Optimal Portfolio Choice with Unknown Benchmark Efficiency

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Abstract

When a benchmark model is inefficient, including test assets in addition to the benchmark portfolios can improve the performance of the optimal portfolio. In reality, the efficiency of a benchmark model relative to the test assets is *ex ante* unknown; moreover, the optimal portfolio is constructed based on estimated parameters. Therefore, whether and how to include the test assets becomes a critical question faced by real world investors. For such a setting, we propose a combining portfolio strategy, optimally balancing the value of including test assets and the effect of estimation errors. The proposed combining strategy can work together with some existing estimation risk reduction strategies. In both empirical datasets and simulations, we show that our proposed combining strategy performs well.

1. Introduction

Starting with the capital asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965), many asset pricing models have been proposed by finance researchers. All these models are now routinely used for performance evaluation, but none of them can completely explain the cross-section of expected asset returns. As a result, a typical problem faced by real world investors is a portfolio choice problem with a benchmark model of *ex ante* unknown efficiency. In this paper, we focus on such a setting. Specifically, we consider a portfolio choice problem for an investor with mean-variance preferences and in a universe with a set of test assets as well as a set of tradable factors suggested by a benchmark model (which we term as the benchmark portfolios). The efficiency of the benchmark model relative to the given set of test assets is, however, unknown to the investor.

When the benchmark model is efficient with respect to the test assets, it is well known that the optimal portfolio only consists of the benchmark portfolios. When the benchmark model is inefficient, including additional test assets can increase the maximum Sharpe ratio (e.g., Dybvig and Ross, 1985). In the case that the mean and the covariance matrix of asset returns (including both the test assets and the benchmark portfolios) are known, there is no need for the investor to explicitly decide whether to include the test assets into the optimal portfolio, and he can simply construct the optimal portfolio according to the theory regardless of the efficiency of the benchmark. This is because when the benchmark is efficient, the constructed optimal portfolio automatically assigns zero weights to the test assets.

In reality, the mean and the covariance matrix of asset returns are unavailable and need to be estimated from a sample of historical data. When estimated parameters are used to construct the optimal portfolio, estimation errors are introduced, which lower the portfolio performance, and the effect of estimation errors increases with the number of assets involved.[1] In addition, even if the benchmark model is *ex ante* efficient, the estimated optimal portfolio is unlikely to assign zero weights to the test assets. Therefore, the investor needs to decide whether and how to include the additional test assets. It is only beneficial to include the

[1]For example, DeMiguel, Garlappi, and Uppal (2009) show that due to estimation errors, many optimization based portfolio rules underperform the naïve equal-weighted portfolio which is free of estimation errors.

test assets when the value in terms of improved maximum Sharpe ratio outweighs the cost associated with additional estimation errors.

We consider eight popular benchmark models and five sets of widely used test assets to examine the value of including the additional test assets into the optimal portfolio when there are estimation errors. If we ignore estimation errors and simply plug in the sample mean and the sample covariance matrix (which we term as the sample optimal portfolio), we find that the effect of estimation errors is so large that it is never beneficial to include the test assets into the optimal portfolio, regardless of the efficiency of the benchmark model relative to the test assets.

Adopting some strategies proposed in the literature to address the estimation errors, we start to observe the value of including the additional test assets in some (but not all) of the cases examined. Specifically, we consider the two-fund rule of Kan and Zhou (2007), the portfolio rule that optimally combines the sample optimal portfolio and the $1/N$ rule (which we term as portfolio PEW), and the use of the shrinkage covariance matrix of Ledoit and Wolf (2004). All of these strategies successfully alleviate the effect of estimation errors in the sample optimal portfolio; but after applying these strategies, whether the optimal portfolio including all the assets outperforms the one with only benchmark portfolios can only be figured out *ex post*. For a real world investor, such *ex post* performance results are not particularly relevant. An investor needs to decide when and how to include the test assets *ex ante*, i.e., using only the information available at the time of his portfolio decision.

Given historical return data, the popular Gibbons-Ross-Shanken (1989) test (hereafter the GRS test) provides a means to understand the efficiency of the benchmark model relative to the test assets. One plausible solution is to use the GRS test results to guide the portfolio choice decision (which we term as the switching strategy). We examine the performance of such strategy and find that its overall performance is mediocre. This suggests that using only the efficiency information from the GRS test (without considering the effect of estimation errors) is not enough to provide a good solution to the portfolio choice problem with unknown benchmark efficiency.

Instead of the switching strategy, we propose a combining strategy. Specifically, in the mean-variance framework, we obtain a portfolio that optimally combines two component portfolios, and prove theoretically that the combining strategy offers performance improvement relative to using only one of the component portfolios. In this optimization process, the estimation errors in the component portfolios are taken into account, and the optimal combining coefficients are derived by balancing the effect of estimation errors and the value of including the test assets. The proposed combining strategy can be directly applied to the sample optimal portfolio or it can be applied together with some existing estimation risk reduction strategies.

Under the assumption that the excess returns of the assets follow a multivariate normal distribution and are independent and identically distributed (i.i.d.) over time, we derive the explicit expressions of the combining coefficients when the combining strategy is applied to the sample optimal portfolio and portfolio PEW. However, the obtained optimal combining coefficients depend on some unknown parameters. To implement the combining strategy, the combining coefficients must be constructed using estimated parameters, which introduces another layer of estimation errors. We show that it is crucial to address the second layer of estimation errors for these combining portfolios, and a cross-validation based shrinkage approach is adopted to deal with the issue.

When the shrinkage covariance matrix of Ledoit and Wolf (2004) is used to address the estimation errors in the sample optimal portfolio, explicit expressions of the optimal combining coefficients are not available. We directly apply the coefficients derived from combining the sample optimal portfolio and find that these coefficients work well with the shrinkage covariance matrix. But the second layer of shrinkage is no longer necessary when the shrinkage covariance matrix is used.[2]

In both empirical datasets and simulations, we evaluate the proposed combining strategy and find that it performs well in general. The proposed combining strategy remains effective after taking into account stock level portfolio turnover and transaction costs. Across the various combining portfolios considered, the one applied together with the shrinkage covariance

[2]This is because the use of the shrinkage covariance matrix helps reduce the effect of estimation errors in the sample optimal portfolio, so the combining coefficients derived for the sample optimal portfolio tend to shrink more than the coefficients derived for the case in which the shrinkage covariance matrix is used (that are, however, not available analytically).

matrix of Ledoit and Wolf (2004) tends to perform the best overall, both with and without the transaction costs.

Portfolio choice problem with benchmark is a setting that is used in many of the existing studies. For example, Black and Litterman (1992) suggest using the CAPM as a benchmark toward which the investor shrinks his views about expected returns. Pástor (2000) approaches the portfolio selection problem in a Bayesian framework that incorporates a prior degree of belief in an asset pricing model, and infers investors' degree of confidence in the asset pricing model by comparing the optimal asset allocation derived from the model and the empirically observed asset allocation (i.e., the market portfolio). These papers do not examine the out-of-sample performance of the portfolios, which is the focus of our paper.

Some other studies use the implications from an asset pricing model to reduce the dimensionality of the portfolio choice problem. For example, MacKinlay and Pástor (2000) assume a risk factor is missing from an asset pricing model, and exploit the relation between the mispricing and the residual covariance matrix to estimate expected returns for portfolio selection. Extending the work of MacKinlay and Pástor (2000), a recent study by Raponi, Uppal, and Zaffaroni (2021) develops a normative theory for constructing mean-variance portfolios that are robust to model misspecification, allowing for both omitted risk factors and asset-specific pricing errors. Brandt, Santa-Clara, and Valkanov (2009) illustrate the use of characteristics based portfolios suggested by a given factor model in constructing the optimal portfolio.[3] In addition, injecting the factor structure implied by an asset pricing model can help estimate the covariance matrix in the high-dimensional case. De Nard, Ledoit, and Wolf (2021) is a recent attempt along this direction. Under the assumption that only a small subset of the test assets has non-zero alphas, Ao, Li, and Zheng (2019) propose a lasso based approach to reduce the number of test assets to be included with the benchmark portfolios. The above studies focus on reducing the dimensionality and do not address the estimation risk explicitly. Our paper, on the other hand, explicitly takes into account the effect of estimation risk in the optimization process and focuses on the expected out-of-sample portfolio performance.

[3]In our empirical analysis, one of the benchmark models (i.e., DMNU-7) and one set of test assets (i.e., DMNU-48) are built on the characteristics based portfolios.

Our paper also differs from most of the existing portfolio choice studies in that we treat the benchmark portfolios and the test assets differently. Even though the portfolio choice problem in a universe with a benchmark model of unknown efficiency is a typical situation faced by the real world investors, most of the existing portfolio choice studies do not focus on such a setting, and therefore, do not differentiate the benchmark portfolios from the test assets and treat them equally. Given the extensive asset pricing studies, it is reasonable to expect that the identified factors are different from the test assets. Our combining strategy deals with such a difference explicitly, with one component portfolio using only the benchmark portfolios and the other using all available assets. We show that the combining strategy outperforms the portfolio that includes all assets without drawing a distinction between the benchmark portfolios and the test assets.

We adopt a similar theoretical framework as in Kan and Zhou (2007), Tu and Zhou (2011), and Kan, Wang, and Zhou (2022), explicitly taking into account the effect of estimation risk in the optimization process with the assumption that the asset returns are i.i.d. multivariate normally distributed. In theory, the obtained optimal portfolio will generate the best-performing portfolio in the examined setting. However, in practice, the obtained optimal portfolio is not directly implementable because the combining coefficients must be constructed using estimated parameters. None of the existing studies explicitly addresses the estimation errors in the implementable combining coefficients. We show that for the portfolio strategies derived in such theoretical framework (e.g., the two-fund rule of Kan and Zhou (2007), portfolio PEW), explicitly addressing the second layer of estimation errors helps improve portfolio performance; and the proposed combining strategy also generates robust performance for the cases in which the i.i.d. normality assumption seems to be violated.[4] In addition, the obtained optimal combining coefficients can be readily applied together with the shrinkage covariance matrix (Ledoit and Wolf, 2004).

The remainder of the paper is organized as follows. Section 2 introduces the portfolio choice problem with unknown benchmark efficiency, discusses the effect of estimation errors,

[4]Both the two-fund rule and portfolio PEW are designed to address the estimation risk in the i.i.d. normality setting; and no consideration is given to the potential departure from such assumption. The proposed cross-validation based second layer of shrinkage does not depend on the i.i.d. normality assumption. In our empirical analysis, we find that the second layer shrinkage performs particularly well for the cases with evidence of violation of the i.i.d. normality assumption.

examines some existing strategies to deal with estimation risk, and evaluates the performance of a plausible solution, i.e., the switching strategy. Section 3 introduces the proposed combining strategy, illustrates how to apply it together with the existing estimation risk reduction strategies, and evaluates the performance of the combining portfolios. Section 4 investigates the stock level turnover of various portfolios. Section 5 concludes and discusses future research opportunities. The proofs are presented in the appendix. Some additional results are reported in the Online Appendix.

2. Portfolio Choice with Unknown Benchmark Efficiency

2.1 The Setup

Consider a portfolio choice problem of an investor in a universe with a risk-free asset, K benchmark portfolios, and N test assets. Let $r_t = [r'_{1,t},\ r'_{2,t}]'$, where $r_{1,t}$ and $r_{2,t}$ are the excess returns of the benchmark portfolios and the test assets at time t, respectively. The mean and the covariance matrix of r_t are given by

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}, \tag{1}$$

where V is assumed to be non-singular.[5] The investor is assumed to choose a portfolio q in order to maximize the mean-variance utility function

$$U_q = \mu_q - \frac{\gamma}{2}\sigma_q^2, \tag{2}$$

where γ is the investor's risk aversion coefficient, and μ_q and σ_q^2 are the mean and the variance of portfolio q. It is well known that the optimal portfolio, denoted as p^*, has weights

$$w_{p^*} = \frac{1}{\gamma} V^{-1}\mu \tag{3}$$

on the $M \equiv K + N$ assets. The utility of holding portfolio p^* is $U_{p^*} = \frac{\theta^2}{2\gamma}$ where $\theta^2 = \mu' V^{-1}\mu$ is the maximum squared Sharpe ratio from using the M assets.

[5] We assume that none of the benchmark portfolios is linear combination of the test assets. However, if k of the K benchmark portfolios are linear combinations of the test assets with $k \leq K$ and $k < N$, we can simply delete k redundant assets from the N test assets and replace N by $N - k$, and the portfolio strategies as well as the analytical results derived in this paper can be readily applied.

When the benchmark is efficient, i.e., the N test assets have zero alphas, $\alpha = \mu_2 - V_{21}V_{11}^{-1}\mu_1 = 0_N$ with 0_N being an $N \times 1$ vector of zeros, it can be shown that portfolio p^* has zero weights on the N test assets, and the weights on the K benchmark portfolios are

$$w_{s^*} = \frac{1}{\gamma}V_{11}^{-1}\mu_1. \tag{4}$$

We call w_{s^*} the benchmark optimal portfolio and denote it as s^*. The utility of holding portfolio s^* is $U_{s^*} = \frac{\theta_1^2}{2\gamma}$ where $\theta_1^2 = \mu_1' V_{11}^{-1}\mu_1$ is the maximum squared Sharpe ratio based on the K benchmark portfolios, and $\theta^2 = \theta_1^2$ when the benchmark is efficient.

When the benchmark is inefficient, i.e., $\alpha \neq 0_N$ and $\theta^2 > \theta_1^2$, portfolio p^* has non-zero weights on the N test assets and it outperforms portfolio s^*. The utility improvement is given by $U_{p^*} - U_{s^*} = \frac{\delta^2}{2\gamma}$ with $\delta^2 = \theta^2 - \theta_1^2 = \mu' V^{-1}\mu - \mu_1' V_{11}^{-1}\mu_1 = \alpha'\Sigma^{-1}\alpha$ and $\Sigma = V_{22} - V_{21}V_{11}^{-1}V_{12}$.[6]

Note that portfolio p^* is identical to portfolio s^* when the benchmark is efficient, and it outperforms portfolio s^* when the benchmark is inefficient. Therefore, it is always optimal for the investor to hold portfolio p^* regardless of the efficiency of the benchmark.

2.2 Estimation Errors and Sample Optimal Portfolios

In practice, μ and V are unknown, and portfolios p^* and s^* are unattainable to the investor. We assume that the investor estimates μ and V using a window of h months of historical data of excess returns, and the estimates of μ and V at time t are given by

$$\hat{\mu}_t = \begin{bmatrix} \hat{\mu}_{1,t} \\ \hat{\mu}_{2,t} \end{bmatrix} = \frac{1}{h}\sum_{\tau=t-h+1}^{t} r_\tau, \quad \hat{V}_t = \begin{bmatrix} \hat{V}_{11,t} & \hat{V}_{12,t} \\ \hat{V}_{21,t} & \hat{V}_{22,t} \end{bmatrix} = \frac{1}{h}\sum_{\tau=t-h+1}^{t} (r_\tau - \hat{\mu}_t)(r_\tau - \hat{\mu}_t)'. \tag{5}$$

Natural estimators of w_{p^*} and w_{s^*} are the sample counterparts of (3) and (4):

$$\hat{w}_{p,t} = \frac{1}{\gamma}\hat{V}_t^{-1}\hat{\mu}_t, \tag{6}$$

$$\hat{w}_{s,t} = \frac{1}{\gamma}\hat{V}_{11,t}^{-1}\hat{\mu}_{1,t}. \tag{7}$$

We call these two portfolios the sample optimal portfolio and the benchmark sample optimal portfolio, and denote them as portfolio p and portfolio s, respectively. The out-of-sample

[6] See Jobson and Korkie (1982) and Gibbons, Ross, and Shanken (1989).

returns of these two portfolios at time $t+1$ are denoted by $r_{p,t+1} = \hat{w}_{p,t}' r_{t+1}$ and $r_{s,t+1} = \hat{w}_{s,t}' r_{1,t+1}$. Note that when $\hat{\mu}_t$ and $\hat{V}_t$ are used instead of the true parameter values, estimation errors are introduced in portfolios p and s.

Without estimation errors, it is always optimal to hold portfolio p^* regardless of the efficiency of the benchmark model. It is no longer the case when there are estimation errors. With more assets involved, portfolio p contains more estimation errors than portfolio s does, and estimation errors will lower out-of-sample portfolio performance *ceteris paribus*. Therefore, it is only preferable to hold portfolio p over portfolio s when the benefit of including the N test assets (in terms of $\theta^2 > \theta_1^2$) outweighs the cost associated with the additional estimation errors.

Unlike w_{p^*} and w_{s^*}, $\hat{w}_{p,t}$ and $\hat{w}_{s,t}$ are random variables as they are functions of the data in the estimation window. For a portfolio with random weights w_t, we follow studies in the existing literature and evaluate its performance based on its expected out-of-sample utility[7]

$$\mathbb{E}[U(w_t)] = \mathbb{E}\left[w_t' \mu - \frac{\gamma}{2} w_t' V w_t \right], \quad (8)$$

which is the utility level that the investor can achieve on average by applying the portfolio rule repeatedly.

Empirically, the expected out-of-sample utility of a portfolio is typically proxied by the certainty equivalent return (CER). We evaluate portfolios based on the CER using a rolling estimation window approach. For each month t, we use data in the most recent h months (up to month t) to compute the weights of a portfolio, and obtain the out-of-sample portfolio return in month $t+1$. This practice generates $T-h$ out-of-sample returns for a given portfolio where T stands for the number of months in the sample period. Based on the $T-h$ out-of-sample returns, we compute the sample mean and the sample variance of the portfolio, $\hat{\mu}$ and $\hat{\sigma}^2$, and the CER of the portfolio is calculated as $CER = \hat{\mu} - \frac{\gamma}{2}\hat{\sigma}^2$. The risk aversion coefficient is assumed to be $\gamma = 3$.[8]

In our empirical analysis, we consider eight benchmark models, and they are: (1) CAPM

[7]For example, Brown (1976), Jorion (1986), Frost and Savarino (1986), Stambaugh (1997), Ter Horst, De Roon, and Werker (2006), Kan and Zhou (2007), Kan, Wang, and Zhou (2022), Tu and Zhou (2011).

[8]The relative ranking of the portfolios studied in this paper is invariant to the value of γ. Nevertheless, in Table OA.5 of the Online Appendix, we also report the CER results for $\gamma = 5$.

using the CRSP value-weighted market portfolio, (2) Fama and French (1993) three-factor model (FF-3), (3) Carhart (1997) four-factor model (Carhart-4), (4) Fama and French (2015) five-factor model (FF-5), (5) Fama and French five-factor plus the momentum factor (FF5-UMD), (6) Hou, Xue, and Zhang (2015) q-factor model (q-factor), (7) Daniel, Hirshleifer, and Sun (2020) three-factor model (DHS), and (8) DeMiguel, Martín-Utrera, Nogales, and Uppal (2020) seven-factor model (DMNU-7). Monthly factor returns of the benchmark models are computed using individual stock return data from the CRSP database. The factors in the first five models are constructed following the approaches specified in Ken French's website. The factors in the remaining three models (i.e., q-factor, DHS, and DMNU-7) are constructed following the approaches in the corresponding papers.[9] For the first three models (i.e., CAPM, FF-3, and Carhart-4), we use the data over the period of 1927/1–2018/12. For FF-5 and FF5-UMD, data are available over the period of 1963/7–2018/12. For q-factor model, DHS, and DMNU-7, the sample periods are 1972/1–2018/12, 1972/7–2018/12, and 1980/1–2018/12, respectively.

We consider five different sets of test assets. The monthly excess returns of the test assets are computed from individual stock returns. The first two sets of test assets are each constructed based on a given anomaly, and they are 10 value-weighted momentum portfolios (MOM-10) and 10 value-weighted idiosyncratic volatility portfolios (IVOL-10). The sample period of these two sets is 1927/1–2018/12. MOM-10 is constructed following the procedure in Ken French's website. To construct IVOL-10, at the beginning of each month from January 1927 to December 2018, idiosyncratic volatility relative to the Fama-French three-factor model is obtained for each stock in CRSP based on daily data in the previous three months. Stocks are sorted into deciles based on the idiosyncratic volatility using the NYSE breakpoints. Stocks with fewer than 20 non-missing daily data in the three-month period are excluded from the portfolios. Value-weighted portfolios are formed and held for one month.

The next two sets of test assets are not based on any anomaly findings, and they are 10 value-weighted industry portfolios (IND-10) and 30 value-weighted industry portfolios (IND-30) over the period of 1927/1–2018/12, constructed following the approach in Ken French's

[9]We thank Alberto Martín-Utrera for sharing with us the monthly firm characteristics data in DeMiguel, Martín-Utrera, Nogales, and Uppal (2020). Based on this dataset, we obtain the factor returns in DMNU-7 as well as the test asset returns in DMNU-48.

website. Industry portfolios are often used as test assets in portfolio choice studies.

The last set of test assets is from DeMiguel, Martín-Utrera, Nogales, and Uppal (2020), containing zero-investment portfolios built on various firm characteristics over 1980/1–2018/12. DeMiguel, Martín-Utrera, Nogales, and Uppal (2020) consider 51 firm-specific characteristics to explain the cross-section of stock returns, taking into account transaction costs. Our last set includes 48 (DMNU-48) out of the 51 firm-characteristic based portfolios in DeMiguel, Martín-Utrera, Nogales, and Uppal (2020), with three portfolios (i.e., *dolvol, lgr, mom6m*) removed due to potential multicollinearity concern following Green, Hand, and Zhang (2017).[10] This last set of test assets contains information from various anomalies, and the recent findings (e.g., DeMiguel, Martín-Utrera, Nogales, and Uppal (2020), and Li, DeMiguel, and Martín-Utrera (2020)) highlight the potential of using these test assets to improve the performance of the benchmark models.

Table 1 reports the CER (in annualized percentage points) for portfolios s and p using the eight benchmark models and the five sets of test assets with $h = 120$ and $\gamma = 3$. We see that for every case in Table 1, $\hat{w}_{p,t}$ greatly underperforms $\hat{w}_{s,t}$, and the underperformance is more apparent when the number of test assets is large (e.g., IND-30, DMNU-48). For the test assets built on the anomaly findings (i.e., MOM-10, IVOL-10, DMNU-48), the existing literature documents that they have non-zero alphas relative to typical asset pricing models, but the findings in Table 1 suggest that the cost associated with the additional estimation errors in test assets greatly outweighs the potential benefit of improved Sharpe ratio. Therefore, if we completely ignore estimation errors and simply plug in the estimated parameters, the effect of estimation errors can be so severe that an investor is almost never better off by including additional test assets in his portfolio, even those with non-zero alphas. At face value, these results cast some doubts on the value of the findings from the anomaly literature.

In terms of the performance of the benchmark sample optimal portfolio s, Table 1 shows that the model performance, in general, increases with the publication date of the model, with the exception of DMNU-7.[11] One main reason for the underperformance of DMNU-7

[10]Note that the six characteristic-based factors in the DMNU-7 model are also in DMNU-48. When DMNU-7 is the benchmark model and DMNU-48 is the test set, the number of test assets becomes 42.

[11]With more studies devoted to identifying better asset pricing models, it is expected that the more recently uncovered asset pricing models are more efficient, e.g., Kan, Wang, and Zheng (2022).

is that among the benchmark models considered, DMNU-7 contains the largest number of factors, and therefore, its portfolio s suffers the highest level of estimation risk.[12]

2.3 Estimation Risk Reduction Strategies

Various strategies have been proposed in the existing literature to deal with estimation risk. In this subsection, we consider several such strategies and examine whether the value of including test assets can be realized after applying these strategies to the sample optimal portfolios.

2.3.1 Kan and Zhou (2007) Two-fund Rule.

Kan and Zhou (2007) propose a two-fund rule to deal with estimation risk,[13] and the proposed portfolio takes the following form:

$$\hat{w}_{p2,t} = \frac{\hat{b}_t}{\gamma}\hat{V}_t^{-1}\hat{\mu}_t = \hat{b}_t\hat{w}_{p,t}, \tag{9}$$

where $\hat{b}_t$ is a newly introduced implementable scalar parameter,[14]

$$\hat{b}_t = \frac{b_1\hat{\theta}_{a,t}^2}{\hat{\theta}_{a,t}^2 + M/h}, \tag{10}$$

with

$$b_1 = \frac{(h-M-1)(h-M-4)}{h(h-2)}, \tag{11}$$

$$\hat{\theta}_{a,t}^2 = \frac{(h-M-2)\hat{\theta}_t^2 - M}{h} + \frac{2(\hat{\theta}_t^2)^{\frac{M}{2}}(1+\hat{\theta}_t^2)^{-\frac{h-2}{2}}}{hB_{\hat{\theta}_t^2/(1+\hat{\theta}_t^2)}(M/2,(h-M)/2)}, \tag{12}$$

[12]We also notice that during the out-of-sample period of DMNU-7 (i.e., 1990/1–2018/12), most of the models tend to perform poorly. See the results of $\hat{w}_{s,t}$ when DMNU-48 is used as the test assets. In this period, all benchmark sample optimal portfolios have negative CER except for the DHS model.

[13]In addition to the two-fund rule, Kan and Zhou (2007) also propose a three-fund rule to deal with estimation errors. The three-fund rule optimally combines the sample optimal portfolio, the sample global minimum variance portfolio, and the risk-free asset to diversify the risk. We find that the performance of the three-fund rule is similar to that of the two-fund rule. For brevity, we skip the three-fund rule in the paper, but the results of the three-fund rule are available upon request.

[14]Under the assumption that r_t is i.i.d. multivariate normally distributed, Kan and Zhou (2007) show that the optimal value of the scalar parameter is $b^* = \frac{b_1\theta^2}{\theta^2+M/h}$, and they recommend $\hat{b}_t$ to implement b^*.

where $B_z(a,b)$ is the incomplete beta function and $\hat{\theta}_t^2 = \hat{\mu}_t' \hat{V}_t^{-1} \hat{\mu}_t$. Note that the newly introduced scalar parameter optimally allocates the weights between the sample optimal portfolio $\hat{w}_{p,t}$ and the risk-free asset, and therefore, this portfolio is called a two-fund rule and we use p_2 to denote it.

Similarly, we can apply the two-fund rule to the benchmark sample optimal portfolio,

$$\hat{w}_{s_2,t} = \frac{\hat{c}_t}{\gamma} \hat{V}_{11,t}^{-1} \hat{\mu}_{1,t} = \hat{c}_t \hat{w}_{s,t}, \tag{13}$$

where

$$\hat{c}_t = \frac{c_1 \hat{\theta}_{1a,t}^2}{\hat{\theta}_{1a,t}^2 + K/h}, \tag{14}$$

and

$$c_1 = \frac{(h-K-1)(h-K-4)}{h(h-2)}, \tag{15}$$

$$\hat{\theta}_{1a,t}^2 = \frac{(h-K-2)\hat{\theta}_{1,t}^2 - K}{h} + \frac{2(\hat{\theta}_{1,t}^2)^{\frac{K}{2}}(1+\hat{\theta}_{1,t}^2)^{-\frac{h-2}{2}}}{hB_{\hat{\theta}_{1,t}^2/(1+\hat{\theta}_{1,t}^2)}(K/2,(h-K)/2)}, \tag{16}$$

with $\hat{\theta}_{1,t}^2 = \hat{\mu}_{1,t}' \hat{V}_{11,t}^{-1} \hat{\mu}_{1,t}$. We call $\hat{w}_{s_2,t}$ the benchmark two-fund portfolio and denote it as portfolio s_2.

2.3.2 Combining with the $1/N$ Rule (PEW).

DeMiguel, Garlappi, and Uppal (2009) show that due to estimation errors, many optimization based portfolio rules underperform the naïve equal-weighted portfolio (i.e., the $1/N$ rule) which is free of estimation errors. Tu and Zhou (2011) use the $1/N$ rule as the shrinkage target and recommend optimally combining the estimated optimal portfolio with the $1/N$ rule to address the estimation risk. We consider a similar portfolio strategy that optimally combines the sample optimal portfolio and the $1/N$ rule,

$$\hat{w}_{pew,t} = \hat{\kappa}_{1,t} \hat{w}_{p,t} + \hat{\kappa}_{2,t} w_{ew}, \tag{17}$$

where $w_{ew} = 1_M/M$ is the equal-weighted portfolio, and $\hat{\kappa}_{1,t}$ and $\hat{\kappa}_{2,t}$ are the implementable optimal combining coefficients,[15]

$$\hat{\kappa}_{1,t} = \frac{\hat{\tau}^2_{a,t}}{\frac{\hat{\tau}^2_{a,t}+\hat{\theta}^2_{ew,t}+M/h}{b_1} - \frac{h\hat{\theta}^2_{ew}}{h-M-2}}, \quad \hat{\kappa}_{2,t} = \frac{\hat{\mu}_{ew,t}}{\gamma\hat{\sigma}^2_{ew,t}}\left(1 - \frac{h\hat{\kappa}_{1,t}}{h-M-2}\right), \tag{18}$$

with $\hat{\mu}_{ew,t} = w'_{ew}\hat{\mu}_t$, $\hat{\sigma}^2_{ew,t} = w'_{ew}\hat{V}_t w_{ew}$, $\hat{\theta}^2_{ew,t} = \hat{\mu}^2_{ew,t}/\hat{\sigma}^2_{ew,t}$, and

$$\hat{\tau}^2_{a,t} = \frac{(h-M-2)\hat{\tau}^2_t - (M-1)(1+\hat{\theta}^2_{ew,t})}{h} + \frac{2(1+\hat{\theta}^2_{ew,t})\Delta^{\frac{M-1}{2}}(1+\Delta)^{-\frac{h-3}{2}}}{hB_{\Delta/(1+\Delta)}((M-1)/2,(h-M)/2)}, \tag{19}$$

where $\hat{\tau}^2_t = \hat{\theta}^2_t - \hat{\theta}^2_{ew,t}$ and $\Delta = \hat{\tau}^2_t/(1+\hat{\theta}^2_{ew,t})$. We denote this implementable portfolio as PEW.

2.3.3 Shrinkage Covariance Matrix of Ledoit and Wolf (2004).

When the number of assets is large relative to the length of the estimation window, the sample covariance matrix is typically not well-conditioned. To address this issue, Ledoit and Wolf (2004) introduce a shrinkage estimator which is a linear combination of the sample covariance matrix and the identity matrix,[16]

$$\hat{V}^{LW2004}_t = (1-\rho_t)\hat{V}_t + \rho_t\nu_t I_M, \tag{20}$$

where I_M is an $M \times M$ identity matrix, ν_t is the shrinkage target which equals to the average of the eigenvalues of $\hat{V}_t$, and ρ_t is the shrinkage intensity

$$\rho_t = \frac{\min\left[\frac{1}{h^2}\sum_{s=t-h+1}^{t}||(r_s-\hat{\mu}_t)(r_s-\hat{\mu}_t)' - \hat{V}_t||^2, ||\hat{V}_t - \nu_t I_M||^2\right]}{||\hat{V}_t - \nu_t I_M||^2} \tag{21}$$

[15] The optimal combining coefficients κ^*_1 and κ^*_2 are derived to maximize the expected out-of-sample utility with the assumption that r_t are i.i.d. multivariate normally distributed; $\hat{\kappa}_{1,t}$ and $\hat{\kappa}_{2,t}$ are the corresponding implementable combining coefficients. The derivation of κ^*_1 and κ^*_2 is available in the appendix. Unlike Tu and Zhou (2011), we do not require the two combining coefficients to sum up to one so that the expected out-of-sample utility of the optimal portfolio is proportional to $1/\gamma$ and the performance ranking of this portfolio relative to other portfolios studied in this paper is *invariant* to the value of γ. Under the i.i.d. normality assumption, we can prove theoretically that the optimal portfolio without requiring the two combining coefficients to sum up to one outperforms the one with the restriction. Empirically, we find that the two optimal portfolios perform similarly, and the combining strategy proposed in Section 3 can improve the performance of both portfolios.

[16] Ledoit and Wolf (2017) propose the use of a nonlinear shrinkage estimator of the covariance matrix, and Ledoit and Wolf (2020) present an analytical formula for the nonlinear shrinkage estimator. Our empirical analyses show that the optimal portfolio using the nonlinear shrinkage estimator tends to underperform the one using the linear shrinkage estimator. For brevity, we skip the optimal portfolio with the nonlinear shrinkage estimator in the paper, but the corresponding results are available upon request.

with $||A|| = \sqrt{\text{tr}(AA')/M}$ being the Frobenius norm. When $\hat{V}_t^{LW2004}$ instead of $\hat{V}_t$ is used in (6), we obtain the sample optimal portfolio with the shrinkage covariance matrix, and denote it as $\hat{w}_{p,t}^{LW2004}$. Similar shrinkage covariance matrix can be applied in the benchmark sample optimal portfolio to obtain the benchmark sample optimal portfolio with the shrinkage covariance matrix, $\hat{w}_{s,t}^{LW2004}$.

2.3.4 Portfolio Performance with Estimation Risk Reduction Strategies.

Table 2 reports the CER of the optimal portfolios after adopting various estimation risk reduction strategies based on the same eight benchmark models and the five sets of test assets for $h = 120$ and $\gamma = 3$. In Panel A, the two-fund rule is applied to both the benchmark sample optimal portfolio and the sample optimal portfolio (i.e., $\hat{w}_{s_2,t}$ and $\hat{w}_{p_2,t}$). In Panel B, we only apply the PEW rule to the sample optimal portfolio (i.e., $\hat{w}_{pew,t}$), and continue to use the two-fund rule for the benchmark sample optimal portfolio (i.e., $\hat{w}_{s_2,t}$).[17] In Panel C, the linear shrinkage covariance matrix of Ledoit and Wolf (2004) are adopted for both the sample optimal portfolio p and the benchmark sample optimal portfolio s.

Relative to the performance of $\hat{w}_{p,t}$ and $\hat{w}_{s,t}$ in Table 1, the results in Table 2 suggest that the strategies considered in the table are all effective to deal with estimation risk in the sample optimal portfolios. Because $\hat{w}_{p,t}$ contains more estimation errors than $\hat{w}_{s,t}$ does, the performance improvement for $\hat{w}_{p,t}$ is larger than that for $\hat{w}_{s,t}$. As a result, we start to observe the value of including additional test assets into the optimal portfolios for some cases in Table 2.

Since the more recent models (e.g., q-factor, DHS, and DMNU-7) are likely to be more efficient, it is harder to identify test assets that can further improve the performance of those benchmark models. In Table 2, the value of including additional test assets is only shown for the earlier models. When MOM-10 is included as test assets, the performance of the CAPM, FF-3, and FF-5 are improved in Panels A and B, and that of the CAPM and FF-3 are improved in Panel C. The fact that including MOM-10 does not improve the performance

[17] Since the number of factors in the benchmark models is typically small and the estimation errors in the benchmark sample optimal portfolio s are low, compared to the benchmark two-fund portfolio (i.e., $\hat{w}_{s_2,t}$), it makes small difference whether to apply the PEW rule to the benchmark sample optimal portfolio.

of Carhart-4 and FF5-UMD is likely because momentum is one of the model factors. When IVOL-10 is included as test assets, the performance of the CAPM, FF-3, and Carhart-4 are improved in Panels A and B, and that of the CAPM, FF-3, Carhart-4, FF-5, and FF5-UMD are improved in Panel C.

Including industry portfolios (i.e., IND-10 and IND-30), in general, does not improve the performance of the benchmark models (except for IND-10 with FF-3 and Carhart-4 in Panel B). This is likely because industry portfolios are not constructed based on anomaly findings, and limited improvement is expected from including the industry portfolios.

DMNU-48 contains information from various anomaly findings. It is somewhat surprising to notice that when DMNU-48 is used as test assets, the performance is not improved even for the earlier models in Table 2. One major reason for the poor performance is the high estimation risk due to the large number of test assets in DMNU-48. Another likely reason for such results is that the underlying parameters (i.e., the mean and the covariance matrix) of the anomaly portfolios are time varying, and therefore, the sample mean and the sample covariance matrix from the estimation window do not provide a good estimate of the underlying parameters out-of-sample.[18]

In summary, Table 2 presents evidence that when adopting some popular strategies to reduce the impact of estimation errors, including the test assets into the optimal portfolio can be beneficial in some cases. However, the results in Table 2 are *ex post* portfolio performance, and such information is not available to the investors at the time of portfolio construction. Whether and how to include the test assets into their optimal portfolios is a decision that

[18]Existing studies document time varying stock return predictability, e.g., McLean and Pontiff (2016) and Green, Hand, and Zhang (2017). In Section OA.2 of the Online Appendix, we examine the results based on simulated data using the stationary block bootstrap procedure of Politis and Romano (1994). When the bootstrap procedure is applied, the underlying data generating process becomes stationary, and we start to observe performance improvement in some cases.

investors need to make *ex ante*.[19]

2.4 A Plausible Solution: the Switching Strategy

Whether to include the test assets into the optimal portfolio depends on the tradeoff between the additional estimation errors and the potential efficiency improvement introduced by the test assets. We do not have a readily available measure for the estimation errors. In terms of efficiency improvement, Gibbons, Ross, and Shanken (1989) propose a widely used test (the GRS test) for the efficiency of a benchmark model with respect to a given set of test assets using sample data. We examine the performance of a switching portfolio strategy that is guided by the GRS test results using the data in the estimation window. The performance results and the corresponding discussion are reported in Section OA.1 of the Online Appendix.

Overall, we find that the performance of the switching strategy is mediocre, suggesting that using only the efficiency information from the GRS test (without considering the effect of estimation risk) to guide the portfolio choice decision does not seem to be a good solution to the portfolio choice problem with unknown benchmark efficiency.

3. The Combining Strategy

In this section, we propose a combining portfolio strategy, which takes into account both the efficiency information and the effect of estimation errors. The proposed combining strategy is a general one in the sense that it can be applied to different component portfolios (i.e., the portfolios to be combined). The general combining framework is introduced in Section 3.1, and we show theoretically the usefulness of the combining strategy. In Section 3.2, the

[19] Based on the results in Table 2, one may argue that the best strategy is to build optimal portfolios based on those recently uncovered asset pricing models (e.g., q-factor, DHS, and DMNU-7). Given that these models are recently uncovered and we do not have enough out-of-sample data, the empirical results in Table 2 may not provide a full assessment of the performance of the models. In this paper, we take an agnostic stand and assume that the efficiency of the model relative to the given set of test assets is unknown to the investor. We admit that using the empirical findings from the existing literature to identify the best set of assets (both the benchmark portfolios and the test assets) to form optimal portfolios is an interesting question to explore, but it is not the focus of this paper. We will leave it for future study.

combining strategy is applied to various component portfolios, and some implementation issues are discussed. Section 3.3 reports the empirical performance of the combining strategy.

3.1 The General Framework

Let $\hat{w}_{1,t}$ and $\hat{w}_{2,t}$ be the weights on the M assets of two component portfolios, and the weights are obtained by using information from the estimation window so that they can contain estimation errors. Consider a portfolio strategy that combines $\hat{w}_{1,t}$ and $\hat{w}_{2,t}$, i.e.,

$$\hat{w}_{c,t} = \lambda_1 \hat{w}_{1,t} + \lambda_2 \hat{w}_{2,t}, \tag{22}$$

where λ_1 and λ_2 are two scalar combining coefficients. The optimal combining coefficients are determined by maximizing the expected out-of-sample utility of the combining portfolio

$$\mathbb{E}\left[U_c(\lambda_1,\lambda_2)\right] = \mathbb{E}\left[\hat{w}_{c,t}'\mu\right] - \frac{\gamma}{2}\mathbb{E}\left[\hat{w}_{c,t}'V\hat{w}_{c,t}\right] = \lambda_1\bar{\mu}_1 + \lambda_2\bar{\mu}_2 - \frac{\gamma}{2}\left(\lambda_1^2\bar{\sigma}_1^2 + \lambda_2^2\bar{\sigma}_2^2 + 2\lambda_1\lambda_2\bar{\sigma}_{12}\right), \tag{23}$$

where

$$\bar{\mu}_1 = \mathbb{E}\left[\mu_{1,t}\right] = \mathbb{E}\left[\hat{w}_{1,t}'\mu\right], \tag{24}$$

$$\bar{\mu}_2 = \mathbb{E}\left[\mu_{2,t}\right] = \mathbb{E}\left[\hat{w}_{2,t}'\mu\right], \tag{25}$$

$$\bar{\sigma}_1^2 = \mathbb{E}\left[\sigma_{1,t}^2\right] = \mathbb{E}\left[\hat{w}_{1,t}'V\hat{w}_{1,t}\right], \tag{26}$$

$$\bar{\sigma}_2^2 = \mathbb{E}\left[\sigma_{2,t}^2\right] = \mathbb{E}\left[\hat{w}_{2,t}'V\hat{w}_{2,t}\right], \tag{27}$$

$$\bar{\sigma}_{12} = \mathbb{E}\left[\sigma_{12,t}\right] = \mathbb{E}\left[\hat{w}_{1,t}'V\hat{w}_{2,t}\right], \tag{28}$$

with $\mu_{1,t}$, $\mu_{2,t}$, $\sigma_{1,t}^2$, $\sigma_{2,t}^2$ being the conditional mean and variance of portfolios $\hat{w}_{1,t}$ and $\hat{w}_{2,t}$ and $\sigma_{12,t}$ being the conditional covariance between the two portfolios. Differentiating $\mathbb{E}\left[U_c(\lambda_1,\lambda_2)\right]$ with respect to λ_1 and λ_2 and setting them equal to zero, we obtain the optimal values of λ_1 and λ_2 by solving the first order conditions. The following proposition presents the results.

Proposition 1 *The λ_1 and λ_2 that maximize $\mathbb{E}\left[U_c(\lambda_1,\lambda_2)\right]$ are given by*

$$\lambda_1^* = \frac{\bar{\mu}_1}{\gamma\bar{\sigma}_1^2}\left(\frac{1-\left(\bar{\theta}_2/\bar{\theta}_1\right)\bar{\rho}_{12}}{1-\bar{\rho}_{12}^2}\right), \tag{29}$$

$$\lambda_2^* = \frac{\bar{\mu}_2}{\gamma\bar{\sigma}_2^2}\left(\frac{1-\left(\bar{\theta}_1/\bar{\theta}_2\right)\bar{\rho}_{12}}{1-\bar{\rho}_{12}^2}\right), \tag{30}$$

where $\bar{\theta}_i = \bar{\mu}_i/\bar{\sigma}_i$ and $\bar{\sigma}_i = \sqrt{\bar{\sigma}_i^2}$ for $i = 1, 2$ and $\bar{\rho}_{12} = \bar{\sigma}_{12}/(\bar{\sigma}_1\bar{\sigma}_2)$. Plug (29) and (30) in (23), we get the maximum expected out-of-sample utility

$$\mathbb{E}\left[U_c(\lambda_1^*, \lambda_2^*)\right] = \frac{\bar{\theta}_1^2 + \bar{\theta}_2^2 - 2\bar{\theta}_1\bar{\theta}_2\bar{\rho}_{12}}{2\gamma\left(1-\bar{\rho}_{12}^2\right)} > \max\left[\frac{\bar{\theta}_1^2}{2\gamma}, \frac{\bar{\theta}_2^2}{2\gamma}\right]. \tag{31}$$

The term before the parentheses in (29) and (30), i.e., $\bar{\mu}_i/(\gamma\bar{\sigma}_i^2)$ for $i = 1, 2$, represents the optimal weight allocated to portfolio $\hat{w}_{i,t}$ without combining the other portfolio. The expected out-of-sample utility of this optimal portfolio is $\bar{\theta}_i^2/(2\gamma)$. When the two component portfolios are optimally combined, (31) shows that the combining strategy always outperforms the optimal portfolio using only one component portfolio.[20] (29), (30), and (31) suggest that the optimal weights allocated to the component portfolios and the performance improvement of the combining strategy depend on the relative efficiency (i.e., $\bar{\theta}_2/\bar{\theta}_1$) as well as the potential diversification effect (captured by $\bar{\rho}_{12}$) between the two component portfolios.[21] It can be shown that $\mathbb{E}[U_c(\lambda_1^*, \lambda_2^*)]$ increases with $\bar{\theta}_2$ if and only if $\bar{\theta}_2 > \bar{\rho}_{12}\bar{\theta}_1$, and it increases with $\bar{\rho}_{12}$ if and only if $\bar{\rho}_{12} > \min[\bar{\theta}_1/\bar{\theta}_2, \bar{\theta}_2/\bar{\theta}_1]$.

3.2 Applying the Combining Strategy

We now apply the combining strategy to various component portfolios. In Section 3.2.1, the combining strategy is applied to the sample optimal portfolio. In Section 3.2.2 and 3.2.3, we discuss how to apply the combining strategy when some estimation risk reduction strategies are first applied to the sample optimal portfolio. Specifically, we consider the same set of portfolios examined in Section 2.3.

[20]Using the Cauchy-Schwarz inequality, it can be shown that $|\bar{\rho}_{12}| = \frac{|E[\sigma_{12,t}]|}{\sqrt{E[\sigma_{1,t}^2]E[\sigma_{2,t}^2]}} \leq \frac{E[\sigma_{1,t}\sigma_{2,t}]}{\sqrt{E[\sigma_{1,t}^2]E[\sigma_{2,t}^2]}} \leq 1$. Unless the two component portfolios are perfectly positively or negatively conditionally correlated and $\sigma_{1,t}$ is proportional to $\sigma_{2,t}$, we have $|\bar{\rho}_{12}| < 1$.

[21]Note that $\bar{\theta}_i$ contains both the true efficiency (i.e., the Sharpe ratio without estimation errors) and the effect of estimation errors in $\hat{w}_{i,t}$. Estimation errors also affect $\bar{\rho}_{12}$.

3.2.1 The Sample Optimal Portfolio.

We first derive the optimal combining strategy based on the sample optimal portfolio and the benchmark sample optimal portfolio,[22]

$$\hat{w}_{c,t} = \lambda_1 \hat{w}_{p,t} + \lambda_2 \tilde{w}_{s,t}, \tag{32}$$

where $\tilde{w}_{s,t} = [\hat{w}'_{s,t},\ 0'_N]'$. Under the assumption that r_t follows a multivariate normal distribution and is i.i.d. over time, we get

$$\bar{\mu}_p = \mathbb{E}[\mu_{p,t}] = \mathbb{E}[\hat{w}'_{p,t}\mu] = \frac{h\theta^2}{\gamma(h-M-2)}, \tag{33}$$

$$\bar{\mu}_s = \mathbb{E}[\mu_{s,t}] = \mathbb{E}[\hat{w}'_{s,t}\mu_1] = \frac{h\theta_1^2}{\gamma(h-K-2)}, \tag{34}$$

$$\bar{\sigma}_p^2 = \mathbb{E}[\sigma_{p,t}^2] = \mathbb{E}[\hat{w}'_{p,t}V\hat{w}_{p,t}] = \frac{h(h-2)(M+h\theta^2)}{\gamma^2(h-M-1)(h-M-2)(h-M-4)}, \tag{35}$$

$$\bar{\sigma}_s^2 = \mathbb{E}[\sigma_{s,t}^2] = \mathbb{E}[\hat{w}'_{s,t}V_{11}\hat{w}_{s,t}] = \frac{h(h-2)(K+h\theta_1^2)}{\gamma^2(h-K-1)(h-K-2)(h-K-4)}, \tag{36}$$

$$\bar{\sigma}_{ps} = \mathbb{E}[\sigma_{ps,t}] = \mathbb{E}[\hat{w}'_{p,t}V\tilde{w}_{s,t}] = \frac{h(h-2)(K+h\theta_1^2)}{\gamma^2(h-K-1)(h-K-4)(h-M-2)}. \tag{37}$$

The proof of the above expressions is in the appendix. Applying (29) and (30) from Proposition 1, we obtain

$$\lambda_1^* = \frac{(h-M-2)\delta^2}{B-C}, \tag{38}$$

$$\lambda_2^* = (h-K-2)\left(\frac{\theta_1^2}{C} - \frac{\delta^2}{B-C}\right), \tag{39}$$

where

$$B = \frac{(h-M-2)(\theta^2 + \frac{M}{h})}{b_1}, \tag{40}$$

$$C = \frac{(h-K-2)(\theta_1^2 + \frac{K}{h})}{c_1}. \tag{41}$$

The expected out-of-sample utility of the optimal combining portfolio is given by

$$\mathbb{E}[U_c(\lambda_1^*, \lambda_2^*)] = \frac{h\theta_1^4}{2\gamma C} + \frac{h\delta^4}{2\gamma(B-C)} = \frac{h\theta^4}{2\gamma B} + \frac{h(B-C)}{2\gamma BC}\left(\theta^2 - \frac{B\delta^2}{B-C}\right)^2. \tag{42}$$

[22]Given that the two-fund rule scales up the sample optimal portfolio, optimally combining $\hat{w}_{p,t}$ and $\hat{w}_{s,t}$ is equivalent to optimally combining the two-fund portfolios with the true optimal scaling parameters.

When $h > M + 4$, it is easy to see that both B and C are positive. In addition, it can be verified that $\bar{\theta}_p^2 = \bar{\mu}_p^2/\bar{\sigma}_p^2 = h\theta^4/B$, $\bar{\theta}_s^2 = \bar{\mu}_s^2/\bar{\sigma}_s^2 = h\theta_1^4/C$, and $\bar{\rho}_{ps}^2 = \bar{\sigma}_{ps}^2/(\bar{\sigma}_p^2\bar{\sigma}_s^2) = C/B$. Because $0 < \bar{\rho}_{ps}^2 < 1$, we have $B > C$. When the benchmark portfolios are not *ex ante* efficient (i.e., $\delta > 0$), (38) suggests that the optimal weight on the sample optimal portfolio (i.e., λ_1^*) is positive, and λ_1^* increases with δ but decreases with $N = M - K$. On the other hand, the optimal weight on the benchmark sample optimal portfolio λ_2^* is positive when $\theta_1^2/\theta^2 > \bar{\rho}_{ps}^2$ and it decreases with δ and increases with N.[23]

Implementable Combining Rule

Note that λ_1^* and λ_2^* in (38) and (39) depend on unknown parameters θ_1^2 and δ^2, so the optimal combining portfolio using λ_1^* and λ_2^* is unattainable. Natural estimators of θ_1^2 and δ^2 are their sample counterparts

$$\hat{\theta}_{1,t}^2 = \hat{\mu}_{1,t}'\hat{V}_{11,t}^{-1}\hat{\mu}_{1,t}, \quad \hat{\delta}_t^2 = \hat{\mu}_t'\hat{V}_t^{-1}\hat{\mu}_t - \hat{\mu}_{1,t}'\hat{V}_{11,t}^{-1}\hat{\mu}_{1,t}. \tag{43}$$

However, it is known that these estimators can be heavily biased, especially when h is small, so we adopt an adjusted estimator suggested by Kubokawa, Robert, and Saleh (1993) to estimate θ_1^2 and δ^2. The adjusted estimator of θ_1^2 is given in (16) and that of δ^2 is[24]

$$\hat{\delta}_{a,t}^2 = \frac{(h - M - 2)\hat{\delta}_t^2 - N(1 + \hat{\theta}_{1,t}^2)}{h} + \frac{2(1 + \hat{\theta}_{1,t}^2)\Xi^{\frac{N}{2}}(1 + \Xi)^{-\frac{h-K-2}{2}}}{hB_{\Xi/(1+\Xi)}(N/2, (h - M)/2)}, \tag{44}$$

where $\Xi = \hat{\delta}_t^2/(1 + \hat{\theta}_{1,t}^2)$. With $\hat{\theta}_{1a,t}^2$ and $\hat{\delta}_{a,t}^2$ available, we can obtain the adjusted estimators of λ_1^* and λ_2^*,

$$\hat{\lambda}_{1a,t} = \frac{(h - M - 2)\hat{\delta}_{a,t}^2}{\hat{B}_{a,t} - \hat{C}_{a,t}}, \tag{45}$$

$$\hat{\lambda}_{2a,t} = (h - K - 2)\left(\frac{\hat{\theta}_{1a,t}^2}{\hat{C}_{a,t}} - \frac{\hat{\delta}_{a,t}^2}{\hat{B}_{a,t} - \hat{C}_{a,t}}\right), \tag{46}$$

where

$$\hat{B}_{a,t} = \frac{(h - M - 2)\left[\hat{\theta}_{1a,t}^2 + \hat{\delta}_{a,t}^2 + \frac{M}{h}\right]}{b_1}, \quad \hat{C}_{a,t} = \frac{(h - K - 2)\left(\hat{\theta}_{1a,t}^2 + \frac{K}{h}\right)}{c_1}. \tag{47}$$

[23] $\lambda_2^* > 0 \Leftrightarrow \frac{\theta_1^2}{C} - \frac{\delta^2}{B-C} = \frac{B\theta_1^2 - C\theta^2}{C(B-C)} > 0 \Leftrightarrow \theta_1^2/\theta^2 > C/B = \bar{\rho}_{ps}^2$.

[24] We do not use the difference between $\hat{\theta}_{a,t}^2$ and $\hat{\theta}_{1a,t}^2$ to estimate δ^2 because this difference can be negative but the true value of δ^2 is always positive.

We denote the combining portfolio that uses $\hat{\lambda}_{1a,t}$ and $\hat{\lambda}_{2a,t}$ as portfolio $\tilde{c}$ and its weight at time t is

$$\hat{w}_{\tilde{c},t} = \hat{\lambda}_{1a,t}\hat{w}_{p,t} + \hat{\lambda}_{2a,t}\tilde{w}_{s,t}, \tag{48}$$

which is implementable.

Modified Combining Rule

The optimal value of λ_1 and λ_2 in (38) and (39) are obtained by explicitly taking into account the estimation errors in $\hat{w}_{p,t}$ and $\hat{w}_{s,t}$. When $\hat{\lambda}_{1a,t}$ and $\hat{\lambda}_{2a,t}$, instead of λ_1^* and λ_2^*, are used to implement the combining rule, additional estimation errors are introduced. We consider a modified combining portfolio to deal with this second layer of estimation errors. The revised strategy is similar to the two-fund rule of Kan and Zhou (2007) in the sense that an additional parameter is introduced to address the estimation errors by adjusting the weights between portfolio $\tilde{c}$ and the risk-free asset. The modified portfolio takes the following form,

$$\hat{w}_{c_2,t} = \eta\hat{w}_{\tilde{c},t}, \tag{49}$$

where η is a scalar parameter. The optimal value of η is the one that maximizes the portfolio expected out-of-sample utility,

$$\mathbb{E}\left[U_{c_2}(\eta)\right] = \eta\mathbb{E}\left[\hat{w}'_{\tilde{c},t}\mu\right] - \frac{\gamma}{2}\eta^2\mathbb{E}\left[\hat{w}'_{\tilde{c},t}V\hat{w}_{\tilde{c},t}\right], \tag{50}$$

and it can be shown that

$$\eta^* = \frac{\mathbb{E}\left[\hat{w}'_{\tilde{c},t}\mu\right]}{\gamma\mathbb{E}\left[\hat{w}'_{\tilde{c},t}V\hat{w}_{\tilde{c},t}\right]}. \tag{51}$$

Instead of analytically deriving the expression of η^*, we adopt a three-fold cross-validation approach.[25] Specifically, we divide the h monthly data in the estimation window into three folds. The data in two folds are used to compute portfolio weights $\hat{w}_{\tilde{c},t}$. The obtained weights

[25] Existing theory does not provide a clear guide how to choose the number of folds for a given dataset. The choice is basically a tradeoff between bias and variance. A larger number of folds means less bias but higher variance and also longer running time. In unreported results, we also examine the portfolio performance using five-fold or ten-fold cross-validation. The performance is similar in general; and in many cases, the three-fold tends to perform slightly better.

$\hat{w}_{\tilde{c},t}$, together with the data in the remaining fold, are used to calculate the out-of-sample returns of portfolio $\tilde{c}$. Rotating across folds, we obtain h out-of-sample returns of portfolio $\tilde{c}$. The mean and the variance of these returns are used to estimate $\mathbb{E}\left[\hat{w}_{\tilde{c},t}'\mu\right]$ and $\mathbb{E}\left[\hat{w}_{\tilde{c},t}'V\hat{w}_{\tilde{c},t}\right]$, and the estimated value of η^*, i.e., $\tilde{\eta}$, is set according to (51). In addition, we require $\tilde{\eta}$ to be in the range of $[0,1]$. This is because to deal with the estimation errors in $\hat{w}_{\tilde{c},t}$, we expect further shrinkage of portfolio $\tilde{c}$ toward the risk-free asset. We call this revised combining portfolio as the modified combining portfolio, and denote it as portfolio $\tilde{c}_2$ and its weights as $\hat{w}_{\tilde{c}_2,t} = \tilde{\eta}\hat{w}_{\tilde{c},t}$.

3.2.2 Portfolio PEW.

Now we use portfolio PEW as one of the component portfolios, and apply the proposed combining strategy:

$$\hat{w}_{c,t}^{pew} = \lambda_1^{pew}\hat{w}_{pew,t}^* + \lambda_2^{pew}\tilde{w}_{s,t}, \tag{52}$$

where $\hat{w}_{pew,t}^* = \kappa_1^*\hat{w}_{p,t} + \kappa_2^* w_{ew}$, and the explicit expressions of κ_1^* and κ_2^* are derived in (A15) and (A16). Under the assumption that r_t follows a multivariate normal distribution and is i.i.d. over time, we have

$$\bar{\mu}_{pew} = \mathbb{E}\left[\hat{w}_{pew,t}^*{}'\mu\right] = \frac{1}{\gamma}\left(\theta_{ew}^2 + \frac{h\tau^2\kappa_1^*}{h-M-2}\right), \tag{53}$$

$$\bar{\sigma}_{pew}^2 = \mathbb{E}\left[\hat{w}_{pew,t}^*{}'V\hat{w}_{pew,t}^*\right] = \frac{1}{\gamma^2}\left(\theta_{ew}^2 + \frac{h\tau^2\kappa_1^*}{h-M-2}\right), \tag{54}$$

$$\bar{\sigma}_{pew,s} = \mathbb{E}\left[\hat{w}_{pew,t}^*{}'V\tilde{w}_{s,t}\right] = \frac{\kappa_1^* h(K/h+\theta_1^2)}{\gamma^2 c_1(h-M-2)} + \frac{\kappa_2^* h(\mu_{ew}-\alpha_{ew})}{\gamma(h-K-2)}, \tag{55}$$

where $\mu_{ew} = w_{ew}'\mu$, $\sigma_{ew}^2 = w_{ew}'Vw_{ew}$, $\theta_{ew}^2 = \mu_{ew}^2/\sigma_{ew}^2$, $\alpha_{ew} = 1_N'\alpha/M$ and $\tau^2 = \theta^2 - \theta_{ew}^2$. The proof of the above expressions are in the appendix. Based on the above expressions, together with (34) and (36), we obtain the optimal values of the combining coefficients λ_1^{pew} and λ_2^{pew} using (29) and (30). Note that the optimal combining coefficients depend on the true values of μ_{ew}, σ_{ew}^2, θ_{ew}^2, τ^2, θ_1^2, and α. The implementable combining coefficients $\hat{\lambda}_{1a,t}^{pew}$ and $\hat{\lambda}_{2a,t}^{pew}$ are constructed using $\hat{\mu}_{ew,t}$, $\hat{\sigma}_{ew,t}^2$, $\hat{\theta}_{ew}^2$, $\hat{\tau}_{a,t}^2$, $\hat{\theta}_{1a,t}^2$, and $\hat{\alpha}_t = \hat{\mu}_{2,t} - \hat{V}_{21,t}\hat{V}_{11,t}^{-1}\hat{\mu}_{1,t}$. The implementable combining strategy is given by

$$\hat{w}_{\tilde{c},t}^{pew} = \hat{\lambda}_{1a,t}^{pew}\hat{w}_{pew,t} + \hat{\lambda}_{2a,t}^{pew}\tilde{w}_{s,t}. \tag{56}$$

Following a similar cross-validation procedure as in Section 3.2.1, we obtain the modified combining portfolio based on portfolio PEW and the benchmark sample optimal portfolio, and the weights of this portfolio are denoted as $\hat{w}^{pew}_{\tilde{c}_2,t} = \tilde{\eta}^{pew}\hat{w}^{pew}_{\tilde{c},t}$.

3.2.3 Shrinkage Covariance Matrix of Ledoit and Wolf (2004).

Next, we consider the combining strategy using the sample optimal portfolios with shrinkage covariance matrix of Ledoit and Wolf (2004):

$$\hat{w}^{LW2004}_{c,t} = \lambda^{LW2004}_1 \hat{w}^{LW2004}_{p,t} + \lambda^{LW2004}_2 \tilde{w}^{LW2004}_{s,t}, \tag{57}$$

where $\tilde{w}^{LW2004}_{s,t} = [\hat{w}^{LW2004\prime}_{s,t},\ 0'_N]'$. When the shrinkage covariance matrix estimator is used, explicit expressions of the expected out-of-sample utility of the sample optimal portfolio are not available. Therefore, we are unable to derive explicit expressions of the corresponding optimal combining coefficients. As an alternative, we use the implementable optimal combining coefficients in (45) and (46),

$$\hat{w}^{LW2004}_{\tilde{c},t} = \hat{\lambda}_{1a,t}\hat{w}^{LW2004}_{p,t} + \hat{\lambda}_{2a,t}\tilde{w}^{LW2004}_{s,t}. \tag{58}$$

A similar cross-validation procedure as in Section 3.2.1 is applied to $\hat{w}^{LW2004}_{\tilde{c},t}$ to obtain the modified combining portfolio based on the sample optimal portfolios with shrinkage covariance matrix estimators, i.e., $\hat{w}^{LW2004}_{\tilde{c}_2,t} = \tilde{\eta}^{LW2004}\hat{w}^{LW2004}_{\tilde{c},t}$.

3.3 Empirical Performance of the Combining Strategy

Table 3 reports the CER of the combining portfolio $\tilde{c}$ and that of the modified combining portfolio $\tilde{c}_2$ using the same eight benchmark models and the five sets of test assets for $h = 120$ and $\gamma = 3$. Panels A, B, and C report the CER results of the combining strategy applied to the sample optimal portfolio, to portfolio PEW, and to the optimal portfolio with Ledoit and Wolf (2004) shrinkage covariance matrix, respectively.

For every case in Panels A and B, we notice that the modified combining portfolios (i.e., $\hat{w}_{\tilde{c}_2,t}$ and $\hat{w}^{pew}_{\tilde{c}_2,t}$) outperform the implementable combining portfolios (i.e., $\hat{w}_{\tilde{c},t}$ and $\hat{w}^{pew}_{\tilde{c},t}$), suggesting that the second layer of shrinkage successfully addresses the estimation errors contained in the implementable combining coefficients, i.e., $\hat{\lambda}_{1a,t}$, $\hat{\lambda}_{2a,t}$, $\hat{\lambda}^{pew}_{1a,t}$, $\hat{\lambda}^{pew}_{2a,t}$. In addition, we find that the performance improvement is especially large when DMNU-48 is used as the test assets or DMNU-7 is the benchmark model. As mentioned previously, the underlying parameters of these anomaly portfolios are likely to be time varying, and the results in Panels A and B suggest that the use of the non-parametric cross-validation approach helps improve portfolio performance in the cases with the potential departure from the i.i.d. normality assumption.

Comparing the performance of the modified combining portfolios (i.e., $\hat{w}_{\tilde{c}_2}$ and $\hat{w}^{pew}_{\tilde{c}_2}$) in Panels A and B of Table 3 with the performance of the portfolios in Panels A and B of Table 2, we find that the modified combining portfolio almost always outperforms the portfolio including all assets (i.e., $\hat{w}_{p_2,t}$ and $\hat{w}_{pew,t}$) in Table 2. For the cases in which $\hat{w}_{s_2,t}$ outperforms the portfolio including all assets in Table 2, the modified combining portfolio either beats $\hat{w}_{s_2,t}$ or performs similarly.

In Panel C of Table 3, the optimal portfolios with the shrinkage covariance matrix are used as the component portfolios in the combining strategy. Unlike the results in Panels A and B, the modified combining portfolio $\tilde{c}_2$ under-performs the combining portfolio $\tilde{c}$ for most cases in Panel C, suggesting that when the shrinkage covariance matrix is used to deal with the estimation risk in the sample optimal portfolios, the second layer of shrinkage in the combining portfolio is of limited value. This is because the combining coefficients derived for the sample optimal portfolios (i.e., $\hat{\lambda}_{1a,t}$ and $\hat{\lambda}_{2a,t}$) tend to shrink more than the combining coefficients derived for the sample optimal portfolios with shrinkage covariance matrix (that are, however, not available analytically). As a result, further shrinkage is not necessary when $\hat{\lambda}_{1a,t}$ and $\hat{\lambda}_{2a,t}$ are applied to the optimal portfolios with shrinkage covariance matrix. Nevertheless, the combining portfolio itself remains useful. Note that $\hat{w}^{LW2004}_{\tilde{c},t}$ outperforms both $\hat{w}^{LW2004}_{s,t}$ and $\hat{w}^{LW2004}_{p,t}$ for all the cases in Panel C of Table 2 except for IND-10 or IND-30 with the CAPM or DHS as the benchmark model. For the four exceptions, $\hat{w}^{LW2004}_{p,t}$ greatly underperforms $\hat{w}^{LW2004}_{s,t}$; and $\hat{w}^{LW2004}_{\tilde{c},t}$ generates a significant performance improvement relative to $\hat{w}^{LW2004}_{p,t}$, providing a performance comparable to that of $\hat{w}^{LW2004}_{s,t}$.

Comparing the modified combining portfolio $\tilde{c}_2$ in Panels A and B with the combining portfolio $\tilde{c}$ in Panel C, we find that the combining portfolio with the shrinkage covariance matrix tends to perform better. Out of the 40 benchmark model/test assets combinations considered in Table 3, the combining portfolio with the shrinkage covariance matrix performs the best in 38 cases. For the remaining two cases (i.e., IND-10 with CAPM, IND-30 with FF5-UMD), the performance of the combining portfolio with the shrinkage covariance matrix is close to that of the best performing combining portfolio.

In sum, the results in Table 3 suggest that the proposed combining portfolio is an effective strategy to deal with the portfolio choice problem of unknown benchmark efficiency. When the two-fund rule or the PEW rule is first applied to address the estimation errors in the sample optimal portfolio, addressing the second layer of estimation errors in the proposed combining strategy is crucial to the portfolio performance. When the shrinkage covariance matrix is used to address the estimation errors in the sample optimal portfolio, the second layer of shrinkage is no longer necessary, but the proposed combining strategy remains effective. Overall, we find that the combining strategy with the shrinkage covariance matrix (LW2004) tends to perform the best.[26]

To gain some additional insights, we examine the combining coefficients (of both the first and second layer) in our proposed combining strategy from the empirical analysis and report the results in Table 4. The cross-month average values are reported and the corresponding standard deviations are shown in the parentheses. As the combination of the two-fund portfolio is equivalent to the combination of the sample optimal portfolio, the coefficient in the two-fund portfolio (i.e., $\hat{b}_t$) and that in the benchmark two-fund portfolio (i.e., $\hat{c}_t$) are also presented for comparison. When the shrinkage covariance matrix is used, we use the same combining coefficients derived for the sample optimal portfolio (i.e., $\hat{\lambda}_{1a,t}$ and $\hat{\lambda}_{2a,t}$) for the first layer. Therefore, only the results of the coefficient from the second layer of shrinkage (i.e., $\tilde{\eta}^{LW2004}$) are reported.

Table 4 shows that relative to the coefficients from the two-fund portfolio and the benchmark two-fund portfolio (i.e., $\hat{b}_t$, $\hat{c}_t$), the combining coefficients from our proposed strategy (i.e. $\hat{\lambda}_{1a,t}$, $\hat{\lambda}_{2a,t}$) are smaller. For a given set of test assets, the coefficient associated with the sample optimal portfolio ($\hat{\lambda}_{1a,t}$) tends to decrease with more recent models, and that of the

[26]In Section OA.2 of the Online Appendix, we also examine the usefulness of the proposed combining strategy based on simulated data, and reach similar conclusion.

benchmark sample optimal portfolio ($\hat{\lambda}_{2a,t}$) tends to increase with the more recent models. This pattern is expected. As the benchmark model becomes more efficient, the value of including the test assets is reduced and therefore, the weight of the sample optimal portfolio decreases and that of the benchmark sample optimal portfolio increases. The second layer of shrinkage, with $\tilde{\eta} < 1$, shifts more weights to the risk-free asset to further reduce the estimation risk. Comparing the results of $\tilde{\eta}$ and $\tilde{\eta}^{LW2004}$, we find that the value of $\tilde{\eta}^{LW2004}$ is larger in general. This is consistent with the argument that the combining coefficients derived for the sample optimal portfolio tends to shrink more than the combining coefficients derived for the case when the shrinkage covariance matrix is used.

4. Portfolio Turnover

To evaluate the portfolio performance in practice, portfolio turnover needs to be considered. A higher turnover suggests higher transaction costs, which lowers portfolio performance. The test assets and the benchmark portfolios used to construct the optimal portfolios in this paper are all stock portfolios, which means potential trading diversification as documented in DeMiguel, Martín-Utrera, Nogales, and Uppal (2020) is involved. Therefore, in this section, we study the stock level turnover of the various optimal portfolios.

Let n be the number of stocks available to the investor, B_t be the $n \times M$ matrix that captures the weights on individual stocks of the M assets at the end of month t and $w_t = [w_{1,t}, \ldots, w_{M,t}]'$ be the optimal weights on the M assets at the end of month t. The stock level weights of the optimal portfolio at the end of month t are obtained as $s_t = B_t \times w_t = [s_{1,t}, \ldots, s_{n,t}]'$, and the stock level turnover of the optimal portfolio at the end of month t is computed as

$$Turn_t = \sum_{i=1}^{n} \left| s_{i,t} - s_{i,t-1} R_{i,t} \right|, \tag{59}$$

where $R_{i,t}$ is the gross return of stock i in month t. In Table 5, we report the cross-month average stock level turnover of various optimal portfolios using the same empirical data as in Tables 1 to 3 for $h = 120$ and $\gamma = 3$.

Panel A of Table 5 presents the turnover results for the portfolios based on the sample optimal portfolios and the KZ two-fund rule. Of the portfolios examined in Panel A, the sample optimal portfolio ($\hat{w}_{p,t}$) has the highest level of turnover. The turnover of the benchmark sample optimal portfolio ($\hat{w}_{s,t}$) is much lower than that of $\hat{w}_{p,t}$ due to the smaller number of assets involved (i.e., only the benchmark portfolios but not the test assets). Across the test assets examined, we notice that the turnover of $\hat{w}_{p,t}$ is lower when the industry portfolios (IND-10 and IND-30) are included as test assets. This is because the compositions of the industry portfolios are relatively stable over time, which leads to lower stock level portfolio turnover.[27] Across the benchmark models, we notice that the turnover of $\hat{w}_{s,t}$ increases with the more recent models, and it does not necessarily increase with the number of factors in the models.[28]

The two-fund rule of Kan and Zhou (2007) addresses the estimation risk in $\hat{w}_{p,t}$ and $\hat{w}_{s,t}$ by shifting more portfolio weight to the risk-free asset, and such shift helps reduce the portfolio turnover. With more estimation errors in $\hat{w}_{p,t}$ than in $\hat{w}_{s,t}$, we expect a stronger effect to portfolio turnover when applying the two-fund rule to $\hat{w}_{p,t}$ than to $\hat{w}_{s,t}$. The results in Panel A are consistent with our expectation, and we even observe the turnover of $\hat{w}_{p_2,t}$ to be lower than that of $\hat{w}_{s_2,t}$ in some cases (e.g., IND-10 and IND-30 with DHS or DMNU-7 as the benchmark).

Due to a similar shifting effect, the turnover of the modified combining portfolio ($\hat{w}_{\tilde{c}_2,t}$) is always lower than that of the implementable combining portfolio ($\hat{w}_{\tilde{c},t}$) in Panel A. Comparing the turnover of $\hat{w}_{\tilde{c}_2}$ with that of $\hat{w}_{p_2,t}$, we find that $\hat{w}_{\tilde{c}_2}$ has lower turnover in all cases considered expect for IND-30 with DHS as the benchmark.

Panel B reports the turnover results of the portfolios based on PEW, and we observe similar patterns. The PEW rule (i.e., $\hat{w}_{pew,t}$), like the KZ two-fund rule in Panel A, helps reduce the turnover of the sample optimal portfolio (i.e., $\hat{w}_{p,t}$). The modified combining portfolio ($\hat{w}^{pew}_{\tilde{c}_2,t}$) always has lower turnover relative to that of the implementable combining portfolio ($\hat{w}^{pew}_{\tilde{c},t}$); and the turnover of $\hat{w}^{pew}_{\tilde{c}_2,t}$ is lower than that of $\hat{w}_{pew,t}$ in all cases except for

[27]In unreported results, we also examine the turnover at the portfolio level (instead of the stock level), and do not observe lower turnover when industry portfolios are included.

[28]For example, the turnover of DHS (a three-factor model) is much higher than that of FF5-UMD (a six-factor model). This is because that DHS includes a monthly updated factor with many changing compositions, which contributes to the higher stock level turnover.

IND-30 with DHS as the benchmark.

Comparing the results of $\hat{w}_{p,t}$ and $\hat{w}_{s,t}$ in Panel A with those of $\hat{w}_{p,t}^{LW2004}$ and $\hat{w}_{s,t}^{LW2004}$ in Panel C, we find that the use of the shrinkage covariance matrix also helps reduce the portfolio turnover. Applying the implementable combining rule (i.e., $\hat{w}_{\tilde{c},t}^{LW2004}$) can significantly further reduce the turnover of $\hat{w}_{p,t}^{LW2004}$. Comparing the turnover of $\hat{w}_{\tilde{c},t}^{LW2004}$ with that of the modified combining portfolio $\tilde{c}_2$ in Panels A and B, we find that $\hat{w}_{\tilde{c},t}^{LW2004}$ has a lower turnover in general. From Table 3, we know that $\hat{w}_{\tilde{c},t}^{LW2004}$ performs well without transaction costs. The low turnover of $\hat{w}_{\tilde{c},t}^{LW2004}$ shown in Table 5 further supports the usefulness of such combining strategy after transaction cost.[29]

5. Conclusion

In this paper, we examine the portfolio choice problem with a benchmark model of *ex ante* unknown efficiency. The unknown efficiency, together with the estimation risk associated with estimated parameters, makes the portfolio choice problem a challenge in practice. In order to address this problem, we propose a combining portfolio strategy that optimally balances the value of including test assets and the cost associated with the additional estimation errors. The proposed combining strategy is a general one in the sense that it can be readily applied together with some existing estimation risk reduction strategies. Specifically, we discuss how to apply it to the sample optimal portfolio, the portfolio rule that optimally combines the sample optimal portfolio and the $1/N$ rule (i.e., PEW), and the sample optimal portfolio with the shrinkage covariance matrix of Ledoit and Wolf (2004). Both empirically and in simulated data, we show that our proposed combining strategy performs well.

Estimation errors can significantly undermine the out-of-sample performance of the optimal portfolio in practice. This raises a potential question on the value of existing empirical asset pricing studies as well as the various anomaly findings because they mostly focus on the in-sample results. From an investor's perspective, the out-of-sample performance is more relevant. How to evaluate and compare asset pricing models out-of-sample is a question that deserves further study. In addition, instead of adopting an agnostic perspective as in the

[29]In Section OA.3 of the Online Appendix, we report the portfolio performance with transaction cost.

current paper, how to select the benchmark model and/or the test assets *ex ante* for portfolio construction is also an interesting question yet to be answered.

Appendix

Proof of (33) to (37):

Under the multivariate normality assumption, it is well known that $\hat{\mu}_t$ and $\hat{V}_t$ are independent of each other and have the following distributions:

$$\hat{\mu}_t \sim \mathcal{N}(\mu, V/h), \quad \hat{V}_t \sim \mathcal{W}_M(h-1, V/h), \tag{A1}$$

where $\mathcal{W}_M(h-1, V/h)$ is a Wishart distribution with $h-1$ degrees of freedom and covariance matrix V/h. Define an $M \times M$ orthonormal matrix $P = [P_1,\ P_2]$ with its first K columns equal to

$$P_1 = V^{\frac{1}{2}} \begin{bmatrix} I_K \\ 0_{N\times K} \end{bmatrix} V_{11}^{-\frac{1}{2}}. \tag{A2}$$

Transform $\hat{\mu}_t$ and $\hat{V}_t$ to

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \sqrt{h}P'V^{-\frac{1}{2}}\hat{\mu}_t \sim \mathcal{N}(\sqrt{h}P'V^{-\frac{1}{2}}\mu, I_M), \tag{A3}$$

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} = hP'V^{-\frac{1}{2}}\hat{V}_t V^{-\frac{1}{2}}P \sim \mathcal{W}_M(h-1, I_M), \tag{A4}$$

where z_1 is a $K \times 1$ vector and W_{11} is the upper left $K \times K$ submatrix of W, and z and W are independent of each other.

We can write the weights of the sample optimal portfolio p as

$$\hat{w}_{p,t} = \frac{\sqrt{h}}{\gamma}V^{-\frac{1}{2}}PW^{-1}z, \tag{A5}$$

and it follows that

$$\bar{\mu}_p = \mathbb{E}[\mu_{p,t}] = \mathbb{E}\left[\frac{\sqrt{h}}{\gamma}\mu'V^{-\frac{1}{2}}PW^{-1}z\right] = \frac{\sqrt{h}}{\gamma}\mu'V^{-\frac{1}{2}}P\mathbb{E}[W^{-1}]\mathbb{E}[z] = \frac{h\theta^2}{\gamma(h-M-2)}, \tag{A6}$$

$$\bar{\sigma}_p^2 = \mathbb{E}[\sigma_{p,t}^2] = \frac{h}{\gamma^2}\mathbb{E}\left[z'W^{-2}z\right] = \frac{h(h-2)(M+h\theta^2)}{\gamma^2(h-M-1)(h-M-2)(h-M-4)}. \tag{A7}$$

(A6) and (A7) are obtained by applying the expressions for the inverse moments of a Wishart distribution from Haff (1979)

$$\mathbb{E}[W^{-1}] = \frac{1}{h-M-2} I_M, \quad \mathbb{E}[W^{-2}] = \frac{h-2}{(h-M-1)(h-M-2)(h-M-4)} I_M, \tag{A8}$$

and the fact that $z'z \sim \chi^2_M(h\theta^2)$ and $\mathbb{E}[z'z] = M + h\theta^2$.

Replacing M by K and θ^2 by θ_1^2 in (A6) and (A7), we get

$$\bar{\mu}_s = \mathbb{E}[\mu_{s,t}] = \frac{h\theta_1^2}{\gamma(h-K-2)}, \tag{A9}$$

$$\bar{\sigma}_s^2 = \mathbb{E}[\sigma_{s,t}^2] = \frac{h(h-2)(K+h\theta_1^2)}{\gamma^2(h-K-1)(h-K-2)(h-K-4)}. \tag{A10}$$

Finally, we can write

$$\bar{\sigma}_{ps} = \mathbb{E}[\sigma_{ps,t}] = \frac{1}{\gamma^2}\mathbb{E}\left[\hat{\mu}_t'\hat{V}_t^{-1}V\begin{bmatrix} I_K \\ 0_{N\times K}\end{bmatrix}\hat{V}_{11,t}^{-1}\hat{\mu}_{1,t}\right] = \frac{h}{\gamma^2}\mathbb{E}\left[z'W^{-1}\begin{bmatrix} I_K \\ 0_{N\times K}\end{bmatrix}W_{11}^{-1}z_1\right]. \tag{A11}$$

Applying the partitioned matrix inverse formula, we have

$$\bar{\sigma}_{ps} = \frac{h}{\gamma^2}\mathbb{E}\left[z_1'W_{11}^{-2}z_1 + (z_1'W_{11}^{-1}W_{12} - z_2')W_{22\cdot 1}^{-1}W_{21}W_{11}^{-2}z_1\right], \tag{A12}$$

where $W_{22\cdot 1} = W_{22} - W_{21}W_{11}^{-1}W_{12}$. Using Theorem 3.2.10 of Muirhead (1982), we have $W_{22\cdot 1} \sim \mathcal{W}_N(h-K-1, I_N)$, $\text{vec}\left(Z \equiv W_{21}W_{11}^{-\frac{1}{2}}\right) \sim \mathcal{N}(0_{NK}, I_{NK})$, $W_{11} \sim \mathcal{W}_K(h-1, I_K)$, and $W_{22\cdot 1}$, W_{11}, and Z are independent of each other. Therefore, $\mathbb{E}[z_2'W_{22\cdot 1}^{-1}W_{21}W_{11}^{-2}z_1] = \mathbb{E}[z_2'W_{22\cdot 1}^{-1}ZW_{11}^{-\frac{3}{2}}z_1] = 0$ because $\mathbb{E}[Z] = 0_{N\times K}$ and Z is independent of W_{11}, $W_{22\cdot 1}$ and z. The explicit expression of $\mathbb{E}[\sigma_{ps,t}]$ can be written as

$$\begin{aligned}
\bar{\sigma}_{ps} &= \frac{h}{\gamma^2}\left(\mathbb{E}[z_1'W_{11}^{-2}z_1] + \mathbb{E}[z_1'W_{11}^{-1}W_{12}W_{22\cdot 1}^{-1}W_{21}W_{11}^{-2}z_1]\right) \\
&= \frac{h}{\gamma^2}\left(\mathbb{E}[z_1'W_{11}^{-2}z_1] + \frac{\mathbb{E}[z_1'W_{11}^{-1}W_{12}W_{21}W_{11}^{-2}z_1]}{h-M-2}\right) \\
&= \frac{h}{\gamma^2}\left(\mathbb{E}[z_1'W_{11}^{-2}z_1] + \frac{\mathbb{E}[z_1'W_{11}^{-\frac{1}{2}}Z'ZW_{11}^{-\frac{3}{2}}z_1]}{h-M-2}\right) \\
&= \frac{h}{\gamma^2}\left(\mathbb{E}[z_1'W_{11}^{-2}z_1] + \frac{N\mathbb{E}[z_1'W_{11}^{-2}z_1]}{h-M-2}\right) \\
&= \frac{h(h-2)(K+h\theta_1^2)}{\gamma^2(h-K-1)(h-K-4)(h-M-2)}.
\end{aligned} \tag{A13}$$

The fourth equality is obtained because $Z'Z \sim \mathcal{W}_K(N, I_K)$ and $\mathbb{E}[Z'Z] = NI_K$. The last equality is obtained using (A10). $\square$

Derivation of κ_1^* and κ_2^* in the PEW rule:

We can apply the results in Proposition 1 to obtain κ_1^* and κ_2^*. Specifically, plug $\mu_{ew} = w_{ew}'\mu$, $\sigma_{ew}^2 = w_{ew}'Vw_{ew}$, (A6), (A7), and

$$\mathbb{E}\left[\hat{w}_{p,t}'Vw_{ew}\right] = \frac{h\mu_{ew}}{\gamma(h-M-2)} \tag{A14}$$

into (29) and (30), we obtain

$$\kappa_1^* = \frac{\theta^2 - \theta_{ew}^2}{\frac{\theta^2+M/h}{b_1} - \frac{h\theta_{ew}^2}{h-M-2}} = \frac{\tau^2}{\frac{\theta^2+M/h}{b_1} - \frac{h\theta_{ew}^2}{h-M-2}}, \tag{A15}$$

$$\kappa_2^* = \frac{\mu_{ew}}{\gamma\sigma_{ew}^2}\left(1 - \frac{h\kappa_1^*}{h-M-2}\right). \tag{A16}$$

The expected out-of-sample utility of $\hat{w}_{pew,t}^* = \kappa_1^*\hat{w}_{p,t} + \kappa_2^* w_{ew}$ is

$$\mathbb{E}\left[U(\hat{w}_{pew,t}^*)\right] = \frac{\theta^2}{2\gamma} - \left(\frac{\theta^2+M/h}{2\gamma b_1} - \frac{\theta^2}{2\gamma}\right)\frac{\theta^2/(2\gamma) - \theta_{ew}^2/(2\gamma)}{(\theta^2+M/h)/(2\gamma b_1) - \theta_{ew}^2/(2\gamma)}. \tag{A17}$$

$\square$

Proof of (53), (54), and (55):

For (53), we have

$$\begin{aligned}
\mathbb{E}[\hat{w}_{pew,t}^{*}{}'\mu] &= \frac{\kappa_1^* h\theta^2}{\gamma(h-M-2)} + \kappa_2^*\mu_{ew} \\
&= \frac{\kappa_1^* h\theta^2}{\gamma(h-M-2)} + \frac{\theta_{ew}^2}{\gamma}\left(1 - \frac{h\kappa_1^*}{h-M-2}\right) \\
&= \frac{1}{\gamma}\left(\theta_{ew}^2 + \frac{h\kappa_1^*(\theta^2-\theta_{ew}^2)}{h-M-2}\right).
\end{aligned} \tag{A18}$$

The first equality is obtained using (A6), and the second equality is obtained using (A16).

For (54), we have

$$\mathbb{E}[\hat{w}_{pew,t}^{*}{}'V\hat{w}_{pew,t}^*]$$

$$
\begin{aligned}
&= \kappa_1^{*2}\mathbb{E}[\sigma_{p,t}^2] + \kappa_2^{*2}\sigma_{ew}^2 + 2\kappa_1^*\kappa_2^*\mathbb{E}[w_{ew}'V\hat{w}_{p,t}] \\
&= \frac{\kappa_1^{*2}(M+h\theta^2)}{\gamma^2 b_1(h-M-2)} + \kappa_2^{*2}\sigma_{ew}^2 + \frac{2\kappa_1^*\kappa_2^* h\mu_{ew}}{\gamma(h-M-2)} \\
&= \frac{1}{\gamma^2}\left[\frac{\kappa_1^{*2}(M+h\theta^2)}{b_1(h-M-2)} + \theta_{ew}^2\left(1-\frac{h\kappa_1^*}{h-M-2}\right)^2 + \frac{2\kappa_1^* h\theta_{ew}^2}{h-M-2}\left(1-\frac{h\kappa_1^*}{h-M-2}\right)\right] \\
&= \frac{1}{\gamma^2}\left[\theta_{ew}^2 + \frac{h\kappa_1^{*2}}{h-M-2}\left(\frac{M/h+\theta^2}{b_1} - \frac{h\theta_{ew}^2}{h-M-2}\right)\right] \\
&= \frac{1}{\gamma^2}\left(\theta_{ew}^2 + \frac{h\tau^2\kappa_1^*}{h-M-2}\right). \qquad \text{(A19)}
\end{aligned}
$$

The second equality is obtained using (A7) and (A14). The third equality is obtained using the expression of κ_2^* in (A16). The last equality is obtained using the expression of κ_1^* in (A15).

For (55), we have

$$
\begin{aligned}
\mathbb{E}[\hat{w}_{pew,t}^{*}{}'V\tilde{w}_{s,t}] &= \kappa_1^*\mathbb{E}[\sigma_{ps,t}] + \kappa_2^*\mathbb{E}[w_{ew}'V\tilde{w}_{s,t}] \\
&= \frac{\kappa_1^*(K+h\theta_1^2)}{\gamma^2 c_1(h-M-2)} + \frac{\kappa_2^* h}{\gamma(h-K-2)} w_{ew}'V\begin{bmatrix} V_{11}^{-1}\mu_1 \\ 0_{N\times K}\end{bmatrix} \\
&= \frac{\kappa_1^*(K+h\theta_1^2)}{\gamma^2 c_1(h-M-2)} + \frac{\kappa_2^* h}{\gamma(h-K-2)}\frac{1}{M}\left(1_K'\mu_1 + 1_N'V_{21}V_{11}^{-1}\mu_1\right) \\
&= \frac{\kappa_1^*(K+h\theta_1^2)}{\gamma^2 c_1(h-M-2)} + \frac{\kappa_2^* h}{\gamma(h-K-2)}\frac{1}{M}\left(1_M'\mu - 1_N'\mu_2 + 1_N'V_{21}V_{11}^{-1}\mu_1\right) \\
&= \frac{\kappa_1^*(K+h\theta_1^2)}{\gamma^2 c_1(h-M-2)} + \frac{\kappa_2^* h(\mu_{ew} - 1_N'\alpha/M)}{\gamma(h-K-2)} \\
&= \frac{\kappa_1^* h(K/h+\theta_1^2)}{\gamma^2 c_1(h-M-2)} + \frac{\kappa_2^* h(\mu_{ew}-\alpha_{ew})}{\gamma(h-K-2)}. \qquad \text{(A20)}
\end{aligned}
$$

The second equality is obtained using (A13). □

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Table 1: **CER of Sample Optimal Portfolios**

This table reports the certainty equivalent returns (in annualized percentage points) of the sample optimal portfolio p and the benchmark sample optimal portfolio s with $h = 120$ and $\gamma = 3$. Five sets of test assets are examined against eight benchmarks. At the end of a given month t, the sample optimal portfolio p and the benchmark sample optimal portfolio s are constructed using excess returns from month $t - 119$ to month t and are held for one month (i.e., month $t + 1$). The out-of-sample excess returns (i.e., month $t+1$) are used to compute the certainty equivalent returns of the portfolios.

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\hat{w}_{s,t}$	2.83	2.70	5.32	8.73	8.71	14.23	25.05	−34.22
$\hat{w}_{p,t}$	−14.88	−27.74	−48.36	−43.02	−81.61	−62.10	−40.85	−247.20
IVOL-10:								
$\hat{w}_{s,t}$	2.83	2.70	5.32	8.73	8.71	14.23	25.05	−34.22
$\hat{w}_{p,t}$	−28.61	−48.20	−55.63	−112.39	−121.00	−103.60	−67.28	−312.96
IND-10:								
$\hat{w}_{s,t}$	2.83	2.70	5.32	8.73	8.71	14.23	25.05	−34.22
$\hat{w}_{p,t}$	−24.06	−27.87	−40.84	−59.51	−72.99	−57.10	−37.59	−224.00
IND-30:								
$\hat{w}_{s,t}$	2.83	2.70	5.32	8.73	8.71	14.23	25.05	−34.22
$\hat{w}_{p,t}$	−184.91	−233.40	−273.61	−396.08	−417.19	−369.99	−298.08	−1141.24
DMNU-48:								
$\hat{w}_{s,t}$	−0.45	−1.62	−4.41	−3.02	−8.40	−2.44	11.29	−34.22
$\hat{w}_{p,t}$	−3424.82	−4384.52	−4679.40	−4934.98	−5387.40	−4588.02	−3800.61	−3424.82

Table 2: **CER of Portfolios with Estimation Risk Reduction Strategies**

This table compares the certainty equivalent returns (in annualized percentage points) of the optimal portfolios using all assets (benchmark portfolios and test assets) with those of the optimal portfolios using only the benchmark portfolios when various estimation risk reduction strategies are adopted. In Panels A and B, the two-fund rule and the PEW rule are applied to the sample optimal portfolio (i.e., $\hat{w}_{p_2,t}$, $\hat{w}_{pew,t}$), and they are compared with the benchmark two-fund portfolio (i.e., $\hat{w}_{s_2,t}$). In Panel C, the Ledoit and Wolf (2004) shrinkage covariance matrix is used for both the sample optimal portfolio and the benchmark sample optimal portfolio. Five sets of test assets are examined against eight benchmarks. The estimation window is set to $h = 120$ and the risk aversion coefficient is set to $\gamma = 3$. At the end of a given month t, various portfolios are constructed using excess returns from month $t - 119$ to month t and are held for one month (i.e., month $t + 1$). The out-of-sample excess returns (i.e., month $t + 1$) are used to compute the certainty equivalent returns of the portfolios.

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
				A. KZ two-fund rule				
MOM-10:								
$\hat{w}_{s_2,t}$	3.11	4.98	10.83	16.79	20.29	21.47	30.73	27.51
$\hat{w}_{p_2,t}$	9.15	11.29	9.14	17.87	11.21	15.20	23.01	−1.87
IVOL-10:								
$\hat{w}_{s_2,t}$	3.11	4.98	10.83	16.79	20.29	21.47	30.73	27.51
$\hat{w}_{p_2,t}$	6.81	10.20	18.25	8.00	14.08	4.51	20.17	−26.88
IND-10:								
$\hat{w}_{s_2,t}$	3.11	4.98	10.83	16.79	20.29	21.47	30.73	27.51
$\hat{w}_{p_2,t}$	−0.19	4.19	8.28	7.80	7.59	9.48	15.63	2.86
IND-30:								
$\hat{w}_{s_2,t}$	3.11	4.98	10.83	16.79	20.29	21.47	30.73	27.51
$\hat{w}_{p_2,t}$	−4.09	−1.42	4.07	−1.66	0.36	−0.72	3.36	−16.72
DMNU-48:								
$\hat{w}_{s_2,t}$	−0.28	1.72	3.07	9.23	8.45	8.00	18.35	27.51
$\hat{w}_{p_2,t}$	−49.07	−77.67	−80.00	−73.90	−80.68	−77.39	−42.43	−49.07

Table 2 CER of Portfolios with Estimation Risk Reduction Strategies (Cont'd)

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
				B. Portfolio PEW				
MOM-10:								
$\hat{w}_{s_2,t}$	3.11	4.98	10.83	16.79	20.29	21.47	30.73	27.51
$\hat{w}_{pew,t}$	9.84	12.03	10.63	19.42	14.14	18.12	24.20	0.64
IVOL-10:								
$\hat{w}_{s_2,t}$	3.11	4.98	10.83	16.79	20.29	21.47	30.73	27.51
$\hat{w}_{pew,t}$	8.05	11.37	20.29	8.98	16.03	5.87	21.51	−27.27
IND-10:								
$\hat{w}_{s_2,t}$	3.11	4.98	10.83	16.79	20.29	21.47	30.73	27.51
$\hat{w}_{pew,t}$	1.72	5.95	11.65	9.65	10.90	13.08	17.30	3.81
IND-30:								
$\hat{w}_{s_2,t}$	3.11	4.98	10.83	16.79	20.29	21.47	30.73	27.51
$\hat{w}_{pew,t}$	−1.78	0.98	7.04	−0.05	2.49	1.72	5.70	−16.53
DMNU-48:								
$\hat{w}_{s_2,t}$	−0.28	1.72	3.07	9.23	8.45	8.00	18.35	27.51
$\hat{w}_{pew,t}$	−47.16	−76.25	−77.36	−71.03	−77.12	−72.33	−37.23	−47.16
			C. LW2004 shrinkage covariance matrix					
MOM-10:								
$\hat{w}_{s,t}^{LW2004}$	2.83	4.19	10.28	15.60	17.69	25.11	36.97	54.61
$\hat{w}_{p,t}^{LW2004}$	7.87	6.32	5.95	14.79	13.01	23.66	29.56	29.46
IVOL-10:								
$\hat{w}_{s,t}^{LW2004}$	2.83	4.19	10.28	15.60	17.69	25.11	36.97	54.61
$\hat{w}_{p,t}^{LW2004}$	6.49	16.31	20.51	23.50	22.89	16.17	25.22	23.66
IND-10:								
$\hat{w}_{s,t}^{LW2004}$	2.83	4.19	10.28	15.60	17.69	25.11	36.97	54.61
$\hat{w}_{p,t}^{LW2004}$	−11.50	−1.64	0.48	3.68	4.04	15.44	13.77	21.27
IND-30:								
$\hat{w}_{s,t}^{LW2004}$	2.83	4.19	10.28	15.60	17.69	25.11	36.97	54.61
$\hat{w}_{p,t}^{LW2004}$	−75.83	−60.75	−62.07	−52.99	−57.43	−36.55	−51.86	−57.99
DMNU-48:								
$\hat{w}_{s,t}^{LW2004}$	−0.45	0.39	0.96	8.49	7.80	14.24	26.85	54.61
$\hat{w}_{p,t}^{LW2004}$	−327.45	−321.84	−247.96	−329.40	−267.88	−325.88	−375.69	−327.45

Table 3: **CER of the Combining Strategy**

This table reports the certainty equivalent returns (in annualized percentage points) of the combining portfolio $\tilde{c}$ and the modified combining portfolio $\tilde{c}_2$ for $h = 120$ and $\gamma = 3$. Five sets of test assets are examined against eight benchmarks. Panels A to C report the results of the combining strategy applied to the sample optimal portfolio, to portfolio PEW, and to the optimal portfolio with Ledoit and Wolf (2004) shrinkage covariance matrix, respectively. At the end of a given month t, various portfolios are constructed using excess returns from month $t - 119$ to month t and are held for one month (i.e., month $t+1$). The out-of-sample excess returns (i.e., month $t+1$) are used to compute the certainty equivalent returns of the portfolios.

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
				A. Sample optimal portfolio				
MOM-10:								
$\hat{w}_{\tilde{c},t}$	9.55	10.62	8.64	19.70	13.65	17.81	25.83	3.30
$\hat{w}_{\tilde{c}_2,t}$	10.50	11.78	9.45	22.50	17.66	28.84	32.76	35.54
IVOL-10:								
$\hat{w}_{\tilde{c},t}$	7.86	9.80	17.88	9.79	15.93	9.45	25.88	−7.04
$\hat{w}_{\tilde{c}_2,t}$	12.40	14.77	25.72	26.02	33.74	25.92	38.26	25.57
IND-10:								
$\hat{w}_{\tilde{c},t}$	0.64	3.76	8.67	10.83	11.84	11.70	23.63	9.97
$\hat{w}_{\tilde{c}_2,t}$	1.73	4.37	10.34	14.66	18.31	21.58	26.32	43.41
IND-30:								
$\hat{w}_{\tilde{c},t}$	−2.51	0.37	8.26	5.49	11.04	7.65	22.80	−6.21
$\hat{w}_{\tilde{c}_2,t}$	0.27	4.19	9.71	15.13	19.50	20.70	25.39	39.35
DMNU-48:								
$\hat{w}_{\tilde{c},t}$	−46.81	−78.68	−88.46	−75.90	−92.72	−92.88	−45.85	−37.63
$\hat{w}_{\tilde{c}_2,t}$	32.34	38.49	41.23	19.93	19.85	4.78	26.87	47.98

Table 3 CER of the Combining Strategy (Cont'd)

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
				B. Portfolio PEW				
MOM-10:								
$\hat{w}^{pew}_{\tilde{c},t}$	9.39	10.82	8.88	20.99	16.60	20.84	26.05	3.25
$\hat{w}^{pew}_{\tilde{c}_2,t}$	10.64	13.31	10.33	23.04	19.72	29.75	33.00	40.56
IVOL-10:								
$\hat{w}^{pew}_{\tilde{c},t}$	6.65	8.30	17.61	8.99	16.97	9.13	25.90	−6.25
$\hat{w}^{pew}_{\tilde{c}_2,t}$	11.80	15.01	26.24	26.65	34.22	26.70	38.62	27.44
IND-10:								
$\hat{w}^{pew}_{\tilde{c},t}$	−1.51	3.40	8.85	12.70	14.36	12.03	24.60	8.22
$\hat{w}^{pew}_{\tilde{c}_2,t}$	2.03	4.47	10.20	15.89	19.05	21.68	27.17	43.95
IND-30:								
$\hat{w}^{pew}_{\tilde{c},t}$	−3.59	−0.40	7.81	6.46	12.17	8.29	24.41	−9.13
$\hat{w}^{pew}_{\tilde{c}_2,t}$	0.31	4.83	9.82	15.35	19.18	20.32	26.17	39.34
DMNU-48:								
$\hat{w}^{pew}_{\tilde{c},t}$	−44.72	−76.98	−85.88	−72.07	−86.37	−84.06	−39.57	−38.16
$\hat{w}^{pew}_{\tilde{c}_2,t}$	34.49	40.38	41.73	23.65	22.68	10.63	29.49	48.76
				C. LW2004 shrinkage covariance matrix				
MOM-10:								
$\hat{w}^{LW2004}_{\tilde{c},t}$	12.91	16.13	18.19	27.32	27.98	33.37	42.18	61.08
$\hat{w}^{LW2004}_{\tilde{c}_2,t}$	12.77	15.02	15.73	27.08	26.88	32.36	40.38	57.67
IVOL-10:								
$\hat{w}^{LW2004}_{\tilde{c},t}$	13.32	23.32	31.27	38.92	42.56	35.98	44.44	56.49
$\hat{w}^{LW2004}_{\tilde{c}_2,t}$	13.05	21.51	26.97	38.07	41.60	34.87	40.45	55.90
IND-10:								
$\hat{w}^{LW2004}_{\tilde{c},t}$	1.06	6.85	13.54	17.43	20.64	26.03	34.76	57.27
$\hat{w}^{LW2004}_{\tilde{c}_2,t}$	1.74	4.16	9.73	14.21	15.93	23.42	31.10	56.80
IND-30:								
$\hat{w}^{LW2004}_{\tilde{c},t}$	0.85	5.93	12.29	15.87	18.76	25.19	35.39	58.29
$\hat{w}^{LW2004}_{\tilde{c}_2,t}$	1.71	4.16	8.69	13.60	13.38	22.81	30.94	57.71
DMNU-48:								
$\hat{w}^{LW2004}_{\tilde{c},t}$	89.27	82.43	70.17	79.12	67.44	76.33	83.29	87.70
$\hat{w}^{LW2004}_{\tilde{c}_2,t}$	89.27	82.31	66.09	78.93	63.66	72.63	79.22	87.38

Table 4: **The Combining Coefficients**

Based on similar empirical exercises as in Table 3, this table reports the cross-month average combining coefficients and the corresponding standard deviations (in parentheses) for the various combining portfolios. $\hat{\lambda}_{1a,t}$ and $\hat{\lambda}_{2a,t}$ are the implementable combining coefficients in (45) and (46) when the combining strategy is applied to the sample optimal portfolio; $\tilde{\eta}$ is the estimated parameter of the second layer shrinkage specified in Section 3.2.1. $\hat{\lambda}^{pew}_{1a,t}$, $\hat{\lambda}^{pew}_{2a,t}$, and $\tilde{\eta}^{pew}$ are the corresponding estimated parameters specified in Section 3.2.2 when the combining strategy is applied to portfolio PEW. When the combining strategy is applied to the optimal portfolios with Ledoit and Wolf (2004) shrinkage covariance estimator, the same combining coefficients (i.e, $\hat{\lambda}_{1a,t}$ and $\hat{\lambda}_{2a,t}$) are used; and the coefficient from the second layer (i.e., $\tilde{\eta}^{LW2004}$) is reported in the last row of each panel in the table. For comparison, the coefficients from the two-fund portfolio and the benchmark two-fund portfolio (i.e., $\hat{b}_t$ and $\hat{c}_t$) as specified in (10) and (14) are also reported in the table.

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\hat{b}_t$	0.43	0.44	0.44	0.47	0.46	0.50	0.54	0.59
	(0.15)	(0.15)	(0.16)	(0.12)	(0.13)	(0.17)	(0.16)	(0.06)
$\hat{c}_t$	0.54	0.58	0.61	0.68	0.67	0.73	0.79	0.78
	(0.29)	(0.19)	(0.20)	(0.11)	(0.09)	(0.15)	(0.13)	(0.04)
$\hat{\lambda}_{1a,t}$	0.40	0.40	0.35	0.35	0.34	0.36	0.35	0.30
	(0.15)	(0.16)	(0.16)	(0.16)	(0.17)	(0.18)	(0.15)	(0.11)
$\hat{\lambda}_{2a,t}$	0.11	0.14	0.23	0.29	0.30	0.33	0.40	0.45
	(0.30)	(0.21)	(0.21)	(0.18)	(0.14)	(0.19)	(0.16)	(0.12)
$\tilde{\eta}$	0.63	0.54	0.51	0.67	0.65	0.62	0.61	0.80
	(0.35)	(0.30)	(0.32)	(0.25)	(0.28)	(0.34)	(0.30)	(0.14)
$\hat{\lambda}^{pew}_{1a,t}$	1.20	1.01	0.85	0.57	0.55	0.58	0.61	0.38
	(0.32)	(0.37)	(0.37)	(0.65)	(0.61)	(0.51)	(0.30)	(0.40)
$\hat{\lambda}^{pew}_{2a,t}$	−0.41	−0.09	0.11	0.31	0.32	0.33	0.38	0.53
	(0.30)	(0.31)	(0.31)	(0.30)	(0.25)	(0.31)	(0.25)	(0.25)
$\tilde{\eta}^{pew}$	0.60	0.56	0.53	0.66	0.64	0.64	0.63	0.80
	(0.33)	(0.29)	(0.31)	(0.24)	(0.26)	(0.30)	(0.28)	(0.13)
$\tilde{\eta}^{LW2004}$	0.79	0.73	0.69	0.88	0.81	0.76	0.76	0.94
	(0.36)	(0.33)	(0.36)	(0.25)	(0.30)	(0.36)	(0.36)	(0.14)

Table 4 The Combining Coefficients (Cont'd)

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
IVOL-10:								
$\hat{b}_t$	0.44	0.46	0.49	0.54	0.54	0.54	0.57	0.60
	(0.18)	(0.17)	(0.17)	(0.11)	(0.11)	(0.14)	(0.15)	(0.05)
$\hat{c}_t$	0.54	0.58	0.61	0.68	0.67	0.73	0.79	0.78
	(0.29)	(0.19)	(0.20)	(0.11)	(0.09)	(0.15)	(0.13)	0.04
$\hat{\lambda}_{1a,t}$	0.41	0.43	0.45	0.49	0.49	0.46	0.46	0.33
	(0.19)	(0.18)	(0.16)	(0.15)	(0.14)	(0.15)	(0.14)	(0.08)
$\hat{\lambda}_{2a,t}$	0.10	0.11	0.12	0.14	0.14	0.22	0.28	0.42
	(0.36)	(0.23)	(0.18)	(0.16)	(0.11)	(0.12)	(0.10)	(0.10)
$\tilde{\eta}$	0.55	0.50	0.52	0.72	0.65	0.60	0.59	0.82
	(0.33)	(0.32)	(0.31)	(0.20)	(0.23)	(0.32)	(0.30)	(0.11)
$\hat{\lambda}^{pew}_{1a,t}$	1.22	1.06	1.01	0.88	0.90	0.86	0.83	0.53
	(0.36)	(0.29)	(0.21)	(0.16)	(0.12)	(0.18)	(0.12)	(0.18)
$\hat{\lambda}^{pew}_{2a,t}$	−0.43	−0.12	−0.02	0.12	0.11	0.17	0.23	0.44
	(0.35)	(0.30)	(0.24)	(0.16)	(0.10)	(0.17)	(0.13)	(0.13)
$\tilde{\eta}^{pew}$	0.51	0.51	0.53	0.70	0.63	0.60	0.58	0.80
	(0.33)	(0.31)	(0.31)	(0.21)	(0.24)	(0.31)	(0.29)	(0.11)
$\tilde{\eta}^{LW2004}$	0.69	0.65	0.64	0.92	0.84	0.76	0.75	0.99
	(0.37)	(0.36)	(0.37)	(0.15)	(0.26)	(0.33)	(0.33)	(0.03)
IND-10:								
$\hat{b}_t$	0.38	0.43	0.46	0.49	0.48	0.53	0.53	0.59
	(0.15)	(0.10)	(0.11)	(0.06)	(0.07)	(0.11)	(0.10)	(0.05)
$\hat{c}_t$	0.54	0.58	0.61	0.68	0.67	0.73	0.79	0.78
	(0.29)	(0.19)	(0.20)	(0.11)	(0.09)	(0.15)	(0.13)	(0.04)
$\hat{\lambda}_{1a,t}$	0.35	0.38	0.39	0.33	0.32	0.34	0.22	0.27
	(0.14)	(0.11)	(0.11)	(0.14)	(0.13)	(0.12)	(0.13)	(0.10)
$\hat{\lambda}_{2a,t}$	0.16	0.16	0.19	0.31	0.33	0.35	0.54	0.49
	(0.30)	(0.22)	(0.23)	(0.22)	(0.19)	(0.22)	(0.26)	(0.14)
$\tilde{\eta}$	0.48	0.46	0.42	0.66	0.53	0.61	0.63	0.79
	(0.38)	(0.30)	(0.26)	(0.25)	(0.24)	(0.33)	(0.24)	(0.15)
$\hat{\lambda}^{pew}_{1a,t}$	1.46	1.01	0.90	0.52	0.50	0.64	0.08	0.34
	(0.90)	(0.26)	(0.27)	(0.81)	(0.78)	(0.36)	(0.83)	(0.31)
$\hat{\lambda}^{pew}_{2a,t}$	−0.72	−0.09	0.06	0.37	0.38	0.32	0.73	0.57
	(0.88)	(0.33)	(0.33)	(0.51)	(0.47)	(0.34)	(0.61)	(0.23)
$\tilde{\eta}^{pew}$	0.46	0.48	0.44	0.64	0.52	0.62	0.65	0.79
	(0.37)	(0.30)	(0.26)	(0.24)	(0.23)	(0.31)	(0.21)	(0.16)
$\tilde{\eta}^{LW2004}$	0.31	0.44	0.42	0.78	0.62	0.67	0.73	0.99
	(0.35)	(0.36)	(0.36)	(0.26)	(0.32)	(0.36)	(0.32)	(0.03)

Table 4 The Combining Coefficients (Cont'd)

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
IND-30:								
$\hat{b}_t$	0.19	0.23	0.24	0.26	0.25	0.27	0.25	0.33
	(0.08)	(0.07)	(0.07)	(0.05)	(0.05)	(0.08)	(0.07)	(0.05)
$\hat{c}_t$	0.54	0.58	0.61	0.68	0.67	0.73	0.79	0.78
	(0.29)	(0.19)	(0.20)	(0.11)	(0.09)	(0.15)	(0.13)	(0.04)
$\hat{\lambda}_{1a,t}$	0.18	0.20	0.20	0.19	0.17	0.17	0.11	0.15
	(0.08)	(0.07)	(0.06)	(0.08)	(0.08)	(0.07)	(0.06)	(0.06)
$\hat{\lambda}_{2a,t}$	0.30	0.31	0.35	0.42	0.44	0.50	0.64	0.57
	(0.28)	(0.20)	(0.21)	(0.19)	(0.16)	(0.19)	(0.19)	(0.09)
$\tilde{\eta}$	0.35	0.32	0.34	0.61	0.51	0.58	0.55	0.78
	(0.35)	(0.29)	(0.27)	(0.26)	(0.26)	(0.37)	(0.29)	(0.13)
$\hat{\lambda}_{1a,t}^{pew}$	1.19	0.94	0.81	0.58	0.55	0.56	0.19	0.37
	(0.25)	(0.22)	(0.26)	(0.44)	(0.43)	(0.35)	(0.54)	(0.28)
$\hat{\lambda}_{2a,t}^{pew}$	−0.41	0.06	0.22	0.41	0.44	0.47	0.69	0.61
	(0.30)	(0.30)	(0.32)	(0.26)	(0.23)	(0.28)	(0.34)	(0.14)
$\tilde{\eta}^{pew}$	0.33	0.33	0.36	0.60	0.50	0.59	0.56	0.78
	(0.33)	(0.27)	(0.26)	(0.24)	(0.24)	(0.35)	(0.26)	(0.13)
$\tilde{\eta}^{LW2004}$	0.29	0.34	0.39	0.76	0.59	0.69	0.71	0.99
	(0.33)	(0.34)	(0.36)	(0.26)	(0.33)	(0.37)	(0.35)	(0.03)
DMNU-48:								
$\hat{b}_t$	0.25	0.23	0.23	0.22	0.21	0.23	0.23	0.25
	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)
$\hat{c}_t$	0.57	0.58	0.57	0.70	0.67	0.70	0.76	0.78
	(0.22)	(0.16)	(0.21)	(0.08)	(0.09)	(0.16)	(0.14)	(0.04)
$\hat{\lambda}_{1a,t}$	0.25	0.23	0.22	0.21	0.20	0.21	0.22	0.17
	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)
$\hat{\lambda}_{2a,t}$	0.14	0.18	0.19	0.34	0.32	0.33	0.39	0.50
	(0.21)	(0.13)	(0.16)	(0.04)	(0.04)	(0.12)	(0.10)	(0.05)
$\tilde{\eta}$	0.61	0.54	0.47	0.58	0.51	0.57	0.61	0.69
	(0.26)	(0.25)	(0.23)	(0.24)	(0.24)	(0.25)	(0.27)	(0.19)
$\hat{\lambda}_{1a,t}^{pew}$	1.00	0.99	0.98	0.94	0.94	0.95	0.93	0.68
	(0.01)	(0.01)	(0.02)	(0.03)	(0.03)	(0.03)	(0.04)	(0.12)
$\hat{\lambda}_{2a,t}^{pew}$	0.15	0.17	0.16	0.32	0.29	0.31	0.37	0.51
	(0.21)	(0.14)	(0.15)	(0.05)	(0.04)	(0.12)	(0.10)	(0.05)
$\tilde{\eta}^{pew}$	0.61	0.54	0.46	0.58	0.50	0.57	0.61	0.68
	(0.26)	(0.24)	(0.24)	(0.24)	(0.24)	(0.25)	(0.27)	(0.19)
$\tilde{\eta}^{LW2004}$	1.00	1.00	0.86	1.00	0.87	0.92	0.91	1.00
	(0.00)	(0.02)	(0.28)	(0.02)	(0.26)	(0.20)	(0.22)	(0.02)

Table 5: **Stock Level Portfolio Turnover**

This table reports the stock level turnover of the portfolios examined in Tables 1 to 3 for $h = 120$ and $\gamma = 3$ using the same empirical data. At the end of a given month t, various portfolios are constructed using excess returns from month $t - 119$ to month t and are held for one month (i.e., month $t+1$). Average stock level portfolio turnovers across the $T-h$ months in the sample period are reported in the table.

A. Sample optimal portfolio and KZ two-fund rule

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\hat{w}_{s,t}$	0.07	0.84	3.62	3.58	4.85	7.45	11.40	21.02
$\hat{w}_{p,t}$	25.72	29.15	32.12	31.95	34.50	38.68	41.65	54.66
$\hat{w}_{s_2,t}$	0.06	0.57	2.63	2.61	3.43	5.92	9.68	16.90
$\hat{w}_{p_2,t}$	12.20	14.18	15.91	16.60	17.96	22.19	24.69	33.74
$\hat{w}_{\tilde{c},t}$	11.55	13.30	13.59	13.86	15.11	18.98	20.84	26.70
$\hat{w}_{\tilde{c}_2,t}$	8.25	8.68	8.98	9.89	11.23	14.57	15.52	21.71
IVOL-10:								
$\hat{w}_{s,t}$	0.07	0.84	3.62	3.58	4.85	7.45	11.40	21.02
$\hat{w}_{p,t}$	27.55	32.49	35.71	45.02	45.89	45.88	47.07	59.55
$\hat{w}_{s_2,t}$	0.06	0.57	2.63	2.61	3.43	5.92	9.68	16.90
$\hat{w}_{p_2,t}$	14.15	17.58	19.79	26.26	26.35	27.51	29.37	36.86
$\hat{w}_{\tilde{c},t}$	13.43	16.90	18.80	24.80	24.91	25.39	27.07	29.48
$\hat{w}_{\tilde{c}_2,t}$	9.54	12.02	12.80	19.95	18.69	19.61	19.67	24.83
IND-10:								
$\hat{w}_{s,t}$	0.07	0.84	3.62	3.58	4.85	7.45	11.40	21.02
$\hat{w}_{p,t}$	2.29	3.57	6.99	6.83	8.04	11.99	13.99	26.73
$\hat{w}_{s_2,t}$	0.06	0.57	2.63	2.61	3.43	5.92	9.68	16.90
$\hat{w}_{p_2,t}$	1.03	1.71	3.65	3.55	4.10	6.93	8.20	16.50
$\hat{w}_{\tilde{c},t}$	0.97	1.70	3.93	3.65	4.33	7.39	10.07	18.02
$\hat{w}_{\tilde{c}_2,t}$	0.70	1.13	2.27	2.84	2.75	5.85	7.40	14.89
IND-30:								
$\hat{w}_{s,t}$	0.07	0.84	3.62	3.58	4.85	7.45	11.40	21.02
$\hat{w}_{p,t}$	8.71	11.16	15.17	16.49	17.74	23.00	23.03	45.05
$\hat{w}_{s_2,t}$	0.06	0.57	2.63	2.61	3.43	5.92	9.68	16.90
$\hat{w}_{p_2,t}$	2.01	2.85	4.16	4.64	4.85	7.00	6.61	16.09
$\hat{w}_{\tilde{c},t}$	1.90	2.76	4.72	4.83	5.27	8.27	10.59	19.63
$\hat{w}_{\tilde{c}_2,t}$	1.17	1.39	2.31	3.67	3.29	6.31	7.21	16.04
DMNU-48:								
$\hat{w}_{s,t}$	0.07	0.84	2.60	3.70	4.67	6.84	11.01	21.02
$\hat{w}_{p,t}$	115.93	127.37	132.54	135.16	141.08	132.96	128.99	115.93
$\hat{w}_{s_2,t}$	0.07	0.57	1.81	2.75	3.35	5.34	9.29	16.90
$\hat{w}_{p_2,t}$	30.77	31.90	32.11	31.69	32.01	32.06	32.10	30.77
$\hat{w}_{\tilde{c},t}$	30.67	31.64	31.38	30.50	30.45	31.18	32.96	28.17
$\hat{w}_{\tilde{c}_2,t}$	20.89	19.44	17.74	20.34	19.02	20.84	22.43	20.76

Table 5 Stock Level Portfolio Turnover (Cont'd)

B. Portfolio PEW

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\hat{w}_{pew,t}$	11.30	13.29	14.92	16.30	17.61	21.84	24.29	33.62
$\hat{w}^{pew}_{\tilde{c},t}$	12.52	13.77	13.47	13.87	15.01	18.67	20.38	26.54
$\hat{w}^{pew}_{\tilde{c}_2,t}$	8.55	9.08	9.00	9.59	10.90	14.42	15.24	21.36
IVOL-10:								
$\hat{w}_{pew,t}$	13.34	16.85	19.13	25.97	26.03	27.19	29.04	36.74
$\hat{w}^{pew}_{\tilde{c},t}$	14.16	17.29	18.89	24.74	24.88	25.27	26.95	29.09
$\hat{w}^{pew}_{\tilde{c}_2,t}$	9.58	12.21	12.87	19.45	18.30	19.46	19.30	24.00
IND-10:								
$\hat{w}_{pew,t}$	0.91	1.58	3.45	3.47	4.01	6.80	7.99	16.39
$\hat{w}^{pew}_{\tilde{c},t}$	1.10	1.69	3.71	3.79	4.42	7.26	10.37	18.10
$\hat{w}^{pew}_{\tilde{c}_2,t}$	0.74	1.12	2.22	2.90	2.80	5.80	7.62	14.79
IND-30:								
$\hat{w}_{pew,t}$	1.86	2.68	3.95	4.56	4.76	6.86	6.44	15.96
$\hat{w}^{pew}_{\tilde{c},t}$	2.03	2.72	4.43	4.81	5.24	8.08	10.72	19.59
$\hat{w}^{pew}_{\tilde{c}_2,t}$	1.17	1.38	2.27	3.57	3.23	6.24	7.30	15.86
DMNU-48:								
$\hat{w}_{pew,t}$	30.78	31.93	32.07	31.68	31.85	32.04	32.14	30.78
$\hat{w}^{pew}_{\tilde{c},t}$	30.67	31.65	31.39	30.51	30.38	31.15	32.84	28.14
$\hat{w}^{pew}_{\tilde{c}_2,t}$	20.73	19.24	17.66	20.16	18.82	20.74	22.29	20.58

Table 5 Stock Level Portfolio Turnover (Cont'd)

C. LW2004 shrinkage covariance matrix

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\hat{w}_{s,t}^{LW2004}$	0.07	0.71	3.14	2.25	3.44	5.58	8.24	13.04
$\hat{w}_{p,t}^{LW2004}$	14.30	16.14	16.71	17.22	17.22	19.19	21.69	22.68
$\hat{w}_{\tilde{c},t}^{LW2004}$	6.42	7.36	7.08	7.28	7.19	9.59	11.59	12.47
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	5.86	6.42	5.93	6.94	6.59	8.92	10.54	12.20
IVOL-10:								
$\hat{w}_{s,t}^{LW2004}$	0.07	0.71	3.14	2.25	3.44	5.58	8.24	13.04
$\hat{w}_{p,t}^{LW2004}$	12.31	14.42	15.86	18.72	18.89	19.30	21.19	21.26
$\hat{w}_{\tilde{c},t}^{LW2004}$	5.65	7.17	8.38	9.98	10.10	10.64	12.56	12.06
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	4.89	6.07	6.85	9.64	9.30	9.52	11.12	12.02
IND-10:								
$\hat{w}_{s,t}^{LW2004}$	0.07	0.71	3.14	2.25	3.44	5.58	8.24	13.04
$\hat{w}_{p,t}^{LW2004}$	1.31	2.19	4.26	3.26	4.47	6.39	8.50	7.91
$\hat{w}_{\tilde{c},t}^{LW2004}$	0.56	1.10	2.65	1.95	2.69	4.55	6.89	8.82
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	0.35	0.81	1.70	1.84	2.11	3.95	6.08	8.79
IND-30:								
$\hat{w}_{s,t}^{LW2004}$	0.07	0.71	3.14	2.25	3.44	5.58	8.24	13.04
$\hat{w}_{p,t}^{LW2004}$	4.39	5.26	6.86	5.79	6.72	8.14	9.81	7.60
$\hat{w}_{\tilde{c},t}^{LW2004}$	0.97	1.41	2.76	2.13	2.73	4.50	6.86	8.70
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	0.57	0.90	1.70	2.04	2.13	3.98	5.95	8.68
DMNU-48:								
$\hat{w}_{s,t}^{LW2004}$	0.07	0.68	2.18	2.34	3.17	5.01	7.68	13.04
$\hat{w}_{p,t}^{LW2004}$	34.24	32.79	28.95	33.51	30.08	35.25	41.24	34.24
$\hat{w}_{\tilde{c},t}^{LW2004}$	9.23	8.31	7.14	8.02	6.98	9.52	12.51	12.03
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	9.23	8.30	6.87	8.02	6.72	9.27	12.25	12.03

Online Appendix:

“Optimal Portfolio Choice with Unknown Benchmark Efficiency”

May 2023

This online appendix presents some additional results. Section OA.1. discusses the performance of the switching strategy, and the corresponding results are reported in Table OA.1. Section OA.2. presents the simulation results. Table OA.2 and Table OA.3 report, respectively, the average CER and the standard error of the CER of various portfolios based on simulated data. Section OA.3. presents the portfolio performance with transaction costs, and the corresponding results are in Table OA.4. Finally, Table OA.5 shows the certainty equivalent returns of the portfolios examined in Tables 1 to 3 of the paper based on the same empirical datasets with $h = 120$ and $\gamma = 5$.

OA.1. The Switching Strategy

In this section, we study whether the efficiency information provided by the GRS test can lead to an improved portfolio strategy. Specifically, we examine a portfolio strategy that uses the GRS test result to determine when to include the test assets. At the end of each month t, we assume that the investor uses the data in the estimation window (i.e., the most recent h months) to conduct the GRS test. If the efficiency of the benchmark model is rejected at the pre-selected significance level, then the investor will hold the portfolio that also includes the test assets in month $t+1$. Otherwise, he will hold the portfolio using only the benchmark portfolios in month $t+1$. The GRS test result can change from month to month, and so does the portfolio that the investor holds. Therefore, we call this portfolio strategy the switching strategy, and denote it as $\hat{w}_{sw,t}$. Table OA.1 reports the results of the switching strategy based on the same empirical data as in the paper with $h = 120$ and $\gamma = 3$. Three typical significance levels, i.e., $\alpha = 1\%$, 5%, and 10%, are examined. Panel A of the table reports the proportion of the $T - h$ months (in percentage points) in which the null of the GRS test is rejected at various significance levels. Panels B to D report the CER results of the switching strategy with various estimation risk reduction strategies adopted.[1]

Table OA.1 about here

Panel A shows that across the test assets considered, the rejection proportions tend to be low for the industry portfolios (IND-10 and IND-30) but are very high for DMNU-48. Across the asset pricing models, we notice relatively low rejection proportions for the more recent models, especially for $\alpha = 1\%$. Results in Panel A are, in general, consistent with our expectations about the efficiency of the asset pricing models and the test assets. The CER results in Panels B to D, however, suggest that the switching strategy based on the efficiency information from the GRS test does not perform well in general. In many cases, the GRS test results do not seem to effectively guide when to include the additional test assets. For example, when DMNU-48 is the test assets, the rejection proportion is 100% for many of the asset pricing models (as shown in Panel A), and thus the switching strategy always chooses the optimal portfolio that also includes the test assets.

[1]The switching strategy is applied to the two portfolios in the corresponding panels of Table 2 in the paper. For example, when portfolio PEW is adopted (i.e., Panel B), the switching strategy is applied to $\hat{w}_{s_2,t}$ and $\hat{w}_{pew,t}$.

The *ex post* results in Table 2 of the paper, however, suggest that for these cases, the portfolio including the test assets underperforms the one using only the benchmark portfolios.

Comparing the CER results in Table OA.1 with those in Table 2 of the paper, we see that in many cases, the performance of the switching strategy is between that of the two portfolios (i.e., the portfolio with and without the test assets) in Table 2 of the paper. This suggests that the switching strategy, to some extent, helps to balance between the two portfolios so that the investor can avoid the poorer one. But we also notice that in some cases, the switching strategy can underperform both portfolios, likely due to the ineffective switching. Another issue with the switching strategy is that, *ex ante*, it is not clear which significance level should be used. Comparing the portfolio performance across the three significance levels in Table OA.1, we cannot identify a consistent winner; and in some cases, the change in portfolio performance is not monotonic across the significance level.

In sum, the results of the switching strategy in Table OA.1 indicate that using only the efficiency information from the GRS test to guide the portfolio choice decision does not seem to be a good solution to the portfolio choice problem with unknown benchmark efficiency.

OA.2. Simulation Results

When empirical data are used to evaluate portfolio performance, the results are subject to potential sampling issues. To address this concern, we assess the usefulness of the proposed combining strategy based on simulated data in this section.

We apply the stationary block bootstrap procedure proposed in Politis and Romano (1994) to simulate the data, with an expected length of the block set to 10 months. The data generated from such procedure keep the stationary property of the original data. Specifically, for an empirical dataset (including both the benchmark portfolios and the test assets) containing T months, we generate T monthly data using the bootstrap procedure, and apply various portfolio rules on the simulated data using the rolling window approach with $h = 120$ and $\gamma = 3$. The CER of the portfolios are calculated accordingly.[2] We conduct 1000 simulations in total, and report the average

[2]Though the combining strategy is derived under the assumption that asset returns are i.i.d. normally distributed,

CER of the portfolios across simulations in Table OA.2 and the corresponding standard errors in Table OA.3. In general, the simulation results are consistent with the empirical findings, confirming the usefulness of the proposed combining strategy.

Table OA.2 about here

Table OA.3 about here

Panel A of Table OA.2 reports the results based on the sample optimal portfolio and the two-fund rule. The CER results of $\hat{w}_{p,t}$ and $\hat{w}_{s,t}$ in Panel A present a similar pattern as observed in Table 1 of the paper: due to the large estimation errors, the sample optimal portfolio p always underperforms the benchmark sample optimal portfolio s, never realizing the value of including additional test assets. When the two-fund rule is applied to the sample optimal portfolio and the benchmark sample optimal portfolio, the estimation risk is alleviated and the portfolio performance improves. We start to observe the value of including the additional test assets in some cases. The relative performance of $\hat{w}_{p_2,t}$ and $\hat{w}_{s_2,t}$ based on simulated data is however, not always consistent with the empirical results shown in Panel A of Table 2 in the paper.[3] Nevertheless, the proposed combining strategy remains effective with the simulated data. The modified combining portfolio $\hat{w}_{\tilde{c}_2,t}$ outperforms the two-fund portfolio $\hat{w}_{p_2,t}$. In the cases in which $\hat{w}_{s_2,t}$ outperforms $\hat{w}_{p_2,t}$, the modified combining portfolio provides a significant performance improvement relative to $\hat{w}_{p_2,t}$, generating a performance level better than or similar to that of $\hat{w}_{s_2,t}$. In Panel B, the results based on PEW are reported, and we observe similar patterns as those in Panel A. The simulation results in Panel C confirm the previous empirical findings in that the combining strategy works well with the shrinkage covariance matrix (i.e., $\hat{w}_{\tilde{c},t}^{LW2004}$ performs well against other portfolios) but the second layer of shrinkage is not necessary when the shrinkage covariance matrix is used.

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we do not assume normality in our simulation because many evidence from the existing literature suggests that the data generating process underlying actual asset returns departs from i.i.d. normality. With the bootstrap procedure, we can evaluate the portfolios in a setting that is closer to what the investor will experience in reality.

[3]Most noticeable with simulated data is that when IVOL-10 is included as test assets, $\hat{w}_{p_2,t}$ no longer outperforms $\hat{w}_{s_2,t}$ for the CAPM, FF-3 or Carhart-4. When IVOL-10 is included as test assets, we find that the potential efficiency improvement relative to the CAPM, FF-3, and Carhart-4 is only apparent in the period after 1963/7, i.e., the starting point of the sample period used in Ang, Hodrick, Xing, and Zhang (2006); and the efficiency improvement is much weaker in the period before 1963/7. A similar finding is documented in Detzel, Duarte, Kamara, Siegel, and Sun (2019). If instead of the entire sample period, we bootstrap using data from the two sub-periods (i.e., 1927/1–1963/6 and 1963/7–2018/12) separately, then the relative performance of $\hat{w}_{p_2,t}$ and $\hat{w}_{s_2,t}$ from simulation is similar to what is observed empirically.

The standard error results in Table OA.3 present a pattern similar to the observations from Table OA.2. The sample optimal portfolio ($\hat{w}_{p,t}$) has the highest level of standard error. Applying the various estimation risk reduction strategies helps reduce the standard errors. Relative to the portfolios with all assets (i.e., $\hat{w}_{p_2,t}$, $\hat{w}_{pew,t}$, $\hat{w}_{p,t}^{LW2004}$), the proposed combining portfolios (i.e., $\hat{w}_{\tilde{c}_2,t}$, $\hat{w}_{\tilde{c}_2,t}^{pew}$, $\hat{w}_{\tilde{c},t}^{LW2004}$) always have lower standard errors. Relative to the portfolio with only the benchmark portfolios (i.e., $\hat{w}_{s_2,t}$, $\hat{w}_{s,t}^{LW2004}$), the proposed combining portfolios have similar level of standard errors in many cases (and even lower in some cases).

OA.3. Portfolio CER with Transaction Cost

In Table OA.4, we examine the portfolio performance assuming a transaction cost of 10 basis points.[4] With transaction costs, the performance of all the portfolios decreases. Because the modified combining portfolios ($\hat{w}_{\tilde{c}_2,t}$ and $\hat{w}_{\tilde{c}_2,t}^{pew}$) have lower turnover than the portfolios involving all assets (i.e., $\hat{w}_{p_2,t}$ and $\hat{w}_{pew,t}$) as shown in Table 5 of the paper, in Panels A and B of Table OA.4, we continue to observe the modified combining portfolios to beat the portfolios involving all assets after transaction costs.

Table OA.4 about here

Given the low turnover of the portfolio involving only the benchmark portfolios (i.e., $\hat{w}_{s_2,t}$), it is more difficult for the proposed modified combining portfolio to beat $\hat{w}_{s_2,t}$ when transaction costs are considered. When the industry portfolios (IND-10 and IND-30) are included as test assets, the relative performance of the modified combining portfolio and $\hat{w}_{s_2,t}$ remains similar with and without transaction costs due to the fact that industry portfolios have low turnovers. When DMNU-48 is included as test assets or DMNU-7 is the benchmark, because the potential performance improvement is large as shown in Table 3 of the paper, we continue to observe the modified combining portfolio to outperform $\hat{w}_{s_2,t}$ in most cases after the transaction costs.

[4]There is no consensus as to how to incorporate transaction cost and what value of transaction cost to be used in the portfolio choice problem. However, the belief that the average transaction cost is decreasing over time is well accepted. Frazzini et al. (2018) show that the mean market impact of the trades in their sample is about 10 basis points. Based on orders executed by Morgan Stanley in 2004, Engle et al. (2012) find similar level of transaction costs. We, therefore, adopt a linear transaction cost of 10 basis points to illustrate portfolio performance after transaction cost.

The use of the shrinkage covariance matrix and the combining strategy both help reduce portfolio turnover as shown in Panel C of Table 5 of the paper. With transaction costs, the proposed combining portfolio with shrinkage covariance matrix (i.e., $\hat{w}_{\tilde{c},t}^{LW2004}$) continues to perform well against the other portfolios. Panel C of Table OA.4 shows that $\hat{w}_{\tilde{c},t}^{LW2004}$ always outperforms $\hat{w}_{p,t}^{LW2004}$; and it beats $\hat{w}_{s,t}^{LW2004}$ in 37 out of the total 40 cases. Because $\hat{w}_{\tilde{c}_2,t}^{LW2004}$ has a lower turnover than $\hat{w}_{\tilde{c},t}^{LW2004}$, the relative performance of $\hat{w}_{\tilde{c}_2,t}^{LW2004}$ to $\hat{w}_{\tilde{c},t}^{LW2004}$ improves with transaction costs; but the magnitude of improvement is limited and therefore, we continue to observe that $\hat{w}_{\tilde{c},t}^{LW2004}$ outperforms in most cases.

Table 3 of the paper shows that the combining strategy with LW2004 ($\hat{w}_{\tilde{c},t}^{LW2004}$), in general, performs the best without transaction costs. Table 5 presents evidence that across the proposed combining strategies, $\hat{w}_{\tilde{c},t}^{LW2004}$ has the lowest level of turnover. As a result, it is not surprising to observe in Table OA.4 that with transaction costs, $\hat{w}_{\tilde{c},t}^{LW2004}$ continues to be the best performing combining portfolio overall.

References

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Table OA.1: **GRS Test Results and CER of the Switching Strategy**

This table reports the GRS test results and the certainty equivalent returns (in annualized percentage points) of the switching strategy with $h = 120$ and $\gamma = 3$. Five sets of test assets are examined against eight benchmarks. At the end of a given month t, excess returns from month $t-119$ to month t are used to conduct the GRS test, and three significance levels (i.e., 1%, 5%, 10%) are examined. When the null is rejected, the optimal portfolio using all assets (benchmark portfolios and test assets) is held in month $t+1$. Otherwise, the optimal portfolio using only benchmark portfolios is held in month $t+1$. Panel A reports the proportion (in percentage points) of the months in which the null of the GRS test is rejected. Panels B to D report the certainty equivalent returns of the switching strategy applied to the two portfolios in Panels A to C of Table 2 in the paper.

A. Rejection proportion of the GRS test (in percentage points)

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\alpha = 1\%$	38.21	41.97	26.12	33.88	33.33	39.41	30.59	11.49
$\alpha = 5\%$	56.50	60.57	41.77	49.45	36.81	54.50	47.26	40.23
$\alpha = 10\%$	62.70	67.68	53.96	59.34	47.07	59.23	54.11	59.20
IVOL-10:								
$\alpha = 1\%$	33.03	41.06	47.97	64.29	67.40	54.95	50.46	16.67
$\alpha = 5\%$	54.57	56.81	64.84	68.32	75.09	66.89	67.81	41.95
$\alpha = 10\%$	63.62	66.57	76.32	74.36	84.25	75.90	78.77	61.78
IND-10:								
$\alpha = 1\%$	26.73	25.30	35.37	23.99	17.22	28.15	9.59	9.20
$\alpha = 5\%$	41.46	54.07	58.13	48.72	42.67	43.69	16.89	21.26
$\alpha = 10\%$	54.07	65.85	69.21	64.10	58.61	57.21	22.37	30.46
IND-30:								
$\alpha = 1\%$	19.51	29.17	33.94	34.43	26.56	21.40	0.91	12.93
$\alpha = 5\%$	39.13	57.42	63.21	59.34	55.31	45.27	13.70	46.84
$\alpha = 10\%$	51.52	73.07	74.70	67.58	63.00	59.91	22.83	54.60
DMNU-48:								
$\alpha = 1\%$	100.00	100.00	100.00	95.40	94.83	100.00	100.00	72.70
$\alpha = 5\%$	100.00	100.00	100.00	100.00	100.00	100.00	100.00	83.33
$\alpha = 10\%$	100.00	100.00	100.00	100.00	100.00	100.00	100.00	91.67

Table OA.1: **GRS Test Results and CER of the Switching Strategy (Cont'd)**

B. KZ two-fund rule

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\alpha = 1\%$	6.12	6.58	7.15	14.95	12.69	11.03	19.27	1.56
$\alpha = 5\%$	8.40	9.51	6.69	15.17	12.38	13.00	17.87	1.34
$\alpha = 10\%$	8.42	10.14	7.17	18.35	11.75	11.43	18.26	−2.48
IVOL-10:								
$\alpha = 1\%$	5.18	7.81	13.38	10.12	16.07	3.62	12.78	2.65
$\alpha = 5\%$	5.65	7.83	12.92	8.43	14.33	3.00	18.05	−0.96
$\alpha = 10\%$	6.87	8.89	16.43	6.81	14.23	2.40	18.32	−39.15
IND-10:								
$\alpha = 1\%$	0.33	3.23	8.89	8.21	8.28	10.69	28.15	24.98
$\alpha = 5\%$	0.26	1.59	7.03	7.62	9.79	8.64	27.39	10.93
$\alpha = 10\%$	−0.18	1.09	7.33	10.85	10.42	7.21	24.42	4.97
IND-30:								
$\alpha = 1\%$	−1.85	−0.34	4.03	5.50	7.60	7.92	29.99	0.34
$\alpha = 5\%$	−2.79	−2.46	2.78	1.22	4.39	−0.86	25.61	−8.27
$\alpha = 10\%$	−2.26	−2.86	3.29	5.11	5.05	−0.24	22.04	−12.62
DMNU-48:								
$\alpha = 1\%$	−49.07	−77.67	−80.00	−80.09	−86.40	−77.39	−42.43	−39.27
$\alpha = 5\%$	−49.07	−77.67	−80.00	−73.90	−80.68	−77.39	−42.43	−43.01
$\alpha = 10\%$	−49.07	−77.67	−80.00	−73.90	−80.68	−77.39	−42.43	−48.42

Table OA.1: **GRS Test Results and CER of the Switching Strategy (Cont'd)**

C. Portfolio PEW

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\alpha = 1\%$	5.79	5.97	7.82	17.00	15.45	13.62	20.11	3.74
$\alpha = 5\%$	8.04	9.01	7.38	15.99	15.35	16.65	19.07	4.63
$\alpha = 10\%$	8.08	10.17	7.79	19.58	14.79	15.20	19.26	1.84
IVOL-10:								
$\alpha = 1\%$	5.68	8.30	14.28	11.04	18.38	4.80	13.62	1.67
$\alpha = 5\%$	6.48	9.28	14.94	9.40	16.66	5.77	19.24	−1.11
$\alpha = 10\%$	7.51	9.83	17.77	8.55	17.26	5.00	20.20	−38.75
IND-10:								
$\alpha = 1\%$	0.56	3.28	9.98	9.05	9.56	12.62	29.18	25.72
$\alpha = 5\%$	0.98	3.12	10.39	9.89	13.55	11.26	28.86	10.57
$\alpha = 10\%$	1.07	2.67	10.62	12.59	14.32	9.92	25.65	4.62
IND-30:								
$\alpha = 1\%$	−2.33	−0.26	4.09	5.02	6.72	8.52	29.66	0.64
$\alpha = 5\%$	−1.44	−0.81	5.47	2.97	6.50	0.59	26.21	−10.25
$\alpha = 10\%$	−0.67	−0.50	6.71	7.47	7.68	1.27	23.62	−13.93
DMNU-48:								
$\alpha = 1\%$	−47.16	−76.25	−77.36	−77.07	−82.80	−72.33	−37.23	−37.14
$\alpha = 5\%$	−47.16	−76.25	−77.36	−71.03	−77.12	−72.33	−37.23	−41.55
$\alpha = 10\%$	−47.16	−76.25	−77.36	−71.03	−77.12	−72.33	−37.23	−46.49

Table OA.1: **GRS Test Results and CER of the Switching Strategy (Cont'd)**

D. LW2004 shrinkage covariance matrix

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\alpha = 1\%$	4.12	2.04	9.22	11.38	16.29	19.87	28.14	44.86
$\alpha = 5\%$	6.43	4.31	5.02	9.79	15.59	23.39	28.35	40.02
$\alpha = 10\%$	6.02	5.20	5.86	14.30	15.08	20.98	28.83	38.33
IVOL-10:								
$\alpha = 1\%$	6.91	16.48	19.86	27.60	28.48	21.02	23.98	42.30
$\alpha = 5\%$	6.70	15.60	15.82	25.06	23.72	20.58	24.41	35.90
$\alpha = 10\%$	7.88	15.67	21.06	24.83	24.46	17.62	24.15	27.51
IND-10:								
$\alpha = 1\%$	−7.27	−2.13	4.62	6.90	6.44	21.06	32.98	48.53
$\alpha = 5\%$	−7.43	−3.90	0.70	7.79	7.97	19.09	32.23	44.64
$\alpha = 10\%$	−8.54	−4.93	−0.28	8.16	6.82	16.04	30.65	42.27
IND-30:								
$\alpha = 1\%$	−32.89	−31.46	−27.89	−11.70	−9.15	5.00	33.87	39.28
$\alpha = 5\%$	−42.21	−47.73	−41.47	−28.46	−25.76	−9.44	31.50	15.32
$\alpha = 10\%$	−49.27	−51.27	−48.73	−28.83	−31.39	−19.30	20.94	−3.69
DMNU-48:								
$\alpha = 1\%$	−327.45	−321.84	−247.96	−348.68	−282.18	−325.88	−375.69	−359.90
$\alpha = 5\%$	−327.45	−321.84	−247.96	−329.40	−267.88	−325.88	−375.69	−354.42
$\alpha = 10\%$	−327.45	−321.84	−247.96	−329.40	−267.88	−325.88	−375.69	−349.03

Table OA.2: **Average Portfolio CER based on Simulated Data**

This table reports the average certainty equivalent returns of various portfolios across 1000 simulations with $h = 120$ and $\gamma = 3$. For a given empirical datasets (test assets and benchmark portfolios) containing T months, we bootstrap T monthly data using the stationary block bootstrap approach of Politis and Romano (1994) with the expected block length set to 10 months. Various portfolio rules are applied to the bootstrapped data, and the certainty equivalent returns are computed based on the $T-h$ out-of-sample portfolio returns.

A. Sample optimal portfolio and KZ two-fund rule

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\hat{w}_{s,t}$	0.48	−3.37	0.40	3.18	1.77	15.43	33.91	68.26
$\hat{w}_{p,t}$	−27.89	−38.89	−51.71	−46.38	−59.99	−37.27	−10.01	−43.98
$\hat{w}_{s_2,t}$	1.14	1.34	6.91	11.27	14.12	20.78	36.37	89.55
$\hat{w}_{p_2,t}$	2.96	4.16	3.32	11.70	11.65	13.89	28.32	67.38
$\hat{w}_{\tilde{c},t}$	2.54	3.04	3.87	12.24	12.40	18.21	35.31	82.22
$\hat{w}_{\tilde{c}_2,t}$	3.01	4.16	5.72	13.84	14.57	18.93	34.46	84.70
IVOL-10:								
$\hat{w}_{s,t}$	0.48	−3.37	0.40	3.18	1.77	15.43	33.91	68.26
$\hat{w}_{p,t}$	−46.56	−64.28	−72.55	−68.93	−82.77	−55.62	−26.59	−47.09
$\hat{w}_{s_2,t}$	1.14	1.34	6.91	11.27	14.12	20.78	36.37	89.55
$\hat{w}_{p_2,t}$	−3.04	−3.72	−1.66	7.78	8.67	12.52	26.43	72.57
$\hat{w}_{\tilde{c},t}$	−3.07	−4.46	−0.03	7.76	8.94	15.55	33.14	84.69
$\hat{w}_{\tilde{c}_2,t}$	−0.36	0.16	4.06	11.96	13.76	18.26	33.62	87.73
IND-10:								
$\hat{w}_{s,t}$	0.48	−3.37	0.40	3.18	1.77	15.43	33.91	68.26
$\hat{w}_{p,t}$	−29.81	−38.40	−41.80	−35.51	−45.59	−28.16	−9.44	−33.08
$\hat{w}_{s_2,t}$	1.14	1.34	6.91	11.27	14.12	20.78	36.37	89.55
$\hat{w}_{p_2,t}$	0.09	2.30	6.25	13.72	14.47	18.06	25.92	71.45
$\hat{w}_{\tilde{c},t}$	−0.03	1.36	6.24	13.57	14.52	20.73	34.16	83.98
$\hat{w}_{\tilde{c}_2,t}$	0.72	2.29	6.89	13.92	15.12	20.42	33.05	85.89

Table OA.2: **Average Portfolio CER based on Simulated Data (Cont'd)**

A. Sample optimal portfolio and KZ two-fund rule (cont'd)

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
IND-30:								
$\hat{w}_{s,t}$	0.48	−3.37	0.40	3.18	1.77	15.43	33.91	68.26
$\hat{w}_{p,t}$	−218.61	−265.68	−298.17	−318.03	−353.79	−306.08	−270.48	−663.49
$\hat{w}_{s_2,t}$	1.14	1.34	6.91	11.27	14.12	20.78	36.37	89.55
$\hat{w}_{p_2,t}$	−1.61	0.21	2.94	8.70	9.62	10.33	16.81	38.52
$\hat{w}_{\tilde{c},t}$	−1.25	−0.10	4.96	11.55	13.05	18.27	33.15	73.78
$\hat{w}_{\tilde{c}_2,t}$	0.67	2.22	6.82	13.74	15.66	19.42	32.85	82.24
DMNU-48:								
$\hat{w}_{s,t}$	1.42	−0.07	0.25	5.96	2.34	15.38	30.09	68.26
$\hat{w}_{p,t}$	−2672.67	−3211.10	−3504.54	−3906.74	−4293.39	−3543.12	−3261.40	−2672.67
$\hat{w}_{s_2,t}$	1.74	3.47	6.50	14.40	15.20	20.52	32.67	89.55
$\hat{w}_{p_2,t}$	11.59	0.27	−5.45	−13.73	−21.12	−6.05	1.70	11.59
$\hat{w}_{\tilde{c},t}$	12.48	0.45	−6.32	−11.14	−20.05	−1.82	12.65	52.92
$\hat{w}_{\tilde{c}_2,t}$	57.83	54.42	51.78	51.80	48.22	54.84	64.13	89.31

Table OA.2: **Average Portfolio CER based on Simulated Data (Cont'd)**

B. Portfolio PEW

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\hat{w}_{s_2,t}$	1.14	1.34	6.91	11.27	14.12	20.78	36.37	89.55
$\hat{w}_{pew,t}$	2.24	3.46	3.06	11.41	11.75	13.71	27.66	67.32
$\hat{w}_{\tilde{c},t}^{pew}$	1.78	2.11	2.50	11.10	11.50	16.99	33.73	80.64
$\hat{w}_{\tilde{c}_2,t}^{pew}$	2.94	4.10	5.19	13.05	14.00	17.91	33.05	83.62
IVOL-10:								
$\hat{w}_{s_2,t}$	1.14	1.34	6.91	11.27	14.12	20.78	36.37	89.55
$\hat{w}_{pew,t}$	−3.41	−4.08	−1.53	7.88	9.08	12.59	25.96	72.29
$\hat{w}_{\tilde{c},t}^{pew}$	−4.67	−6.00	−1.53	6.91	8.23	14.33	31.73	82.57
$\hat{w}_{\tilde{c}_2,t}^{pew}$	−0.55	0.09	3.57	11.62	13.43	17.46	32.34	86.06
IND-10:								
$\hat{w}_{s_2,t}$	1.14	1.34	6.91	11.27	14.12	20.78	36.37	89.55
$\hat{w}_{pew,t}$	−0.10	1.71	6.07	13.66	14.77	18.10	25.39	71.35
$\hat{w}_{\tilde{c},t}^{pew}$	−1.07	0.12	4.68	12.46	13.48	19.31	32.66	82.03
$\hat{w}_{\tilde{c}_2,t}^{pew}$	1.28	2.55	6.39	13.21	14.48	19.34	31.67	84.53
IND-30:								
$\hat{w}_{s_2,t}$	1.14	1.34	6.91	11.27	14.12	20.78	36.37	89.55
$\hat{w}_{pew,t}$	−1.38	0.08	3.23	8.91	9.99	10.58	16.42	38.92
$\hat{w}_{\tilde{c},t}^{pew}$	−2.02	−1.12	3.68	10.16	11.77	16.81	31.37	72.19
$\hat{w}_{\tilde{c}_2,t}^{pew}$	1.08	2.43	6.34	12.82	14.77	18.28	31.33	81.15
DMNU-48:								
$\hat{w}_{s_2,t}$	1.74	3.47	6.50	14.40	15.20	20.52	32.67	89.55
$\hat{w}_{pew,t}$	10.90	−0.31	−5.77	−13.67	−20.83	−5.89	2.69	10.90
$\hat{w}_{\tilde{c},t}^{pew}$	11.90	−0.29	−7.00	−11.43	−20.32	−1.99	12.60	51.88
$\hat{w}_{\tilde{c}_2,t}^{pew}$	57.51	54.05	51.34	51.70	48.08	54.65	63.89	88.66

Table OA.2: **Average Portfolio CER based on Simulated Data (Cont'd)**

C. LW 2004 shrinkage covariance matrix

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\hat{w}_{s,t}^{LW2004}$	0.48	−2.06	4.26	9.24	10.38	20.41	36.14	79.61
$\hat{w}_{p,t}^{LW2004}$	−1.72	−2.35	−2.58	3.73	4.37	8.63	27.75	47.16
$\hat{w}_{\tilde{c},t}^{LW2004}$	5.99	8.04	9.60	18.56	19.29	23.28	37.85	73.32
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	5.11	7.04	8.47	17.32	17.72	21.69	36.56	72.34
IVOL-10:								
$\hat{w}_{s,t}^{LW2004}$	0.48	−2.06	4.26	9.24	10.38	20.41	36.14	79.61
$\hat{w}_{p,t}^{LW2004}$	−8.64	−12.09	−9.34	5.06	2.27	12.13	31.15	56.99
$\hat{w}_{\tilde{c},t}^{LW2004}$	1.99	3.17	8.51	19.36	21.49	25.71	40.18	75.49
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	1.75	2.91	7.43	18.18	19.85	23.97	38.65	74.68
IND-10:								
$\hat{w}_{s,t}^{LW2004}$	0.48	−2.06	4.26	9.24	10.38	20.41	36.14	79.61
$\hat{w}_{p,t}^{LW2004}$	−12.28	−11.42	−9.04	3.28	0.02	9.38	20.62	38.83
$\hat{w}_{\tilde{c},t}^{LW2004}$	1.23	3.67	8.80	16.10	17.73	23.58	35.85	71.05
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	1.12	3.11	7.47	14.56	15.54	21.55	34.33	70.03
IND-30:								
$\hat{w}_{s,t}^{LW2004}$	0.48	−2.06	4.26	9.24	10.38	20.41	36.14	79.61
$\hat{w}_{p,t}^{LW2004}$	−69.10	−67.60	−70.54	−55.68	−62.19	−49.34	−37.71	−39.97
$\hat{w}_{\tilde{c},t}^{LW2004}$	2.29	4.48	9.36	16.04	18.01	23.34	36.40	71.87
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	1.80	3.65	7.83	14.56	15.84	21.23	34.89	71.09
DMNU-48:								
$\hat{w}_{s,t}^{LW2004}$	1.42	1.61	3.85	12.39	12.03	20.01	32.54	79.61
$\hat{w}_{p,t}^{LW2004}$	−59.23	−43.37	−15.03	−55.97	−26.86	−55.38	−63.08	−59.23
$\hat{w}_{\tilde{c},t}^{LW2004}$	91.79	84.34	74.62	80.93	72.11	84.09	95.43	106.50
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	91.53	83.91	72.70	80.36	70.26	83.26	94.73	105.89

Table OA.3: **Standard Errors of Portfolio CER based on Simulated Data**

This table reports the standard errors of the certainty equivalent returns of various portfolios across 1000 simulations with $h = 120$ and $\gamma = 3$. For a given empirical datasets (test assets and benchmark portfolios) containing T months, we bootstrap T monthly data using the stationary block bootstrap approach of Politis and Romano (1994) with the expected block length set to 10 months. Various portfolio rules are applied to the bootstrapped data, and the certainty equivalent returns are computed based on the $T - h$ out-of-sample portfolio returns.

A. Sample optimal portfolio and KZ two-fund rule

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\hat{w}_{s,t}$	2.46	3.91	6.13	8.69	10.11	10.44	12.58	32.55
$\hat{w}_{p,t}$	7.95	10.34	14.09	22.59	27.32	24.18	19.73	64.80
$\hat{w}_{s_2,t}$	2.12	2.78	4.78	6.74	7.43	9.25	11.95	27.40
$\hat{w}_{p_2,t}$	3.73	4.41	5.60	9.14	10.81	10.98	12.10	30.85
$\hat{w}_{\tilde{c},t}$	3.91	4.72	6.18	9.18	10.95	11.29	12.94	30.72
$\hat{w}_{\tilde{c}_2,t}$	3.39	3.91	5.13	7.68	9.12	10.01	12.13	28.34
IVOL-10:								
$\hat{w}_{s,t}$	2.46	3.91	6.13	8.69	10.11	10.44	12.58	32.55
$\hat{w}_{p,t}$	16.59	23.23	25.31	42.27	46.62	40.77	36.79	77.63
$\hat{w}_{s_2,t}$	2.12	2.78	4.78	6.74	7.43	9.25	11.95	27.40
$\hat{w}_{p_2,t}$	5.35	7.51	8.74	16.53	17.26	17.33	18.88	35.63
$\hat{w}_{\tilde{c},t}$	5.55	7.78	8.74	16.37	17.07	16.92	18.10	33.64
$\hat{w}_{\tilde{c}_2,t}$	4.04	5.32	6.50	12.96	13.63	14.52	16.46	29.27
IND-10:								
$\hat{w}_{s,t}$	2.46	3.91	6.13	8.69	10.11	10.44	12.58	32.55
$\hat{w}_{p,t}$	7.08	10.27	11.76	17.40	19.93	19.61	17.67	58.95
$\hat{w}_{s_2,t}$	2.12	2.78	4.78	6.74	7.43	9.25	11.95	27.40
$\hat{w}_{p_2,t}$	2.78	3.79	4.95	7.63	8.22	9.75	10.78	29.71
$\hat{w}_{\tilde{c},t}$	3.13	4.17	5.64	8.18	8.88	10.22	12.23	30.25
$\hat{w}_{\tilde{c}_2,t}$	2.63	3.48	4.87	7.26	7.83	9.60	11.79	27.96

Table OA.3: **Standard Errors of Portfolio CER based on Simulated Data (Cont'd)**

A. Sample optimal portfolio and KZ two-fund rule (cont'd)

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
IND-30:								
$\hat{w}_{s,t}$	2.46	3.91	6.13	8.69	10.11	10.44	12.58	32.55
$\hat{w}_{p,t}$	32.89	40.94	48.49	68.97	77.27	82.47	69.83	257.14
$\hat{w}_{s_2,t}$	2.12	2.78	4.78	6.74	7.43	9.25	11.95	27.40
$\hat{w}_{p_2,t}$	3.61	4.27	5.15	8.98	9.43	10.78	11.54	36.11
$\hat{w}_{\tilde{c},t}$	4.12	4.88	6.46	9.95	10.67	11.93	13.95	37.27
$\hat{w}_{\tilde{c}_2,t}$	3.12	3.71	5.00	8.01	8.29	9.85	12.54	30.50
DMNU-48:								
$\hat{w}_{s,t}$	3.97	7.52	10.61	13.34	14.66	12.45	14.00	32.55
$\hat{w}_{p,t}$	1086.24	1406.32	1586.78	1783.56	2038.10	1606.83	1414.73	1086.24
$\hat{w}_{s_2,t}$	3.42	5.93	8.23	10.32	10.82	10.94	13.18	27.40
$\hat{w}_{p_2,t}$	73.05	88.92	96.76	101.72	111.93	95.85	89.47	73.05
$\hat{w}_{\tilde{c},t}$	72.44	88.19	96.37	101.59	111.80	94.68	88.53	70.22
$\hat{w}_{\tilde{c}_2,t}$	33.35	32.58	34.15	39.43	47.64	38.59	36.86	37.91

Table OA.3: **Standard Errors of Portfolio CER based on Simulated Data (Cont'd)**

B. Portfolio PEW

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\hat{w}_{s_2,t}$	2.12	2.78	4.78	6.74	7.43	9.25	11.95	27.40
$\hat{w}_{pew,t}$	4.02	4.68	5.82	9.21	10.82	10.94	12.10	30.50
$\hat{w}^{pew}_{\tilde{c},t}$	4.09	4.87	6.20	9.24	10.91	11.25	12.91	30.67
$\hat{w}^{pew}_{\tilde{c}_2,t}$	3.37	3.86	5.00	7.57	8.91	9.88	12.03	27.90
IVOL-10:								
$\hat{w}_{s_2,t}$	2.12	2.78	4.78	6.74	7.43	9.25	11.95	27.40
$\hat{w}_{pew,t}$	5.63	7.71	8.86	16.37	17.06	17.17	18.83	35.28
$\hat{w}^{pew}_{\tilde{c},t}$	5.86	8.03	8.93	16.25	16.96	16.89	18.13	33.51
$\hat{w}^{pew}_{\tilde{c}_2,t}$	4.07	5.31	6.41	12.63	13.30	14.25	16.30	28.77
IND-10:								
$\hat{w}_{s_2,t}$	2.12	2.78	4.78	6.74	7.43	9.25	11.95	27.40
$\hat{w}_{pew,t}$	3.46	4.30	5.28	7.73	8.33	9.69	10.83	29.33
$\hat{w}^{pew}_{\tilde{c},t}$	3.69	4.50	5.70	8.17	8.88	10.25	12.26	30.17
$\hat{w}^{pew}_{\tilde{c}_2,t}$	2.82	3.57	4.75	7.10	7.65	9.41	11.67	27.57
IND-30:								
$\hat{w}_{s_2,t}$	2.12	2.78	4.78	6.74	7.43	9.25	11.95	27.40
$\hat{w}_{pew,t}$	4.33	4.87	5.58	9.19	9.63	10.73	11.68	35.40
$\hat{w}^{pew}_{\tilde{c},t}$	4.32	5.10	6.43	9.88	10.60	11.76	14.01	36.74
$\hat{w}^{pew}_{\tilde{c}_2,t}$	3.12	3.71	4.87	7.74	8.06	9.60	12.39	29.96
DMNU-48:								
$\hat{w}_{s_2,t}$	3.42	5.93	8.23	10.32	10.82	10.94	13.18	27.40
$\hat{w}_{pew,t}$	72.93	88.52	96.13	101.03	111.05	95.27	88.84	72.93
$\hat{w}^{pew}_{\tilde{c},t}$	72.39	87.87	95.95	100.71	110.60	93.90	87.72	70.17
$\hat{w}^{pew}_{\tilde{c}_2,t}$	33.19	32.22	33.68	38.83	46.84	38.05	36.36	37.64

Table OA.3: **Standard Errors of Portfolio CER based on Simulated Data (Cont'd)**

C. LW 2004 shrinkage covariance matrix

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\hat{w}_{s,t}^{LW2004}$	2.46	3.85	5.42	7.60	8.48	9.43	11.58	24.34
$\hat{w}_{p,t}^{LW2004}$	5.37	6.49	7.21	12.76	14.13	13.78	14.44	25.55
$\hat{w}_{\tilde{c},t}^{LW2004}$	3.54	4.06	4.85	7.43	8.29	9.43	11.72	20.89
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	3.40	3.95	4.64	7.44	8.18	9.53	11.79	21.10
IVOL-10:								
$\hat{w}_{s,t}^{LW2004}$	2.46	3.85	5.42	7.60	8.48	9.43	11.58	24.34
$\hat{w}_{p,t}^{LW2004}$	5.67	8.45	9.27	14.59	15.75	15.94	16.67	23.51
$\hat{w}_{\tilde{c},t}^{LW2004}$	3.16	4.38	5.50	9.05	9.49	11.17	13.20	20.25
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	2.89	4.09	5.19	9.06	9.42	11.37	13.33	20.42
IND-10:								
$\hat{w}_{s,t}^{LW2004}$	2.46	3.85	5.42	7.60	8.48	9.43	11.58	24.34
$\hat{w}_{p,t}^{LW2004}$	4.48	6.05	7.10	9.34	11.22	11.25	12.71	23.15
$\hat{w}_{\tilde{c},t}^{LW2004}$	2.71	3.50	4.80	6.52	7.24	8.67	11.02	20.20
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	2.44	3.21	4.59	6.52	7.16	8.91	11.21	20.41
IND-30:								
$\hat{w}_{s,t}^{LW2004}$	2.46	3.85	5.42	7.60	8.48	9.43	11.58	24.34
$\hat{w}_{p,t}^{LW2004}$	12.84	13.58	15.39	17.80	20.07	19.96	21.39	31.95
$\hat{w}_{\tilde{c},t}^{LW2004}$	3.05	3.48	4.80	6.77	7.34	8.69	11.30	20.10
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	2.80	3.32	4.67	6.78	7.30	8.95	11.52	20.33
DMNU-48:								
$\hat{w}_{s,t}^{LW2004}$	3.97	7.08	9.42	11.30	11.80	11.43	12.96	24.34
$\hat{w}_{p,t}^{LW2004}$	116.35	106.00	90.77	110.72	94.92	110.82	118.73	116.35
$\hat{w}_{\tilde{c},t}^{LW2004}$	24.70	23.46	22.87	22.81	22.34	24.25	26.49	27.70
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	24.77	23.57	23.33	22.90	22.70	24.49	26.61	27.86

Table OA.4: **Portfolio CER with Transaction Cost**

This table reports the certainty equivalent returns (in annualized percentage points) of the portfolios examined in Tables 1 to 3 of the paper for $h = 120$ and $\gamma = 3$ using the same empirical data and assuming a transaction cost of 10 bps. At the end of a given month t, various portfolios are constructed using excess returns from month $t-119$ to month t and are held for one month (i.e., month $t+1$). The out-of-sample excess returns (i.e., month $t+1$) are used to compute the certainty equivalent returns of the portfolios.

A. Sample optimal portfolio and KZ two-fund rule

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\hat{w}_{s,t}$	2.74	1.70	1.32	4.37	3.02	6.10	12.72	−53.46
$\hat{w}_{p,t}$	−44.79	−61.00	−83.69	−78.22	−116.81	−101.56	−84.55	−292.60
$\hat{w}_{s_2,t}$	3.03	4.31	7.93	13.65	16.28	15.03	20.27	11.66
$\hat{w}_{p_2,t}$	−4.96	−4.82	−8.28	−0.41	−7.30	−7.58	−2.94	−32.42
$\hat{w}_{\tilde{c},t}$	−3.88	−4.48	−6.12	4.50	−1.68	−1.40	4.03	−20.42
$\hat{w}_{\tilde{c}_2,t}$	0.81	1.97	−0.15	11.47	6.07	14.12	16.61	15.34
IVOL-10:								
$\hat{w}_{s,t}$	2.74	1.70	1.32	4.37	3.02	6.10	12.72	−53.46
$\hat{w}_{p,t}$	−59.82	−82.37	−93.92	−160.06	−169.90	−150.27	−115.68	−363.43
$\hat{w}_{s_2,t}$	3.03	4.31	7.93	13.65	16.28	15.03	20.27	11.66
$\hat{w}_{p_2,t}$	−8.96	−8.01	−2.75	−19.86	−14.12	−23.48	−10.00	−60.48
$\hat{w}_{\tilde{c},t}$	−7.05	−7.54	−1.97	−16.25	−10.54	−16.09	−1.71	−33.94
$\hat{w}_{\tilde{c}_2,t}$	1.89	3.31	12.91	5.62	14.49	7.14	18.63	2.91
IND-10:								
$\hat{w}_{s,t}$	2.74	1.70	1.32	4.37	3.02	6.10	12.72	−53.46
$\hat{w}_{p,t}$	−26.79	−32.14	−48.51	−68.01	−82.48	−70.43	−52.93	−247.68
$\hat{w}_{s_2,t}$	3.03	4.31	7.93	13.65	16.28	15.03	20.27	11.66
$\hat{w}_{p_2,t}$	−1.41	2.15	4.23	3.50	2.78	1.70	6.48	−12.87
$\hat{w}_{\tilde{c},t}$	−0.51	1.74	4.32	6.38	6.75	3.54	12.68	−7.00
$\hat{w}_{\tilde{c}_2,t}$	0.89	3.03	7.89	11.22	15.10	15.32	18.24	29.43

Table OA.4: **Portfolio CER with Transaction Cost (Cont'd)**

A. Sample optimal portfolio and KZ two-fund rule (cont'd)

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
IND-30:								
$\hat{w}_{s,t}$	2.74	1.70	1.32	4.37	3.02	6.10	12.72	−53.46
$\hat{w}_{p,t}$	−195.48	−247.20	−290.96	−418.47	−440.57	−396.34	−324.36	−1176.74
$\hat{w}_{s_2,t}$	3.03	4.31	7.93	13.65	16.28	15.03	20.27	11.66
$\hat{w}_{p_2,t}$	−6.50	−4.86	−0.76	−7.43	−5.61	−8.83	−4.35	−32.32
$\hat{w}_{\tilde{c},t}$	−4.80	−2.95	2.90	−0.58	4.60	−1.61	11.12	−24.55
$\hat{w}_{\tilde{c}_2,t}$	−1.13	2.53	7.18	10.68	15.58	14.09	17.54	24.54
DMNU-48:								
$\hat{w}_{s,t}$	−0.54	−2.67	−7.39	−7.62	−13.84	−9.78	−0.42	−53.46
$\hat{w}_{p,t}$	−3445.27	−4373.05	−4660.12	−4907.76	−5347.07	−4574.43	−3819.13	−3445.27
$\hat{w}_{s_2,t}$	−0.35	1.01	0.99	5.83	4.54	2.31	8.50	11.66
$\hat{w}_{p_2,t}$	−73.41	−100.42	−102.64	−96.23	−102.75	−100.48	−68.21	−73.41
$\hat{w}_{\tilde{c},t}$	−71.07	−101.35	−110.58	−97.43	−113.48	−114.95	−71.49	−58.49
$\hat{w}_{\tilde{c}_2,t}$	15.90	23.06	27.54	4.29	6.54	−10.27	9.11	32.73

Table OA.4: **Portfolio CER with Transaction Cost (Cont'd)**

B. Portfolio PEW

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\hat{w}_{s_2,t}$	3.03	4.31	7.93	13.65	16.28	15.03	20.27	11.66
$\hat{w}_{pew,t}$	−3.28	−3.10	−5.72	1.47	−4.01	−4.31	−1.31	−29.81
$\hat{w}^{pew}_{\tilde{c},t}$	−5.18	−4.86	−5.78	5.76	1.35	1.99	4.86	−20.43
$\hat{w}^{pew}_{\tilde{c}_2,t}$	0.61	3.00	0.63	12.31	8.44	15.18	17.08	20.55
IVOL-10:								
$\hat{w}_{s_2,t}$	3.03	4.31	7.93	13.65	16.28	15.03	20.27	11.66
$\hat{w}_{pew,t}$	−6.78	−5.96	0.06	−18.55	−11.82	−21.73	−8.29	−60.74
$\hat{w}^{pew}_{\tilde{c},t}$	−9.16	−9.51	−2.37	−16.93	−9.47	−16.22	−1.54	−32.69
$\hat{w}^{pew}_{\tilde{c}_2,t}$	1.26	3.24	13.26	6.86	15.40	8.03	19.35	5.57
IND-10:								
$\hat{w}_{s_2,t}$	3.03	4.31	7.93	13.65	16.28	15.03	20.27	11.66
$\hat{w}_{pew,t}$	0.63	4.06	7.81	5.42	6.18	5.42	8.37	−11.81
$\hat{w}^{pew}_{\tilde{c},t}$	−2.82	1.37	4.73	8.06	9.14	4.06	13.49	−8.73
$\hat{w}^{pew}_{\tilde{c}_2,t}$	1.14	3.13	7.79	12.40	15.78	15.48	18.94	30.22
IND-30:								
$\hat{w}_{s_2,t}$	3.03	4.31	7.93	13.65	16.28	15.03	20.27	11.66
$\hat{w}_{pew,t}$	−4.01	−2.27	2.43	−5.74	−3.38	−6.25	−1.80	−32.03
$\hat{w}^{pew}_{\tilde{c},t}$	−6.03	−3.69	2.74	0.38	5.72	−0.72	12.65	−27.41
$\hat{w}^{pew}_{\tilde{c}_2,t}$	−1.08	3.20	7.32	11.00	15.32	13.76	18.17	24.58
DMNU-48:								
$\hat{w}_{s_2,t}$	−0.35	1.01	0.99	5.83	4.54	2.31	8.50	11.66
$\hat{w}_{pew,t}$	−71.52	−99.05	−99.93	−93.33	−98.93	−95.41	−62.92	−71.52
$\hat{w}^{pew}_{\tilde{c},t}$	−69.02	−99.67	−107.94	−93.58	−107.01	−106.15	−65.18	−58.89
$\hat{w}^{pew}_{\tilde{c}_2,t}$	18.12	25.00	28.33	8.08	9.82	−4.38	11.79	33.66

Table OA.4: **Portfolio CER with Transaction Cost (Cont'd)**

C. LW2004 shrinkage covariance matrix

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\hat{w}_{s,t}^{LW2004}$	2.74	3.35	6.77	12.91	13.71	18.83	27.90	41.25
$\hat{w}_{p,t}^{LW2004}$	−8.89	−12.18	−12.90	−5.19	−6.66	2.01	5.72	5.05
$\hat{w}_{\tilde{c},t}^{LW2004}$	5.35	7.67	10.23	18.91	19.89	22.68	29.47	47.82
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	5.90	7.70	9.16	19.14	19.60	22.60	29.14	44.88
IVOL-10:								
$\hat{w}_{s,t}^{LW2004}$	2.74	3.35	6.77	12.91	13.71	18.83	27.90	41.25
$\hat{w}_{p,t}^{LW2004}$	−8.05	−0.21	2.72	1.92	1.36	−5.56	1.83	−0.36
$\hat{w}_{\tilde{c},t}^{LW2004}$	6.73	15.24	21.90	27.62	31.16	24.30	30.81	43.22
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	7.42	14.83	19.58	27.25	31.34	24.80	28.83	42.69
IND-10:								
$\hat{w}_{s,t}^{LW2004}$	2.74	3.35	6.77	12.91	13.71	18.83	27.90	41.25
$\hat{w}_{p,t}^{LW2004}$	−13.06	−4.23	−4.25	−0.37	−1.32	8.16	4.39	12.47
$\hat{w}_{\tilde{c},t}^{LW2004}$	0.39	5.56	10.58	15.06	17.48	20.90	27.18	47.94
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	1.33	3.20	7.87	12.00	13.45	19.08	24.53	47.52
IND-30:								
$\hat{w}_{s,t}^{LW2004}$	2.74	3.35	6.77	12.91	13.71	18.83	27.90	41.25
$\hat{w}_{p,t}^{LW2004}$	−80.99	−67.03	−69.85	−60.47	−65.92	−46.12	−62.83	−66.85
$\hat{w}_{\tilde{c},t}^{LW2004}$	−0.30	4.24	9.16	13.25	15.51	20.09	27.80	48.94
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	1.05	3.10	6.81	11.13	10.88	18.43	24.50	48.41
DMNU-48:								
$\hat{w}_{s,t}^{LW2004}$	−0.54	−0.45	−1.54	5.66	4.20	8.65	18.42	41.25
$\hat{w}_{p,t}^{LW2004}$	−343.97	−337.31	−260.10	−345.95	−280.51	−342.95	−391.27	−343.97
$\hat{w}_{\tilde{c},t}^{LW2004}$	80.68	74.59	63.50	71.43	60.88	67.54	72.02	76.63
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	80.68	74.48	59.90	71.25	57.55	64.31	68.51	76.33

Table OA.5: **Portfolio CER with $\gamma = 5$**

This table reports the certainty equivalent returns of the portfolios examined in Tables 1 to 3 of the paper bases on the same empirical datasets with $h = 120$ and $\gamma = 5$.

A. Sample optimal portfolio and KZ two-fund rule

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\hat{w}_{s,t}$	1.70	1.62	3.19	5.24	5.22	8.54	15.03	−20.53
$\hat{w}_{p,t}$	−8.93	−16.64	−29.02	−25.81	−48.96	−37.26	−24.51	−148.32
$\hat{w}_{s_2,t}$	1.87	2.99	6.50	10.07	12.18	12.88	18.44	16.50
$\hat{w}_{p_2,t}$	5.49	6.77	5.48	10.72	6.72	9.12	13.81	−1.12
$\hat{w}_{\tilde{c},t}$	5.73	6.37	5.19	11.82	8.19	10.69	15.50	1.98
$\hat{w}_{\tilde{c}_2,t}$	6.30	7.07	5.67	13.50	10.59	17.30	19.65	21.32
IVOL-10:								
$\hat{w}_{s,t}$	1.70	1.62	3.19	5.24	5.22	8.54	15.03	−20.53
$\hat{w}_{p,t}$	−17.16	−28.92	−33.38	−67.44	−72.60	−62.16	−40.37	−187.77
$\hat{w}_{s_2,t}$	1.87	2.99	6.50	10.07	12.18	12.88	18.44	16.50
$\hat{w}_{p_2,t}$	4.08	6.12	10.95	4.80	8.45	2.70	12.10	−16.13
$\hat{w}_{\tilde{c},t}$	4.72	5.88	10.73	5.88	9.56	5.67	15.53	−4.22
$\hat{w}_{\tilde{c}_2,t}$	7.44	8.86	15.43	15.61	20.25	15.55	22.96	15.34
IND-10:								
$\hat{w}_{s,t}$	1.70	1.62	3.19	5.24	5.22	8.54	15.03	−20.53
$\hat{w}_{p,t}$	−14.43	−16.72	−24.50	−35.71	−43.79	−34.26	−22.56	−134.40
$\hat{w}_{s_2,t}$	1.87	2.99	6.50	10.07	12.18	12.88	18.44	16.50
$\hat{w}_{p_2,t}$	−0.12	2.51	4.97	4.68	4.55	5.69	9.38	1.72
$\hat{w}_{\tilde{c},t}$	0.39	2.26	5.20	6.50	7.10	7.02	14.18	5.98
$\hat{w}_{\tilde{c}_2,t}$	1.04	2.62	6.21	8.79	10.99	12.95	15.79	26.05

Table OA.5: **Portfolio CER with $\gamma = 5$ (Cont'd)**

A. Sample optimal portfolio and KZ two-fund rule (cont'd)

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
IND-30:								
$\hat{w}_{s,t}$	1.70	1.62	3.19	5.24	5.22	8.54	15.03	−20.53
$\hat{w}_{p,t}$	−110.94	−140.04	−164.17	−237.65	−250.31	−221.99	−178.85	−684.74
$\hat{w}_{s_2,t}$	1.87	2.99	6.50	10.07	12.18	12.88	18.44	16.50
$\hat{w}_{p_2,t}$	−2.45	−0.85	2.44	−0.99	0.21	−0.43	2.01	−10.03
$\hat{w}_{\tilde{c},t}$	−1.51	0.22	4.96	3.29	6.63	4.59	13.68	−3.72
$\hat{w}_{\tilde{c}_2,t}$	0.16	2.51	5.83	9.08	11.70	12.42	15.24	23.61
DMNU-48:								
$\hat{w}_{s,t}$	−0.27	−0.97	−2.65	−1.81	−5.04	−1.46	6.78	−20.53
$\hat{w}_{p,t}$	−2054.89	−2630.71	−2807.64	−2960.99	−3232.44	−2752.81	−2280.36	−2054.89
$\hat{w}_{s_2,t}$	−0.17	1.03	1.84	5.54	5.07	4.80	11.01	16.50
$\hat{w}_{p_2,t}$	−29.44	−46.60	−48.00	−44.34	−48.41	−46.43	−25.46	−29.44
$\hat{w}_{\tilde{c},t}$	−28.09	−47.21	−53.08	−45.54	−55.63	−55.73	−27.51	−22.58
$\hat{w}_{\tilde{c}_2,t}$	19.40	23.09	24.74	11.96	11.91	2.87	16.12	28.79

Table OA.5: **Portfolio CER with $\gamma = 5$ (Cont'd)**

B. Portfolio PEW

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\hat{w}_{pew,t}$	5.90	7.22	6.38	11.65	8.49	10.87	14.52	0.39
$\hat{w}^{pew}_{\tilde{c},t}$	5.63	6.49	5.33	12.60	9.96	12.50	15.63	1.95
$\hat{w}^{pew}_{\tilde{c}_2,t}$	6.38	7.99	6.20	13.82	11.83	17.85	19.80	24.33
IVOL-10:								
$\hat{w}_{pew,t}$	4.83	6.82	12.17	5.39	9.62	3.52	12.91	−16.36
$\hat{w}^{pew}_{\tilde{c},t}$	3.99	4.98	10.57	5.39	10.18	5.48	15.54	−3.75
$\hat{w}^{pew}_{\tilde{c}_2,t}$	7.08	9.01	15.75	15.99	20.53	16.02	23.17	16.47
IND-10:								
$\hat{w}_{pew,t}$	1.03	3.57	6.99	5.79	6.54	7.85	10.38	2.28
$\hat{w}^{pew}_{\tilde{c},t}$	−0.91	2.04	5.31	7.62	8.61	7.22	14.76	4.93
$\hat{w}^{pew}_{\tilde{c}_2,t}$	1.22	2.68	6.12	9.54	11.43	13.01	16.30	26.37
IND-30:								
$\hat{w}_{pew,t}$	−1.07	0.59	4.22	−0.03	1.49	1.03	3.42	−9.92
$\hat{w}^{pew}_{\tilde{c},t}$	−2.15	−0.24	4.69	3.87	7.30	4.97	14.65	−5.48
$\hat{w}^{pew}_{\tilde{c}_2,t}$	0.19	2.90	5.89	9.21	11.51	12.19	15.70	23.60
DMNU-48:								
$\hat{w}_{pew,t}$	−28.30	−45.75	−46.42	−42.62	−46.27	−43.40	−22.34	−28.30
$\hat{w}^{pew}_{\tilde{c},t}$	−26.83	−46.19	−51.53	−43.24	−51.82	−50.43	−23.74	−22.90
$\hat{w}^{pew}_{\tilde{c}_2,t}$	20.69	24.23	25.04	14.19	13.61	6.38	17.69	29.25

Table OA.5: **Portfolio CER with $\gamma = 5$ (Cont'd)**

C. LW 2004 shrinkage covariance matrix

	CAPM	FF-3	Carhart-4	FF-5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10:								
$\hat{w}_{s,t}^{LW2004}$	1.70	2.51	6.17	9.36	10.61	15.07	22.18	32.77
$\hat{w}_{p,t}^{LW2004}$	4.72	3.79	3.57	8.88	7.81	14.19	17.74	17.68
$\hat{w}_{\tilde{c},t}^{LW2004}$	7.74	9.68	10.91	16.39	16.79	20.02	25.31	36.65
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	7.66	9.01	9.44	16.25	16.13	19.41	24.23	34.60
IVOL-10:								
$\hat{w}_{s,t}^{LW2004}$	1.70	2.51	6.17	9.36	10.61	15.07	22.18	32.77
$\hat{w}_{p,t}^{LW2004}$	3.89	9.79	12.30	14.10	13.74	9.70	15.13	14.19
$\hat{w}_{\tilde{c},t}^{LW2004}$	7.99	13.99	18.76	23.35	25.53	21.59	26.67	33.90
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	7.83	12.91	16.18	22.84	24.96	20.92	24.27	33.54
IND-10:								
$\hat{w}_{s,t}^{LW2004}$	1.70	2.51	6.17	9.36	10.61	15.07	22.18	32.77
$\hat{w}_{p,t}^{LW2004}$	−6.90	−0.99	0.29	2.21	2.43	9.26	8.26	12.76
$\hat{w}_{\tilde{c},t}^{LW2004}$	0.64	4.11	8.12	10.46	12.39	15.62	20.85	34.36
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	1.04	2.50	5.84	8.53	9.56	14.05	18.66	34.08
IND-30:								
$\hat{w}_{s,t}^{LW2004}$	1.70	2.51	6.17	9.36	10.61	15.07	22.18	32.77
$\hat{w}_{p,t}^{LW2004}$	−45.50	−36.45	−37.24	−31.79	−34.46	−21.93	−31.11	−34.80
$\hat{w}_{\tilde{c},t}^{LW2004}$	0.51	3.56	7.37	9.52	11.26	15.11	21.24	34.97
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	1.03	2.50	5.22	8.16	8.03	13.69	18.57	34.63
DMNU-48:								
$\hat{w}_{s,t}^{LW2004}$	−0.27	0.23	0.57	5.10	4.68	8.54	16.11	32.77
$\hat{w}_{p,t}^{LW2004}$	−196.47	−193.10	−148.78	−197.64	−160.73	−195.53	−225.41	−196.47
$\hat{w}_{\tilde{c},t}^{LW2004}$	53.56	49.46	42.10	47.47	40.46	45.80	49.97	52.62
$\hat{w}_{\tilde{c}_2,t}^{LW2004}$	53.56	49.39	39.65	47.36	38.19	43.58	47.53	52.43