aligned}$$

[6] There exist explicit expressions of $E[(y'B_k y)^s]$ in the literature (see Magnus (1986) and Holmquist (1996)). However, as pointed out by Kan (2008), these explicit expressions are unsuitable for computational purpose except when s is very small.

$$
\begin{aligned}
& -\frac{6(n-3k)^+}{n(n^2-1)(n+3)}, && (40)\\
E[\hat{\rho}(k)^4] \;=\; & \frac{3(n-k)[n^3(n+1)(n+3)-n^2(n^2+8n+21)k+3n(2n+15)k^2-35k^3]}{n^4(n^2-1)(n+3)(n+5)} - \\
& \frac{12[2n^2-n(n+6)k+15k^2](n-2k)^+}{n^3(n^2-1)(n+3)(n+5)} + \frac{24([(n-3k)^+]^2-n(n-4k)^+)}{n^2(n^2-1)(n+3)(n+5)}, && (41)
\end{aligned}
$$

where a^+ stands for $\max[a,0]$. The mean of $\hat{\rho}(k)$ was derived by Moran (1948). The second moment of $\hat{\rho}(k)$ was derived by Dufour and Roy (1985, 1989) and Anderson (1990). The third and fourth moments of $\hat{\rho}(k)$ were given in Anderson (1990, 1993). Provost and Rudiuk (1995) also provide the analytical expression of the fifth moment of $\hat{\rho}(k)$. For higher order moments of $\hat{\rho}(k)$, analytical expressions are not available. The greatest difficulty in generating explicit formula of higher order moments of $\hat{\rho}(k)$ is the extremely involved algebra needed in evaluating $\mathrm{tr}(B_k^j)$; its complexity increases exponentially with increasing j. In contrast, our approach of evaluating $\mathrm{tr}(B_k^j)$ is relatively simple, and it can be used to efficiently compute the moments of $\hat{\rho}(k)$ up to any order.

3.3. Exact joint moments of $\hat{\rho}(k)$

Most tests of randomness are linear combination of $\hat{\rho}(k)$ or $\hat{\rho}(k)^2$. In order to compute the exact mean and variance of these test statistics, we need to have expressions for $\mathrm{Var}[\hat{\rho}(k)]$ and $\mathrm{Var}[\hat{\rho}(k)^2]$ as well as for $\mathrm{Cov}[\hat{\rho}(j),\hat{\rho}(k)]$ and $\mathrm{Cov}[\hat{\rho}(j)^2,\hat{\rho}(k)^2]$. $\mathrm{Var}[\hat{\rho}(k)]$ and $\mathrm{Var}[\hat{\rho}(k)^2]$ can be easily obtained using the results in Proposition 3, but evaluation of the covariances requires analytical expressions for the joint moments. The following Proposition presents a general result for the mixed moments of p sample autocorrelation coefficients.

Proposition 4. *Suppose $x = [x_1,\ldots,x_n]'$ is multivariate elliptically distributed with mean $\mu 1_n$ and variance-covariance matrix $\sigma^2 I_n$. The expectation of the product $\hat{\rho}(k_1)^{s_1}\cdots\hat{\rho}(k_p)^{s_p}$ is given by*

$$
E\left[\prod_{i=1}^{p}\hat{\rho}(k_i)^{s_i}\right] = \frac{1}{(n-1)(n+1)\cdots(n-3+2s)} E\left[\prod_{i=1}^{p}(y'B_{k_i}y)^{s_i}\right], \qquad (42)
$$

where $s = s_1+\cdots+s_p$ and $y \sim N(0_{n-1}, I_{n-1})$. The expectation of the product of quadratic *forms can be evaluated using*

$$E\left[\prod_{i=1}^{p}(y'B_{k_i}y)^{s_i}\right] = \frac{1}{s!}\sum_{\nu_1=0}^{s_1}\cdots\sum_{\nu_p=0}^{s_p}(-1)^{\sum_{i=1}^{p}\nu_i}\binom{s_1}{\nu_1}\cdots\binom{s_p}{\nu_p}E[(y'B_\nu y)^s], \tag{43}$$

where $B_\nu = \left(\frac{s_1}{2}-\nu_1\right)B_1 + \left(\frac{s_2}{2}-\nu_2\right)B_2 + \cdots + \left(\frac{s_p}{2}-\nu_p\right)B_p$ *and* $E[(y'B_\nu y)^s]$ *can be computed using the recursive relation in (36).*

Note that in Proposition 4, we present the moments of a product of quadratic forms in normal random variables using a new formula from Kan (2008). Unlike existing expressions (e.g., Magnus (1978, 1979) and Holmquist (1996)), this expression is computationally very efficient, and it allows us to compute the mixed moments of $\hat{\rho}(k_i)$ even for fairly large s.[7]

When s is small, we can derive analytical expressions of the mixed moments of $\hat{\rho}(k_i)$. The following lemma presents the analytical expressions of $E[\hat{\rho}(j)\hat{\rho}(k)]$ and $E[\hat{\rho}(j)^2\hat{\rho}(k)^2]$. Together with the expressions of $E[\hat{\rho}(k)]$ and $E[\hat{\rho}(k)^2]$ in (38) and (39), we can then obtain analytical expressions for $\mathrm{Cov}[\hat{\rho}(j), \hat{\rho}(k)]$ and $\mathrm{Cov}[\hat{\rho}(j)^2, \hat{\rho}(k)^2]$.

Lemma 4. *Suppose* $x = [x_1, \ldots, x_n]'$ *is multivariate elliptically distributed with mean* $\mu 1_n$ *and variance-covariance matrix* $\sigma^2 I_n$*. Denote* $a_j = \mathrm{tr}(B_j)$, $a_{jk} = \mathrm{tr}(B_jB_k)$, $a_{jkl} = \mathrm{tr}(B_jB_kB_l)$, *and* $a_{jklm} = \mathrm{tr}(B_jB_kB_lB_m)$*. Then for* $j < k$*, the expectation of* $\hat{\rho}(j)\hat{\rho}(k)$ *and* $\hat{\rho}(j)^2\hat{\rho}(k)^2$ *are given by*

$$\begin{aligned}
E[\hat{\rho}(j)\hat{\rho}(k)] &= \frac{a_ja_k + 2a_{jk}}{n^2-1} = \frac{(n-3j)(n-k)}{n^2(n^2-1)} - \frac{2(n-j-k)^+}{n(n^2-1)}, \qquad (44)\\
E[\hat{\rho}(j)^2\hat{\rho}(k)^2] &= \frac{a_j^2a_k^2 + 16a_ja_{jkk} + 16a_ka_{kjj} + 4a_{jj}a_{kk} + 8a_{jk}^2}{(n^2-1)(n+3)(n+5)}\\
&\quad + \frac{2a_j^2a_{kk} + 2a_k^2a_{jj} + 8a_ja_ka_{jk} + 16a_{jkjk} + 32a_{jjkk}}{(n^2-1)(n+3)(n+5)}\\
&= \frac{-(n-k)}{n^4(n^2-1)(n+3)(n+5)}\big[105j^2k - (75j^2 + 60jk)n\\
&\quad + (36j - 3j^2 + 15k - 3jk)n^2 + (5j + 3k - 5)n^3 + (j-6)n^4 - n^5\big]\\
&\quad - \frac{2}{n^3(n^2-1)(n+3)(n+5)}\big[(15j^2 + 9n^2 - jn^2 + n^3)(n-2k)^+
\end{aligned}$$

[7] A set of Matlab programs to implement this and other results in the paper is available at `http://www.rotman.utoronto.ca/~kan/research.htm`.

$$
\begin{aligned}
&+ (15k^2 - 24kn + 9n^2 - kn^2 + n^3)(n-2j)^+ \\
&+ (60jk - 24jn - 4n^3)(n-j-k)^+\big] \\
&+ \frac{2}{n^2(n^2-1)(n+3)(n+5)}\big[6(n-3k)(n-2j-k)^+ \\
&+ 6(n-3j)(n-j-2k)^+ + 2(n-3j)(n+j-2k)^+ \\
&- 12n(n-2j-2k)^+ + 4(2n-3k)(n-\max[2j,k])^+ \\
&- 2n(n+2j-2\max[2j,k])^+ - 2n(n-k)^2\delta_{2j,k}\big]\,,
\end{aligned}
\tag{45}
$$

where $\delta_{2j,k} = 1$ *if* $2j = k$ *and zero otherwise.*

The analytical expression of $\mathrm{Cov}[\hat{\rho}(j), \hat{\rho}(k)]$ is already available in Dufour and Roy (1985, 1989) and Anderson (1990), but our analytical expression of $\mathrm{Cov}[\hat{\rho}(j)^2, \hat{\rho}(k)^2]$ is new.

4. Evaluation of various approximate tests

Most specification tests of randomness are based on the linear combination of either $\hat{\rho}(k)$ or $\hat{\rho}(k)^2$. Statistical inferences are usually based on the approximate distributions of these test statistics. Due to the lack of knowledge of the exact distribution, exact tests are rarely seen, and performance of the approximate tests are hard to evaluate. Using our results from Sections 3.2 and 3.3, we are able to obtain the exact mean and variance for these test statistics, which provide a means to evaluate these approximate tests and enhance our understanding of the exact properties of these test statistics.

4.1. Evaluation of approximate tests based on a linear combination of $\hat{\rho}(k)$

In this subsection, we evaluate three different specification tests of randomness that are linear combinations of $\hat{\rho}(k)$: Knoke's test, the variance-ratio test, and the long-horizon regression test. Knoke's test was suggested by Knoke (1977) and its test statistic is given by

$$
T = \sum_{k=1}^{n-1} \frac{\hat{\rho}(k)}{k}. \tag{46}
$$

Knoke's test is a normal approximation test that is based on the exact mean of T but the variance of T is obtained by simulation. With our results from Section 3, we are able to provide the exact variance of Knoke's statistic. In the following lemma, both the exact mean and variance of Knoke's statistic are presented.

Lemma 5. *Suppose $x = [x_1, \ldots, x_n]'$ is multivariate elliptically distributed with mean $\mu 1_n$ and variance-covariance matrix $\sigma^2 I_n$. The exact mean of Knoke's statistic is given by*

$$E[T] = \sum_{k=1}^{n-1} -\frac{n-k}{n(n-1)k} = \frac{1}{n} - \frac{1}{n-1}\sum_{k=1}^{n-1}\frac{1}{k} = \frac{1-H_n}{n-1}, \tag{47}$$

where $H_n = \sum_{k=1}^{n}\frac{1}{k}$ is the nth harmonic number. The exact variance of Knoke's statistic is given by

$$\begin{aligned} \mathrm{Var}[T] &= \sum_{j=1}^{n-1}\sum_{k=1}^{n-1}\frac{\mathrm{Cov}[\hat\rho(j),\hat\rho(k)]}{jk} \\ &= \frac{(n+2)n(n-1)H_n^{(2)} - 2n^2H_n^2 + (3n^2+3n-2)H_n - 2n(3n-2)}{(n+1)n(n-1)^2}, \end{aligned} \tag{48}$$

where $H_n^{(2)} = \sum_{k=1}^{n}\frac{1}{k^2}$.

For the variance-ratio and long-horizon regression tests, several asymptotic distributions are currently available in the literature, but the most popular test is still the normal approximation test due to its simplicity. If these normal approximation tests are accurate, we should expect the exact mean and variance of these test statistics to be close to the approximate mean and variance.

The variance-ratio test statistic is given by

$$\hat\theta(m) = 1 + 2\sum_{k=1}^{m-1}\left(1-\frac{k}{m}\right)\hat\rho(k), \tag{49}$$

where m is the length of the long-horizon returns chosen in the variance ratio test, and $2 \le m \le n$.[8] The most popular asymptotic distribution of $\hat\theta(m)$, suggested by Lo and

[8]Note that there are many ways of defining the sample variance ratio (see, e.g., Cochrane (1988), Lo and MacKinlay (1988), and Poterba and Summers (1988)). Nevertheless, they are all asymptotically equivalent under the null hypothesis.

Mackinlay (1988), is

$$\sqrt{n}(\hat{\theta}(m) - 1) \overset{A}{\sim} N\left(0, \frac{2(m-1)(2m-1)}{3m}\right), \tag{50}$$

which is derived under the assumption that m is fixed and $n \to \infty$.

By providing the exact mean and variance of $\hat{\theta}(m)$, we can evaluate how well the above asymptotic normal distribution approximates the exact distribution. The explicit expressions for the exact mean and variance of the variance ratio test statistic are given in the following lemma.

Lemma 6. *Suppose $x = [x_1, \ldots, x_n]'$ is multivariate elliptically distributed with mean $\mu 1_n$ and variance-covariance matrix $\sigma^2 I_n$. The exact mean and variance of the variance ratio test statistic are given by*

$$E[\hat{\theta}(m)] = 1 - 2\sum_{k=1}^{m-1}\left(1 - \frac{k}{m}\right)\frac{n-k}{n(n-1)} = 1 - \frac{(3n-m-1)(m-1)}{3n(n-1)}, \tag{51}$$

$$\begin{aligned}\mathrm{Var}[\hat{\theta}(m)] &= 4\sum_{j=1}^{m-1}\sum_{k=1}^{m-1}\left(1 - \frac{j}{m}\right)\left(1 - \frac{k}{m}\right)\mathrm{Cov}[\hat{\rho}(j), \hat{\rho}(k)] \\ &= \frac{(m-1)[(2m-1)(2n^2 + m(m+1)) - (7m^2 - m - 2)n]}{3m(n+1)(n-1)^2} \\ &\quad + \frac{2(m+1)(m-1)^2[(m+4)n - 2m - 2]}{9(n+1)n^2(n-1)^2} \\ &\quad + \frac{2(m-2)_4(2m-1) - (2m-2-n)_5^+}{15m^2(n-1)_3},\end{aligned} \tag{52}$$

where $(a)_r = a(a+1)\cdots(a+r-1)$ and $(a)_r^+ = \max[a, 0](a+1)\cdots(a+r-1)$.

According to (50), the approximate mean and variance of $\hat{\theta}(m)$ are 1 and $2(m-1)(2m-1)/(3mn)$, respectively. (51) suggests that the exact mean of $\hat{\theta}(m)$ is always less than its approximate mean of 1. In Figure 4, we plot the ratio of the approximate variance of $\hat{\theta}(m)$ to its exact variance as a function of m using the solid lines for four different sample sizes ($n = 60$, 240, 600, and 2400). We can see that the approximate variance can be much higher than the exact variance, especially when m is large relative to n. In addition, the variance-ratio test statistic is bounded from below.[9] Therefore, the normal approximation test based on the asymptotic distribution in (50) can have a serious size problem.

Figure 4 about here

The long-horizon regression test statistic is given by

$$\hat{\beta}(m) = \sum_{k=1}^{2m-1} \frac{\min[k, 2m-k]}{m} \hat{\rho}(k), \tag{54}$$

where m is the length of the long-horizon returns chosen, and $1 \leq m \leq n/2$. The conventional asymptotic distribution of $\hat{\beta}(m)$, provided by Richardson and Smith (1991), is

$$\sqrt{n}\hat{\beta}(m) \overset{A}{\sim} N\left(0, \frac{2m^2+1}{3m}\right), \tag{55}$$

which is derived under the assumption that m is fixed and $n \to \infty$. In Lemma 7, we provide explicit expressions for the exact mean and the exact variance of the long-horizon regression test statistic, which can help us to evaluate the approximate mean and variance provided by Richardson and Smith (1991).

Lemma 7. *Suppose $x = [x_1, \ldots, x_n]'$ is multivariate elliptically distributed with mean $\mu 1_n$ and variance-covariance matrix $\sigma^2 I_n$. The exact mean and variance of the long-horizon regression test statistic are given by*

$$\begin{aligned} E[\hat{\beta}(m)] &= -\sum_{k=1}^{2m-1} \frac{\min[k, 2m-k]}{m} \frac{n-k}{n(n-1)} = -\frac{m(n-m)}{n(n-1)}, \\ \mathrm{Var}[\hat{\beta}(m)] &= \sum_{j=1}^{2m-1}\sum_{k=1}^{2m-1} \frac{\min[j, 2m-j]\min[k, 2m-k]}{m^2} \mathrm{Cov}[\hat{\rho}(j), \hat{\rho}(k)] \\ &= \frac{(2m^2+1)(n-m)}{3m(n^2-1)} - \frac{2m^2(n-m)[n+(n-2)m]}{(n-1)^2 n^2 (n+1)} \end{aligned} \tag{56}$$

[9] It can be shown that the variance-ratio test statistic can be expressed as

$$\hat{\theta}(m) = \frac{1_m' \hat{\Gamma} 1_m}{m}, \tag{53}$$

where $\hat{\Gamma} = (\hat{\gamma}_{ij})$ is an $m \times m$ sample autocorrelation matrix with $\hat{\gamma}_{ij} = \hat{\rho}(|i-j|)$. It is well known that the sample autocorrelation matrix is positive semi-definite (see e.g., McLeod and Jimenez (1984)); it follows that $\hat{\theta}(m) \geq 0$.

$$+\frac{(m^2-1)(7m^2+2)-30m^3(n-2m)}{15m(n-1)_3}$$
$$+\frac{4(3m-n-2)_5^+-(4m-n-2)_5^+}{60m^2(n-1)_3}. \tag{57}$$

According to (55), the approximate mean and variance of $\hat{\beta}(m)$ are 0 and $(2m^2+1)/(3mn)$, respectively. From (56), it is easy to see that the exact mean of $\hat{\beta}(m)$ is always less than the approximate mean of 0. In Figure 4, we also plot the ratio of approximate variance to exact variance of $\hat{\beta}(m)$ as a function of m using the dashed lines for four different sample sizes (n = 60, 240, 600, and 2400). Just like the variance-ratio test statistic, the approximate variance of $\hat{\beta}(m)$ can be much higher than its exact variance, especially when m is large relative to n. Therefore, the normal approximation test based on the asymptotic distribution in (55) can also have serious size problem.

4.2. Evaluation of approximate tests based on a linear combination of $\hat{\rho}(k)^2$

In this subsection, we examine the finite sample property of two well-known portmanteau statistics, Box-Pierce Q-statistic and Ljung-Box Q-statistic. These two statistics are constructed based on linear combinations of $\hat{\rho}(k)^2$ and are given by

$$Q_{BP} = n\sum_{k=1}^{m}\hat{\rho}(k)^2, \tag{58}$$
$$Q_{LB} = n(n+2)\sum_{k=1}^{m}\frac{\hat{\rho}(k)^2}{n-k}. \tag{59}$$

In practice, the distributions of Q_{BP} and Q_{LB} under the null hypothesis of uncorrelatedness are approximated by the distribution of χ^2_m. Therefore, if these two test statistics are well behaved, they should have mean close to m and variance close to $2m$.

Using the explicit expressions for the exact moments of $\hat{\rho}(k)$, we can readily obtain the analytical expressions for the exact mean of the two Q-statistics, which are provided in the following lemma.

Lemma 8. *Suppose $x=[x_1,\dots,x_n]'$ is multivariate elliptically distributed with mean $\mu 1_n$ and variance-covariance matrix $\sigma^2 I_n$. Let $h=(m-\lfloor n/2\rfloor)^+$. The exact mean of Q_{BP} and*

Q_{LB} are given by[10]

$$
\begin{aligned}
E[Q_{BP}] &= \frac{m[2n^3-(m+3)n^2+(m+1)(2m+1)]-4nh(2m-n+1-h)}{2n(n^2-1)}, &(60)\\
E[Q_{LB}] &= \frac{(n+2)\left\{m[2n(n-3)-3(m+1)]+4n[2h+n(H_{n-1}-H_{n-m+h-1})]\right\}}{2n(n^2-1)}, &(61)
\end{aligned}
$$

where $H_n=\sum_{k=1}^{n}\frac{1}{k}$ is the nth harmonic number.

Using (41) and (45), it is easy to numerically compute $\mathrm{Var}[Q_{BP}]$ and $\mathrm{Var}[Q_{LB}]$. However, analytical expressions of $\mathrm{Var}[Q_{BP}]$ and $\mathrm{Var}[Q_{LB}]$ are much harder to obtain. After laborious algebra, we can obtain the following analytical expression for $E[Q_{BP}^2]$:

$$
\begin{aligned}
E[Q_{BP}^2] &= n^2\sum_{k=1}^{m}E[\hat\rho(k)^4]+2n^2\sum_{j=1}^{m-1}\sum_{k=j+1}^{m}E[\hat\rho(j)^2\hat\rho(k)^2]\\
&= \frac{1}{(n^2-1)(n+3)(n+5)}\left\{\frac{m}{12n^2}\left[24n(-2-2n-n^2+3n^3-2n^4+n^5)\right.\right.\\
&\quad+(35+114n^2+60n^3-261n^4+84n^5+12n^6)m\\
&\quad-6(-35-80n-70n^2-14n^3+17n^4+2n^5)m^2\\
&\quad\left.+\ (455+720n+282n^2+24n^3+3n^4)m^3-12(-35-24n+n^2)m^4+140m^5\right]\\
&\quad+[4n(n-m)(n-m-1)+6m-2n+3]\zeta_m+4nq_4(q_4+1)(4q_4-12m+3n-4)\\
&\quad+4q_3(q_3+1)[2q_3(q_3+1)-(3m-n+1)(3m-n+2)-1]\\
&\quad+\frac{q_2}{6n}\Big(-(2m-n+2)\left[(24-84n-150n^2+59n^3+16n^4)\right.\\
&\quad\left.-\ 2m(-63-54n+63n^2+10n^3)-6m^2(-39-18n+n^2)+156m^3\right]\\
&\quad+\zeta_n\left[48-27n-111n^2+43n^3+16n^4-2m(-87-111n+63n^2+10n^3)\right.\\
&\quad\left.\left.-\ 6m^2(-47-18n+n^2)+156m^3\right]\Big)\right\}, \qquad (62)
\end{aligned}
$$

where ζ_x equals 1 if x is odd and 0 otherwise, $q_2=\lfloor(2m-n+1)^+/2\rfloor$, $q_3=\lfloor(3m-n+1)^+/2\rfloor$, and $q_4=\lfloor(4m-n+2)^+/2\rfloor$. In order to conserve space, we do not provide the proof of this expression but its proof is available upon request. With the analytical expressions of $E[Q_{BP}]$ and $E[Q_{BP}^2]$ available, we can easily obtain an analytical expression of $\mathrm{Var}[Q_{BP}]$.

With the ability to compute the exact mean and variance of the two portmanteau statistics, we can now evaluate how well the asymptotic χ^2_m distribution approximates the exact

[10] Dufour and Roy (1986) provide the analytical expression of $E[Q_{BP}]$ for $m\le n/2$.

distributions of Q_{BP} and Q_{LB}. In Figure 5, we plot the exact mean of the two Q-statistics as a function of m for four different sample sizes ($n = 60$, 240, 600, and 2400). From Figure 5, it can be seen that the exact mean of Q_{LB} is very close to the approximate mean of m. However, the exact mean of the Q_{BP} deviates significantly from m except when m is very small. The deviation increases as m increases. In Figure 6, we plot the exact variance of the two Q-statistics as a function of m for the four different sample sizes. It can be seen that the approximate variance of $2m$ does not provide a good approximation of the exact variance for either one of the two Q-statistics. Even for moderately small values of m, the exact variance of the Ljung-Box Q-statistic exceeds the approximate variance significantly. As for the Box-Pierce Q-statistic, its exact variance is close to the approximate variance when m is small to moderate relative to n. However, when m is large relative to n, $\mathrm{Var}[Q_{BP}]$ is significantly below the approximate variance. Figures 5 and 6 confirm the results from previous studies (e.g., Davies, Triggs, and Newbold (1977), and Dufour and Roy (1986)) which find that the Box-Pierce test suffers a location bias, and that the variance of the Ljung-Box statistic can be much larger than the asymptotic variance even though its mean is close to the approximate mean.

Figure 5 about here

Figure 6 about here

Since the exact mean and variance of the two Q-statistics are not close to the mean and variance of the asymptotic χ^2_m distribution, there can be a potential problem with the size of these two tests. We use Monte-Carlo simulation to evaluate the size property of Q_{BP} and Q_{LB}. For each of the four different sample sizes ($n = 60$, 240, 600, and 2400), we generate a time series of length n independently from a standard normal distribution and compute Q_{BP} and Q_{LB}. We repeat this experiment 1,000,000 times and examine the actual probabilities of rejection of the two Q-statistics based on the χ^2_m approximation. The actual sizes of the two tests are presented in Figures 7a (for a nominal size of 10%) and 7b (for a nominal size of 5%). The dashed lines represent the actual size of the Box-Pierce test, and the solid lines represent the actual size of the Ljung-Box test. Figures 7a and 7b show that both the

Box-Pierce and the Ljung-Box tests have unsatisfactory size properties: the Box-Pierce test leads to under-rejection of the null hypothesis and the Ljung-Box test leads to over-rejection of the null hypothesis. These findings agree with the results of some earlier studies (e.g., Davies, Triggs, and Newbold (1977), and Ansley and Newbold (1979)) and suggest that these two tests can be quite unreliable even when m is moderately large relative to n.

Figure 7a about here

Figure 7b about here

4.3. Modified portmanteau tests

Since there are serious size problems with the Box-Pierce and Ljung-Box tests, there have been some attempts in the literature to propose modified portmanteau tests that aim at correcting this size problem. One of the earlier attempts is by Dufour and Roy (1986). They suggest the test statistic

$$Q_{DR} = \sum_{k=1}^{m} \frac{(\hat{\rho}(k) - E[\hat{\rho}(k)])^2}{\mathrm{Var}[\hat{\rho}(k)]} \overset{A}{\sim} \chi_m^2, \tag{63}$$

where $E[\hat{\rho}(k)]$ and $\mathrm{Var}[\hat{\rho}(k)]$ are the exact mean and variance of $\hat{\rho}(k)$ based on (38) and (39). Note that unlike Q_{BP} and Q_{LB}, we have $E[Q_{DR}] = m$, so Q_{DR} may be closer to the χ_m^2 distribution.

Another attempt is by Kwan and Sim (1996) who propose the following test statistic

$$Q_{KS} = \sum_{k=1}^{m} (n - k - 3) z_k^2, \tag{64}$$

where

$$z_k = \frac{1}{2} \ln \left(\frac{1 + \hat{\rho}(k)}{1 - \hat{\rho}(k)} \right) \tag{65}$$

is Fisher's transformation of $\hat{\rho}(k)$ and z_k is supposed to be closer to a normal distribution than $\hat{\rho}(k)$. In addition, their test is performed by comparing Q_{KS} with χ_α^2, where α is the approximate mean of Q_{KS}, and it is given by[11]

$$\alpha = \sum_{k=1}^{m} (n-k-3) \left(E[\hat{\rho}(k)^2] + \frac{2}{3} E[\hat{\rho}(k)^4] \right). \tag{66}$$

Note that Dufour and Roy (1986) and Kwan and Sim (1996) also propose other more complicated versions of their modified portmanteau tests. We have also analyzed the other modified portmanteau tests and they perform similarly to the tests reported in here. In order to conserve space, we do not report the size analysis of the other modified portmanteau tests but the results are available upon request.

We now propose our version of a modified portmanteau test, which we call the adjusted Box-Pierce test. It is defined as

$$Q_{BP}^a = m + \sqrt{\frac{2m}{\mathrm{Var}[Q_{BP}]}} (Q_{BP} - E[Q_{BP}]) \overset{A}{\sim} \chi_m^2. \tag{67}$$

Since we have the analytical expressions of $E[Q_{BP}]$ and $\mathrm{Var}[Q_{BP}]$, the adjusted Box-Pierce test statistic can be computed with minimal additional effort.[12] Note that with this adjustment, we have $E[Q_{BP}^a] = m$ and $\mathrm{Var}[Q_{BP}^a] = 2m$, so the first two moments of Q_{BP}^a are exactly the same as those from the χ_m^2 distribution.

In Figures 8a and 8b, we report the actual probabilities of rejection for the three modified portmanteau tests using the same simulation experiment as in Figures 7a and 7b. The nominal size of the test is set to be 10% in Figures 8a and 5% in Figures 8b. The dashed lines represent the actual probabilities of rejection for Q_{DR}. The figures show that there is still an over-rejection problem with Q_{DR}, and this over-rejection problem is also documented by Dufour and Roy (1986) and Kwan and Sim (1996). The dotted lines represent the actual probabilities of rejection for Q_{KS}. Even though there is still an over-rejection problem, it shows a remarkable improvement over Q_{BP}, Q_{LB} and Q_{DR}. Finally, the solid lines represent the actual probabilities of rejection for our adjusted Box-Pierce test. Despite the simplicity of the adjusted Box-Pierce test, its size is almost correct and compares quite favorably with

[11]Instead of using (39) and (41), Kwan and Sim (1996) use approximate formulas from Davies, Triggs, and Newbold (1977) to obtain $E[\hat{\rho}(k)^2]$ and $E[\hat{\rho}(k)^4]$, who assume that the mean of the data is known.

[12]We can similarly define an adjusted Ljung-Box test. Simulation evidence shows that both adjusted tests have similar size properties (results are available upon request). We prefer the adjusted Box-Pierce test because it is easier to compute.

the other modified portmanteau tests. Therefore, we believe the adjusted Box-Pierce test statistic is a better choice than the modified portmanteau tests by Dufour and Roy (1986) and Kwan and Sim (1996).

Figure 8a about here

Figure 8b about here

5. Conclusions

Sample autocorrelation coefficients are widely used for testing randomness. Though it is common knowledge that the asymptotic test is unsatisfactory, exact tests based on sample autocorrelation coefficients are largely unavailable in the current literature due to the lack of an efficient approach to computing the exact distribution of the sample autocorrelation coefficients. The main obstacle is in obtaining the eigenvalues for an $n \times n$ symmetric matrix that characterizes a given autocorrelation coefficient. In this paper, we provide an efficient algorithm for obtaining the eigenvalues for this matrix. Under the assumption of a multivariate elliptical distribution, we provide an efficient numerical algorithm for evaluating the cumulative density function of the sample autocorrelation coefficients, as well as explicit expressions for the exact moments and joint moments of the sample autocorrelation coefficients. This enables us to evaluate the size property of the asymptotic normal test and the normal approximation test based on the exact mean and variance. We find that the asymptotic test has a serious under-rejection problem and that the normal approximation test is a remarkable improvement over the asymptotic test for certain popular significance levels. However, there can be cases in which the normal approximation test does not work well, especially in the extreme end of the two tails. With our efficient algorithm, exact tests are now practical to conduct.

In addition, we provide explicit expressions for the exact mean and variance of various autocorrelation-based test statistics to help evaluate these tests. Actual size properties of the Box-Pierce and Ljung-Box tests are investigated, and they are shown to have poor size properties when the number of lags is relative large to the sample size. A simple adjusted

Box-Pierce test based on the exact mean and variance of the traditional Box-Pierce test statistic is proposed, and it is shown to have a far superior size property than the traditional Box-Pierce and Ljung-Box tests as well as other modified portmanteau tests proposed in the literature.

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Appendix A

Proof of Proposition 1. Let T_m be an $m \times m$ symmetric tridiagonal Toeplitz matrix with its first row as $[0,\ \frac{1}{2},\ 0'_{m-2}]$. It is well known that $\cos(\theta)$ and Q_b are the eigenvalues and eigenvectors of T_m (see e.g., Trench (1985)), so we have $T_m = Q_b \Lambda_b Q_b'$, where $\Lambda_b = \mathrm{Diag}(\cos(\theta))$. Similarly, we have $T_{m+1} = Q_a \Lambda_a Q_a'$, where $\Lambda_a = \mathrm{Diag}(\cos(\phi))$. Denoting $\Lambda = \mathrm{Diag}(\lambda)$ and $e_m = [0'_{m-1}, \frac{1}{2}]'$, we have

$$\begin{aligned}
& Q\Lambda Q' \\
= \;& [I_n,\ 0_{n\times(k-l)}]\left(Q_a\Lambda_a Q_a' \otimes E_L E_L' + \begin{bmatrix} Q_b \\ 0'_m \end{bmatrix} \Lambda_b [Q_b',\ 0_m] \otimes E_R E_R'\right)[I_n,\ 0_{n\times(k-l)}]' \\
= \;& [I_n,\ 0_{n\times(k-l)}]\left(T_{m+1} \otimes E_L E_L' + \begin{bmatrix} T_m & 0_m \\ 0'_m & 0 \end{bmatrix} \otimes E_R E_R'\right)[I_n,\ 0_{n\times(k-l)}]' \\
= \;& [I_n,\ 0_{n\times(k-l)}]\left(T_{m+1} \otimes E_L E_L' + T_{m+1} \otimes E_R E_R' - \begin{bmatrix} 0_{m\times m} & e_m \\ e'_m & 0 \end{bmatrix} \otimes E_R E_R'\right)[I_n,\ 0_{n\times(k-l)}]' \\
= \;& [I_n,\ 0_{n\times(k-l)}](T_{m+1} \otimes I_k)[I_n,\ 0_{n\times(k-l)}]' \\
= \;& A_k.
\end{aligned} \tag{68}$$

The penultimate equality follows because $E_L E_L' + E_R E_R' = I_k$ and the matrix

$$\begin{bmatrix} 0_{m\times m} & e_m \\ e'_m & 0 \end{bmatrix} \otimes E_R E_R' = \begin{bmatrix} 0_{mk\times mk} & e_m \otimes E_R E_R' \\ e'_m \otimes E_R E_R' & 0_{k\times k} \end{bmatrix} \tag{69}$$

has only nonzero elements in its last $k-l$ rows and columns. The last equality in (68) follows because $T_{m+1} \otimes I_k$ is an $(m+1)k \times (m+1)k$ symmetric matrix with entries of $1/2$ in its kth superdiagonal and sub-diagonal, and zero otherwise. It remains to show that $QQ' = I_n$ and $Q'Q = I_n$. Let

$$Q_* = \left[Q_a \otimes E_L,\ \begin{bmatrix} Q_b \\ 0'_m \end{bmatrix} \otimes E_R\right]. \tag{70}$$

Using the fact that $E_L' E_L = I_l$, $E_R' E_R = I_{k-l}$, $E_L' E_R = 0_{l\times(k-l)}$, $Q_a Q_a' = Q_a' Q_a = I_{m+1}$, $Q_b Q_b' = Q_b' Q_b = I_m$, we have

$$Q_*' Q_* = \begin{bmatrix} I_{m+1} \otimes I_l & 0_{(m+1)l\times m(k-l)} \\ 0_{m(k-l)\times(m+1)l} & I_m \otimes I_{k-l} \end{bmatrix} = I_n \tag{71}$$

and

$$Q_*Q_*' = I_{m+1}\otimes E_LE_L'+\left(I_{m+1}-\begin{bmatrix} 0_{m\times m} & 0_m \\ 0_m' & 1 \end{bmatrix}\right)\otimes E_RE_R' = \begin{bmatrix} I_n & 0_{n\times(k-l)} \\ 0_{(k-l)\times n} & 0_{(k-l)\times(k-l)} \end{bmatrix}. \quad (72)$$

Since $Q = [I_n,\ 0_{n\times(k-l)}]Q^*$, we have $QQ' = I_n$ and

$$Q'Q = Q_*'[I_n,\ 0_{n\times(k-l)}]'[I_n,\ 0_{n\times(k-l)}]Q_* = Q_*'Q_*Q_*'Q_* = I_n. \quad (73)$$

This completes the proof.

Proof of Lemma 1. Writing

$$\begin{aligned} B_k &= P_n'Q\Lambda Q'P_n = P_n'Q(I_n+\Lambda)Q'P_n - P_n'Q'QP_n = P_n'Q(I_n+\Lambda)Q'P_n - I_{n-1}, \quad (74) \\ \Lambda - \frac{1}{n}\tilde{q}\tilde{q}' &= (I_n+\Lambda) - \frac{1}{n}(I_n+\Lambda)^{\frac{1}{2}}Q'1_n1_n'Q(I_n+\Lambda)^{\frac{1}{2}} - I_n \\ &= (I_n+\Lambda)^{\frac{1}{2}}Q'M_nQ(I_n+\Lambda)^{\frac{1}{2}} - I_n \\ &= (I_n+\Lambda)^{\frac{1}{2}}Q'P_nP_n'Q(I_n+\Lambda)^{\frac{1}{2}} - I_n, \quad (75) \end{aligned}$$

we can see that the eigenvalues of B_k are equal to the eigenvalues of $P_n'Q(I_n+\Lambda)Q'P_n$ minus one, and the eigenvalues of $\Lambda-\frac{1}{n}\tilde{q}\tilde{q}'$ are equal to the eigenvalues of $(I_n+\Lambda)^{\frac{1}{2}}Q'P_nP_n'Q(I_n+\Lambda)^{\frac{1}{2}}$ minus one. Since AB and BA share the same nonzero eigenvalues, $(I_n+\Lambda)^{\frac{1}{2}}Q'P_nP_n'Q(I_n+\Lambda)^{\frac{1}{2}}$ and $P_n'Q(I_n+\Lambda)Q'P_n$ share the same eigenvalues except that $(I_n+\Lambda)^{\frac{1}{2}}Q'P_nP_n'Q(I_n+\Lambda)^{\frac{1}{2}}$ has one more eigenvalue of zero. Therefore, B_k and $\Lambda - \frac{1}{n}\tilde{q}\tilde{q}'$ share the same eigenvalues except that $\Lambda - \frac{1}{n}\tilde{q}\tilde{q}'$ has one more eigenvalue of -1. It remains to show that the eigenvalues of B_k are greater than -1. This follows because the eigenvalues of A_k are all greater than -1, and by the Poincaré separation theorem (see, e.g., Magnus and Neudecker (1999, p.209)), the $n-1$ eigenvalues of $B_k = P_n'A_kP_n$ must lie between the n eigenvalues of A_k, and as a result the eigenvalues of B_k must also be greater than -1. This completes the proof.

Proof of Proposition 2. Let $q_a = Q_a'1_{m+1}$ and $q_b = Q_b'1_m$. It is straightforward to show that the elements of q_a and q_b are given by

$$q_{a,i} = \begin{cases} \left(\frac{2}{m+2}\right)^{\frac{1}{2}}\cot(\phi_i/2) & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even,} \end{cases} \quad (76)$$

$$q_{b,i} = \begin{cases} \left(\frac{2}{m+1}\right)^{\frac{1}{2}}\cot(\theta_i/2) & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even.} \end{cases} \quad (77)$$

Since the last $k-l$ rows of Q_* in (70) are zero, we have

$$q = Q'1_n = Q_*'1_{(m+1)k} = \begin{bmatrix} Q_a'1_{m+1} \otimes E_L'1_k \\ Q_b'1_m \otimes E_R'1_k \end{bmatrix} = \begin{bmatrix} q_a \otimes 1_l \\ q_b \otimes 1_{k-l} \end{bmatrix}. \tag{78}$$

It follows that $\tilde{q} = (I_n + \Lambda)^{\frac{1}{2}} Q'1_n$ is given by

$$\tilde{q} = \begin{bmatrix} \tilde{q}_a \otimes 1_l \\ \tilde{q}_b \otimes 1_{k-l} \end{bmatrix}, \tag{79}$$

where

$$\tilde{q}_{a,i} = [1+\cos(\phi_i)]^{\frac{1}{2}} q_{a,i} = \sqrt{2}\cos(\phi_i/2) q_{a,i} = \begin{cases} \frac{2[\csc(\phi_i/2)-\sin(\phi_i/2)]}{\sqrt{m+2}} & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even,} \end{cases} \tag{80}$$

$$\tilde{q}_{b,i} = [1+\cos(\theta_i)]^{\frac{1}{2}} q_{b,i} = \sqrt{2}\cos(\theta_i/2) q_{b,i} = \begin{cases} \frac{2[\csc(\theta_i/2)-\sin(\theta_i/2)]}{\sqrt{m+1}} & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even.} \end{cases} \tag{81}$$

Note that for $\tilde{q}_i = 0$, the corresponding λ_i of A_k remains as an eigenvalue of B_k, so we just focus on the cases with nonzero $\tilde{q}_i$. Define $\tilde{q}_a^o$ and $\tilde{q}_b^o$ as the vectors of odd elements of $\tilde{q}_a$ and $\tilde{q}_b$, respectively. Let

$$\tilde{q}^o = \begin{bmatrix} \tilde{q}_a^o \otimes 1_l \\ \tilde{q}_b^o \otimes 1_{k-l} \end{bmatrix}, \qquad \lambda^o = \begin{bmatrix} \cos(\phi^o) \otimes 1_l \\ \cos(\theta^o) \otimes 1_{k-l} \end{bmatrix}, \qquad \lambda^e = \begin{bmatrix} \cos(\phi^e) \otimes 1_l \\ \cos(\theta^e) \otimes 1_{k-l} \end{bmatrix}, \tag{82}$$

and $\Lambda^o = \mathrm{Diag}(\lambda^o)$; the eigenvalues of B_k are given by λ^e and the eigenvalues of

$$\Lambda^o - \frac{1}{n}\tilde{q}^o \tilde{q}^{o\prime}. \tag{83}$$

As λ^o has many duplicate elements, it allows us to further deflate the problem. Consider the following orthonormal matrix

$$R = \begin{bmatrix} I_{\lfloor (m+2)/2 \rfloor} \otimes R_a & 0_{\lfloor (m+2)/2 \rfloor \times \lfloor (m+1)/2 \rfloor} \\ 0_{\lfloor (m+1)/2 \rfloor \times \lfloor (m+2)/2 \rfloor} & I_{\lfloor (m+1)/2 \rfloor} \otimes R_b \end{bmatrix}, \tag{84}$$

where $R_a = [l^{-\frac{1}{2}} 1_l,\ P_l]$ and $R_b = [(k-l)^{-\frac{1}{2}} 1_{k-l},\ P_{k-l}]$ and they are both orthonormal matrices. Using the fact that $R_a'1_l = [l^{\frac{1}{2}},\ 0_{l-1}']'$ and $R_b'1_{k-l} = [(k-l)^{\frac{1}{2}},\ 0_{k-l-1}']'$, we obtain

$$r \equiv R'\tilde{q}^o = \begin{bmatrix} \tilde{q}_a^o \otimes [l^{\frac{1}{2}},\ 0_{l-1}']' \\ \tilde{q}_b^o \otimes [(k-l)^{\frac{1}{2}},\ 0_{k-l-1}']' \end{bmatrix} \tag{85}$$

and it has only nonzero elements of $l^{\frac{1}{2}}\tilde{q}_a^o$ and $(k-l)^{\frac{1}{2}}\tilde{q}_b^o$. Note that $\Lambda^o - \frac{1}{n}\tilde{q}^o\tilde{q}^{o\prime}$ and $R'\Lambda^o R - \frac{1}{n}R'\tilde{q}^o\tilde{q}^{o\prime}R = \Lambda^o - \frac{1}{n}rr'$ share the same eigenvalues, so we can obtain the eigenvalues of $\Lambda^o - \frac{1}{n}rr'$ instead. For the zero elements of r, the corresponding elements of λ^o are also the eigenvalues of B_k, so $[\cos(\phi^o)' \otimes 1'_{l-1},\ \cos(\theta^o)' \otimes 1'_{k-l-1}]$ are eigenvalues of B_k. Eliminating the zero elements in r, the rest of the eigenvalues of B_k can be obtained as the eigenvalues of

$$\mathrm{Diag}([\cos(\phi^o)',\ \cos(\theta^o)']) - \frac{1}{n}\tilde{r}\tilde{r}', \tag{86}$$

where $\tilde{r} = [l^{\frac{1}{2}}\tilde{q}_a^{o\prime},\ (k-l)^{\frac{1}{2}}\tilde{q}_b^{o\prime}]'$ is the nonzero elements of $\tilde{r}$. This completes the proof.

Proof of Lemma 2. We only provide the proof for the case that $l > 0$. The proof for the case of $l = 0$ is similar. For even i, $g(x)$ strictly increases in x when $\cos\phi_{i+1} < x < \cos\theta_{i-1}$, so in order to show $\cos\theta_i > \xi_{bi}$, we need to prove that $g(\cos\theta_i) > 0$ for even i. Defining $h(j) = g(\cos\theta_{2j})$, we have

$$h'(j) = -g'(\cos\theta_{2j})\sin\theta_{2j}\frac{2\pi}{m+1} < 0. \tag{87}$$

The above inequality holds because $g'(\cos\theta_{2j}) > 0$ and $\sin\theta_{2j} > 0$ when $0 < \theta_{2j} < \pi$. This implies $g(\cos\theta_2) > g(\cos\theta_4) > \cdots > g(\cos\theta_{2p})$, where $p = \lfloor m/2 \rfloor$. Therefore, in order to show $g(\cos\theta_i) > 0$ for even i, it suffices to show $g(\cos\theta_{2p}) > 0$.

By defining

$$g_1(x) = \sum_{i=1}^{\lfloor (m+2)/2 \rfloor} \frac{\cot^2(\phi_{2i-1}/2)}{\cos\phi_{2i-1} - x}, \tag{88}$$

$$g_2(x) = \sum_{i=1}^{\lfloor (m+1)/2 \rfloor} \frac{\cot^2(\theta_{2i-1}/2)}{\cos\theta_{2i-1} - x}, \tag{89}$$

we can write

$$g(x) = \frac{2(1-m\delta)}{m+2}g_1(x) + \frac{2[(m+1)\delta - 1]}{m+1}g_2(x). \tag{90}$$

Note that $m\delta < 1$ and $(m+1)\delta > 1$, so if we can prove that $g_1(\cos\theta_{2p}) > 0$ and $g_2(\cos\theta_{2p}) > 0$, it implies $g(\cos\theta_{2p}) > 0$.

We first consider the case that m is even. When m is even, $2p = m$. Note that $g_2(\cos\theta_m) > 0$ because all of its terms are positive. For $g_1(\cos\theta_m)$, all of its terms are positive except for the last term. Keeping only the first and the last term, we have

$$\begin{aligned} g_1(\cos\theta_m) &\geq \frac{\cot^2(\phi_1/2)}{\cos\phi_1 - \cos\theta_m} + \frac{\cot^2(\phi_{m+1}/2)}{\cos\phi_{m+1} - \cos\theta_m} \\ &= \frac{\cot^2(\phi_1/2)}{\cos\phi_1 + \cos\theta_1} + \frac{\tan^2(\phi_1/2)}{-\cos\phi_1 + \cos\theta_1} \\ &= \frac{2\left[\cos^2\phi_1(2-\cos\theta_1) - \cos\theta_1\right]}{(\cos^2\phi_1 - \cos^2\theta_1)\sin^2\phi_1}. \end{aligned} \tag{91}$$

The denominator of the above expression is positive, so we just need to show that the numerator is also positive. Since both $\cos^2 x$ and $\cos x$ are concave function for $0 < x < \pi/2$, we have

$$\begin{aligned} \phi_1 &= \frac{1}{m+2} \times 0 + \frac{(m+1)}{m+2} \times \theta_1 \\ \Rightarrow \cos^2\phi_1 &\geq \frac{1}{m+2} + \frac{(m+1)\cos^2\theta_1}{m+2} = \frac{1+(m+1)\cos^2\theta_1}{m+2}, \end{aligned} \tag{92}$$

$$\begin{aligned} \theta_1 &= \frac{(m-2)}{m+1} \times 0 + \frac{3}{m+1} \times \frac{\pi}{3} \\ \Rightarrow \cos\theta_1 &\geq \frac{m-2}{m+1} + \frac{3}{2(m+1)} = \frac{2m-1}{2m+2} > \frac{m-1}{m+1}. \end{aligned} \tag{93}$$

Therefore,

$$\begin{aligned} \cos^2\phi_1(2-\cos\theta_1) - \cos\theta_1 &\geq \left[\frac{1+(m+1)\cos^2\theta_1}{m+2}\right](2-\cos\theta_1) - \cos\theta_1 \\ &= \frac{(1-\cos\theta_1)\left[2-(m+1)(\cos\theta_1 - \cos^2\theta_1)\right]}{m+2} \\ &\geq \frac{(1-\cos\theta_1)\left[2\cos\theta_1-(m+1)(\cos\theta_1 - \cos^2\theta_1)\right]}{m+2} \\ &= \frac{(1-\cos\theta_1)\cos\theta_1\left[(m+1)\cos\theta_1 - (m-1)\right]}{m+2} > 0. \end{aligned} \tag{94}$$

For the case that m is odd, $2p = m-1$. When $m = 1$, $\xi_{b1} = \delta - \frac{1}{2}$ (see Table 1) and it is between 0 and 1/2, so (21) holds. It remains to show that for $m \geq 3$, $g_1(\cos\theta_{m-1}) > 0$ and $g_2(\cos\theta_{m-1}) > 0$. It is trivial to see that all the terms in $g_2(\cos\theta_{m-1})$ are positive except for the last term, so keeping only the first and the last term, we have

$$\begin{aligned} g_2(\cos\theta_{m-1}) &\geq \frac{\cot^2(\theta_1/2)}{\cos\theta_1 - \cos\theta_{m-1}} + \frac{\cot^2(\theta_m/2)}{\cos\theta_m - \cos\theta_{m-1}} \\ &= \frac{\cot^2(\theta_1/2)}{\cos\theta_1 + \cos\theta_2} + \frac{\tan^2(\theta_1/2)}{-\cos\theta_1 + \cos\theta_2} \\ &= \frac{2(2+\cos\theta_2)}{\sin^2\theta_1(1+2\cos\theta_2)} > 0. \end{aligned} \tag{95}$$

Similarly, keeping only the first and the last term of $g_1(\cos\theta_{m-1})$, we have

$$\begin{aligned} g_1(\cos\theta_{m-1}) &\geq \frac{\cot^2(\phi_1/2)}{\cos\phi_1-\cos\theta_{m-1}}+\frac{\cot^2(\phi_m/2)}{\cos\phi_m-\cos\theta_{m-1}} \\ &= \frac{\cot^2(\phi_1/2)}{\cos\phi_1+\cos\theta_2}+\frac{\tan^2(\phi_1)}{-\cos\phi_2+\cos\theta_2} \\ &= \frac{[(1-\cos\phi_1)^3+2\cos^4\phi_1-2\cos^2\theta_1(1-2\cos\phi_1+2\cos^2\phi_1)]}{\tan^2(\phi_1/2)\cos^2\phi_1(\cos\phi_2-\cos\theta_2)(\cos\phi_1+\cos\theta_2)}. \end{aligned} \tag{96}$$

The denominator of the above expression is obviously positive, so we only need to show that its numerator is also positive. Denoting $u=\cos\phi_1$ and applying (92), we can write the numerator as

$$\begin{aligned} & (1-u)^3+2u^4-2\cos^2\theta_1(1-2u+2u^2) \\ \geq\ & (1-u)^3+2u^4-\frac{2[(m+2)u^2-1]}{m+1}(1-2u+2u^2) \\ =\ & \frac{(1-u^2)[m+3-(3m+7)u+(2m+6)u^2]}{m+1}. \end{aligned} \tag{97}$$

Let $q(x)=m+3-(3m+7)x+(2m+6)x^2$. When $0\leq m<3+4\sqrt{2}$, the quadratic equation $q(x)=0$ has no real roots and $q(u)>0$. For $m\geq 3+4\sqrt{2}$, we use (93) to show that

$$u>\cos\theta_1>\frac{2m-1}{2m+2}>\frac{3m+7+\sqrt{m^2-6m+9}}{4(m+3)}>\frac{3m+7+\sqrt{m^2-6m-23}}{4(m+3)} \tag{98}$$

and the last term is the larger root of $q(x)=0$. As $q(x)$ is an increasing function of x for x greater than the larger root of $q(x)=0$, we also have $q(u)>0$ when $m\geq 3+4\sqrt{2}$. This completes the proof.

Proof of Lemma 3. When $k=1$, $l=0$ and $m=n$. So, $\xi_a=[\cos(\theta^e)'\otimes 1_k']'$. From Lemma 2, we know that

$$\cos\left(\frac{2\pi}{n+1}\right) \geq \xi_b^{**} \geq \cos\left(\frac{3\pi}{n+1}\right), \tag{99}$$

$$\begin{aligned} \cos\left(\frac{(n-2)\pi}{n+1}\right) &\geq \xi_b^{*} \geq \cos\left(\frac{(n-1)\pi}{n+1}\right) && \text{if } n \text{ is even,} \\ \cos\left(\frac{(n-1)\pi}{n+1}\right) &\geq \xi_b^{*} \geq \cos\left(\frac{n\pi}{n+1}\right) && \text{if } n \text{ is odd.} \end{aligned} \tag{100}$$

Therefore,

$$\cos\left(\frac{2\pi}{n+1}\right) \geq \hat{\rho}(1) \geq -\cos\left(\frac{\pi}{n+1}\right) \qquad \text{if } n \text{ is even,} \tag{101}$$

$$\cos\left(\frac{2\pi}{n+1}\right) \geq \ \hat{\rho}(1) \ \geq \ \xi_b^* \qquad\qquad \text{if } n \text{ is odd.} \tag{102}$$

When $k > 1$ and $l = 0$, we have $\xi_a = \left[\cos(\theta^o)' \otimes 1'_{k-1}, \cos(\theta^e)' \otimes 1'_k\right]'$. Since it must be true that $\cos\theta_1 > \xi_{bi} > \cos\theta_m$, we have

$$\cos\left(\frac{\pi}{m+1}\right) \geq \hat{\rho}(k) \geq \cos\left(\frac{m}{m+1}\right) = -\cos\left(\frac{\pi}{m+1}\right). \tag{103}$$

When $k > 1$ and $l = 1$, we have $\xi_a = \left[\cos(\phi^e)', \cos(\theta^o)' \otimes 1'_{k-2}, \cos(\theta^e)' \otimes 1'_{k-1}\right]'$. And from Lemma 2, we have

$$\cos\left(\frac{\pi}{m+2}\right) \ > \ \xi_b^{**} \ > \ \cos\left(\frac{\pi}{m+1}\right), \tag{104}$$

$$\cos\left(\frac{m\pi}{m+1}\right) \ > \ \xi_b^* \ > \ \cos\left(\frac{(m+1)\pi}{m+2}\right) \quad \text{if } m \text{ is even,} \tag{105}$$

$$\cos\left(\frac{m\pi}{m+2}\right) \ > \ \xi_b^* \ > \ \cos\left(\frac{m\pi}{m+1}\right) \quad \text{if } m \text{ is odd.} \tag{106}$$

Therefore,

$$\xi_b^{**} \geq \ \hat{\rho}(k) \ \geq \ \xi_b^* \qquad\qquad \text{if } m \text{ is even,} \tag{107}$$

$$\xi_b^{**} \geq \ \hat{\rho}(k) \ \geq \ -\cos\left(\frac{\pi}{m+2}\right) \quad \text{if } m \text{ is odd.} \tag{108}$$

When $k > 1$ and $l > 1$, $\xi_a = \left[\cos(\phi^o)' \otimes 1'_{l-1}, \cos(\phi^e)' \otimes 1'_{l-1}, \cos(\theta^o)' \otimes 1'_{k-l-1}, \cos(\theta^e)' \otimes 1'_{k-l}\right]'$. Since it must be true that $\cos(\phi_1) > \xi_{bi} > \cos(\phi_{m+1})$, we have

$$\cos\left(\frac{\pi}{m+2}\right) \geq \hat{\rho}(k) \geq \cos\left(\frac{(m+1)\pi}{m+2}\right) = -\cos\left(\frac{\pi}{m+2}\right). \tag{109}$$

This completes the proof.

Proof of Propositions 3 and 4. It is well known that when y has a spherical distribution, $u = y/(y'y)^{\frac{1}{2}}$ and $y'y$ are independent of each other (see, e.g., Theorem 1.5.6 of Muirhead (1982)). Since the distribution of u is the same for all spherical distributions y, we can assume normality of y without loss of generality. Therefore,

$$E\left[\prod_{i=1}^{p}(y'B_iy)^{s_i}\right] \ = \ E\left[(y'y)^s \prod_{i=1}^{p}\left(\frac{y'B_{k_i}y}{y'y}\right)^{s_i}\right]$$

$$
\begin{aligned}
&= E\left[(y'y)^s \prod_{i=1}^{p} (u'B_{k_i}u)^{s_i}\right] \\
&= E[(y'y)^s]E\left[\prod_{i=1}^{p} (u'B_{k_i}u)^{s_i}\right] \\
&= (n-1)(n+1)\cdots(n-3+2s)E\left[\prod_{i=1}^{p} \left(\frac{y'B_{k_i}y}{y'y}\right)^{s_i}\right]. \qquad (110)
\end{aligned}
$$

The third equality follows from the independence of u and $y'y$. The last equality follows because $y'y \sim \chi^2_{n-1}$, so its sth moment is given by $(n-1)(n+1)\cdots(n-3+2s)$ (see, for example, Johnson, Kotz, and Balakrishnan (1995, Eq. 18.8)). The recursive relation in (36) is based on a recursive relation between moments and cumulants and can be found in Mathai and Provost (1992, Eq. 3.2b.8). Finally, (43) is obtained from Proposition 4 of Kan (2008). This completes the proof.

Proof of Lemma 4. The first equalities of $E[\hat{\rho}(j)\hat{\rho}(k)]$ and $E[\hat{\rho}(j)^2\hat{\rho}(k)^2]$ follow from our Proposition 4 and Lemma 6.2 of Magnus (1978). To obtain the second equalities, we need to have expressions of all the traces. The expressions of a_j and a_{jk} are easy to obtain and they are given by

$$
\begin{aligned}
a_j &= -\frac{n-j}{n}, \qquad (111) \\
a_{jk} &= -\frac{j(n-k)}{n^2} - \frac{(n-j-k)^+}{n} + \frac{(n-k)\delta_{j,k}}{2}, \qquad j \leq k. \qquad (112)
\end{aligned}
$$

For $j < k$, the other traces are given by

$$
\begin{aligned}
a_{jjk} &= -\frac{(n-k)(n^2-2jn+4j^2)}{4n^3} + \frac{(n-k)(n-2j)^+}{2n^2} - \frac{j(n-j-k)^+}{n^2} \\
&\quad - \frac{3(n-2j-k)^+}{4n} - \frac{(n-\max[2j,k])^+}{2n} + \frac{(n-k)\delta_{2j,k}}{4}, \qquad (113) \\
a_{jkk} &= \frac{(n-k)(n-2k)j}{2n^3} - \frac{j(n-2k)^+}{2n^2} - \frac{k(n-j-k)^+}{n^2} \\
&\quad - \frac{3(n-j-2k)^+ + (n+j-2k)^+}{4n}, \qquad (114) \\
a_{jjkk} &= -\frac{(n-k)(8j^2k-4j^2n-4jkn+2jn^2+2kn^2-n^4)}{8n^4} \\
&\quad - \frac{(4k^2-6kn+2n^2)(n-2j)^+}{8n^3} - \frac{(8jk-2jn+2kn-n^3)(n-j-k)^+}{8n^3}
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{(4j^2+2jn+3n^2)(n-2k)^+}{8n^3} - \frac{k(n-2j-k)^+ + j(n-j-2k)^+}{2n^2} \\
& - \frac{(n-2j-2k)^+}{2n} - \frac{(n+2j-2\max[2j,k])^+}{8n} \\
& - \frac{(2k-n)(n-\max[2j,k])^+}{4n^2}, \qquad (115)
\end{aligned}
$$

$$
\begin{aligned}
a_{jkjk} \;=\; & \frac{j^2(n-k)^2}{n^4} - \frac{[8jk+2n(k-j)-n^3](n-j-k)^+}{4n^3} - \frac{k(n-2j-k)^+}{2n^2} \\
& - \frac{j(n+j-2k)^+ + j(n-j-2k)^+}{2n^2} - \frac{(n-2k)^+ + (n-2j-2k)^+}{2n}. \qquad (116)
\end{aligned}
$$

The proof of the above expressions are available upon request. Using these expressions and upon simplification, we obtain our explicit expressions of $E[\hat{\rho}(j)\hat{\rho}(k)]$ and $E[\hat{\rho}(j)^2\hat{\rho}(k)^2]$. This completes the proof.

Proof of Lemma 5. By substituting in the expression of the exact mean of $\hat{\rho}(k)$ which is given in (38), the expression for the exact mean of Knoke's statistic is straightforward. To obtain the explicit expression for the exact variance, we first need to get the explicit expression for $\mathrm{Var}[\hat{\rho}(k)]$ and $\mathrm{Cov}[\hat{\rho}(j),\hat{\rho}(k)]$, which can be derived from (38), (39) and (44). The expressions are given by

$$
\mathrm{Var}[\hat{\rho}(k)] = \frac{(n-2)(n-k)(n^2+n-2k)}{(n+1)n^2(n-1)^2} - \frac{2(n-2k)^+}{n(n^2-1)}, \qquad (117)
$$

$$
\mathrm{Cov}[\hat{\rho}(j),\hat{\rho}(k)] = -\frac{2(n-k)[n+(n-2)j]}{(n+1)n^2(n-1)^2} - \frac{2(n-j-k)^+}{n(n^2-1)} \quad \text{for } j<k. \qquad (118)
$$

In addition, we need two more identities to obtain $\mathrm{Var}[T]$,

$$
2\sum_{j=1}^{n-2}\sum_{k=j+1}^{n-1}\frac{1}{jk} = H_{n-1}^2 - H_{n-1}^{(2)}, \qquad (119)
$$

$$
\sum_{j=1}^{n-2}\sum_{k=1}^{n-1-j}\frac{n-j-k}{jk} = n[(H_n-1)^2-(H_n^{(2)}-1)]. \qquad (120)
$$

Given the above expressions, we can write

$$
\begin{aligned}
\mathrm{Var}[T] &= \sum_{k=1}^{n-1}\frac{\mathrm{Var}[\hat{\rho}(k)]}{k^2} + 2\sum_{j=1}^{n-2}\sum_{k=j+1}^{n-1}\frac{\mathrm{Cov}[\hat{\rho}(j),\hat{\rho}(k)]}{jk} \\
&= \frac{(n-2)\sum_{k=1}^{n-1}\frac{(n-k)(n^2+n-2k)}{k^2} - 4\sum_{j=1}^{n-2}\sum_{k=j+1}^{n-1}\frac{(n-k)[n+(n-2)j]}{jk}}{(n+1)n^2(n-1)^2}
\end{aligned}
$$

$$
\begin{aligned}
& \quad - \frac{2\sum_{j=1}^{n-2}\sum_{k=1}^{n-1-j}\frac{n-j-k}{jk}}{n(n^2-1)} \\
&= \frac{(n-2)[2n-n(n+3)H_n+n^2(n+1)H_n^{(2)}]}{(n+1)n^2(n-1)^2} \\
& \quad - \frac{2n[n(H_n^2-H_n^{(2)})-4(n-1)H_n+(n+2)(n-1)]}{(n+1)n^2(n-1)^2} \\
& \quad - \frac{2n[(H_n-1)^2-(H_n^{(2)}-1)]}{n(n^2-1)} \\
&= \frac{(n+2)n(n-1)H_n^{(2)}-2n^2H_n^2+(3n^2+3n-2)H_n-2n(3n-2)}{(n+1)n(n-1)^2}. \qquad (121)
\end{aligned}
$$

This completes the proof.

Proof of Lemma 6. Using the analytical expression of $E[\hat{\rho}(k)]$ in (38), it is straightforward to obtain the analytical expression of $E[\hat{\theta}(m)]$. For $\mathrm{Var}[\hat{\theta}(m)]$, we use (117) and (118) to obtain

$$
\begin{aligned}
\mathrm{Var}[\hat{\theta}(m)] &= 4\sum_{k=1}^{m-1}\left(1-\frac{k}{m}\right)^2 \mathrm{Var}[\hat{\rho}(k)] \\
& \quad + 8\sum_{j=1}^{m-2}\sum_{k=j+1}^{m-1}\left(1-\frac{j}{m}\right)\left(1-\frac{k}{m}\right)\mathrm{Cov}[\hat{\rho}(j),\hat{\rho}(k)] \\
&= 4\sum_{k=1}^{m-1}\left(1-\frac{k}{m}\right)^2\left(\frac{(n-2)(n-k)(n^2+n-2k)}{(n+1)n^2(n-1)^2}\right) + \\
& \quad + 8\sum_{j=1}^{m-2}\sum_{k=j+1}^{m-1}\left(1-\frac{j}{m}\right)\left(1-\frac{k}{m}\right)\left(-\frac{2(n-k)[n+(n-2)j]}{(n+1)n^2(n-1)^2}\right) \\
& \quad + 4\sum_{j=1}^{m-1}\sum_{k=1}^{m-1}\left(1-\frac{j}{m}\right)\left(1-\frac{k}{m}\right)\left(-\frac{2(n-j-k)^+}{n(n^2-1)}\right). \qquad (122)
\end{aligned}
$$

The analytical expressions for the first two terms can be easily obtained. For the last term, we obtain separate analytical expressions for $n \geq 2(m-1)$ and for $n < 2(m-1)$. Then combining these expressions and after simplification, we obtain (52). This completes the proof.

Proof of Lemma 7. Using the analytical expression of $E[\hat{\rho}(k)]$ in (38), it is straightforward to obtain the analytical expression of $E[\hat{\theta}(m)]$. The exact variance of $\hat{\beta}(m)$ can be derived as following

$$\begin{aligned}
\mathrm{Var}[\hat{\beta}(m)] &= \sum_{j=1}^{2m-1}\sum_{k=1}^{2m-1}\frac{\min[j,2m-j]\min[k,2m-k]}{m^2}\mathrm{Cov}[\hat{\rho}(j),\hat{\rho}(k)] \\
&= \sum_{j=1}^{m}\sum_{k=1}^{m}\frac{jk}{m^2}\mathrm{Cov}[\hat{\rho}(j),\hat{\rho}(k)] + 2\sum_{j=1}^{m}\sum_{k=m+1}^{2m-1}\frac{j(2m-k)}{m^2}\mathrm{Cov}[\hat{\rho}(j),\hat{\rho}(k)] \\
&\quad + \sum_{j=m+1}^{2m-1}\sum_{k=m+1}^{2m-1}\frac{(2m-j)(2m-k)}{m^2}\mathrm{Cov}[\hat{\rho}(j),\hat{\rho}(k)]. \qquad (123)
\end{aligned}$$

Then using (117) and (118), we can obtain analytical expression of $\mathrm{Var}[\hat{\beta}(m)]$ for three different cases: (1) $n \geq 4m-2$, (2) $3m-1 \leq n < 4m-2$, and (3) $n < 3m-1$. By combining these three analytical expressions of $\mathrm{Var}[\hat{\beta}(m)]$ and after simplification, we obtain (57). This completes the proof.

Proof of Lemma 8. Using (39), we obtain

$$\begin{aligned}
E[Q_{BP}] &= \sum_{k=1}^{m}\frac{(n-k)(n^2+n-3k)}{n(n^2-1)} - \sum_{k=1}^{m-h}\frac{2(n-2k)}{n^2-1} \\
&= \frac{m[2n^3-(m+3)n^2+(m+1)(2m+1)]-4nh(2m-n+1-h)}{2n(n^2-1)}, \qquad (124) \\
E[Q_{LB}] &= (n+2)\sum_{k=1}^{m}\frac{(n^2+n-3k)}{n(n^2-1)} - (n+2)\sum_{k=1}^{m-h}\frac{2(n-2k)}{(n^2-1)(n-k)} \\
&= \frac{(n+2)\left\{m[2n(n-3)-3(m+1)]+4n[2h+n(H_{n-1}-H_{n-m+h-1})]\right\}}{2n(n^2-1)}. \qquad (125)
\end{aligned}$$

This completes the proof.

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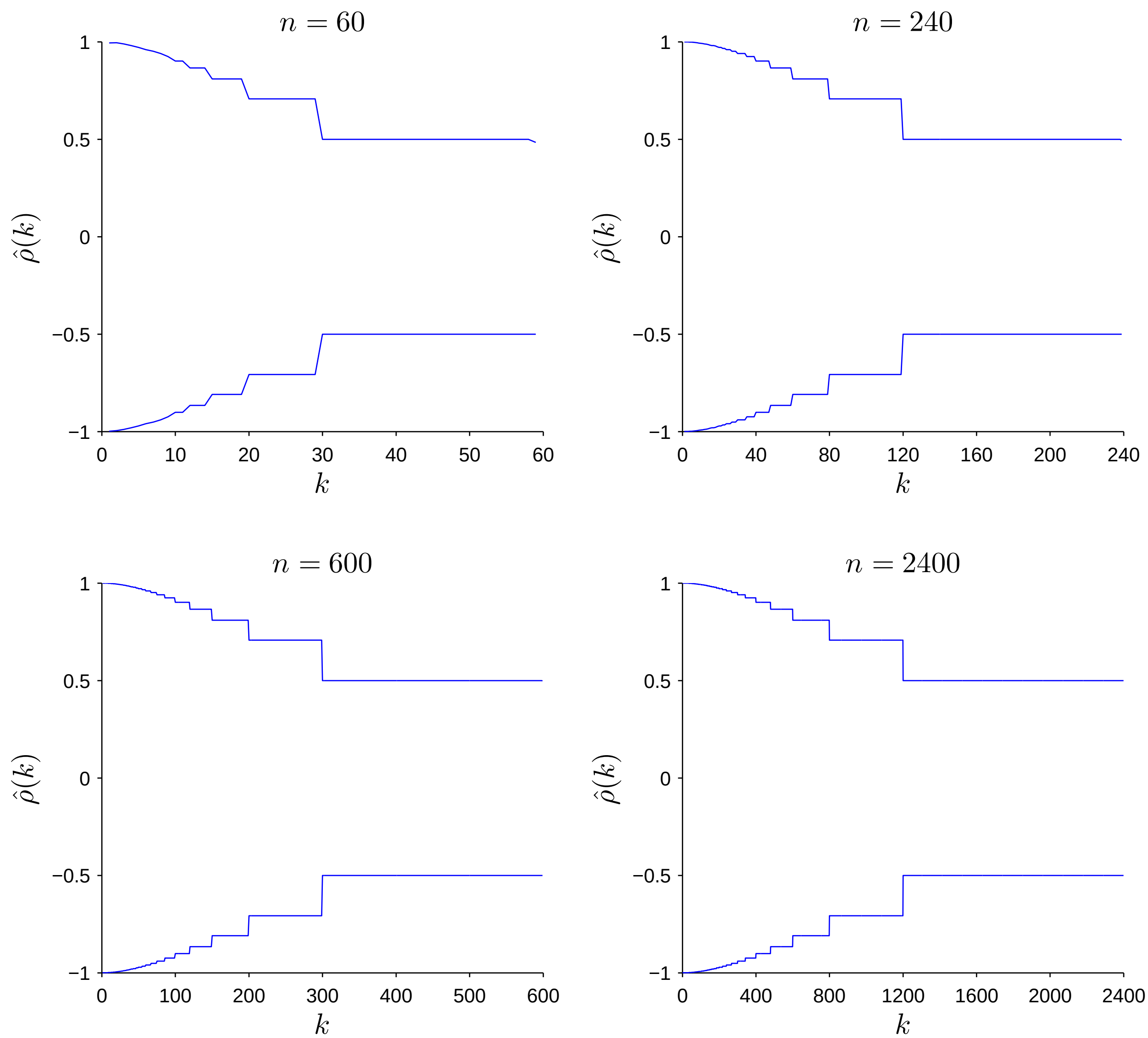

Fig. 1. For four different sample sizes (n), the figure presents the range of the kth lag sample autocorrelation coefficient for $k = 1, \ldots, n-1$.

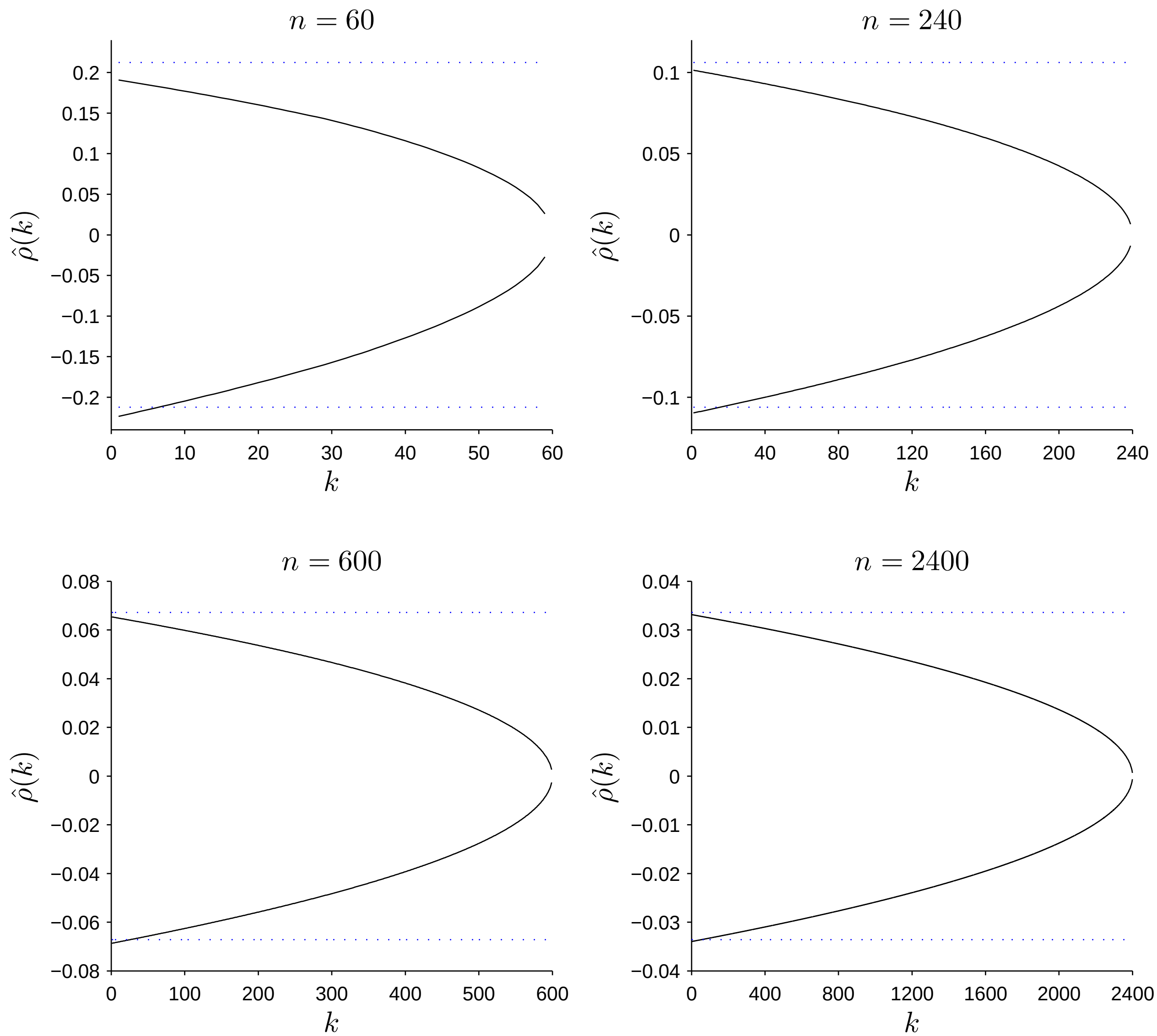

Fig. 2. The figure presents the lower and upper fifth percentiles of both the exact and asymptotic distributions of $\hat{\rho}(k)$ as a function of k for four different sample sizes (n) under the assumption that the data are serially uncorrelated and jointly elliptically distributed with constant mean and variance. The solid lines represent the percentiles based on the exact distribution. The dotted lines represent the percentiles based on the asymptotic normal distribution.

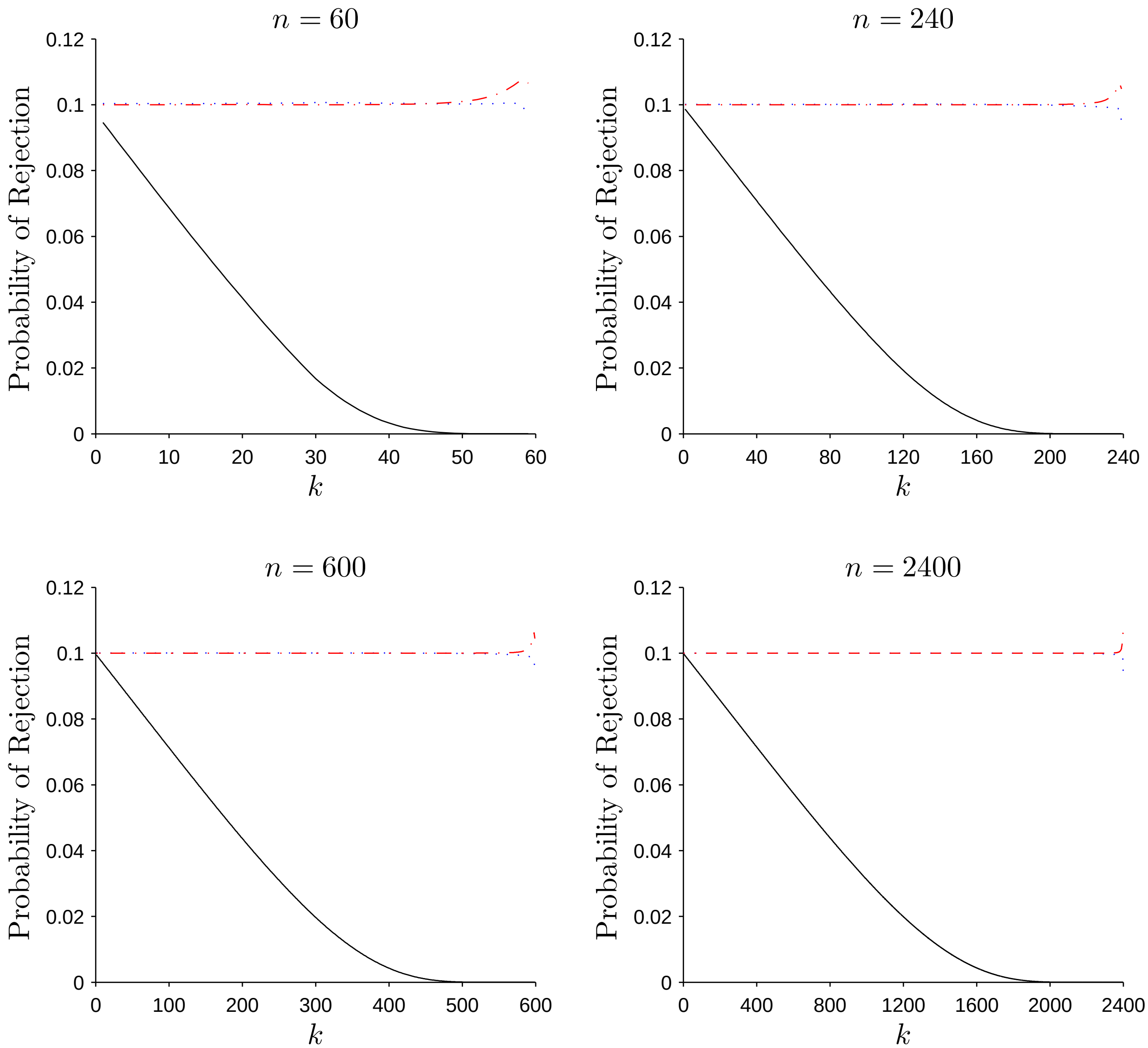

Fig. 3a. The figure presents the actual sizes of the asymptotic test, normal approximation test using exact mean and variance, and test based on Pearson's approximation for $H_0 : \rho(k) = 0$ as a function of k for four different sample sizes (n) when the nominal size of the test is 10%. The solid lines represent the actual size of the asymptotic test. The dotted lines represent the actual size of the normal approximation test using exact mean and variance. The dashed lines represent the actual size of the test based on Pearson's approximation.

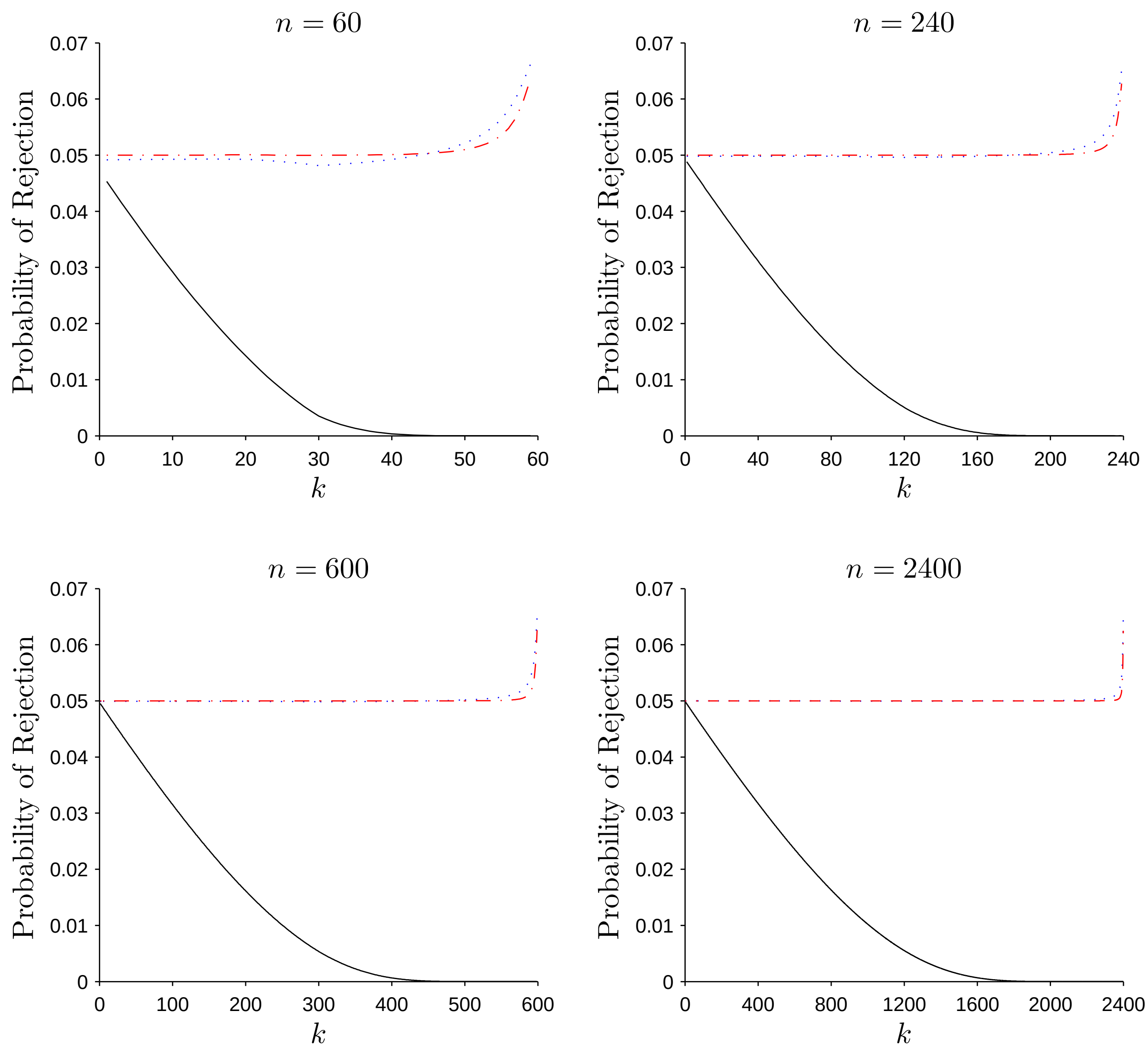

Fig. 3b. The figure presents the actual sizes of the asymptotic test, normal approximation test using exact mean and variance, and test based on Pearson's approximation for $H_0 : \rho(k) = 0$ as a function of k for four different sample sizes (n) when the nominal size of the test is 5%. The solid lines represent the actual size of the asymptotic test. The dotted lines represent the actual size of the normal approximation test using exact mean and variance. The dashed lines represent the actual size of the test based on Pearson's approximation.

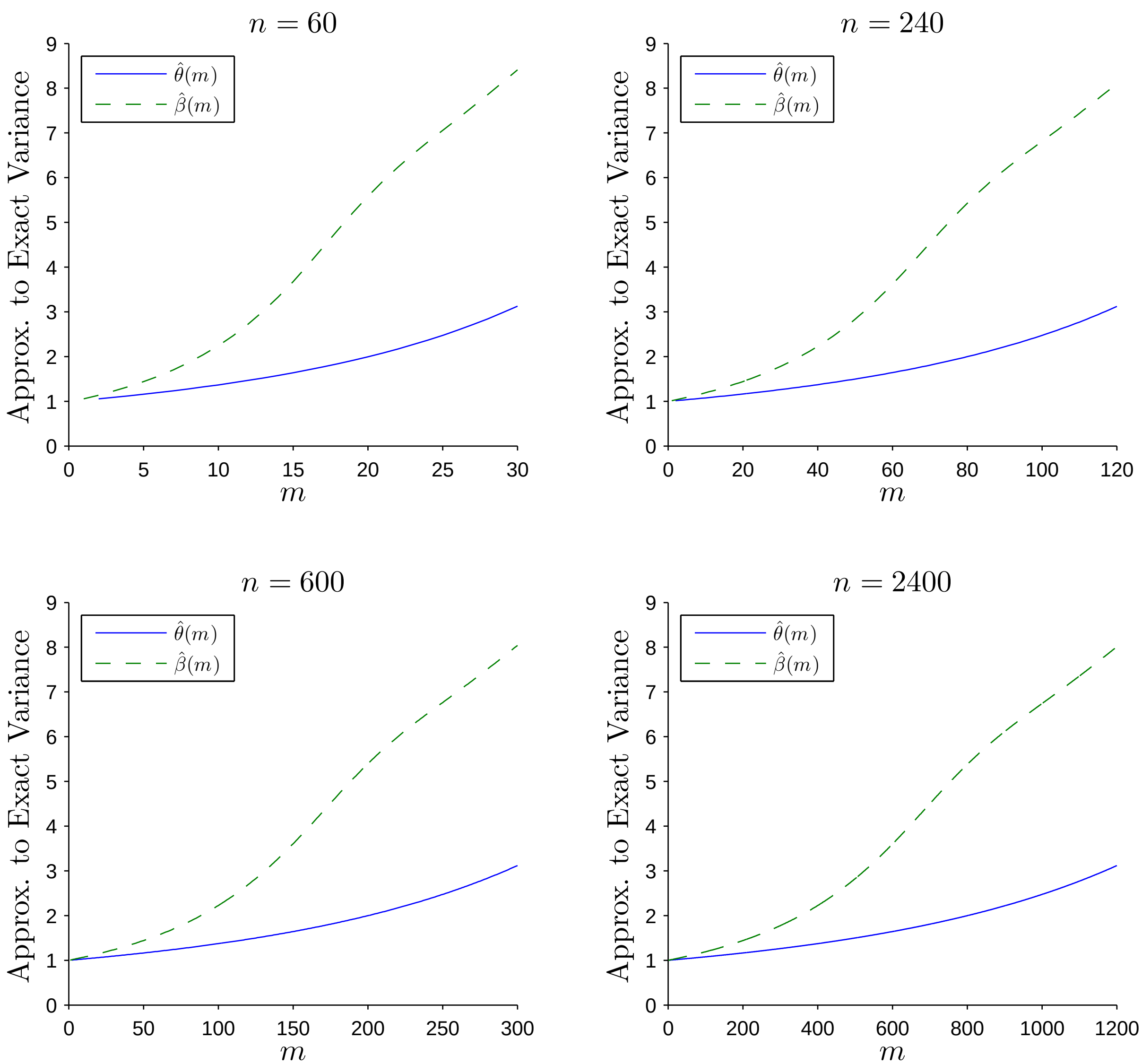

Fig. 4. The figure plots the ratio of approximate to exact variance of the variance-ratio test statistic ($\hat{\theta}(m)$) and long-horizon regression test statistic ($\hat{\beta}(m)$) as a function of m for four different sample sizes (n). The solid lines represent the ratios for the variance-ratio test statistic and dashed lines represent the ratios for the long-horizon regression test statistic.

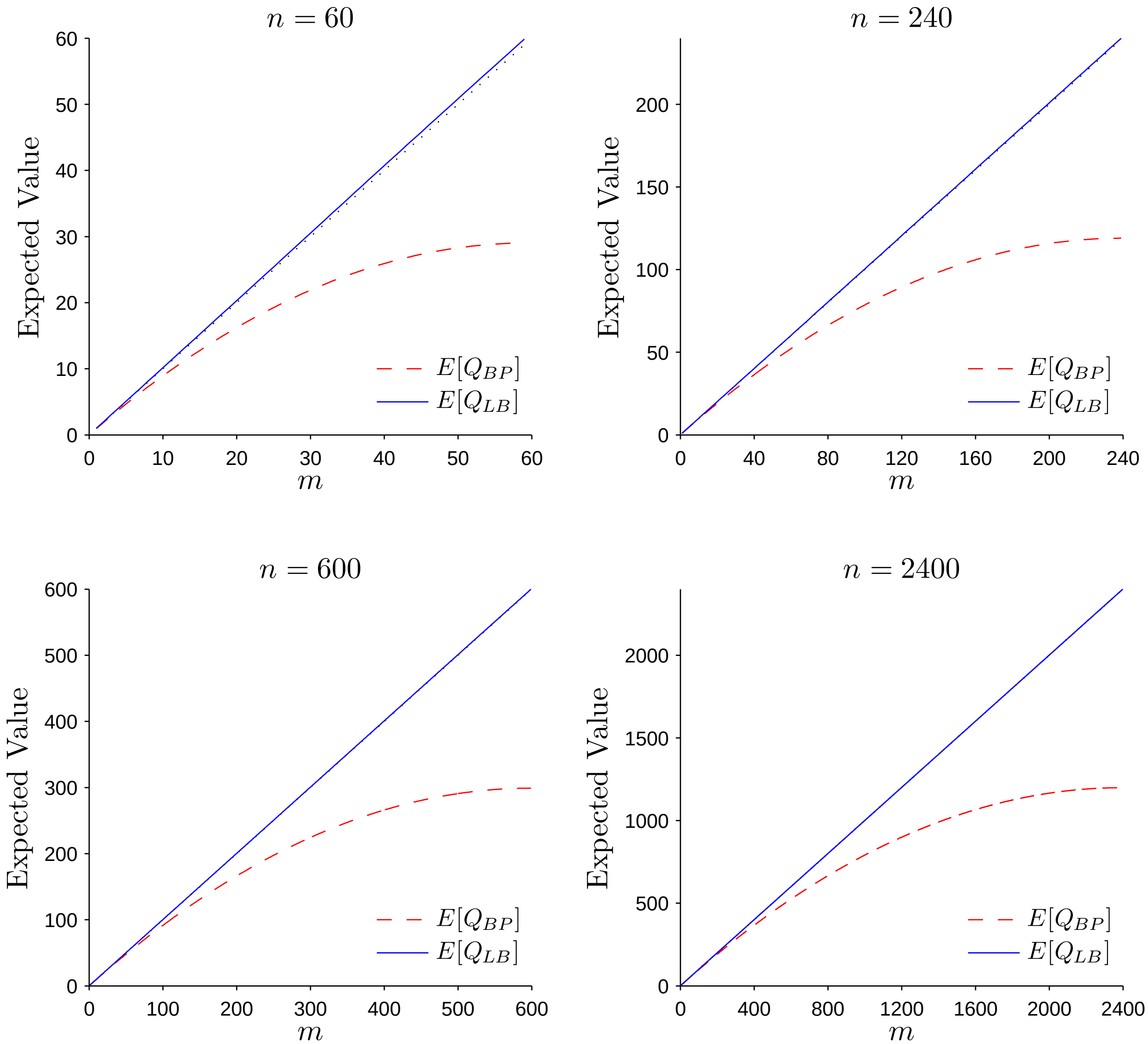

Fig. 5. The figure presents the exact mean of the two Q-statistics as a function of m for four different sample sizes (n). The solid lines represent the exact mean of Ljung-Box Q-statistic. The dashed lines represent the exact mean of Box-Pierce Q-statistic. The dotted lines represent the asymptotic mean of the statistics.

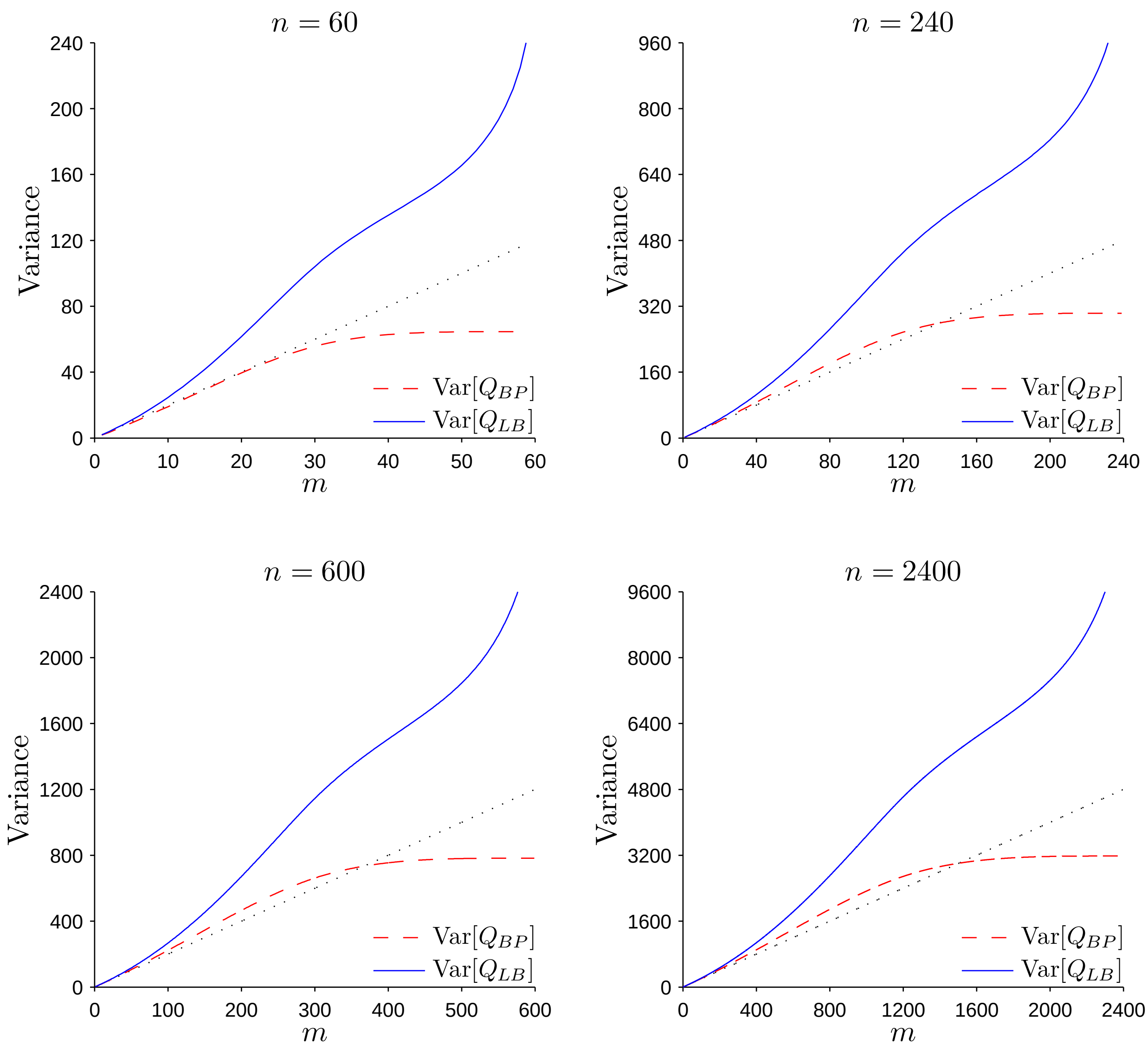

Fig. 6. The figure presents the exact variance of the two Q-statistics as a function of m for four different sample sizes (n). The solid lines represent the exact variance of Ljung-Box Q-statistic. The dashed lines represent the exact variance of Box-Pierce Q-statistic. The dotted lines represent the asymptotic variance of the statistics.

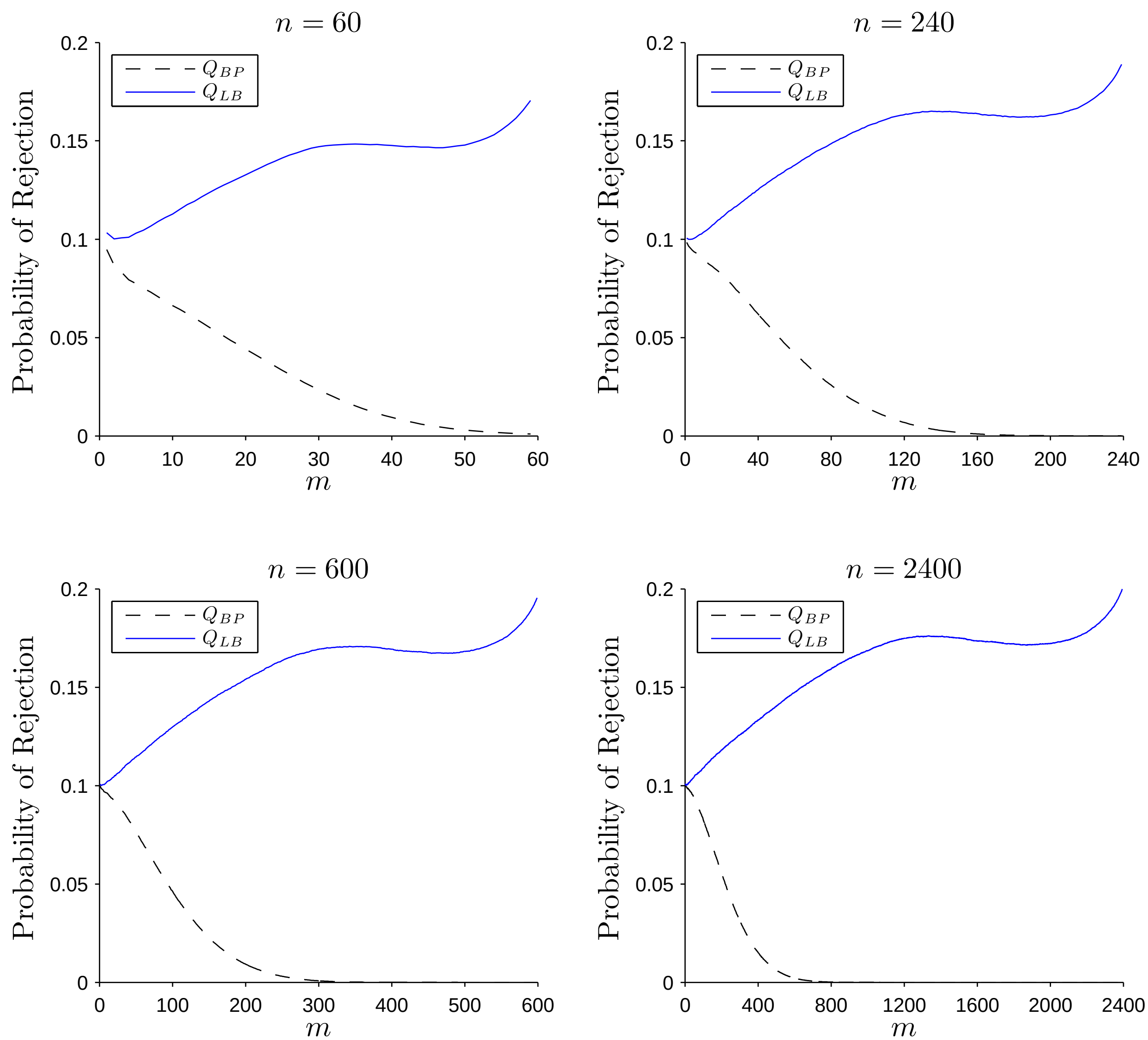

Fig. 7a. The figure presents the actual sizes of the Box-Pierce test (Q_{BP}) and the Ljung-Box test (Q_{LB}) as a function of m for four different sample sizes (n) when the nominal size of the test is 10%. The dashed lines represent the actual size of the Box-Pierce test and the solid lines represent the actual size of the Ljung-Box test.

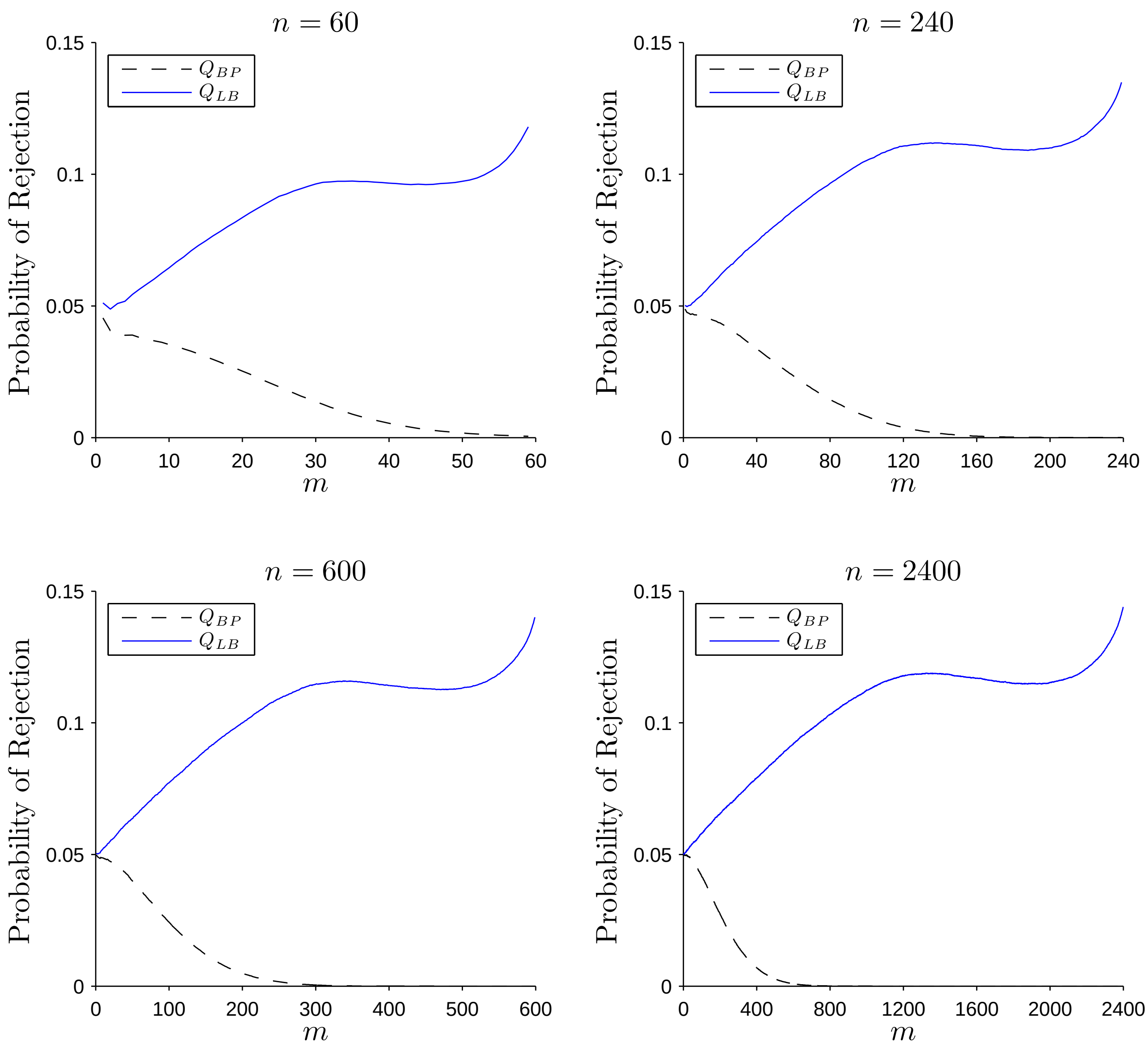

Fig. 7b. The figure presents the actual sizes of the Box-Pierce test (Q_{BP}) and the Ljung-Box test (Q_{LB}) as a function of m for four different sample sizes (n) when the nominal size of the test is 5%. The dashed lines represent the actual size of the Box-Pierce test and the solid lines represent the actual size of the Ljung-Box test.

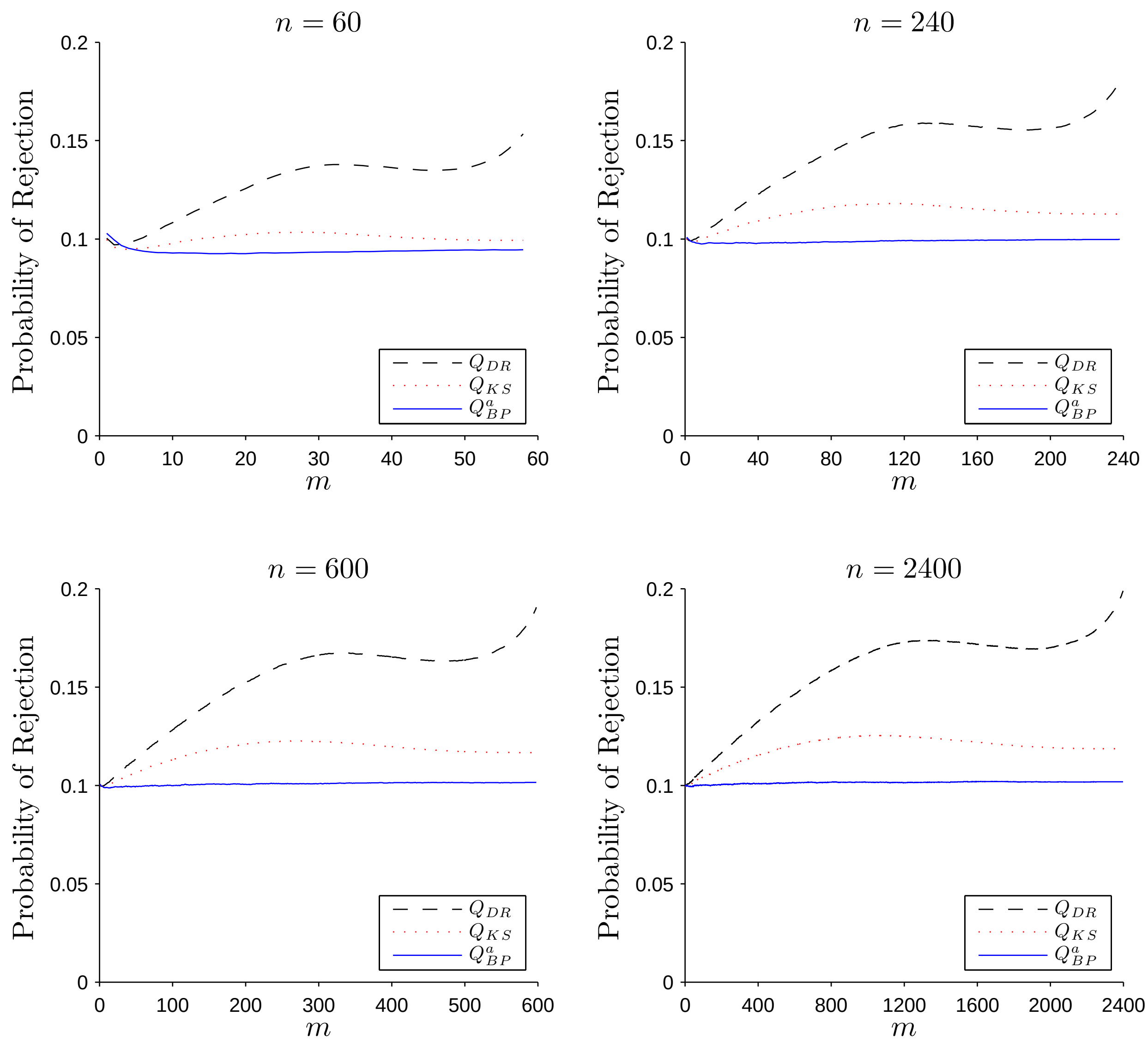

Fig. 8a. The figure presents the actual sizes of the Dufour-Roy test (Q_{DR}), the Kwan-Sim test (Q_{KS}), and the adjusted Box-Pierce test (Q_{BP}^a) as a function of m for four different sample sizes (n) when the nominal size of the test is 10%. The dashed lines represent the actual size of the Dufour-Roy test, the dotted lines represent the actual size of the Kwan-Sim test, and the solid lines represent the actual size of the adjusted Box-Pierce test.

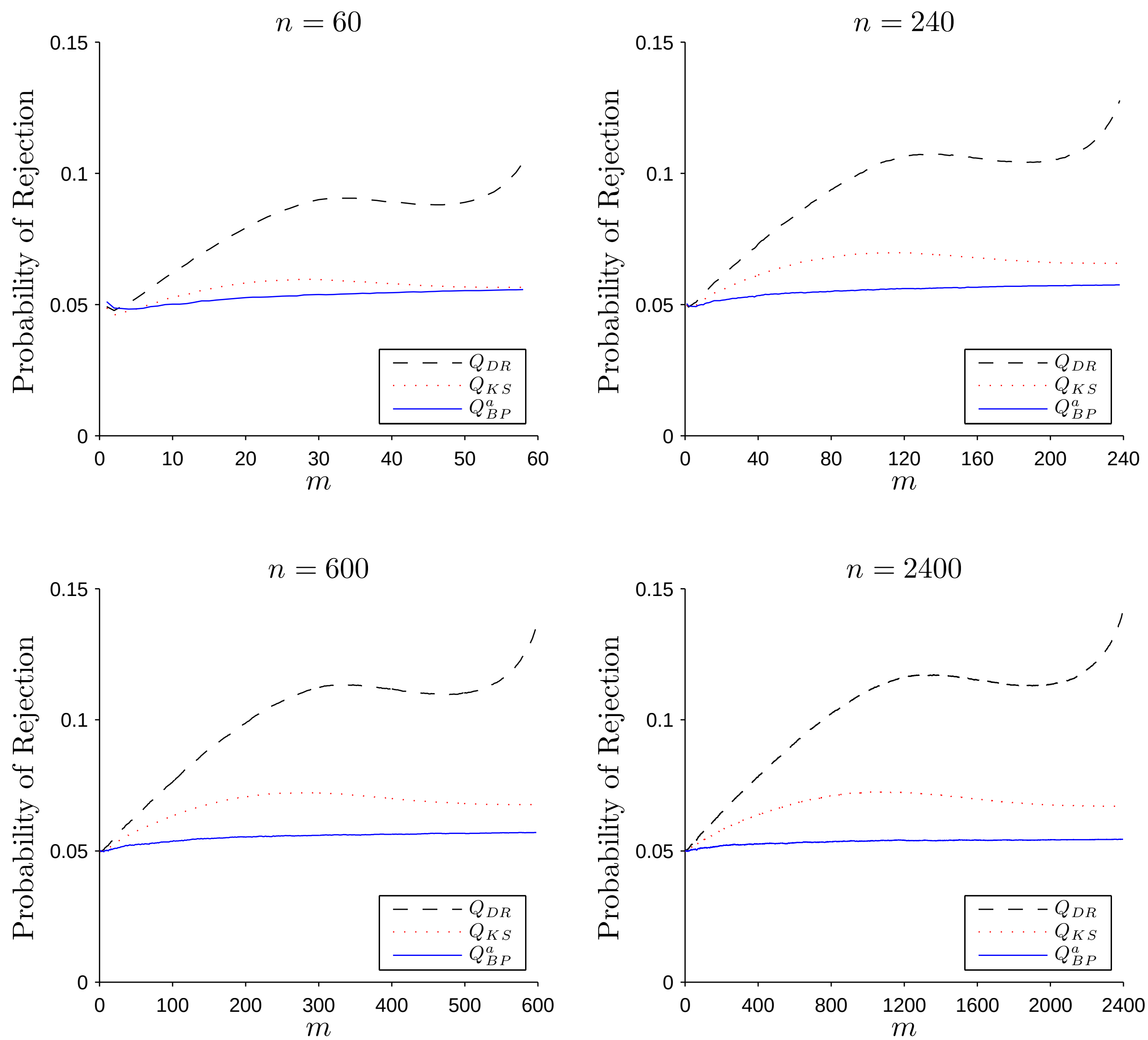

Fig. 8b. The figure presents the actual sizes of the Dufour-Roy test (Q_{DR}), the Kwan-Sim test (Q_{KS}), and the adjusted Box-Pierce test (Q^a_{BP}) as a function of m for four different sample sizes (n) when the nominal size of the test is 5%. The dashed lines represent the actual size of the Dufour-Roy test, the dotted lines represent the actual size of the Kwan-Sim test, and the solid lines represent the actual size of the adjusted Box-Pierce test.

Table 1
List of eigenvalues of $P_n' A_k P_n$ that are distinct from the eigenvalues of A_k

m	ξ_b
	$l = 0$
3	$-2/3$
4	$-1/4$
5	$(-4 \pm \sqrt{21})/10$
6	$(-1 \pm \sqrt{7})/6$
7	Roots of $14x^3 + 12x^2 - 3x - 2 = 0$
8	Roots of $16x^3 + 6x^2 - 6x - 1 = 0$
9	Roots of $144x^4 + 128x^3 - 60x^2 - 48x + 1 = 0$
10	Roots of $80x^4 + 32x^3 - 48x^2 - 12x + 3 = 0$
11	Roots of $176x^5 + 160x^4 - 112x^3 - 96x^2 + 9x + 6 = 0$
12	Roots of $192x^5 + 80x^4 - 160x^3 - 48x^2 + 24x + 3 = 0$
13	Roots of $832x^6 + 768x^5 - 720x^4 - 640x^3 + 120x^2 + 96x - 1 = 0$
14	Roots of $112x^6 + 48x^5 - 120x^4 - 40x^3 + 30x^2 + 6x - 1 = 0$
	$l > 0$
1	$\delta - 1/2$
2	$(2\delta - 1 \pm \sqrt{4\delta^2 + 12\delta + 1})/4$
3	Roots of $x^3 + \left(\frac{1}{2} - \delta\right) x^2 - \left(\frac{1}{4} + \delta\right) x - \frac{\delta}{2} = 0$
4	Roots of $x^4 + \left(\frac{1}{2} - \delta\right) x^3 - \left(\frac{1}{2} + \delta\right) x^2 - \left(\frac{1}{8} + \frac{\delta}{4}\right) x + \left(\frac{1}{16} - \frac{\delta}{4}\right) = 0$
5	Roots of $x^5 + \left(\frac{1}{2} - \delta\right) x^4 - \left(\frac{3}{4} + \delta\right) x^3 - \frac{x^2}{4} + \frac{x}{8} + \left(\frac{1}{32} - \frac{3\delta}{16}\right) = 0$
6	Roots of $x^6 + \left(\frac{1}{2} - \delta\right) x^5 - (1 + \delta) x^4 - \left(\frac{3}{8} - \frac{\delta}{4}\right) x^3 + \left(\frac{1}{4} + \frac{\delta}{4}\right) x^2 + \left(\frac{1}{16} - \frac{\delta}{8}\right) x - \frac{\delta}{8} = 0$
7	Roots of $x^7 + \left(\frac{1}{2} - \delta\right) x^6 - \left(\frac{5}{4} + \delta\right) x^5 - \left(\frac{1}{2} - \frac{\delta}{2}\right) x^4 + \left(\frac{7}{16} + \frac{\delta}{2}\right) x^3$ $+ \left(\frac{1}{8} - \frac{\delta}{8}\right) x^2 - \left(\frac{1}{32} + \frac{\delta}{8}\right) x - \frac{\delta}{16} = 0$
8	Roots of $x^8 + \left(\frac{1}{2} - \delta\right) x^7 - \left(\frac{3}{2} + \delta\right) x^6 - \left(\frac{5}{8} - \frac{3\delta}{4}\right) x^5 + \left(\frac{11}{16} + \frac{3\delta}{4}\right) x^4$ $+ \left(\frac{7}{32} - \frac{3\delta}{16}\right) x^3 - \left(\frac{3}{32} + \frac{3\delta}{16}\right) x^2 - \left(\frac{1}{64} + \frac{\delta}{32}\right) x + \left(\frac{1}{256} - \frac{\delta}{32}\right) = 0$
9	Roots of $x^9 + \left(\frac{1}{2} - \delta\right) x^8 - \left(\frac{7}{4} + \delta\right) x^7 - \left(\frac{3}{4} - \delta\right) x^6 + (1 + \delta) x^5 + \left(\frac{11}{32} - \frac{5\delta}{16}\right) x^4$ $- \left(\frac{13}{64} + \frac{5\delta}{16}\right) x^3 - \frac{3x^2}{64} + \frac{3x}{256} + \left(\frac{1}{512} - \frac{5\delta}{256}\right) = 0$
10	Roots of $x^{10} + \left(\frac{1}{2} - \delta\right) x^9 - (2 + \delta) x^8 - \left(\frac{7}{8} - \frac{5\delta}{4}\right) x^7 + \left(\frac{11}{8} + \frac{5\delta}{4}\right) x^6 + \left(\frac{1}{2} - \frac{\delta}{2}\right) x^5$ $- \left(\frac{3}{8} + \frac{\delta}{2}\right) x^4 - \left(\frac{13}{128} - \frac{3\delta}{64}\right) x^3 + \left(\frac{9}{256} + \frac{3\delta}{64}\right) x^3 + \left(\frac{3}{512} - \frac{3\delta}{256}\right) x^2 + \left(\frac{1}{512} - \frac{3\delta}{256}\right) x - \frac{3\delta}{256} = 0$