On the Explanatory Power of Asset Pricing Models Across and Within Portfolios

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Comments are welcome

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ABSTRACT

We investigate the effect of using portfolios in assessing the explanatory power of an asset pricing model, where the portfolios are formed by sorting firms using a firm-specific variable. We show that the explanatory power of an asset pricing model at the individual firm level can be grossly exaggerated or nullified at the portfolio level, depending on the choice of the sorting variable. We also study the explanatory power of an asset pricing model on the firms within each portfolio, and show that in general the explanatory power of an asset pricing model for firms within a portfolio can increase or decrease with the number of portfolios.

Although asset pricing models are supposed to work for individual firms as well as portfolios, they are often estimated and tested using portfolios only. There are various reasons for using portfolios to assess asset pricing models, most of them statistical in nature. For most applications, we are ultimately interested in how good an asset pricing model is at explaining the expected returns of individual firms. It is therefore important to know what we can learn about asset pricing relations for individual firms from asset pricing studies that are based on portfolios. In the recent literature, there are also studies that examine the performance of asset pricing models within the firms of each portfolio. Unless an asset pricing model is perfect, its performance within the portfolios will be in general different from its performance in the entire sample. It is of interest to understand the theoretical relation between these two measures of performance.[1]

Cautions regarding use of portfolios to test asset pricing models abound in the literature. Roll (1977), for example, suggests that pricing errors of individual firms can disappear in portfolios, and hence tests based on portfolios may produce supporting results even when the model is false. Grauer and Janmaat (2001) provide the condition under which pricing errors of individual firms disappear in portfolios; they also provide an example demonstrating that even when the CAPM has no explanatory power for the expected returns of individual firms, it can perfectly explain the expected returns of some portfolios. While using portfolios can make a bad asset pricing model look good, it can also make a good asset pricing model look bad. Lo and MacKinlay (1990) suggest that when portfolios are formed based on a sorting variable that is known to be correlated with *ex post* pricing errors, then the asset pricing model, despite being true, will be over-rejected at the portfolio level. Kandel and Stambaugh (1995) show that even though the CAPM is almost true for a set of firms, there exist some (repackaged) portfolios of the firms for which the CAPM has almost no explanatory power. Finally, Liang (2000) suggests that if there are measurement errors in the sorting variable that is

[1]Understanding of this theoretical relation is particularly relevant in light of the debate between Berk (2000) and Daniel and Titman (1999) as to whether explanatory power of betas will be reduced after controlling for some stock characteristics like size and book-to-market ratio.

used to form portfolios, then there could be serious biases in the estimation of the asset pricing relation using data at the portfolio level.

In this paper, we focus on portfolios that are formed based on sorting by a firm-specific variable. We address the issue of how the theoretical explanatory power of an asset pricing model on such portfolios is related to its explanatory power for individual firms and to the sorting variable. On an *ex ante* basis, without knowing the pricing errors of an asset pricing model on individual firms, it is not entirely clear how sorting could systematically strengthen or weaken the explanatory power of an asset pricing model for the portfolios. Our analysis shows that the explanatory power of an asset pricing model on such portfolios is only determined by the sorting variable, and that the explanatory power for individual firms plays absolutely no role in determining the explanatory power for portfolios. In extreme cases, an asset pricing model for portfolios can be either perfect or completely incapable of explaining portfolio expected returns, regardless of how good or how bad the asset pricing model is for individual firms. These results cast serious doubts on what we can really learn from empirical asset pricing studies that use portfolios.

In addition, we address the issue of how the explanatory power of an asset pricing model is affected if we examine it only using firms within a portfolio. Berk (2000) suggests that increasing the number of portfolios would decrease the explanatory power of an asset pricing model within each portfolio. Our analysis suggests that his result does not hold in general. In the case where the sorting variable is not perfectly correlated with expected return, increasing the number of portfolios can increase or decrease the explanatory power of an asset pricing model on the firms within a portfolio.

The rest of the paper is organized as follows. Section I discusses the explanatory power of an asset pricing model for individual firms, across portfolios, and within portfolios. It provides an analysis of the relation between the explanatory power of an asset pricing model for individual firms and for portfolios. Section II provides an analysis of how the explanatory power of an asset pricing model for individual firms within a portfolio changes with the number of portfolios. Section III provides simulation evidence to illustrate our theoretical results under some realistic settings. Section IV provides our conclusions. The Appendix contains proofs of all propositions.

I. The Theory

A. Explanatory Power of an Asset Pricing Model for Individual Firms

We assume that there are N individual firms in our sample, where these firms are considered to be randomly drawn from a much larger population, such that the N firms that we choose are representative of the firms in the population. Each firm in the population is characterized by a triplet $y = [\mu, m, s]'$, where μ is the true expected return of the firm, m is the expected return of the firm predicted by an asset pricing model,[2] and s is the value of a firm-specific variable that is used to sort firms into portfolios. We assume that the cross-sectional distribution of this triplet in the population is continuous with density function $f_y(y)$. We also assume that the mean and the variance-covariance matrix of this distribution exist, and the variance-covariance matrix is denoted as

$$\Sigma_y = E[(y - E[y])(y - E[y])'] \equiv \begin{bmatrix} \sigma_\mu^2 & \sigma_{\mu m} & \sigma_{\mu s} \\ \sigma_{\mu m} & \sigma_m^2 & \sigma_{ms} \\ \sigma_{\mu s} & \sigma_{ms} & \sigma_s^2 \end{bmatrix}. \tag{1}$$

We assume Σ_y is positive semidefinite but it does not have to be nonsingular. For example, if the asset pricing model is perfect, i.e., $\mu = m$, or the sorting variable s completely explains the expected return μ, or the sorting variable is a linear transformation of the predicted expected return, then Σ_y is singular.

[2]For our purpose, an asset pricing model is a model that generates a prediction of expected returns for individual firms. It includes both theoretically and empirically motivated models. Furthermore, m does not have to be the predicted expected return; it can simply be a linear transformation of the predicted expected return. For example, we can use β as m for the case of the CAPM.

Since each firm is viewed as a random drawing from the population, it is natural to define the theoretical explanatory power of the asset pricing model on the expected returns of individual firms as the squared correlation coefficient between the true expected return μ and the predicted expected return from the model m:

$$\rho^2_{\mu m} = \frac{\sigma^2_{\mu m}}{\sigma^2_\mu \sigma^2_m}. \tag{2}$$

Note that this theoretical measure is computed based on the population moments. It is different from the measure of explanatory power that is computed using only the N individual firms in the sample, which we define as

$$R^2_{\mu m} = \frac{\left[\sum_{i=1}^N (\mu_i - \bar{\mu})(m_i - \bar{m})\right]^2}{\sum_{i=1}^N (\mu_i - \bar{\mu})^2 \sum_{i=1}^N (m_i - \bar{m})^2}, \tag{3}$$

where $\bar{\mu} = \sum_{i=1}^N \mu_i/N$ and $\bar{m} = \sum_{i=1}^N m_i/N$. This sample measure of explanatory power is similar to the R^2_{OLS} measure defined by Kandel and Stambaugh (1995). However, when $N \to \infty$, $R^2_{\mu m} \to \rho^2_{\mu m}$. In this paper, we are interested in the population measure because we care about not only how good an asset pricing model is at explaining the expected returns of the N firms in the sample, but also about how good it is at explaining the expected returns of all the other firms that are not included in the sample.

Before we move on, we would like to clarify what are random variables and what are constants in our model. The triplet y is a random variable before we select the N firms from the population. Once the N firms in our sample are chosen, however, $y_1, y_2, \ldots, y_N$ are treated as constants. Of the three elements of y, the first element (μ) is typically not observable to an econometrician and hence the measures $\rho^2_{\mu m}$ and $R^2_{\mu m}$ are only theoretical and are not directly observable. In practice, one can use the average return ($\bar{R}$) as a proxy for the expected return and compute $R^2_{\bar{R}m}$, using $\bar{R}$ in lieu of μ. However, since the value of $R^2_{\bar{R}m}$ depends on the realizations of average returns, it is a random variable even after we have chosen the N firms in our sample.[3] As the number

[3]Note that although we assume the firms are random samples drawn from the same population, this does not preclude correlation between realized returns on two firms. Realized returns on all firms are their expected returns plus unexpected returns; the latter are random variables defined in another probability space.

of time series observations used in computing $\bar{R}$ increases, we have $\bar{R} \to \mu$ and hence the measure of explanatory power that is computed using average returns converges to the theoretical measure that is computed based on expected returns. In order to better focus on our main issue, we do not consider measures that are based on average returns in this paper.[4]

It is important to realize that while our definition of theoretical explanatory power is a reasonable one and used by many others, it is not a perfect one. There exist also other measures of the theoretical explanatory power of an asset pricing model (see Chen, Kan, and Zhang (1998) for a discussion of various measures) but to avoid possible confusion, we will limit our discussion to this particular choice.

B. Explanatory Power of an Asset Pricing Model Across Portfolios

Portfolios are often used in tests of asset pricing models. In this paper, we limit our attention to the so-called "equal-number, equal-weighted" portfolios. Namely, the N firms are sorted by s in ascending order into n portfolios, where $n \geq 2$. The n portfolios are nonoverlapping (i.e., each firm can belong to only one portfolio) and each portfolio contains (roughly) the same number of firms. Within each portfolio, the returns as well as the firm-specific variables are equally weighted. Under this scheme of portfolio formation, the population of the ith portfolio consists of all the firms with $s^*_{i-1} \leq s < s^*_i$, where $\int_{-\infty}^{s^*_i} f_s(s) = \frac{i}{n}$.[5] Conditional on a firm belonging to portfolio i, the joint distribution of its triplet y is given by

$$f_y(y|s^*_{i-1} \leq s < s^*_i) = \begin{cases} \frac{f_y(y)}{P[s^*_{i-1} \leq s < s^*_i]} = n f_y(y) & \text{if } s^*_{i-1} \leq s < s^*_i, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

[4]Conditional on N firms being chosen, Chen, Kan, and Zhang (1998) provide an analysis of the sampling distribution of $R^2_{\bar{R}m}$.

[5]In practice, the cutoff points of the n portfolios are determined by the firm-specific variables of the N firms in the sample. We assume N to be large enough so that the population cutoff points are good approximations of the sample cutoff points.

Since the portfolio is equally weighted, the theoretical values of the triplet y for portfolio i are simply their conditional means. Therefore, we have

$$\begin{aligned} \mu_p^i &= E[\mu | s_{i-1}^* \leq s < s_i^*] \\ &= n \int_{s_{i-1}^*}^{s_i^*} \int_{-\infty}^{\infty} \mu f_{\mu,s}(\mu, s) \mathrm{d}\mu \mathrm{d}s \\ &= n \int_{s_{i-1}^*}^{s_i^*} E[\mu|s] f_s(s) \mathrm{d}s, \end{aligned} \tag{5}$$

$$\begin{aligned} m_p^i &= E[m | s_{i-1}^* \leq s < s_i^*] \\ &= n \int_{s_{i-1}^*}^{s_i^*} \int_{-\infty}^{\infty} m f_{m,s}(m, s) \mathrm{d}m \mathrm{d}s \\ &= n \int_{s_{i-1}^*}^{s_i^*} E[m|s] f_s(s) \mathrm{d}s, \end{aligned} \tag{6}$$

$$\begin{aligned} s_p^i &= E[s | s_{i-1}^* \leq s < s_i^*] \\ &= n \int_{s_{i-1}^*}^{s_i^*} s f_s(s) \mathrm{d}s. \end{aligned} \tag{7}$$

Similar to the measure of explanatory power for individual firms, we define the theoretical explanatory power of the asset pricing model on the expected returns of these n portfolios as the squared correlation between μ_p^i and m_p^i:

$$\rho_{\mu m}^2(n) = \frac{\left[\sum_{i=1}^n (\mu_p^i - \bar{\mu}_p)(m_p^i - \bar{m}_p)\right]^2}{\sum_{i=1}^n (\mu_p^i - \bar{\mu}_p)^2 \sum_{i=1}^n (m_p^i - \bar{m}_p)^2}, \tag{8}$$

where $\bar{\mu}_p = \sum_{i=1}^n \mu_p^i / n = E[\mu]$ and $\bar{m}_p = \sum_{i=1}^n m_p^i / n = E[m]$.[6]

By comparing expressions (2) and (8), we can determine whether the explanatory power of an asset pricing model is higher or lower for the individual firms than for the portfolios. From (5) and (6), we can see that the theoretical explanatory power of an asset pricing model on the n portfolios only depends on: (a) the cross-sectional distribution of the sorting variable, $f_s(s)$, (b) the mean of the expected return conditional upon the sorting variable, $E[\mu|s]$, and (c) the mean of the expected return predicted by the asset pricing model conditional upon the sorting variable, $E[m|s]$. In particular, it does

[6] Implicitly, we assume that m_p^i are not constant across portfolios. If μ_p^i is constant across portfolios, $\rho_{\mu m}^2(n)$ is equal to zero by convention.

not depend on the joint distribution of $f_{\mu,m}(\mu, m)$, and hence $\rho^2_{\mu m}$ at the individual firm level does not play any role in determining $\rho^2_{\mu m}(n)$ at the portfolio level.[7] The only possible exception is when we sort the portfolios using $s = m$. However, even when $\rho^2_{sm} = 1$, there is still no obvious relation between $\rho^2_{\mu m}$ and $\rho^2_{\mu m}(n)$ since the former depends on the entire joint distribution $f_{\mu,m}(\mu, m)$, whereas the latter only depends on the conditional expectation $E[\mu|m]$ and the marginal distribution $f_m(m)$. Our observation that $\rho^2_{\mu m}(n)$ for portfolios has no relation to $\rho^2_{\mu m}$ for individual firms applies to more general portfolio formation schemes. For example, s could be a vector of firm-specific variables and portfolios could be sorted on a multi-dimensional basis using different combinations of elements of s. Moreover, the portfolios do not have to be equally weighted, and they do not need to be composed of an equal number of firms.[8]

The observation that $\rho^2_{\mu m}(n)$ has no relation to $\rho^2_{\mu m}$ is surprising. It suggests that we cannot infer the explanatory power of an asset pricing model for individual firms (which presumably is what we care about) from empirical studies that only use portfolio data. The explanatory power of an asset pricing model on the portfolios only depends on the sorting variable and the sorting variable's relation to μ and m. Different sorting schemes could possibly provide a wide range of $\rho^2_{\mu m}(n)$, and none of them can be relied upon to provide information about $\rho^2_{\mu m}$. In what follows in this subsection, we focus on the two extreme situations in which the explanatory power of an asset pricing model for portfolios can appear very high or very low.

Proposition 1 *If there exist scalars a and b such that $E[\mu|s] = a + bE[m|s]$, then for*

[7]In general, we can write $\mu = E[\mu|s] + e_1$ and $m = E[m|s] + e_2$ where e_1 and e_2 are conditionally independent of s. $\rho^2_{\mu m}(n)$ depends only on $E[\mu|s]$ and $E[m|s]$, but $\rho^2_{\mu m}$ also depends on e_1 and e_2. For any given value of $\rho^2_{\mu m}(n)$, we can set the covariance matrix of e_1 and e_2 to make $\rho_{\mu m}$ equal to any value in $(-1, 1)$. Therefore, $\rho^2_{\mu m}$ does not tell us anything about $\rho^2_{\mu m}(n)$.

[8]If the population has a finite number of firms, then when the number of portfolios is equal to the number of firms, $\rho^2_{\mu m}(n) = \rho^2_{\mu m}$ trivially. In our setup, the population from which our firms come from has a continuous distribution, so this scenario will not occur even for a countably infinite number of portfolios.

all $n > 1$,

$$\rho^2_{\mu m}(n) = \begin{cases} 1 & \text{if } b \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

An important case is when a sorting variable s is chosen so that the pricing error $\mu - m$ for individual firms is conditionally independent of s, i.e., $E[\mu - m|s] = 0$, hence $E[\mu|s] = E[m|s]$ (the condition of Proposition 1 is satisfied with $a = 0$ and $b = 1$). Since the pricing errors of individual firms are uncorrelated with the sorting variable, they averaged out in portfolios to zero, and the model thus becomes perfect at the portfolio level.

Another important case is when the conditional expectations $E[\mu|s]$ and $E[m|s]$ are linear in s. In this case, the conditional expectations, $E[\mu|s]$ and $E[m|s]$, must be linear to each other. We immediately obtain a corollary.

Corollary 1 *Suppose* $E[\mu|s]$ *and* $E[m|s]$ *are linear in* s*. We have for all* $n > 1$,

$$\rho^2_{\mu m}(n) = \begin{cases} 1 & \text{if } \sigma_{\mu s} \neq 0 \text{ and } \sigma_{ms} \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

This case is also important because it depends merely on the distribution types without reference to the pricing errors. For example, if the true expected return, the predicted expected return, and the firm-specific variable have a multivariate elliptical distribution, then $E[\mu|s]$ and $E[m|s]$ are linear in s.[9]

Proposition 1 and its corollary suggest that when $E[\mu|s]$ and $E[m|s]$ are linear function of each other, the theoretical explanatory power of an asset pricing model for portfolios can take only two possible values, zero or one. The case of $\rho^2_{\mu m}(n) = 0$ is easy to understand. It happens when the sorting variable s is completely uncorrelated with expected returns (or predicted expected returns) of individual firms. In this case, the resulting portfolios will not display any cross-sectional differences in expected returns

[9] Note that Corollary 1 only requires $E[\mu|s]$ and $E[m|s]$ to be linear in s. It does not require $E[\mu|m]$ to be linear in m (or vice versa), so having multivariate elliptical distribution is sufficient but not necessary for Corollary 1 to hold. Furthermore, the condition that $E[\mu|s]$ and $E[m|s]$ are linear in s can be relaxed to the condition that there exists a monotonic function $u = \psi(s)$ such that $E[\mu|u]$ and $E[m|u]$ are linear in u. This is so because sorting by s is the same as sorting by u.

(or predicted expected returns). Therefore, regardless of how good the asset pricing model is, it has no explanatory power for portfolios.

In practice, it is most likely that one would sort firms into portfolios using a firm-specific variable that would have at least some correlation with the true expected return and possibly some correlation with the predicted expected return. In this case, Proposition 1 suggests that as long as $E[\mu|s]$ and $E[m|s]$ are linear functions of each other, the asset pricing model will always perform perfectly for portfolios, regardless of the number of portfolios used, and regardless of how poorly the asset pricing model actually performs for individual firms.[10] This will occur even when the asset pricing model alone has no explanatory power on the expected returns of individual firms, i.e., $\rho^2_{\mu m} = 0$.

Proposition 1 and its corollary are disturbing. They imply that finding an asset pricing model that performs perfectly for portfolios says nothing about how that asset pricing model performs for individual firms. The result that there exist some portfolios on which a wrong asset pricing model can look good is not entirely new. However, the implications of our Proposition 1 and its corollary are more dramatic than the implications of other studies. Proposition 1 and its corollary suggest that one does not need to know the pricing errors of an asset pricing model in order to find portfolios on which the asset pricing model will have perfect explanatory power. All that is needed is to sort firms using a firm-specific variable s such that $E[\mu|s]$ and $E[m|s]$ are linear functions of each other. The asset pricing model will then perform perfectly on any number of portfolios that are formed based on such a firm-specific variable.

Proposition 1 also provides clues regarding the types of situations in which an asset pricing model would have poor explanatory power on a set of portfolios, depending on the nature of the sorting variable s. If $E[\mu|s]$ and $E[m|s]$ are not highly correlated with each other, portfolios formed based on such a sorting variable s have low $\rho^2_{\mu m}(n)$.

[10] $\rho^2_{\mu m}(n) = 1$ does not imply $\mu^i_p = m^i_p$. However, to the extent that we can find a model the predictions of which are perfectly correlated with expected returns, it is not difficult to impose a linear transformation on it to make $\mu^i_p = m^i_p$.

However, the poor performance of the asset pricing model on these portfolios is only due to the choice of the sorting variable and says nothing about how good or how bad the asset pricing model actually is for individual firms.

Empirically, even when the conditions for Proposition 1 are met, an asset pricing model typically does not provide perfect explanatory power on the average returns of the portfolios. This may be because average returns are noisy measures of expected returns or because the total number of firms in the sample, N, or because the number of firms in each portfolio, N/n, is not large enough to provide us a good approximation of the population. In Section III, we provide simulation evidence to suggest that the second reason is likely to be relatively unimportant in most empirical studies. In Section IV, we provide a formal statistical test of the two hypotheses $\rho^2_{\mu m}(n) = 0$ and $\rho^2_{\mu m}(n) = 1$ using average returns.

II. Explanatory Power of an Asset Pricing Model for Individual Firms within a Portfolio

There are cases (see, for example Daniel and Titman (1997)) in which researchers first sort their sample firms into portfolios and then examine the validity of the asset pricing model using firms within each portfolio. Berk (2000) provides such an analysis and suggests that this procedure is biased in favor of rejecting the asset pricing model under consideration when the sorting variable is correlated with expected returns. In this subsection, we refine his analysis and provide conditions under which his claim is true.

If we limit our attention to firms in portfolio i, then the theoretical explanatory power of an asset pricing model for firms in this portfolio is given by

$$\rho^2_{\mu m \cdot i} = \frac{\sigma^2_{\mu m \cdot i}}{\sigma^2_{\mu \cdot i}\sigma^2_{m \cdot i}}, \tag{11}$$

where

$$\sigma^2_{\mu \cdot i} \quad = \quad E[(\mu - \mu^i_p)^2 | s^*_{i-1} \leq s < s^*_i], \tag{12}$$

$$\sigma^2_{m\cdot i} = E[(m - m_p^i)^2 | s^*_{i-1} \leq s < s^*_i], \tag{13}$$

$$\sigma_{\mu m\cdot i} = E[(\mu - \mu_p^i)(m - m_p^i) | s^*_{i-1} \leq s < s^*_i]. \tag{14}$$

Since, conditional on $s^*_{i-1} \leq s < s^*_i$, the joint density of μ and m is given by

$$f_{\mu m\cdot i}(\mu, m) = n \int_{s^*_{i-1}}^{s^*_i} f_y(\mu, m, s)\mathrm{d}s, \tag{15}$$

we can rewrite the expressions above as

$$\sigma^2_{\mu\cdot i} = n \int_{s^*_{i-1}}^{s^*_i} \int_{-\infty}^{\infty} (\mu - \mu_p^i)^2 f_{\mu,s}(\mu, s)\mathrm{d}\mu\mathrm{d}s, \tag{16}$$

$$\sigma^2_{m\cdot i} = n \int_{s^*_{i-1}}^{s^*_i} \int_{-\infty}^{\infty} (m - m_p^i)^2 f_{m,s}(m, s)\mathrm{d}m\mathrm{d}s, \tag{17}$$

$$\sigma_{\mu m\cdot i} = n \int_{s^*_{i-1}}^{s^*_i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\mu - \mu_p^i)(m - m_p^i) f_y(\mu, m, s)\mathrm{d}\mu\mathrm{d}m\mathrm{d}s. \tag{18}$$

Berk (2000) suggests that as long as s can generate cross-sectional differences in expected returns of the portfolios, then by sorting firms into portfolios, the cross-sectional variations of expected returns within the portfolios are reduced and hence he claims[11]

$$\rho^2_{\mu m\cdot i} < \rho^2_{\mu m}, \quad i = 1, 2, \ldots, n. \tag{19}$$

For the case when n is large enough, he further claims that

$$\rho^2_{\mu m\cdot i} < \epsilon, \qquad i = 1, 2, \ldots, n, \tag{20}$$

for any arbitrary small $\epsilon > 0$. Based on these claims, he suggests that the explanatory power of an asset pricing model within the firms of a portfolio is a decreasing function of the number of portfolios, and that $\rho^2_{\mu m\cdot i}$ understates the true explanatory power of an asset pricing model.

We first show that neither claims is true in general. For example, if $\mu = m + s$ where m and s are independent, then $\rho^2_{\mu m} < 1$ for the individual firms. However, if we sort the

[11] Berk's (2000) definition of explanatory power of an asset pricing model differs slightly from ours in that it is the cross-sectional R^2 between average returns and the expected returns predicted by the asset pricing model. However, it is clear from his analysis that the use of expected returns instead of average returns would not alter his conclusions at all.

firms into portfolios based on s, then there is little variation of s within each portfolio. Consequently, μ and m are almost perfectly correlated within the firms of each portfolio as the number of portfolios increases, and we have $\rho^2_{\mu m \cdot i} \approx 1$ for every portfolio. So the explanatory power of an asset pricing model within each portfolio does not have to be lower than its explanatory power for the entire population of firms. Even in the extreme case when the sorting variable is perfectly correlated with true expected returns, these two claims are not always true. For example, if $s = \mu$ and it has an exponential distribution, then it is easy to show that for the last portfolio, we have $\sigma^2_{\mu \cdot n} = \sigma^2_\mu$ and its value is independent of n.[12] If the pricing error $\alpha = \mu - m$ is independent of μ, then we have $\lim_{n\to\infty} \rho^2_{\mu m \cdot n} = \rho^2_{\mu m}$, so the explanatory power of the asset pricing model within the last portfolio does not go to zero and always stays the same regardless of how many portfolios are formed.[13]

The reason why $\rho^2_{\mu m \cdot i}$ can be greater than $\rho^2_{\mu m}$ can be understood by examining the familiar variance and covariance decomposition formulae,[14]

$$\frac{1}{n}\sum_{i=1}^{n} \sigma^2_{\mu \cdot i} = \sigma^2_\mu - \frac{1}{n}\sum_{i=1}^{N} (\mu_p - \bar{\mu}_p)^2 < \sigma^2_\mu, \tag{21}$$

$$\frac{1}{n}\sum_{i=1}^{n} \sigma^2_{m \cdot i} = \sigma^2_m - \frac{1}{n}\sum_{i=1}^{N} (m_p - \bar{m}_p)^2 < \sigma^2_m, \tag{22}$$

$$\frac{1}{n}\sum_{i=1}^{n} \sigma_{\mu m \cdot i} = \sigma_{\mu m} - \frac{1}{n}\sum_{i=1}^{N} (\mu_p - \bar{\mu_p})(m_p - \bar{m}_p). \tag{23}$$

Therefore, sorting will on average reduce the variation of expected returns and predicted expected returns within a portfolio. Since the variance terms $\sigma^2_{\mu \cdot i}$ and $\sigma^2_{m \cdot i}$ are in the

[12] This result follows from the familiar fact that if X has an exponential distribution, then $P[X > a + b | X > a] = P[X > b]$ and hence the variance of the truncated exponential distribution is the same as the variance of the unconditional distribution.

[13] Under the same conditions, but with μ having a t-distribution, we have $\lim_{n\to\infty} \rho^2_{\mu m \cdot 1} = \lim_{n\to\infty} \rho^2_{\mu m \cdot n} = 1$ and the asset pricing model has almost perfect explanatory power within the first and the last portfolios when n is large enough.

[14] The decomposition formulae state that for random variables X, Y, and Z, we have

$$E[\mathrm{Var}[X|Z]] = \mathrm{Var}[X] - \mathrm{Var}[E[X|Z]],$$
$$E[\mathrm{Cov}[X,Y|Z]] = \mathrm{Cov}[X,Y] - \mathrm{Cov}[E[X|Z], E[Y|Z]].$$

denominator of $\rho^2_{\mu m\cdot i}$, their reduction will actually increase $\rho^2_{\mu m\cdot i}$. The key question is whether $\sigma^2_{\mu m\cdot i}$ will fall enough to cause $\rho^2_{\mu m\cdot i}$ to decrease because of sorting. From (23), we can see that it depends on the cross-sectional covariance between μ_p and m_p. Even though we assume this term to have the same sign as $\sigma_{\mu m}$ and both of them are positive, it only guarantees the average covariance between μ and m within the portfolios to be less than $\sigma_{\mu m}$. It does not guarantee the average $\sigma^2_{\mu m\cdot i}$ to be less than $\sigma^2_{\mu m}$. This is so because the $\sigma_{\mu m\cdot i}$ could all be large but have different signs for different portfolios. Therefore, unlike the average $\sigma^2_{\mu\cdot i}$ and $\sigma^2_{m\cdot i}$, the average $\sigma^2_{\mu m\cdot i}$ does not have to decrease with sorting.

Having shown that Berk's claims are not true in general, we proceed to characterize the conditions for which his claims are true. Analytical expressions for the general case are difficult to derive. Moreover, for any arbitrary distribution of y, its mean and variance-covariance matrix are not sufficient to characterize the entire distribution. Therefore, we focus on the case when y has a multivariate elliptical distribution in the following proposition.[15] By sacrificing generality, we are able to provide analytical results as well as better intuition.

Proposition 2 *If y has a multivariate elliptical distribution with finite mean and variance, then*

$$\rho^2_{\mu m\cdot i} = \frac{[\rho_{\mu m} - \rho_{\mu s}\rho_{ms}h(i)]^2}{[1-\rho^2_{\mu s}h(i)][1-\rho^2_{ms}h(i)]}, \tag{24}$$

where

$$h(i) = 1 - \frac{\int_{s^*_{i-1}}^{s^*_i} n\left(\frac{s-s^i_p}{\sigma_s}\right)^2 f_s(s)\mathrm{d}s}{\int_{s^*_{i-1}}^{s^*_i} ng(s)f_s(s)\mathrm{d}s} \tag{25}$$

and $g(s)$ is a positive function which depends on the class of the elliptical distribution.

[15] y is said to have an elliptical distribution if $f_y(y) \propto |\Sigma_y|^{-\frac{1}{2}}\varphi((y-E[y])'\Sigma_y^{-1}(y-E[y]))$, where $\varphi(u)$ is a positive and nonincreasing function of u for $u \geq 0$. See Kelker (1970) for a discussion of various properties of multivariate elliptical distribution. For Proposition 2, all we need is that there exists a monotonic function $u = \psi(s)$ such that (μ, m, u) has a multivariate elliptical distribution.

For the special case that y has a multivariate normal distribution, we have $g(s) = 1$ and

$$h(i) = n^2 \left[\phi(c_{i-1}) - \phi(c_i)\right]^2 - n \left[c_{i-1}\phi(c_{i-1}) - c_i\phi(c_i)\right], \tag{26}$$

where $\phi(\cdot)$ is the density function of the standard normal distribution and $c_i = \Phi^{-1}(i/n)$, with $\Phi^{-1}(\cdot)$ being the inverse cumulative density function of a standard normal distribution.

Proposition 2 suggests that the explanatory power of an asset pricing model within a portfolio depends on $h(i)$, which is determined by the distribution of s as well as the number of portfolios. In addition, it also depends on $\rho_{\mu m}$, $\rho_{\mu s}$, and ρ_{ms}. Therefore, it is not only the explanatory power of the asset pricing model for individual firms that matters. The sorting variable also matters in determining the performance of an asset pricing model within the firms of a portfolio. We first consider several special cases:

1. $\rho_{ms} = 0$. In this case, we have

$$\rho^2_{\mu m \cdot i} = \frac{\rho^2_{\mu m}}{1 - \rho^2_{\mu s} h(i)}.$$

   For the case of the elliptical distribution, it can be shown that $0 < h(i) < 1$.[16] Therefore, as long as $\rho_{\mu s} \neq 0$, we have $\rho^2_{\mu m \cdot i} > \rho^2_{\mu m}$ and the explanatory power of an asset pricing model within each portfolio is always greater than its explanatory power on all the firms in the population. While sorting may reduce the cross-sectional variation of expected returns within a portfolio, it does not reduce the covariance between μ and m enough for $\rho_{\mu m \cdot i}$ to decrease. Similarly, if $\rho_{\mu s} = 0$, then $\rho^2_{\mu m \cdot i} \geq \rho^2_{\mu m}$.

2. $\rho_{\mu m} = 0$. In this case, as long as $\rho_{\mu s} \neq 0$ and $\rho_{ms} \neq 0$, we have

$$\rho^2_{\mu m \cdot i} = \frac{\rho^2_{\mu s}\rho^2_{ms} h(i)^2}{[1 - \rho^2_{\mu s} h(i)][1 - \rho^2_{ms} h(i)]} > 0$$

[16] Proof is available upon request.

and the asset pricing model will always have some explanatory power within the firms in each portfolio, even though the asset pricing model alone is completely incapable of explaining the expected returns of the firms in the population. Once again, sorting can actually make a poor asset pricing model look good within the portfolios.

3. $\rho_{\mu s} = \pm 1$. In this case, $\rho^2_{\mu m} = \rho^2_{ms}$ and

$$\rho^2_{\mu m \cdot i} = \frac{\rho^2_{\mu m}\left[1 - h(i)\right]}{1 - \rho^2_{\mu m} h(i)}.$$

   Therefore, we have $\rho^2_{\mu m \cdot i} < \rho^2_{\mu m}$ for all the portfolios if $\rho^2_{\mu m} \neq 1$. This example was used by Berk (2000) to show that as long as an asset pricing model is not perfect, its explanatory power is lower within portfolios than for individual firms in the population. However, as shown earlier, this is not always true outside of the class of the multivariate elliptical distribution. Therefore, in general, sorting does not always reduces $\rho^2_{\mu m \cdot i}$ for every portfolio even when the sorting variable is perfectly correlated with expected returns.

In general, when the number of portfolios is finite, $\rho^2_{\mu m \cdot i}$ varies across portfolios and hence comparison between $\rho^2_{\mu m}$ and $\rho^2_{\mu m \cdot i}$ is somewhat difficult. However, as the number of portfolios n goes to infinity,[17] $\rho^2_{\mu m \cdot i}$ converges to a common limit for most of the portfolios and hence we can easily address the question as to whether sorting will eventually improve or reduce the explanatory power of an asset pricing model. This result is given in the following proposition.

Proposition 3 *Suppose y has a multivariate elliptical distribution with finite mean and variance. For any $\epsilon_1 > 0$ and $\epsilon_2 > 0$, there exists n_0 such that for all $n > n_0$, we have $h(i) > 1 - \epsilon_1$ for at least $n(1 - \epsilon_2) - 2$ portfolios. For the case where y has a multivariate*

[17] We can let n go to infinity because the population is characterized by a continuous distribution and has an uncountably infinite number of firms.

normal distribution, we have the stronger result that $h(i) > 1 - \epsilon_1$ *for all portfolios when* $n > n_0$.

Since $h(i) \leq 1$, Proposition 3 suggests that for most of the portfolios, we have $h(i) \to 1$ and

$$\lim_{n\to\infty} \rho^2_{\mu m\cdot i} = \frac{(\rho_{\mu m} - \rho_{\mu s}\rho_{ms})^2}{(1-\rho^2_{\mu s})(1-\rho^2_{ms})} \equiv \rho^2_{\mu m|s}. \tag{27}$$

Proposition 3 does not suggest that all portfolios will have $\rho^2_{\mu m\cdot i}$ approaching to this limit (except for the case of a normal distribution), but rather a majority of them will. However, as $n \to \infty$, the set of portfolios that does not approach this limit is of measure zero.[18] The limit $\rho^2_{\mu m|s}$ is the familiar measure of partial coefficient of determination. It measures the squared correlation of μ and m, conditional upon a particular value of the sorting variable s. For the multivariate elliptical distribution, this partial coefficient of determination is independent of s. As $n \to \infty$, the cutoff point s^*_{i-1} converges to s^*_i for most of the portfolios. Therefore, for most of the portfolios, the within-portfolio explanatory power of the asset pricing model converges to this common limit of $\rho^2_{\mu m|s}$.

From the expression of $\rho^2_{\mu m|s}$, unless the sorting variable is perfectly correlated with expected returns or $\rho_{\mu m} = \rho_{\mu s}\rho_{ms}$,[19] sorting does not completely destroy the explanatory power of an asset pricing model within the portfolios even when the number of portfolios goes to infinity. We therefore focus on the question of when does increasing the number of portfolios reduce the explanatory power of an asset pricing model within the portfolios. The following proposition provides the condition for this to happen.

Proposition 4 *Suppose* y *has a multivariate elliptical distribution with a finite mean*

[18] For example, if y has a multivariate t-distribution with ν ($\nu > 2$) degrees of freedom, we have $\lim_{n\to\infty} h(1) = \lim_{n\to\infty} h(n) = \frac{\nu-2}{\nu-1} < 1$. Sufficient conditions for all the portfolios to approach the limit $\rho^2_{\mu m|s}$ are available upon request.

[19] When μ and s are perfectly correlated, then, conditional upon s, there is no cross-sectional variation in μ and hence $\rho_{\mu m|s} = 0$ by convention.

and variance. For fixed $\rho_{\mu s}$ *and* ρ_{ms}*, we have*

$$\rho^2_{\mu m|s} \leq \rho^2_{\mu m} \quad \text{if} \quad \rho_{\mu m} \ \text{ is between } \ \tfrac{\rho_{\mu s}\rho_{ms}}{1\pm d},$$
$$\rho^2_{\mu m|s} > \rho^2_{\mu m} \quad \text{if} \quad \rho_{\mu m} \ \text{ is between } \ \rho_{\mu s}\rho_{ms} \pm d \ \text{ but not between } \ \tfrac{\rho_{\mu s}\rho_{ms}}{1\pm d},$$

where $d = \left[(1-\rho^2_{\mu s})(1-\rho^2_{ms})\right]^{\frac{1}{2}}$.

Proposition 4 suggests that if we increase the number of portfolios, the explanatory power of an asset pricing model within the portfolios could be eventually higher or lower than the explanatory power for the firms in the entire population. Which is the case depends crucially on $\rho_{\mu m}$, $\rho_{\mu s}$, and ρ_{ms}. Simply because μ and s are highly correlated does not guarantee that an asset pricing model will do a worse job within portfolios than for the entire population. In Figure 1, we plot the feasible regions of $\rho_{\mu m}$ and ρ_{ms} when $\rho_{\mu s} = 0.9$. The grey region is the region where $\rho^2_{\mu m} \geq \rho^2_{\mu m|s}$ and the dark region is the region where $\rho^2_{\mu m} < \rho^2_{\mu m|s}$. Even though μ and s are highly correlated with each other in this case, we can still see that there is quite a wide range of $\rho_{\mu m}$ and ρ_{ms} that will lead to $\rho^2_{\mu m} < \rho^2_{\mu m|s}$. Without knowing the exact values of $\rho_{\mu m}$, $\rho_{\mu s}$, and ρ_{ms}, we cannot conclude that sorting firms into a large number of portfolios makes an asset pricing model look bad within the portfolios.

III. Simulation Evidence

In the previous section, we derived the theoretical explanatory power of an asset pricing model across and within portfolios but only for the population. In empirical studies, we can only use a finite number of firms in our sample, which raises the question of how relevant are our theoretical results in practice. We address this question by simulation. The basic design of our simulation experiment is as follows. We sample 9000 firms from a population that has a multivariate normal distribution for the triplet (μ, m, s). The size of our sample roughly corresponds to the total number of firms listed on the NYSE, AMEX and NASDAQ at the end of 1997. For the marginal distributions, we assume that both μ and m have a mean of 1%/month and a standard deviation of 0.2%/month. Under this assumption, 99% of the firms have expected return and predicted expected return in the range of 0.48%/month to 1.52%/month.[20] We assume that the sorting variable s has a mean of 1.5 and a standard deviation of 2.5. We interpret s as the natural logarithm of market capitalization (in billions of dollars) of common equity. With this assumption, 99% of the firms have market capitalization falling within the range of \$7.2 million to \$281 billion. We have two sets of simulations. In both sets of simulations, we assume $\rho_{\mu s} = -0.8$ and $\rho_{ms} = -0.5$, so the expected return and the predicted expected return are assumed to be negatively correlated with the sorting variable. The choice of negative values for $\rho_{\mu s}$ and ρ_{ms} is motivated by the observation that small firms seem to have higher average returns as well as higher market betas. The only difference between the two sets of simulations is that we assume $\rho_{\mu m} = 0$ for the first set of simulations and $\rho_{\mu m} = 0.8$ for the second set. Therefore, in the first set of simulations, the asset pricing model alone is completely incapable of explaining the expected returns of individual firms whereas in the second set of simulations, the asset pricing model provides strong explanatory power on the expected returns of individual firms. In Figures 2 and 3, we plot the variables against each other for both sets of simulations. Since the 9000 firms are only a sample from the population, the sample correlation coefficients are close, but not equal to their population counterparts.

For each set of simulations, we sort the firms into $n = 10$, 25, 50, and 100 portfolios based on s, each with an equal number of firms. We then form equally weighted as well as value-weighted portfolios (assuming s is the natural logarithm of size) to examine the explanatory power of the asset pricing model across these portfolios. In Panel A of Table I, we present the sample explanatory power of the asset pricing model, $R^2_{\mu m}(n)$, across different sets of portfolios. To help us see the relation between μ_p^i and m_p^i of these portfolios, we also provide a scatter plot of μ_p^i against m_p^i for the cases of 10 and 100 equally weighted and value-weighted portfolios in Figures 4 and 5.

[20] Alternatively, by dropping the "%/month," we can treat m as the CAPM beta.

Table I about here

From both plots as well as Panel A of Table I, we can observe that for both equally weighted or value-weighted portfolios, the asset pricing model provides very strong explanatory power across all sets of portfolios. Although the asset pricing model is completely incapable of explaining expected returns for individual firms in the first set of simulations, its deficiency is hardly detectable at the portfolio level. Compared with the very good asset pricing model in the second set of simulations, the $R^2_{\mu m}(n)$ at the portfolio level only displays minor differences. A researcher looking at these portfolios will find it difficult to distinguish the good model from the bad model. As n increases, $R^2_{\mu m}(n)$ moves further away from its theoretical value of one. This is so because when n is large, the number of firms in each portfolio is not large enough for the average pricing errors to converge to zero. However, even for $n = 100$, the bad asset pricing model still explains more than 97% of the cross-sectional variation of the expected returns for portfolios. With this kind of consistent performance by the bad asset pricing model across different sets of portfolios, one would be easily tempted to conclude that the bad asset pricing model is really a good model. As we have shown, the truth is that $R^2_{\mu m}(n)$ for portfolios tells us absolutely nothing about $\rho^2_{\mu m}$ for individual firms.

In Panel B of Table I, we present the sample explanatory power of the two asset pricing models within the n portfolios, $R^2_{\mu m \cdot i}$. Since there are many portfolios, instead of presenting all of the $R^2_{\mu m \cdot i}$, we simply present the minimum, maximum and average $R^2_{\mu m \cdot i}$ of the n portfolios. For the first set of simulations, we can see that the asset pricing model performs substantially better within the portfolios than for the entire sample. In all cases, we find that the average $R^2_{\mu m \cdot i}$ is always greater than 50%. From the within-portfolio explanatory power numbers, it would be difficult to infer that the bad asset pricing model is in fact completely incapable of explaining expected returns for individual firms. For the second set of simulations, the numbers for the within-portfolio explanatory power of the asset pricing model are mostly lower than that for the population ($\rho^2_{\mu m} = 0.64$). However, the average $R^2_{\mu m \cdot i}$ stays approximately the same as n increases, showing no sign of converging to zero.[21] Although $\rho_{\mu m}$ plays a role in determining $\rho^2_{\mu m|s}$, we still could not use $R^2_{\mu m \cdot i}$ to obtain a meaningful inference about $\rho^2_{\mu m}$. For example, even though the $\rho^2_{\mu m}$ is very different for the two sets of simulations, the average $R^2_{\mu m \cdot i}$ for the first set of simulations is approximately the same or larger than the average $R^2_{\mu m \cdot i}$ for the second set of simulations. Therefore, just like the sample explanatory power of an asset pricing model across portfolios, the sample explanatory power of an asset pricing model within portfolios cannot be used to evaluate asset pricing models.

IV. Concluding Remarks

The message of this paper is clear. The explanatory power of an asset pricing model across and within portfolios does not provide useful information about the explanatory power of the asset pricing model for individual firms. In the across portfolios case, the only relevant factor is the sorting variable. The explanatory power of the asset pricing model is only a function of the sorting variable and its relation to the expected return and the expected return predicted by the asset pricing model. The joint distribution of the expected return and the predicted expected return plays no role in determining the explanatory power of the asset pricing model at the portfolio level. For the within portfolios case, while the correlation between expected return and predicted expected return plays a role in determining the explanatory power of the asset pricing model within portfolios, the correlations between the sorting variable and the expected and predicted expected returns also play important roles. Depending on the correlations between the three variables, the explanatory power of an asset pricing model for individual firms can be higher or lower than its explanatory power within the portfolios. Therefore,

[21] From (27), the theoretical limit for the within-portfolio explanatory power in the second set of simulations is $\rho^2_{\mu m|s} = 0.593$.

neither the range nor the average explanatory power of an asset pricing model within the portfolios are good indicators of the explanatory power of the asset pricing model for individual firms.

The implications of our results are disturbing. Taken at face value, they suggest that we learn nothing from empirical studies that use portfolios, and that we should never use portfolios in empirical asset pricing studies since they provide no meaningful information. Our view is far less pessimistic. There are merits to using portfolios, and researchers will continue to use portfolios in empirical studies in the foreseeable future. While there are problems with using portfolios, there are also problems with using individual firms. The important task for finance researchers is to find ways to mitigate and address these problems.

While we do not have perfect solutions to these problems, we can make one suggestion. It is that we should not simply look at $\rho^2_{\mu m}(n)$ and $\rho^2_{\mu m \cdot i}$ as the only measures of explanatory power of an asset pricing model. Specifically, many theoretical asset pricing models make predictions about the magnitude of the coefficients relating expected returns to risk measures. Imposing additional restrictions like this will help us detect misspecifications even at the portfolio level. However, this will put empirically motivated models at an unfair advantage since they only suggest variables that are correlated with expected returns, without making predictions about the magnitude or even the sign of their slope coefficients.

Given the importance of portfolios in empirical asset pricing studies, we hope that future research will continue to address the important issue of how portfolios should be formed, and we caution about the proper interpretation of empirical results that use portfolios.

Appendix

Proof of Proposition 1: Substituting $E[\mu|s] = a + bE[m|s]$ in the definition of m_p^i, we obtain $\mu_p^i = a + bm_p^i$. As long as $b \neq 0$, μ_p^i is a linear function of m_p^i and we have $\rho^2_{\mu m}(n) = 1$. Otherwise, if $b = 0$, μ_p^i are constant across portfolios and we have $\rho^2_{\mu m}(n) = 0$. This completes the proof.

Proof of Proposition 2: We start off the proof by deriving the expression of $\sigma_{\mu m \cdot i}$ for the case of a multivariate elliptical distribution. From (18), we have

$$\begin{aligned}
\sigma_{\mu m \cdot i} &= n\int_{s_{i-1}^*}^{s_i^*}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mu - \mu_p^i)(m - m_p^i) f_y(\mu, m, s)\mathrm{d}\mu\mathrm{d}m\mathrm{d}s \\
&= \int_{s_{i-1}^*}^{s_i^*}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mu - E[\mu|s] + E[\mu|s] - \mu_p^i)(m - E[m|s] + E[m|s] - m_p^i)\right. \\
&\qquad\qquad \left. f_{\mu m|s}(\mu, m|s)\mathrm{d}\mu\mathrm{d}m\right] n f_s(s)\mathrm{d}s \\
&= \int_{s_{i-1}^*}^{s_i^*}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mu - E[\mu|s])(m - E[m|s]) f_{\mu m|s}(\mu, m|s)\mathrm{d}\mu\mathrm{d}m\right] n f_s(s)\mathrm{d}s \\
&\quad + \int_{s_{i-1}^*}^{s_i^*}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mu - E[\mu|s])(E[m|s] - m_p^i) f_{\mu m|s}(\mu, m|s)\mathrm{d}\mu\mathrm{d}m\right] n f_s(s)\mathrm{d}s \\
&\quad + \int_{s_{i-1}^*}^{s_i^*}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(E[\mu|s] - \mu_p^i)(m - E[m|s]) f_{\mu m|s}(\mu, m|s)\mathrm{d}\mu\mathrm{d}m\right] n f_s(s)\mathrm{d}s \\
&\quad + \int_{s_{i-1}^*}^{s_i^*}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(E[\mu|s] - \mu_p^i)(E[m|s] - m_p^i) f_{\mu m|s}(\mu, m|s)\mathrm{d}\mu\mathrm{d}m\right] n f_s(s)\mathrm{d}s.
\end{aligned}$$

The expression inside the brackets of the first term is the conditional covariance between μ and m. Under the assumption that y has a multivariate elliptical distribution, it is equal to (see, for example, Theorem 1.5.4 of Muirhead (1982))

$$g(s)\left(\sigma_{\mu m} - \frac{\sigma_{\mu s}\sigma_{ms}}{\sigma_s^2}\right),$$

for some positive function $g(s)$. The second term is equal to zero because $E[m|s] - m_p^i$ is not a function of μ and m; it can therefore be taken out of the brackets. Similarly, the third term is also equal to zero. For the last term, we take $(E[\mu|s] - \mu_p^i)(E[m|s] - m_p^i)$ outside of the brackets and it becomes

$$\int_{s_{i-1}^*}^{s_i^*}(E[\mu|s] - \mu_p^i)(E[m|s] - m_p^i) n f_s(s)\mathrm{d}s.$$

Under the multivariate elliptical distribution assumption, we have

$$\begin{aligned} E[\mu|s] &= E[\mu] + \frac{\sigma_{\mu s}}{\sigma_s^2}(s - E[s]) \\ &= \left(E[\mu] - \frac{\sigma_{\mu s}}{\sigma_s^2}E[s]\right) + \frac{\sigma_{\mu s}}{\sigma_s^2}s. \end{aligned} \tag{A1}$$

Substituting (A1) into (5), we have

$$\begin{aligned} \mu_p^i &= n\int_{s_{i-1}^*}^{s_i^*}\left[\left(E[\mu] - \frac{\sigma_{\mu s}}{\sigma_s^2}E[s]\right) + \frac{\sigma_{\mu s}}{\sigma_s^2}s\right] f_s(s)\mathrm{d}s \\ &= \left(E[\mu] - \frac{\sigma_{\mu s}}{\sigma_s^2}E[s]\right) + \frac{\sigma_{\mu s}}{\sigma_s^2}s_p^i, \end{aligned} \tag{A2}$$

where the second equality follows from the fact that

$$\int_{s_{i-1}^*}^{s_i^*} f_s(s)\mathrm{d}s = \frac{1}{n}. \tag{A3}$$

From (A1) and (A2), we have

$$E[\mu|s] - \mu_p^i = \frac{\sigma_{\mu s}}{\sigma_s^2}(s - s_p^i).$$

Similarly,

$$E[m|s] - m_p^i = \frac{\sigma_{ms}}{\sigma_s^2}(s - s_p^i).$$

Substituting these expressions back, we have

$$\begin{aligned} \sigma_{\mu m\cdot i} &= \left(\sigma_{\mu m} - \frac{\sigma_{\mu s}\sigma_{ms}}{\sigma_s^2}\right)\int_{s_{i-1}^*}^{s_i^*} ng(s)f_s(s)\mathrm{d}s + \left(\frac{\sigma_{\mu s}\sigma_{ms}}{\sigma_s^2}\right)\int_{s_{i-1}^*}^{s_i^*} n\left(\frac{s - s_p^i}{\sigma_s}\right)^2 f_s(s)\mathrm{d}s \\ &= \left(\int_{s_{i-1}^*}^{s_i^*} ng(s)f_s(s)\mathrm{d}s\right)\left[\sigma_{\mu m} - \left(\frac{\sigma_{\mu s}\sigma_{ms}}{\sigma_s^2}\right)h(i)\right]. \end{aligned} \tag{A4}$$

Using a similar proof, we can show that

$$\sigma_{\mu\cdot i}^2 = \left(\int_{s_{i-1}^*}^{s_i^*} ng(s)f_s(s)\mathrm{d}s\right)\left[\sigma_\mu^2 - \left(\frac{\sigma_{\mu s}^2}{\sigma_s^2}\right)h(i)\right], \tag{A5}$$

$$\sigma_{m\cdot i}^2 = \left(\int_{s_{i-1}^*}^{s_i^*} ng(s)f_s(s)\mathrm{d}s\right)\left[\sigma_m^2 - \left(\frac{\sigma_{ms}^2}{\sigma_s^2}\right)h(i)\right]. \tag{A6}$$

With these expressions, $\rho_{\mu m\cdot i}^2$ follows trivially from the definition of the correlation coefficient. For the case of multivariate normality, $g(s) = 1$ and we can verify that the denominator of $h(i)$ is equal to one using (A3). The integral in the numerator of $h(i)$ is the variance of a doubly truncated normal distribution. Using (13.135) of Johnson and Kotz (1994), we have

$$\int_{s_{i-1}^*}^{s_i^*} n \left(\frac{s - s_p^i}{\sigma_s} \right)^2 f_s(s) \mathrm{d}s = 1 - n^2 \left[\phi(c_{i-1}) - \phi(c_i) \right]^2 + n \left[c_{i-1}\phi(c_{i-1}) - c_i \phi(c_i) \right], \quad \text{(A7)}$$

where $c_i = (s_i^* - E[s])/\sigma_s = \Phi^{-1}(i/n)$. This completes the proof.

Proof of Proposition 3: Without loss of generality, we assume $E[s] = 0$ and $\sigma_s = 1$. For any $\epsilon_2 > 0$, we can find an $M > 0$ such that $P[-M < s < M] > 1 - \epsilon_2$. Therefore, allowing for possible rounding, there will be at least $n(1 - \epsilon_2) - 2$ portfolios that consist of firms with $s \in (-M, M)$. Define $f^* = \min_{-M \le s \le M} f_s(s)$ and $g^* = \min_{-M \le s \le M} g(s)$. Let $n_0 = \frac{1}{f^* \sqrt{g^* \epsilon_1}}$, then for $n > n_0$, we have

$$(s_i^* - s_{i-1}^*) f^* \le \frac{1}{n} < \frac{1}{n_0} = f^* \sqrt{g^* \epsilon_1},$$

and hence $(s_i^* - s_{i-1}^*)^2 < g^* \epsilon_1$. When $n > n_0$, for every portfolio that consists of firms with $s \in (-M, M)$, we have

$$\begin{aligned} h(i) &= 1 - \frac{\int_{s_{i-1}^*}^{s_i^*} n(s - s_p^i)^2 f_s(s) \mathrm{d}s}{\int_{s_{i-1}^*}^{s_i^*} n g(s) f_s(s) \mathrm{d}s} \\ &> 1 - \frac{\int_{s_{i-1}^*}^{s_i^*} n(s_i^* - s_{i-1}^*)^2 f_s(s) \mathrm{d}s}{\int_{s_{i-1}^*}^{s_i^*} n g^* f_s(s) \mathrm{d}s} \\ &> 1 - \frac{\int_{s_{i-1}^*}^{s_i^*} n g^* \epsilon_1 f_s(s) \mathrm{d}s}{\int_{s_{i-1}^*}^{s_i^*} n g^* f_s(s) \mathrm{d}s} \\ &= 1 - \epsilon_1. \end{aligned}$$

For the case of normality, we need to show that the portfolios of firms with $s \notin (-M, M)$ also have $h(i) \to 1$. By symmetry, we only need to show that the portfolios with $s \in (-\infty, -M]$ have $h(i) \to 1$. Denote $v_1, v_2, \ldots, v_k$ as the variance of s for the first to the kth portfolio that consists of firms with $s \in (-\infty, -M]$. For the case of normality, $g(s) = 1$ and $h(i) = 1 - v_i$, so we only need to show that $\lim_{n \to \infty} v_i = 0$ for this set of portfolios. Since the normal density is increasing over $(-\infty, -M]$, it is easy to show that $v_1 \geq v_2 \cdots \geq v_k$, and hence it suffices to show that $\lim_{n\to\infty} v_1 = 0$. Denoting the upper cutoff point of s for the first portfolio as c_1 and $\Phi(\cdot)$ as the cumulative density function of the standard normal distribution, we have

$$\begin{aligned}
\lim_{n\to\infty} v_1 &= \lim_{c_1\to-\infty} 1 - \frac{c_1\phi(c_1)}{\Phi(c_1)} - \left[\frac{\phi(c_1)}{\Phi(c_1)}\right]^2 \\
&= \lim_{c_1\to-\infty} \frac{\Phi(c_1)^2 - c_1\phi(c_1)\Phi(c_1) - \phi(c_1)^2}{\Phi(c_1)^2} \\
&= \lim_{c_1\to-\infty} \frac{1}{c_1^2 - 1} = 0,
\end{aligned}$$

where the last line is obtained by repeated use of L'Hôpital's rule. This completes the proof.

Proof of Proposition 4: We first establish the feasible range of $\rho_{\mu m}$ for given values of $\rho_{\mu s}$ and ρ_{ms}. It is well known that the necessary and sufficient condition for a matrix to be positive semidefinite is that all of its principal minors are nonnegative. For a 3 by 3 correlation matrix, this condition is equivalent to requiring its determinant to be nonnegative. The determinant of the correlation matrix of y is given by

$$-\rho_{\mu m}^2 + 2\rho_{\mu s}\rho_{ms}\rho_{\mu m} + (1 - \rho_{\mu s}^2 - \rho_{ms}^2), \tag{A8}$$

which is a quadratic equation in $\rho_{\mu m}$. For it to be nonnegative, $\rho_{\mu m}$ must lie between the roots of the quadratic equation, which are

$$\rho_{\mu s}\rho_{ms} \pm \left(\rho_{\mu s}^2\rho_{ms}^2 + 1 - \rho_{\mu s}^2 - \rho_{ms}^2\right)^{\frac{1}{2}} = \rho_{\mu s}\rho_{ms} \pm d. \tag{A9}$$

For $\rho_{\mu m}^2 \geq \rho_{\mu m|s}^2$, we need

$$\begin{aligned}
& d^2\rho_{\mu m}^2 - (\rho_{\mu m} - \rho_{\mu s}\rho_{ms})^2 \geq 0 \\
\Rightarrow\ & (d^2 - 1)\rho_{\mu m}^2 + 2\rho_{\mu s}\rho_{ms}\rho_{\mu m} - \rho_{\mu s}^2\rho_{ms}^2 \geq 0.
\end{aligned} \tag{A10}$$

The left hand side of the equation is also a quadratic equation in $\rho_{\mu m}$ and it will be nonnegative if $\rho_{\mu m}$ is between the two roots:

$$\frac{-\rho_{\mu s}\rho_{ms} \pm \rho_{\mu s}\rho_{ms}d}{d^2 - 1} = \frac{\rho_{\mu s}\rho_{ms}}{1 \pm d}. \tag{A11}$$

Since $0 \leq d \leq 1$, these two roots fall inside the feasible range of $\rho_{\mu m}$. This completes the proof.

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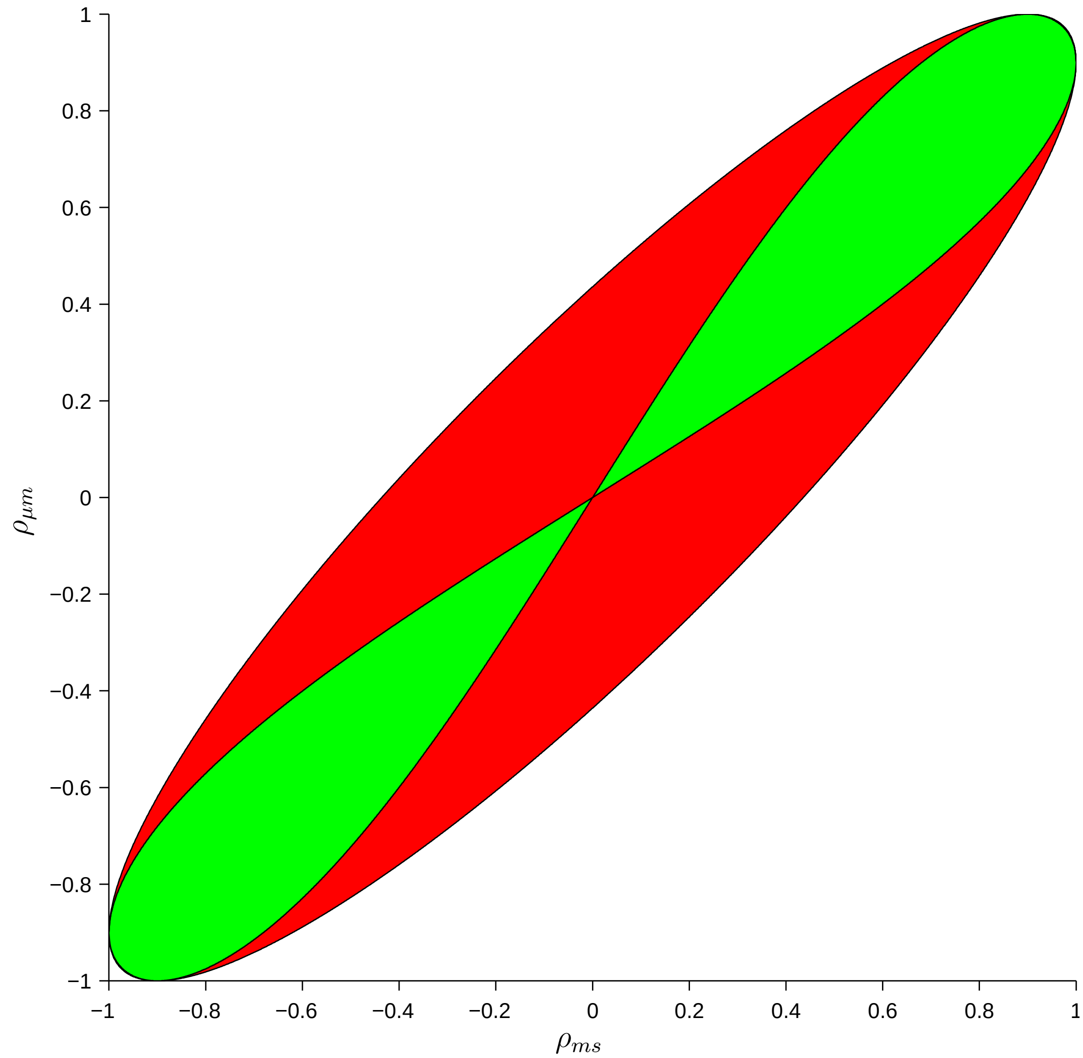

Figure 1: Feasible Regions of $\rho_{\mu m}$ and ρ_{ms} when $\rho_{\mu s} = 0.9$

The figure presents the feasible regions of ρ_{ms} and $\rho_{\mu m}$ when $\rho_{\mu s} = 0.9$. The grey region shows the combinations of $\rho_{\mu m}$ and $\rho_{\mu s}$ that lead to $\rho^2_{\mu m|s} \leq \rho^2_{\mu m}$, and the dark region shows the combinations of $\rho_{\mu m}$ and $\rho_{\mu s}$ that lead to $\rho^2_{\mu m|s} > \rho^2_{\mu m}$. μ is the expected return of a firm, m is its expected return predicted by an asset pricing model, and s is the value of a firm-specific variable which is used to sort firms into portfolios. ρ_{xy} denotes correlation coefficients between variables x and y, and $\rho^2_{\mu m|s}$ is the partial coefficient of determination between μ and m, conditional on the firm-specific variable being equal to s. $\rho^2_{\mu m|s}$ measures the explanatory power of an asset pricing model within a portfolio, when the number of portfolios increases to infinity.

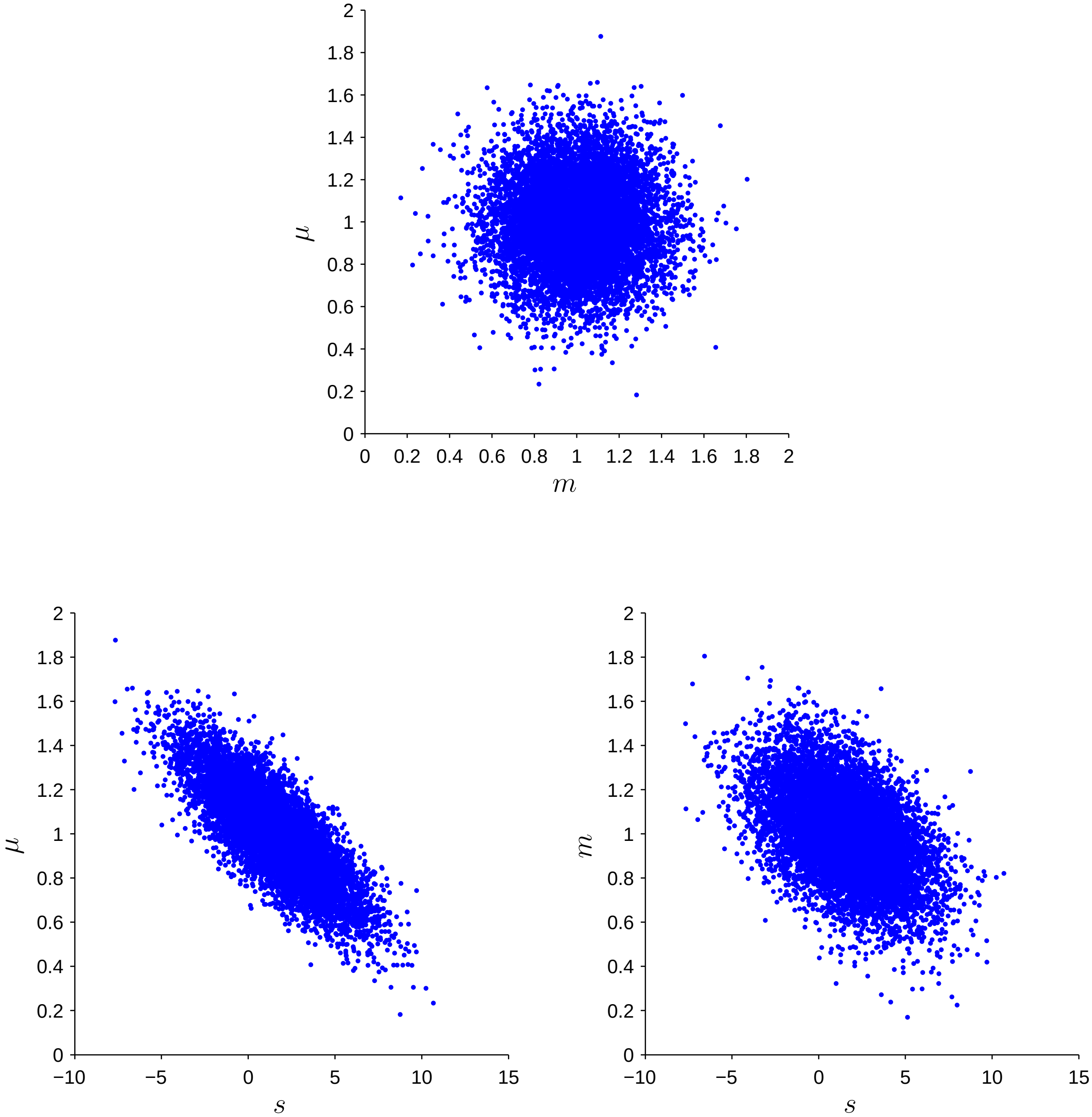

Figure 2: Scatter Plots of Expected Returns, Predicted Expected Returns and Sorting Variables of 9000 Firms for the Bad Model

The figure presents scatter plots of expected return (μ) vs. predicted expected return (m), expected return vs. sorting variable (s), and predicted expected return vs. sorting variable (s) for a sample of 9000 firms. The firms are drawn from a population where the triplet (μ, m, s) is multivariate normally distributed with $\rho_{\mu m} = 0$, $\rho_{\mu s} = -0.8$, and $\rho_{ms} = -0.5$.

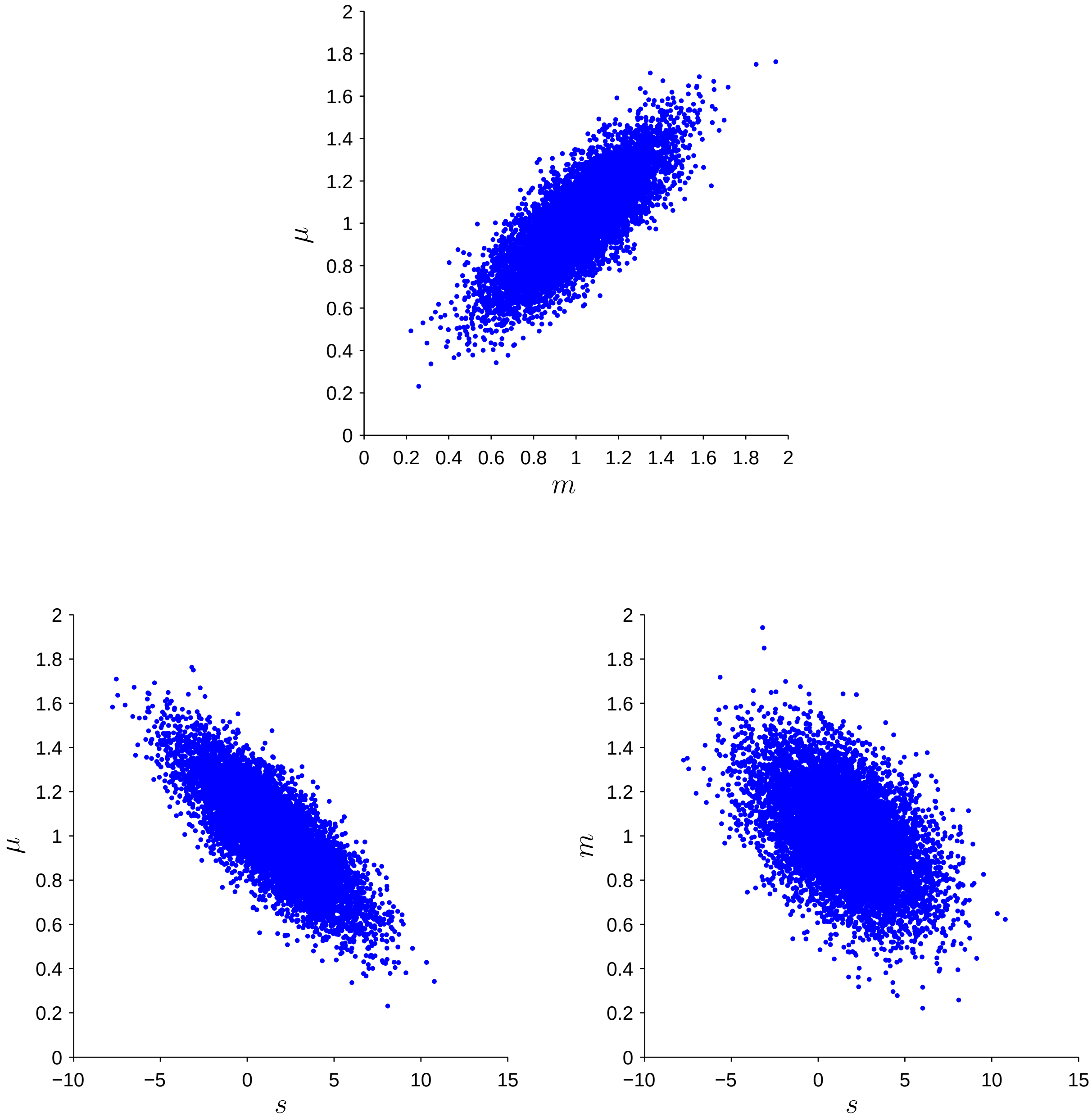

Figure 3: Scatter Plots of Expected Returns, Predicted Expected Returns and Sorting Variables of 9000 Firms for the Good Model

The figure presents scatter plots of expected return (μ) vs. predicted expected return (m), expected return vs. sorting variable (s), and predicted expected return vs. sorting variable for a sample of 9000 firms. The firms are drawn from a population where the triplet (μ, m, s) is multivariate normally distributed with $\rho_{\mu m} = 0.8$, $\rho_{\mu s} = -0.8$, and $\rho_{ms} = -0.5$.

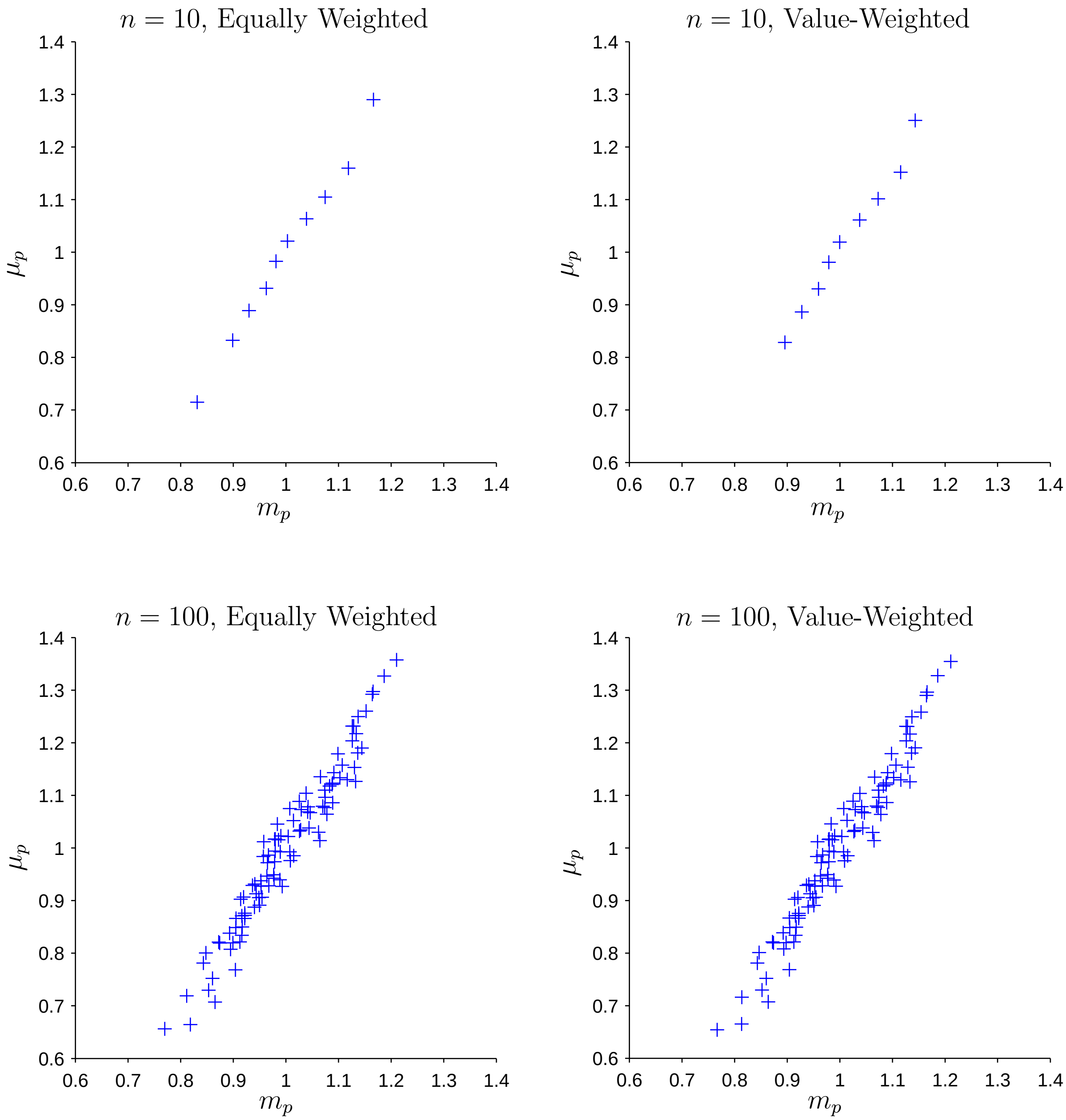

Figure 4: Scatter Plots of Expected Returns and Predicted Expected Returns for 10 and 100 Equally Weighted and Value-Weighted Portfolios for the Bad Model

The figure presents scatter plots of expected portfolio return (μ_p) vs. predicted expected portfolio return (m_p) for 10 and 100 equally weighted and value-weighted portfolios. The portfolios are formed based on 9000 firms which are drawn from a population where the expected return (μ), the predicted expected return (m), and the sorting variable (s) of the individual firms have a multivariate normal distribution with $\rho_{\mu m} = 0$, $\rho_{\mu s} = -0.8$, and $\rho_{ms} = -0.5$. The 9000 firms are sorted into 10 and 100 portfolios based on s, and each portfolio has the same number of firms.

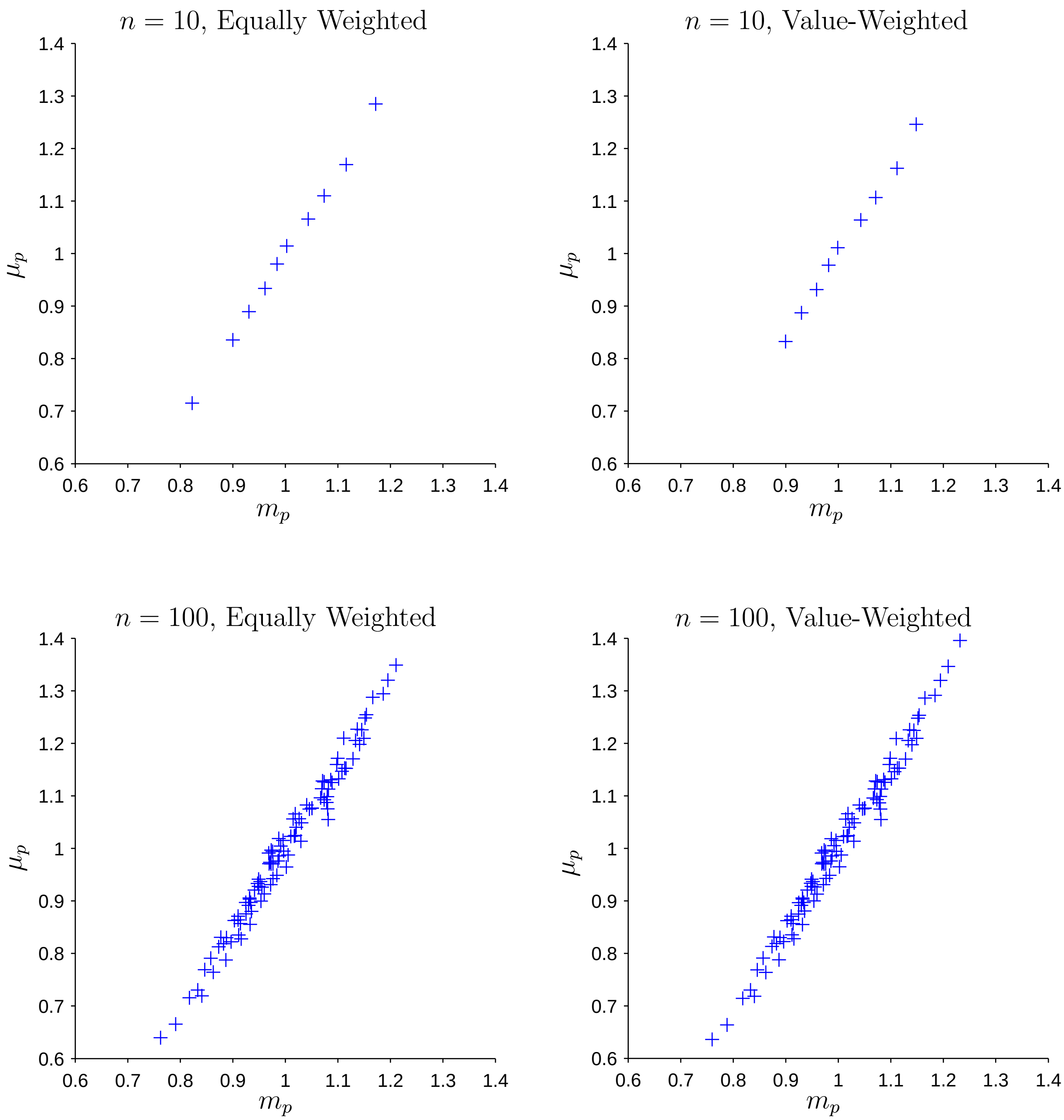

Figure 5: Scatter Plots of Expected Returns and Predicted Expected Returns for 10 and 100 Equally Weighted and Value-Weighted Portfolios for the Good Model

The figure presents scatter plots of expected portfolio return (μ_p) vs. predicted expected portfolio return (m_p) for 10 and 100 equally weighted and value-weighted portfolios. The portfolios are formed based on 9000 firms which are drawn from a population where the expected return (μ), the predicted expected return (m), and the sorting variable (s) of the individual firms have a multivariate normal distribution with $\rho_{\mu m} = 0.8$, $\rho_{\mu s} = -0.8$, and $\rho_{ms} = -0.5$. The 9000 firms are sorted into 10 and 100 portfolios based on s, and each portfolio has the same number of firms.

Table I
Explanatory Power of Asset Pricing Models Across and Within Portfolios

Panel A of the table presents the sample explanatory power of the asset pricing models across different sets of portfolios ($R^2_{\mu m}(n)$). The portfolios are formed based on 9000 firms which are drawn from a population where the expected return (μ), the predicted expected return (m), and the sorting variable (s) of individual firms have a multivariate normal distribution with $\rho_{\mu s} = -0.8$ and $\rho_{ms} = -0.5$. Two sets of results are reported: one under the assumption that $\rho_{\mu m} = 0$ and the other under the assumption that $\rho_{\mu m} = 0.8$. The 9000 firms are sorted into n portfolios based on s, and each portfolio has the same number of firms. We present the cases with n = 10, 20, 50, and 100. For each case, we present the sample explanatory power of the asset pricing model across the equally weighted and the value-weighted portfolios. Panel B of the table presents the sample explanatory power of the asset pricing models within different sets of portfolios ($R^2_{\mu m \cdot i}$) in Panel A. For each case, we present the minimum, the maximum, and the average sample explanatory power of the asset pricing models within the n portfolios.

Panel A: Explanatory Power of Asset Pricing Models Across Portfolios

	$R^2_{\mu m}(n)$			
	$\rho^2_{\mu m} = 0$ (Bad Model)		$\rho^2_{\mu m} = 0.64$ (Good Model)	
n	Equally Weighted	Value-Weighted	Equally Weighted	Value-Weighted
10	0.995	0.994	0.999	0.999
20	0.994	0.993	0.998	0.998
50	0.989	0.988	0.996	0.996
100	0.976	0.975	0.991	0.991

Panel B: Explanatory Power of Asset Pricing Models Within Portfolios

	Distribution of $R^2_{\mu m \cdot i}$					
	$\rho^2_{\mu m} = 0$ (Bad Model)			$\rho^2_{\mu m} = 0.64$ (Good Model)		
n	Minimum	Maximum	Average	Minimum	Maximum	Average
10	0.276	0.627	0.523	0.539	0.611	0.584
20	0.318	0.662	0.570	0.504	0.628	0.583
50	0.278	0.690	0.590	0.469	0.682	0.583
100	0.297	0.738	0.596	0.428	0.727	0.583