A Simple Approach to Bond Option Pricing

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In the last 15 years or so, tremendous efforts and progress have been made in valuing interest rate sensitive derivative securities. Broadly speaking, two different approaches have been used. Some authors have modeled interest rates in an equilibrium setting and derived bond prices and other interest rate derivative securities prices based on the equilibrium movements of the underlying interest rates. Examples include Vasicek (1977) and Cox, Ingersoll, and Ross (1985). Others, pioneered by Ho and Lee (1986) and later generalized by authors such as Black, Derman, and Toy (1990); Hull and White (1990); and Heath, Jarrow, and Morton (1992); have developed models which describe the equilibrium movements of the whole term structure by taking the initial term structures as given. The two approaches differ mainly in that by construction in the second approach discount bonds are always correctly priced.

In both approaches, formulas for European options on discount bonds are given, except for Ho and Lee (1986) and Black, Derman, and Toy (1990) who developed a lattice framework. No attempt was made in any of these studies to derive closed-form formulas for European options on coupon-bonds, or for bond portfolios in general. Implicitly, however,

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Hull and White (1990) realized that Jamshidian's approach (Jamshidian, 1989) can be applied to coupon bonds.

In a seemingly unrelated area, many authors have examined the concept and application of duration and convexity in the context of bond portfolio management. The literature is so abundant that an exhaustive list of studies in a limited space is simply impossible. One article worth mentioning is by Cox, Ingersoll, and Ross (1979). They convincingly show that when interest rates are stochastic, the traditional definition of duration is not adequate. Instead, they propose an alternate definition with time as its unit. Their result becomes the key motivation of the pricing approach proposed in this article. Specifically, the risk measurement (standard deviation of return) of a bond portfolio is identified using the duration definition of Cox, Ingersoll, and Ross (1979). This duration is matched with that of a single discount bond which is used as a proxy for the original bond portfolio, and an option on the discount bond is priced. The resulting option price is a close approximation of the true option price which can be obtained via Jamshidian's approach.

The idea of linking the standard deviation of bond returns to duration has been explicitly explored in another article. By proposing a particular form of bond return standard deviation (i.e., proportional to duration), Schaefer and Schwartz (1987) developed a framework within which options on coupon bonds can be priced. Unfortunately, their approach required the use of numerical procedures. More importantly, their framework depends on an extremely strong and inconsistent assumption: the short-term interest rate is constant.

This article proposes a very simple approach to pricing European options on bond portfolios in a one-factor framework. It is shown that as long as a closed-form formula exists for options on discount bonds, options on bond portfolios can always be priced using the same formula. Besides providing an alternative, simple pricing method for coupon bond options, the article also fills a small gap in the literature by applying the methodology to an environment with a non-Markovian spot rate. Moreover, the proposed approach can be used to examine bond option pricing from an alternative angle: impacts of duration and convexity on bond option prices. This should be valuable to both academics and practitioners who are concerned with the effects of bond characteristics on the bond option prices.

The rest of the article is organized in four sections. It reviews the basics of Jamshidian's approach to pricing options on bond portfolios; introduces an alternative, simple approach; and investigates the performance of the proposed simple approach in a variety of parameter settings. The last section summarizes the article.

JAMSHIDIAN'S APPROACH[1]

Suppose a closed-form formula exists for a European call on a discount bond.[2] Following Jamshidian (1989), let $C(r, t, T, s, K)$ denote the price of a European call on a discount bond which matures at time s. The option matures at T ($T < s$) with an exercise price, K. The spot interest rate at the current time, t, is denoted by r. In addition, let $B(r, t, s)$ denote the price of the discount bond. Part (c) of Jamshidian's proposition roughly states the following:

A European call with maturity T on a bond portfolio consisting of n discount bonds with distinct maturities s_i ($i = 1, 2, \ldots, n$ and $T < s_1 < s_2 < \ldots, < s_n$) and a_i ($a_i > 0$, $i = 1, 2, 3, \ldots, n$) issues of each can be priced as

$$C_a = \sum_{i=1}^{n} a_i C(r, t, T, s_i, K_i) \tag{1}$$

where

$$K_i = B(r^*, t, s_i) \text{ and } r^* \text{ solves } \sum_{i=1}^{n} a_i B(r^*, T, s_i) = K.$$

Although the proposition is cast in the context of Vasicek's interest rate model [Vasicek, (1977)] it is valid for any one-factor term structure model, as stated and proved by Jamshidian (1987, 1989). Jamshidian's approach says that an option on a bond portfolio can be decomposed into a portfolio of options. The individual options all have the same time-to-maturity and are written on the individual discount bonds in the bond portfolio. The decomposition is achieved by properly spreading the original exercise price among all individual options. This exercise price decomposition, in turn, is motivated by the fact that all discount bond prices are instantaneously perfectly correlated.

One disadvantage of Jamshidian's approach is its reliance on an iterative procedure. In addition, one must calculate n option prices if there are n discount bonds in the portfolio. Moreover, it is not immediately

[1]Using a specific interest rate process proposed by CIR (Cox, Ingersoll, and Ross, 1985); Longstaff (1993) derived a closed-form formula for European options on coupon bonds. Longstaff (1993) obtained the formula by directly solving a partial differential equation. Since Jamshidian's approach is not restricted to a particular interest rate process and would lead to Longstaff's formula (after consolidating terms) in the case of CIR interest rate, this study will examines Jamshidian's approach only.

[2]For brevity, only call options are considered throughout the article. Because put–call parity holds for both discount bond options and coupon bond options regardless of the specific pricing models, whatever applies to call options will apply also to puts (after adjusting via put–call parity.)

clear as to how the option can be hedged without holding an array of discount bonds.

If all discount bonds are instantaneously, perfectly correlated, a bond portfolio must also be instantaneously, perfectly correlated with other discount bonds outside the portfolio. There must be a discount bond which exhibits sufficiently similar risk characteristics as the bond portfolio. If one can identify this proxy discount bond, then the pricing of bond portfolio options reduces to the simple pricing of a discount bond option. The following section shows that a proxy bond can, indeed, be identified with the help of duration.

A SIMPLE APPROACH

The concept of duration was initially developed for simple movements of yield curves. The most common definition is

$$D = -\frac{\partial B}{\partial y}\frac{1}{B} \tag{2}$$

where y is the yield to maturity of the bond and B is the bond price. As proved by Ingersoll, Skelton, and Weil (1978), the above duration is a good measure of risk only when the yield curve is flat and moves in a parallel fashion. Cox, Ingersoll, and Ross (1979) realized the limitation of this definition and proposed a different definition of duration, or risk measure.

To formally introduce their definition, assume that in a one-factor framework the short-term interest rate follows the following process

$$dr = \mu_r(r, t)dt + \sigma_r(r, t)dw \tag{3}$$

where $\mu_r(r, t)$ and $\sigma_r(r, t)$ are the drift and instantaneous standard deviation of the changes in interest rate and dw is the increment of a standard Wiener process. To keep the process in (3) sufficiently general, the form of $\mu_r(r, t)$ and $\sigma_r(r, t)$ is not specified. By Ito's lemma, the process followed by a discount bond price, B, is given by

$$\frac{dB}{B} = \mu_B dt + \frac{\partial B}{\partial r}\frac{1}{B}\sigma_r\,(r, t)\,dw \tag{4}$$

where μ_B is the drift of the bond return.

For pricing purposes the form of the drift, μ_B, does not matter.[3] Analogous to the Black–Scholes pricing model, what really matters is the

[3]See footnotes 10 and 11 in Cox, Ingersoll, and Ross (1979).

volatility. Since $\sigma_r(r, t)$ is common to all bonds, the proper risk measure is then $(\partial B/\partial r)(1/B)$. Notice that this expression differs from the definition in (2) only in one respect: the price change is reflected in the spot rate instead of the yield-to-maturity. The alternate measure is advantageous because it is immune to the troubling fact that yields of bonds with different maturities may change by different amounts given a change in the spot rate.

Cox, Ingersoll, and Ross (1979) then propose that duration should be defined as the time to maturity of the discount bond, instead of the quantity $(\partial B/\partial r)(1/B)$. Accordingly, a portfolio's duration is defined as

$$D_p = f^{-1}\left(-\frac{\partial B}{\partial r}\frac{1}{B}\right) = f^{-1}\left(-\frac{\sum a_i \dfrac{\partial B_i}{\partial r}}{\sum a_i B_i}\right) \tag{5}$$

where $f(\tau) = -[\partial B(\tau)/\partial r][1/B(\tau)]$, f^{-1} is the inverse function for time to maturity, τ, and a_i's ($i = 1, 2 \ldots$) are the portfolio weights. The above formula says that the duration of a bond portfolio is the time-to-maturity of a discount bond, τ, whose risk measure, $-[\partial B(\tau)/\partial r][1/B(\tau)]$, is equal to that of the portfolio.[4]

To understand the intuition behind the above definition, notice that the process followed by the portfolio price, given the process in (3), is

$$\begin{aligned}\frac{dP}{P} &= \mu_p dt + \frac{\partial P}{\partial r}\frac{1}{P}\sigma_r(r, t)\, dw \\ &= \mu_p dt + \frac{\sum a_i \dfrac{\partial B_i}{\partial r}}{\sum a_i B_i}\sigma_r(r, t)\, dw \end{aligned} \tag{6}$$

where μ_p is the drift of the portfolio's return and $P = \Sigma a_i B_i$. Comparing (6) with (4), it can be seen that the portfolio has the same risk sensitivity as the discount bond as long as the instantaneous standard deviations are matched. This forms the basis for the definition in (5).[5]

This study's simple approach to pricing European options on portfolios of discount bonds can be stated as follows:

In a one-factor framework, as long as pricing formulas exist for European options on discount bonds, approximate pricing formulas exist for

[4]For brevity, the time-to-maturity argument, τ, is dropped whenever it can be inferred from the context.

[5]The negative sign in the definition of function f simply conforms to the traditional definition of duration.

European options on portfolios of discount bonds. Specifically, the portfolio's duration as defined in (5) is calculated. Then, a discount bond is identified, whose time to maturity is the portfolio's duration and whose face value is such that its current value is equal to the portfolio's value. Finally, a European option on this discount bond is priced. The option's derivatives, such as delta and gamma, can be approximated also using the same approximating formula.

It should be noted that, in general, a portfolio's duration depends on the interest rate level and time. Therefore, the duration matching should be done each time the option is to be priced.

Sometimes researchers work with bond price processes without specifying a process for the underlying short-term interest rate. In such a case, it is possible that the bond price process is a simple one (e.g., lognormal), but the short-term interest rate is non-Markovian.[6] For a non-Markovian interest rate, it is not possible to write down a single dimension diffusion process only in terms of the current value of the Wiener process. As a result, the above duration-matching procedure can not be implemented directly. Fortunately, a simple solution exists. Recall that the essence of the approximation procedure proposed above lies in the matching of bond price volatilities. Hence, when working with discount bond price processes, the above procedure boils down to the following matching equation:

$$\sigma_B(t, \tau) = \frac{\sum a_i \sigma_B(t, t_i) B(t, t_i)}{\sum a_i B(t, t_i)}$$

where $\sigma_B(t, t_i)$ is the volatility of a discount bond with maturity t_i. In other words, one needs to first calculate the weighted average of volatilities of all the discount bonds in the portfolio (i.e., the right-hand side of the above equation), and then match it to that of a discount bond whose maturity, τ, is solved for herewith. This discount bond will be the proxy bond.[7] It is important to note that Jamshidian's approach *does not* apply when the short rate is non-Markovian. The Appendix gives an example of a non-Markovian spot rate where an option on a coupon bond can be priced using this study's approach.

The following questions are answered in the next section. How good is the approximation? Is the approximation robust with respect to parameter inputs? Also, why is it an approximation (as opposed to an accurate solution)?

[6]For more details, see Carverhill (1994a).
[7]The volatility matching is exactly equivalent to duration matching when the short rate is Markovian.

FURTHER INVESTIGATIONS OF THE SIMPLE APPROACH

The next section begins with an in-depth analysis of a typical option contract. The analysis is then broadened by using more parameter combinations. The purposes are to identify the key source of pricing errors and to draw a general conclusion in terms of robustness. The Vasicek [Vasicek (1977)] and CIR [Cox, Ingersoll and Ross (1985)] models are used.

The interest rate process is defined as:

$$dr = \kappa(\mu - r)dt + \sigma_r r^\beta dw \tag{7}$$

where κ, μ, and σ_r are constants. For Vasicek's model $\beta = 0$. For CIR $\beta = 0.5$. In Vasicek's model the price of a discount bond maturing at time s is

$$B(r, t, s) = D(t, s)\, e^{-E(t,s)r}$$

where

$$E(t, s) = \frac{1 - e^{-\kappa(s-t)}}{\kappa}$$
$$D(t, s) = e^{(\mu + \lambda\sigma_r/\kappa - \sigma_r^2/2\kappa^2)(E(t,s)-(s-t)) - \sigma_r^2 E^2(t,\, s)/4\kappa}$$

and λ is the market price of risk. In CIR's model the price of a discount bond maturing at time s is

$$B(r, t, s) = H(t, s)\, e^{-G(t,s)r}$$

where

$$H(t, s) = \left[\frac{2\gamma e^{(\kappa+\lambda+\gamma)(s-t)/2}}{(\kappa + \lambda + \gamma)(e^{\gamma(s-t)} - 1) + 2\gamma}\right]^{2\kappa\mu/\sigma_r^2}$$
$$G(t, s) = \frac{2(e^{\gamma(s-t)} - 1)}{(\kappa + \lambda + \gamma)(e^{\gamma(s-t)} - 1) + 2\gamma}, \quad \gamma = ((\kappa + \lambda)^2 \pm 2\sigma_r^2)^{1/2}.$$

Pricing formulas for European options on discount bonds are given also in the Vasicek and CIR models. Those formulas are not duplicated here. The duration of a bond portfolio can be expressed analytically as:

$$D_{\text{Vasicek}} = -\frac{1}{\kappa} \ln \left|1 + \kappa\left(\frac{\partial P}{\partial r}\frac{1}{P}\right)\right|, \text{ and}$$

$$D_{\mathrm{CIR}} = -\frac{1}{\gamma} \ln \left| 1 - \frac{2\gamma\left(\dfrac{\partial P}{\partial r}\dfrac{1}{P}\right)}{2 + (\kappa + \lambda + \gamma)\left(\dfrac{\partial P}{\partial r}\dfrac{1}{P}\right)} \right|$$

In the above, $\partial P/\partial r\ 1/P$ is the risk measure of the portfolio.

In-Depth Analyses Based on a Single Set of Parameters

A European call on a 15-year bond which pays an annual coupon of 10% on a face value of $100 is priced. The option has an exercise price of $100 and matures in 5 years. In each model, two sets of option prices are calculated at different interest rate levels.[8] One set of option prices is calculated using Jamshidian's approach, and hence are accurate. The other set of prices is based on the proposed simple approach. Pricing errors, both in dollar terms and in percentage forms, are then calculated. The results are summarized in Tables I and II. It can be seen from Panel A of Table I and Table II that the approximation works quite well. For both models pricing errors are within a penny for almost all levels of interest rates or moneyness of the option. The percentage errors tend to be high for deep out-of-the-money options. The percentage errors should be interpreted with caution because the option prices are very small in those cases and the errors in dollar terms are negligible. Another observation is that the approximation performs slightly better with the CIR model.

To see if one can derive a valid hedge ratio based on the approximating single formula, the first derivative of the option's price with respect to interest rate is calculated for both approaches.[9] The results are shown in Panel B of Tables I and II. Again, relative to the magnitude of the derivatives, the errors are small. This is especially true with the CIR model.

Since it is necessary to hold discount bonds to hedge the option, it is important to know how well the proposed approach approximates delta and gamma, the first and second derivatives of the option's price with respect to the value of the coupon bond. The Vasicek model is used to

[8]In each model, a particular set of empirical values for the interest rate process parameters are used. They are chosen based on the estimation results in Chan, Karolyi, Longstaff, and Sanders (1992). The market price of risk is set at zero in both models.
[9]Numerical integration is used to calculate the derivatives in CIR's model.

TABLE I

Accurate Versus Approximate Option Prices and Interest Rate Deltas
Vasicek Model

r	*P/B*	*Accurate*	*Approximate*	*Error ($)*	*Error (%)*
		Panel A. Option Prices			
0.04	116.2254	12.5187	12.5280	0.0093	0.07
0.06	113.3291	9.8515	9.8616	0.0101	0.10
0.08	110.5128	7.5933	7.6033	0.0100	0.13
0.10	107.7740	5.7155	5.7244	0.0089	0.16
0.12	105.1106	4.1885	4.1951	0.0066	0.16
0.14	102.5204	2.9792	2.9828	0.0036	0.12
0.16	100.0013	2.0507	2.0511	0.0004	0.02
0.18	97.5513	1.3620	1.3597	−0.0023	−0.17
0.20	95.1683	0.8706	0.8663	−0.0043	−0.49
0.22	92.8504	0.5342	0.5289	−0.0053	−0.99
0.24	90.5957	0.3139	0.3086	−0.0053	−1.69
0.26	88.4025	0.1764	0.1717	−0.0047	−2.66
0.28	86.2689	0.0945	0.0908	−0.0037	−3.92
0.30	84.1933	0.0483	0.0456	−0.0027	−5.59
		Panel B. First Derivatives with Respect to Interest Rates			
0.04	116.2254	−144.0806	−143.8676	0.2130	−0.15
0.06	113.3291	−122.8966	−122.6783	0.2183	−0.18
0.08	110.5128	−103.1590	−102.9500	0.2090	−0.20
0.10	107.7740	84.8608	84.6706	−0.1902	−0.22
0.12	105.1106	68.1162	67.9467	−0.1695	−0.25
0.14	102.5204	53.1193	52.9647	−0.1546	−0.29
0.16	100.0013	40.0780	39.9288	−0.1492	−0.37
0.18	97.5513	29.1435	28.9910	−0.1525	−0.52
0.20	95.1683	20.3542	20.1954	−0.1588	−0.78
0.22	92.8504	13.6118	13.4503	−0.1615	−1.19
0.24	90.5957	8.6932	8.5376	−0.1556	−1.79
0.26	88.4025	5.2900	5.1504	−0.1396	−2.64
0.28	86.2689	3.0613	2.9454	−0.1159	−3.79
0.30	84.1933	1.6820	1.5931	−0.0889	−5.29

Interest rate process is Vasicek: $dr = k(\mu - r)dt + \sigma dz$ where $k = 0.2$, $\mu = 0.085$, $\sigma = 0.02$. Market price of risk, λ, is zero. Call options with a maturity of 5 years and an exercise price of $100 are written on a 15-year coupon bond with a face value of $100 and a coupon rate of 10%. Coupons are paid annually.
For each level of interest rate (r) and corresponding bond price (divided by the price of a unit discount bond with the same maturity as the option, *P/B*), accurate and approximate option prices (Panel A) and their first derivatives with respect to interest rates (Panel B) are calculated. Absolute and percentage (in terms of the accurate value) errors are shown in the last two columns.

investigate this aspect. The derivatives based on the single approximate formula are used for the approximation approach. For the accurate approach, the chain rule of differentiation is used to calculate the derivatives.[10] The results are in Table III. It is seen that the approximated deltas

[10]Notice that in Jamshidian's approach the value of the coupon bond does not directly appear in each component options. Rather, it is decomposed into current values of individual coupons and the face

TABLE II

Accurate Versus Approximate Option Prices and Interest Rate Delta
CIR Model

r	P/B	*Accurate*	*Approximate*	*Error (\$)*	*Error (%)*
		Panel A. Option Prices			
0.04	112.2878	9.1824	9.1835	0.0011	0.01
0.06	110.4667	7.4475	7.4492	0.0017	0.02
0.08	108.6778	5.9400	5.9423	0.0023	0.04
0.10	106.9205	4.6524	4.6550	0.0025	0.06
0.12	105.1942	3.5744	3.5767	0.0024	0.06
0.14	103.4983	2.6915	2.6933	0.0018	0.07
0.16	101.8323	1.9853	1.9862	0.0008	0.05
0.18	100.1957	1.4341	1.4339	−0.0002	−0.01
0.20	98.5878	1.0144	1.0132	−0.0012	−0.12
0.22	97.0082	0.7028	0.7007	−0.0020	−0.30
0.24	95.4564	0.4769	0.4743	−0.0026	−0.55
0.26	93.9318	0.3171	0.3143	−0.0028	−0.88
0.28	92.4339	0.2067	0.2040	−0.0028	−1.31
0.30	90.9622	0.1322	0.1296	−0.0025	−1.97
		Panel B. First Derivatives with Respect to Interest Rates			
0.04	112.2878	−92.4702	−92.4222	0.0480	−0.05
0.06	110.4667	−80.9267	−80.8649	0.0618	−0.08
0.08	108.6778	−69.7694	−69.7004	0.0690	−0.10
0.10	106.9205	−59.0678	−58.9986	0.0692	−0.12
0.12	105.1942	−48.9779	−48.9131	0.0648	−0.13
0.14	103.4983	−39.6935	−39.6344	0.0591	−0.15
0.16	101.8323	−31.3961	−31.3406	0.0554	−0.18
0.18	100.1957	−24.2147	−24.1598	0.0549	−0.23
0.20	98.5878	−18.2028	−18.1459	0.0568	−0.31
0.22	97.0082	−13.3366	−13.2765	0.0601	−0.45
0.24	95.4564	9.5254	9.4626	0.0628	−0.66
0.26	93.9318	6.6349	6.5715	0.0634	−0.96
0.28	92.4339	4.5098	4.4485	0.0614	−1.36
0.30	90.9622	2.9933	2.9367	0.0566	−1.89

Interest rate process is CIR: $dr = k(\mu - r)dt + \sigma\sqrt{r}dz$ where $k = 0.25$, $\mu = 0.085$, $\sigma = 0.05$. Market price of risk, λ, is zero. Call options with a maturity of 5 years and an exercise price of \$100 are written on a 15-year coupon bond with a face value of \$100 and a coupon rate of 10%. Coupons are paid annually.

For each level of interest rate (r) and corresponding bond price (divided by the price of a unit discount bond with the same maturity as the option, P/B), accurate and approximate option prices (Panel A) and their first derivatives with respect to interest rates (Panel B) are calculated. Absolute and percentage (in terms of the accurate value) errors are shown in the last two columns.

and gammas are very close to their true values for all levels of interest rates or moneyness of the option. (The percentage errors for out-of-the-money options should again be treated with caution.) The significant im-

value. The chain rule is used, realizing that all discount bonds are affected by the same underlying factor, the short-term interest rate.

TABLE III

Accurate Versus Approximate Bond Price Deltas and Gammas
Vasicek Model

r	*P/B*	*Accurate*	*Approximate*	*Error ($)*	*Error (%)*
		Panel A. Option Delta			
0.04	116.2254	0.3702	0.3697	−0.0005	−0.14
0.06	113.3291	0.3453	0.3446	−0.0006	−0.20
0.08	110.5128	0.3168	0.3162	−0.0006	−0.19
0.10	107.7740	0.2849	0.2843	−0.0006	−0.21
0.12	105.1106	0.2500	0.2494	−0.0006	−0.24
0.14	102.5204	0.2131	0.2125	−0.0006	−0.28
0.16	100.0013	0.1757	0.1751	−0.0007	−0.34
0.18	97.5513	0.1397	0.1389	−0.0007	−0.57
0.20	95.1683	0.1066	0.1058	−0.0008	−0.75
0.22	92.8504	0.0779	0.0770	−0.0009	−1.16
0.24	90.5957	0.0544	0.0534	−0.0010	−1.84
0.26	88.4025	0.0361	0.0352	−0.0010	−2.49
0.28	86.2689	0.0229	0.0220	−0.0009	−3.93
0.30	84.1933	0.0137	0.0130	−0.0007	−5.11
		Panel B. Option Gamma			
0.04	116.2254	0.0030	0.0030	0.0000	0.00
0.06	113.3291	0.0037	0.0037	0.0000	0.00
0.08	110.5128	0.0046	0.0046	0.0000	0.00
0.10	107.7740	0.0056	0.0056	−0.0001	0.00
0.12	105.1106	0.0066	0.0066	−0.0001	0.00
0.14	102.5204	0.0075	0.0074	−0.0001	−1.33
0.16	100.0013	0.0081	0.0080	−0.0001	−1.23
0.18	97.5513	0.0083	0.0083	−0.0001	0.00
0.20	95.1683	0.0081	0.0080	−0.0001	−1.23
0.22	92.8504	0.0075	0.0074	−0.0001	−1.33
0.24	90.5957	0.0065	0.0064	−0.0001	−1.54
0.26	88.4025	0.0053	0.0052	−0.0001	−1.89
0.28	86.2689	0.0041	0.0040	−0.0001	−2.44
0.30	84.1933	0.0030	0.0029	−0.0001	−3.33

Interest rate process is Vasicek: $dr = k(\mu - r)dt + \sigma dz$ where $k = 0.2$, $\mu = 0.085$, $\sigma = 0.02$. Market price of risk, λ, is zero. Call options with a maturity of 5 years and an exercise price of $100 are written on a 15-year coupon bond with a face value of $100 and a coupon rate of 10%. Coupons are paid annually.
For each level of interest rate (r) and corresponding bond price (divided by the price of a unit discount bond with the same maturity as the option, P/B), accurate and approximate option deltas (Panel A) and gammas (Panel B) with respect to bond prices are calculated. Absolute and percentage (in terms of the accurate delta and gamma) errors are shown in the last two columns.

plication is that one can hedge an option on a bond portfolio by taking a position in only one discount bond.

To this point it has been shown that the approximation works satisfactorily for the chosen parameters, but, where do the errors come from? It is tempting to think that the proposed approach *is* an accurate approach and the errors shown are simply calculation errors, because the volatility of the coupon bond matches that of the proxy discount bond. Unfortunately, the simple approach is an approximation only. Insights can be gained from looking at the pattern of pricing errors in Panel A of Tables I and II.

In both models (Vasicek and CIR) the simple approach always overprices (underprices) the option when the bond value is above (below) the exercise price.[11] This error pattern shows that some elements have been missed in the approximation scheme which systematically affect the option's value. The missing element is the convexity of the bond portfolio.

Convexity measures the "curvature" of a bond portfolio and, in this context, is defined as

$$C = \frac{\partial^2 B}{\partial r^2}\frac{1}{B} \tag{8}$$

The following relationship holds for both discount bonds and bond portfolios in general:

$$\frac{\partial(-f)}{\partial r} = -f^2 + C \tag{9}$$

where f is defined before. Notice that $(-f)$ is a measure of volatility. In matching the duration, one effectively matches the *current values* of the volatilities of the bond portfolio and the proxy discount bond. The sensitivity of the volatility to interest rate and time is left *unmatched*. This is where the pricing errors come from.

The left-hand side of (9) measures volatility's sensitivity to interest rate. Since the duration has been matched, volatility sensitivity would also be matched if the convexity is matched. Unfortunately, the matching of convexity is not automatic upon the matching of duration. To see this, using the formulas for discount bond prices (given at the end of "A Simple Approach") it can be shown that convexities are $E^2(t, s)$ and $G^2(t, s)$ in the Vasicek and CIR models, respectively. While f is simply $E(t, s)$ and $G(t, s)$, respectively. As a result, for both the Vasicek and CIR models, the sensitivity of a discount bond's volatility to interest rate is zero. However, the sensitivity of a portfolio's volatility to interest rate is *not* zero in either model, and it is always positive (i.e., $C > f^2 > 0$).[12] This amounts to a situation where the underlying asset's volatility is stochastic and is negatively correlated with the value of the underlying asset. (Keep in mind that there is a reverse relationship between bond prices and interest rates.) Thus, when the portfolio's value is high, the volatility tends to be

[11]Because European options are considered, the intrinsic value or moneyness is defined with respect to the present value of the exercise price, which is equivalent to dividing the bond value by the price of a discount bond with the same maturity as the option.

[12]This can be shown via Jensen's inequality.

low, and vice versa. The proxy discount bond used here fails to capture this effect and, hence the observed pattern of pricing errors.[13]

What if one matches both duration and convexity? The size of the pricing errors should definitely shrink. To see this, a portfolio of two discount bonds is used to match the original coupon bond for the Vasicek model.[14] Similar calculations as in Tables I and III are then performed. The results are reported in Tables IV and V. It can be seen that the approximate price is accurate to the fourth decimal place for all levels of interest rates. The accuracy of all derivatives is also dramatically improved. The remaining errors are due to the mismatch of the time-sensitivity of the volatilities, whose impacts are very small.

The above discussions are meant to pinpoint the nature of the proposed simple, approximation approach. They do not necessarily suggest that one ought to use two discount bonds to do the approximation. The reason for doing it here is to illustrate the source of the errors resulting from the approximation scheme.

It should also be pointed out that the failure of matching the convexities (or equivalently, the interest rate sensitivity of the volatility) does not necessarily mean that they are actually far apart. The matching of duration will, in general, also bring the convexity close, albeit not precisely matched. This is apparent in Table VI where for the same set of parameter values used in the previous tables, convexities are calculated for both the coupon bond and the proxy discount bond. The convexities hardly vary across different interest rates, and as a result, the size of mismatch is quite stable.

Analyses Based on a Broader Set of Parameter Values

The above section demonstrates that the approximation tends to work satisfactorily and that pricing errors are mainly caused by the mismatch of convexities. The analysis is broadened in this section to include a variety of different parameter values. Since, the key determining factors for convexity are the mean-reversion speed, k, and the interest rate volatility, σ_r, these two parameters will be varied around their empirical values (used in Tables I through VI) for both the Vasicek and the CIR models. How

[13]For more insights on stochastic volatility and its impacts, see Hull and White (1988).

[14]For convenience, the face value of the discount bond with shorter maturity is set at $50 for all levels of interest rates. For a given portfolio there are an infinite number of two-bond combinations which will match the portfolio's duration and convexity, and, of course, the current value. The matching of both quantities is done by simultaneously solving for the time-to-maturities of both discount bonds. The two-dimension Newton–Raphson method is used.

TABLE IV

Accurate Versus Approximate Option Prices and Interest Rate Deltas
Vasicek Model

r	*P/B*	*Accurate*	*Approximate*	*Error ($)*	*Error (%)*
		Panel A. Option Prices			
0.04	116.2254	12.5187	12.5188	0.0000	0.001
0.06	113.3291	9.8515	9.8515	0.0000	0.000
0.08	110.5128	7.5933	7.5933	0.0000	0.000
0.10	107.7740	5.7155	5.7155	0.0000	0.000
0.12	105.1106	4.1885	4.1884	0.0000	−0.002
0.14	102.5204	2.9792	2.9792	−0.0001	0.000
0.16	100.0013	2.0507	2.0506	−0.0001	−0.005
0.18	97.5513	1.3620	1.3620	−0.0001	0.000
0.20	95.1683	0.8706	0.8706	0.0000	0.000
0.22	92.8504	0.5342	0.5342	0.0000	0.000
0.24	90.5957	0.3139	0.3139	0.0000	0.000
0.26	88.4025	0.1764	0.1764	0.0000	0.000
0.28	86.2689	0.0945	0.0945	0.0000	0.000
0.30	84.1933	0.0483	0.0483	0.0000	0.000
		Panel B. First Derivatives with Respect to Interest			
0.04	116.2254	−144.08061	−144.07938	0.00124	−0.001
0.06	113.3291	−122.89658	−122.89560	0.00098	−0.001
0.08	110.5128	−103.15905	−103.15836	0.00069	−0.001
0.10	107.7740	84.86080	84.86034	0.00046	−0.001
0.12	105.1106	68.11624	68.11591	0.00032	−0.000
0.14	102.5204	53.11927	53.11899	0.00028	−0.001
0.16	100.0013	40.07804	40.07776	0.00028	−0.001
0.18	97.5513	29.14348	29.14325	0.00023	−0.001
0.20	95.1683	20.35419	20.35410	0.00010	−0.000
0.22	92.8504	13.61183	13.61196	−0.00012	0.001
0.24	90.5957	8.69321	8.69357	−0.00036	0.004
0.26	88.4025	5.29002	5.29058	−0.00057	0.011
0.28	86.2689	3.06131	3.06198	−0.00067	0.022
0.30	84.1933	1.68196	1.68263	−0.00067	0.040

Interest rate process is Vasicek: $dr = k(\mu - r)dt + \sigma dz$ where $k = 0.2$, $\mu = 0.085$, $\sigma = 0.02$. Market price of risk, λ, is zero. Call options with a maturity of 5 years and an exercise price of $100 are written on a 15-year coupon bond with a face value of $100 and a coupon rate of 10%. Coupons are paid annually.
For each level of interest rate (r) and corresponding bond price (divided by the price of a unit discount bond with the same maturity as the option, P/B), accurate and approximate option prices (Panel A) and their first derivatives with respect to interest rates (Panel B) are calculated. Absolute and percentage (in terms of the accurate value) errors are shown in the last two columns. Approximations are achieved by pricing options on a portfolio of two discount bonds which match the coupon bond's duration *and* convexity.

pricing errors are related to such key factors as the maturity of the option, the maturity of the underlying bond, and the coupon rate of the bond will be examined also.

When varying one particular parameter or model input, all other inputs are kept at the base level (used in Tables I through VI). For convenience, the base values of the parameters are summarized as follows:

TABLE V

Accurate Versus Approximate Option Deltas and Gammas
Vasicek Model

r	*P/B*	*Accurate*	*Approximate*	*Error*
		Panel A. Option Delta		
0.04	116.2254	0.370219	0.370216	−0.000003
0.06	113.3291	0.345262	0.345259	−0.000003
0.08	110.5128	0.316843	0.316841	−0.000002
0.10	107.7740	0.284934	0.284933	−0.000002
0.12	105.1106	0.250012	0.250011	−0.000001
0.14	102.5204	0.213112	0.213111	−0.000001
0.16	100.0013	0.175743	0.175742	−0.000001
0.18	97.5513	0.139669	0.139668	−0.000001
0.20	95.1683	0.106604	0.106603	−0.000001
0.22	92.8504	0.077905	0.077906	0.000001
0.24	90.5957	0.054366	0.054369	0.000002
0.26	88.4025	0.036148	0.036151	0.000004
0.28	86.2689	0.022854	0.022859	0.000005
0.30	84.1933	0.013718	0.013723	0.000005
		Panel B. Option Gamma		
0.04	116.2254	0.002998	0.002998	0.000000
0.06	113.3291	0.003743	0.003743	0.000000
0.08	110.5128	0.004637	0.004638	0.000000
0.10	107.7740	0.005631	0.005631	0.000000
0.12	105.1106	0.006630	0.006631	0.000000
0.14	102.5204	0.007506	0.007506	0.000001
0.16	100.0013	0.008117	0.008118	0.000001
0.18	97.5513	0.008347	0.008348	0.000001
0.20	95.1683	0.008135	0.008136	0.000001
0.22	92.8504	0.007497	0.007498	0.000001
0.24	90.5957	0.006521	0.006522	0.000001
0.26	88.4025	0.005348	0.005349	0.000001
0.28	86.2689	0.004131	0.004132	0.000001
0.30	84.1933	0.003004	0.003005	0.000001

Interest rate process is Vasicek: $dr = k(\mu - r)dt + \sigma dz$ where $k = 0.2$, $\mu = 0.085$, $\sigma = 0.02$. Market price of risk, λ, is zero. Call options with a maturity of 5 years and an exercise price of $100 are written on a 15-year coupon bond with a face value of $100 and a coupon rate of 10%. Coupons are paid annually.
For each level of interest rate (r) and corresponding bond price (divided by the price of a unit discount bond with the same maturity as the option, *P/B*), accurate and approximate option deltas (Panel A) and gammas (Panel B) are calculated. Absolute errors are shown in the last column. Approximations are achieved by pricing options on a portfolio of two discount bonds which match the coupon bond's duration *and* convexity.

$$\text{Vasicek Model:}\quad \mu_r = 0.085, \sigma_r = 0.02, k = 0.20, \lambda = 0.00;$$
$$\text{CIR Model:}\quad \mu_r = 0.085, \sigma_r = 0.05, k = 0.25, \lambda = 0.00.$$

Unless otherwise stated, the option is a call option with a maturity of five years, written on a 15-year coupon bond with a face value of $100 and a coupon rate of 10%. Coupons are paid annually. The results are summarized in graphs. In each graph, percentage pricing errors are plotted

TABLE VI

Accurate Versus Approximate Convexities

r	*P/B*	*Accurate*	*Approximate*	*Error*
		Panel B. Vasicek Model		
0.04	116.2254	19.7468	19.5728	−0.1739
0.06	113.3291	19.7174	19.5419	−0.1754
0.08	110.5128	19.6877	19.5108	−0.1769
0.10	107.7740	19.6579	19.4794	−0.1784
0.12	105.1106	19.6277	19.4478	−0.1799
0.14	102.5204	19.5974	19.4159	−0.1814
0.16	100.0013	19.5668	19.3838	−0.1829
0.18	97.5513	19.5359	19.3515	−0.1844
0.20	95.1683	19.5048	19.3189	−0.1859
0.22	92.8504	19.4735	19.2862	−0.1873
0.24	90.5957	19.4419	19.2531	−0.1888
0.26	88.4025	19.4101	19.2199	−0.1902
0.28	86.2689	19.3781	19.1864	−0.1917
0.30	84.1933	19.3458	19.1527	−0.1931
		Panel B. CIR Model		
0.04	112.2878	13.427930	13.367290	−0.060640
0.06	110.4667	13.419390	13.358390	−0.060990
0.08	108.6778	13.410790	13.349450	−0.061340
0.10	106.9205	13.402150	13.340460	−0.061690
0.12	105.1942	13.393470	13.331430	−0.062040
0.14	103.4983	13.384730	13.322340	−0.062390
0.16	101.8323	13.375950	13.313210	−0.062730
0.18	100.1957	13.367110	13.304030	−0.063080
0.20	98.5878	13.358230	13.294800	−0.063430
0.22	97.0082	13.349310	13.285530	−0.063780
0.24	95.4564	13.340330	13.276210	−0.064130
0.26	93.9318	13.331310	13.266840	−0.064470
0.28	92.4339	13.322240	13.257420	−0.064820
0.30	90.9622	13.313120	13.247960	−0.065170

Interest rate process is $dr = k(\mu - r)dt + \sigma dz$ for Vasicek where $k = 0.2$, $\mu = 0.085$, $\sigma = 0.02$, and $dr = k(\mu - r)dt + \sigma\sqrt{r}dz$ for CIR where $k = 0.25$, $\mu = 0.085$, $\sigma = 0.05$. Market price of risk, λ, is zero. The underlying bond portfolio is a 15-year coupon bond with a face value of \$100 and a coupon rate of 10%. Coupons are paid annually.
In both models, for each level of interest rate (r) and corresponding bond price (divided by the price of a unit discount bond with the same maturity as the option, P/B), accurate and approximate convexities are calculated. Absolute errors are shown in the last column. The approximate convexities are for the proxy discount bond whose duration is matched with the coupon bond.

against the interest rate levels which go from 1% to 20%.[15] (Percentage pricing errors are calculated in the same way as in, e.g., Table I.) The exercise price of the option is chosen so that the option is always at-the-money when the interest rate is 10%, the middle value of the whole range. Results of the analyses are summarized in eight separate graphs (figures.)

[15]Since options prices vary widely in different parameter settings, reporting absolute pricing errors can be misleading and can make comparisons difficult. Reporting percentage pricing errors avoids both drawbacks. Although not reported, absolute errors are discussed whenever necessary.

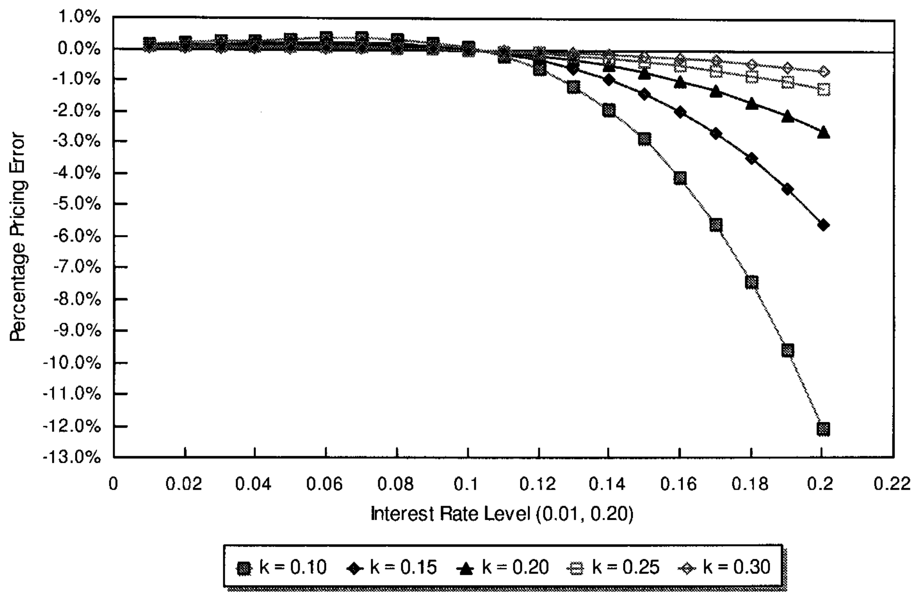

FIGURE 1
Pricing errors under different levels of mean reversion rate, *k*, (Vasicek Model).

Pricing Errors Versus Mean-Reversion Rate, k

The mean-reversion rate is a key determinant for a bond's volatility and convexity. A higher reversion rate means that the spot rate, whenever deviating from the long term level, μ_r, is pulled back quickly. This implies a lower volatility for the bond price, because the interest rate can not move too widely. In this case, one would expect the approximation to work better. The opposite is true with a lower reversion rate. Here, the spot rate is subject to a less powerful pulling force and can fluctuate more, causing a higher bond price volatility. One would expect a bigger pricing error by the approximation model in this case.

The above predictions are confirmed in Figures 1 and 2. Figure 1 is for the Vasicek model where the reversion rate takes five different values (0.10, 0.15, 0.20, 0.25, and 0.30). It can be seen that when the reversion rate is 0.25 or 0.30, the pricing errors are all within 1%.[16] When the reversion speed is 0.20, pricing error is within 2%. But for all three cases

[16]Since the interest rate level of 10% always corresponds to a bond price equal to the exercise price, points to the left of 10% interest rate represent in-the-money options, while those to the right represent out-of-the-money options. It can be seen in all figures that the approximation scheme always overprice, in-the-money options and underprices out-of-the-money options, which is consistent with the findings in the tables.

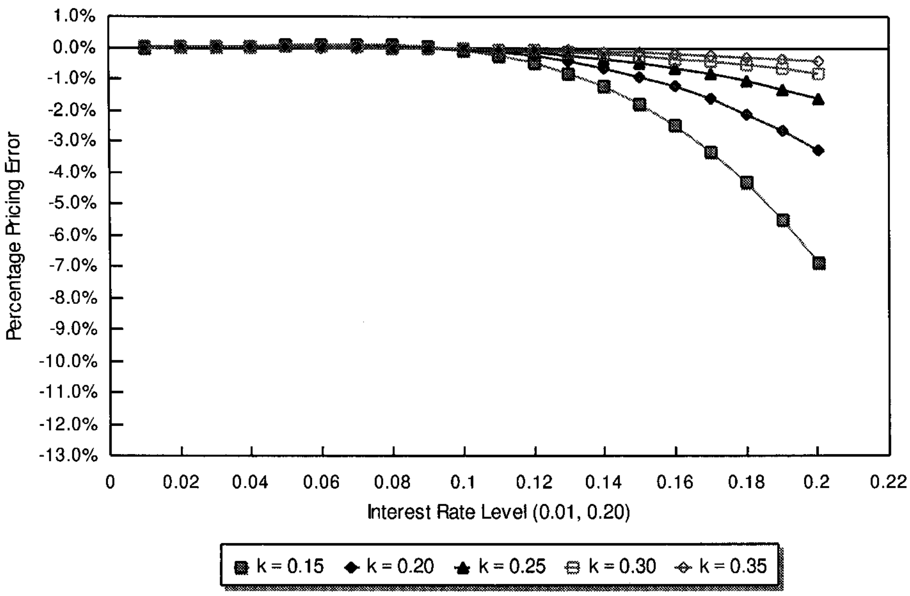

FIGURE 2
Pricing errors under different levels of mean reversion rate, k (CIR Model).

(k = 0.20, 0.25, and 0.30), the absolute pricing errors (not reported here) are all within one penny. For a reversion level of 0.15, the percentage pricing errors are bigger, especially for out-of-the-money options (i.e., when interest rates are high). But again, the size of the absolute pricing errors is generally small. The biggest is 1.3¢ associated with r = 16%. For the deep out-of-the-money option (r = 20%), the percentage pricing error is −5.53%, but the absolute error is only −0.9¢, which is negligible for practical purposes. Finally, the percentage pricing errors are even bigger when the reversion rate is 0.10 (half of the empirical level, 0.20). Again, the bigger errors are associated with deep out-of-the-money options. But the absolute errors are not overly sizable. For example, when r = 20%, the percentage pricing error is −12.07% while the absolute error is only −1.38¢, and the option price is about 10¢. In a situation like this, no model price is "accurate" anyway, because the option price is very close in size to the bid–ask spread.

It is interesting to observe that pricing errors for in-the-money options are well within 1% for all levels of the reversion rate. One can conclude, that therefore, for the Vasicek model, the approximation is practically robust to changes in the mean-reversion speed when the option is in-the-money. When the option is out-of-the-money and when the reversion speed is low, percentage pricing errors are relatively higher. But in those cases, the absolute errors tend to be trivial or the option price is very close to the bid–ask spread. Thus, overall, one can conclude that the approximation is satisfactory for all reasonable levels of the mean-reversion rate in the Vasicek model.

A similar conclusion can be drawn by examining Figure 2 for the CIR model. Indeed, the conclusion can be substantially strengthened, because the biggest percentage error is -7%. In addition, for all reversion rates and degrees of option's moneyness, the absolute pricing errors are all within one penny for the CIR model.

Overall, the performance of the approximation approach is satisfactory in all reasonable levels of the mean-reversion speed. It performs especially well for in-the-money options, or when the reversion speed is high.

Pricing Errors Versus Interest Rate Volatility, σ_r

Interest rate volatility is a direct input for the bond price volatility. On an absolute pricing error basis, one would expect the approximation approach to work better when the interest rate volatility is low. Percentage-wise, it is hard to establish an *a priori* pattern because the option price itself is affected also by the volatility level.

For the Vasicek model, the interest rate volatility is varied from 0.01 to 0.05. For the CIR model, the range is (0.04, 0.08). The results are summarized in Figures 3 and 4.

Figure 3 is for the Vasicek model. It is interesting to observe that, the higher the volatility, the smaller the percentage pricing errors. When the volatility is higher than 0.02, pricing errors are within 2% for out-of-the-money options and within 1% for in-the-money options. When the volatility is at 0.01, out-of-the-money options tend to have a bigger percentage pricing error, but option prices are very small in those cases. Absolute pricing errors are all within 0.2¢. One can safely ignore the misleading percentage errors in this case.

When the volatility increases, the mismatch of convexity increases, causing a bigger pricing error in dollar terms. However, the option price also increases when volatility increases. There are lower percentage pricing errors for a higher volatility level because the option price increases much faster than the pricing errors.

Similar observations can be made in Figure 4 for the CIR model. Here, all percentage pricing errors are within 2% for out-of-the-money

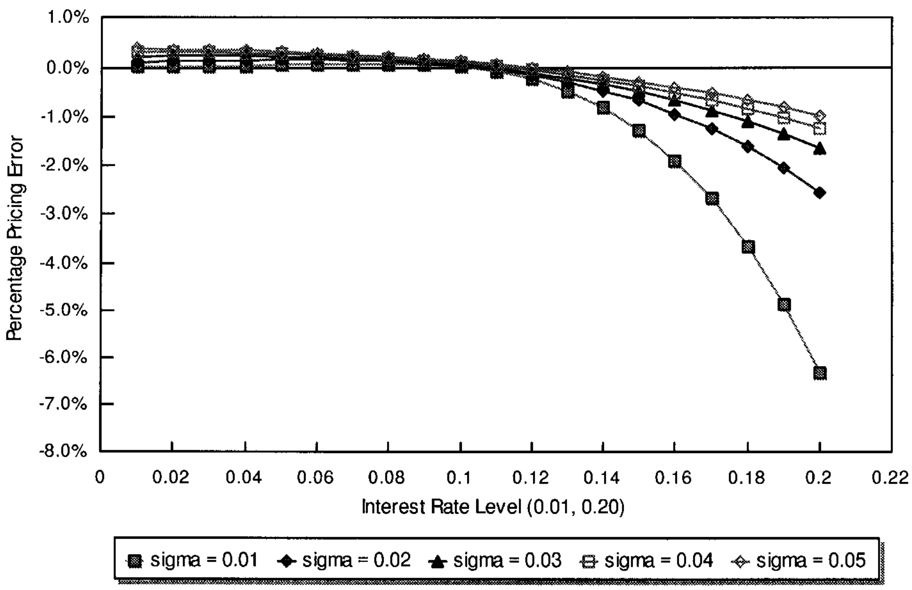

FIGURE 3
Pricing errors under different levels of volatility, sigma (Vasicek Model).

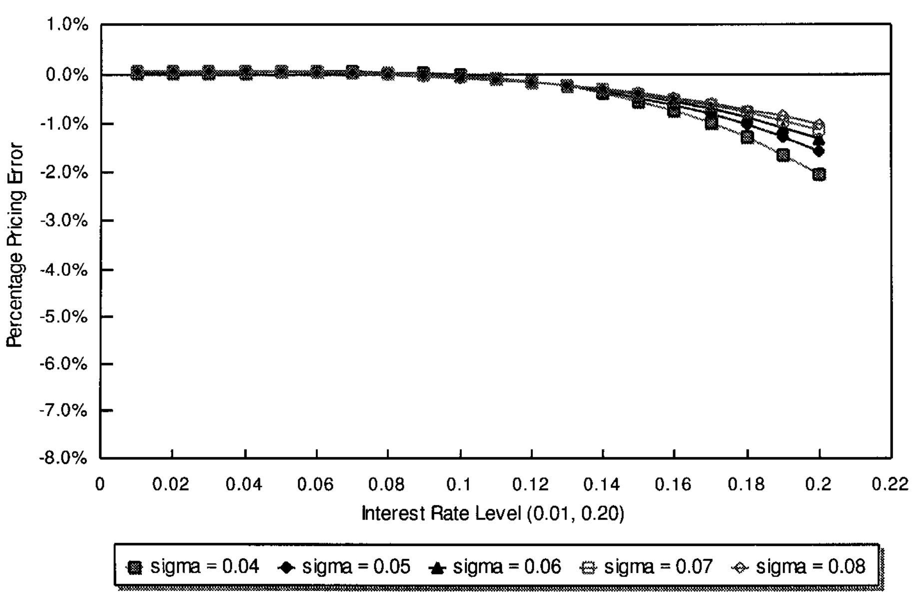

FIGURE 4
Pricing errors under different levels of volatility, sigma (CIR Model).

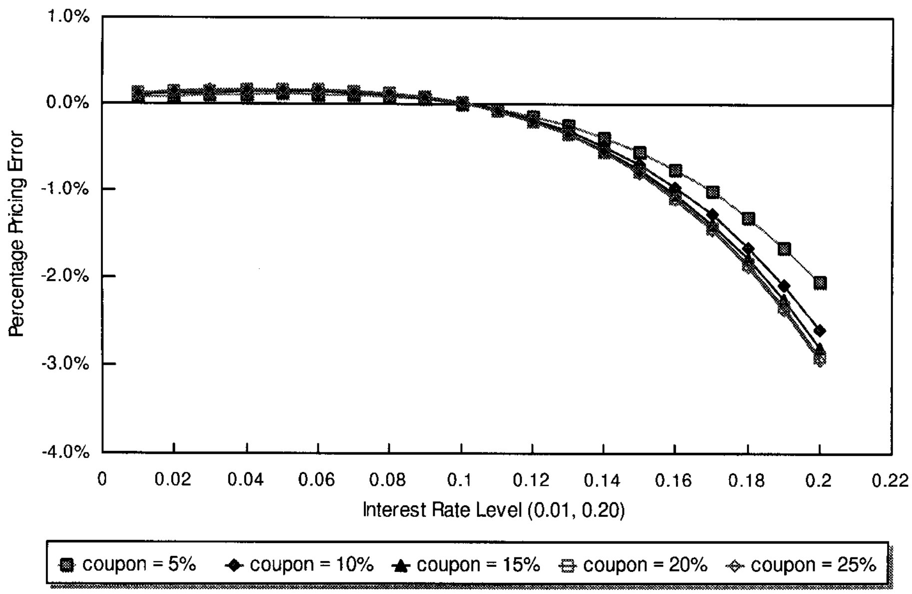

FIGURE 5
Pricing errors for options on bonds with different coupon rates (Vasicek Model).

options and close to zero for in-the-money options. Although not reported, absolute pricing errors are all within a penny.

Overall, the performance of the approximation scheme is quite satisfactory across different levels of interest rate volatilities for both the Vasicek and the CIR models. A higher volatility is of no concern, because the absolute pricing error increases at a much slower rate than the option price itself and, as a result, the percentage error actually decreases in that case.

Pricing Errors Versus Coupon Rate

It is well known that for a given yield and maturity, a bond's convexity is negatively related to its coupon rate. Since the major source of pricing errors of the approximation approach is the mismatch of convexity, one may expect the approximation to be more accurate for higher coupon bonds (which have a smaller convexity). Surprisingly, this is not the case.

Figures 5 and 6 contain the pricing errors for the Vasicek and the CIR models for five different coupon rates (5%, 10%, 15%, 20%, and 25%). (Other parameters again take the base values.) Aside from sharing the common features of other figures such as overpricing in-the-money options and underpricing out-of-the-money options, Figures 5 and 6 re-

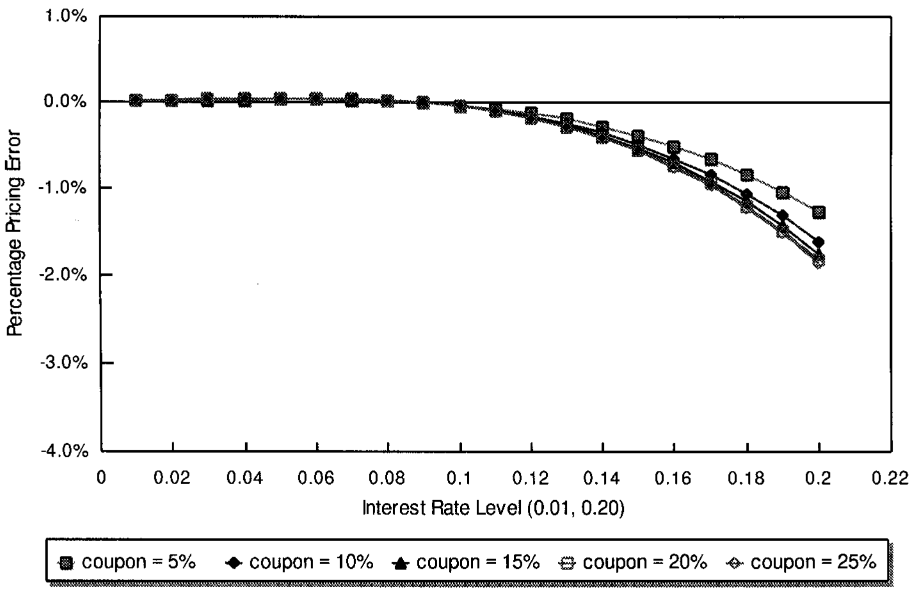

FIGURE 6
Pricing errors for options on bonds with different coupon rates (CIR Model).

veal that pricing errors are indeed positively related to the coupon rate. In other words, percentage pricing errors are higher for low-convexity bonds. Although not reported, the absolute pricing errors exhibit the same pattern. To understand this "puzzle," the accurate and approximate convexities are calculated for all five coupon bonds at different interest rates. It is confirmed that the convexity is negatively related to the coupon rate. However, it is also revealed that the size of the mismatch is actually positively related to the coupon rate. In other words, the mismatch is more severe for low convexity bonds!

To understand this phenomenon, it should be noted that a discount bond is used to approximate a coupon bond, and that a low-coupon bond resembles a discount bond more than a high-coupon bond does. At the extreme, when the coupon rate is close to zero (but not exactly equal to zero), the coupon bond is almost equivalent to a discount bond. Thus, one should expect the mismatch of convexity to become bigger as the coupon rate increases.[17]

[17]It should be pointed out that the above observations do not apply to all coupon ranges. Numerical experiments (not reported here) show that once the coupon rate reaches a certain level (higher than 25% in this case), the mismatch starts to decrease. Again, this should not come as a surprise, because a bond with extremely high coupons has an almost even cash flow pattern. The discrepancy between the coupons and the par value is relatively small, making a weighted average (which is essentially the proxy discount bond) more representative than otherwise.

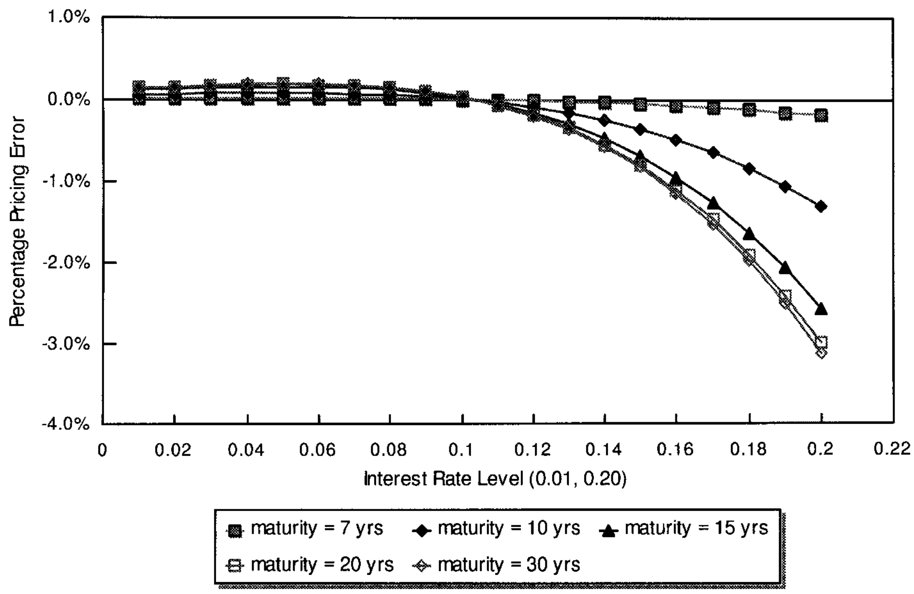

FIGURE 7
Pricing errors for options on bonds with different maturities (Vasicek Model).

Now, the size of pricing errors is considered. Figure 5 reveals that with the Vasicek model, the highest percentage pricing error is associated with deep out-of-the-money options on high coupon bonds. The percentage errors are all within 3%, and the absolute errors for out-of-the-money options (not reported) are all within 1.5¢. As revealed by Figure 6, the approximation again works better with the CIR model. Percentage errors are all within 2%, and the absolute errors (not reported here) are all within one penny.

Overall, caution should be exercised with high-coupon options when using the approximation, but pricing errors both percentage and absolute are, on average, small.

Pricing Errors Versus Bond Maturity

Although most bond options are written on long-term bonds, to derive a conclusive statement about the accuracy of the approximation approach, it is useful to examine the approximation performance over a whole spectrum of bond maturities. To this end, options on bonds with five maturities (7, 10, 15, 20, and 30 years) are priced while keeping the option's maturity at five years in all cases. The results are plotted in Figure 7 for the Vasicek model and Figure 8 for the CIR model.

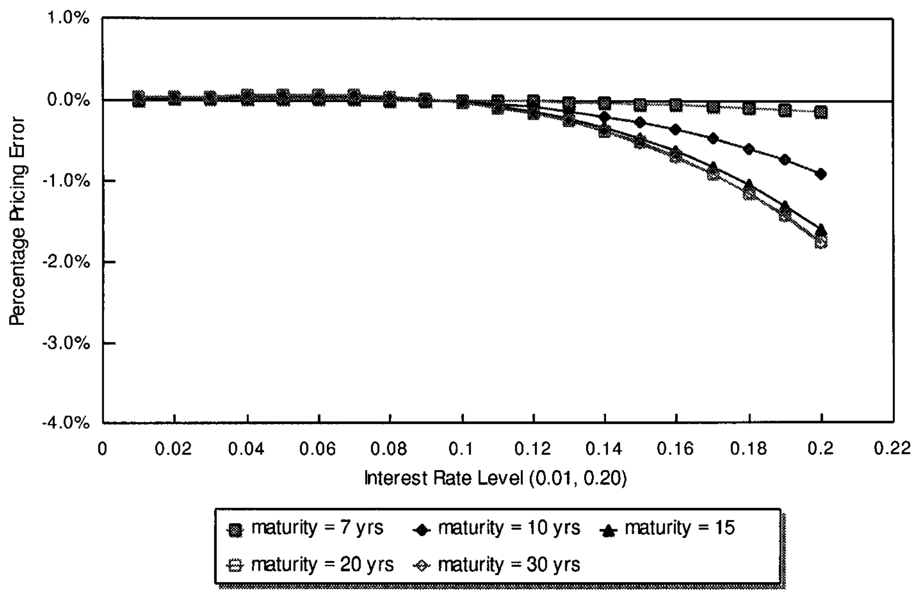

FIGURE 8
Pricing errors for options on bonds with different maturities (CIR Model).

In terms of overall pricing error patterns, the two figures are similar to the previous ones. Here, for both models, the percentage pricing errors are bigger for options on long-term bonds. (The same also holds for absolute pricing errors.) The intuition is obvious: a discount bond approximates a short-term bond better than a long-term bond. At the extreme, when there is only one coupon on the bond beyond the option's maturity, then the approximation becomes a perfect representation.

In terms of size, all percentage errors are within 3% for the Vasicek model and 2% for the CIR model. In terms of absolute size, all errors are trivial for out-of-the-money options. For the Vasicek model, all absolute errors are within a penny. For the CIR model, no error is bigger than a half of a penny.

Therefore, it is concluded that bond maturities are not a concern for the accuracy of the approximation approach.

Pricing Errors Versus Option's Maturity

So far, all analyses are for options with a five-year maturity. To see how the approximation performs for options with shorter maturities, five option maturities of 1, 2, 3, 4, and 5 years are examined. All options are written on the same 15-year bond with a 10% coupon (i.e., the base case bond). For both models (Vasicek and CIR), it is found that the performance of the approximation is better with short term options. For in-the-money options, *all percentage errors are within 1%.* For out-of-the-money options, all absolute errors are well within one penny. Since Panel A of Table I and Table II represents the "worst case" (i.e., longest option's maturity), the figures are not reported.

A summary of the results of an examination of five factors (which may potentially affect the performance of the approximation scheme. (1) the mean-reversion rate, (2) the interest rate volatility, (3) the coupon rate, (4) the bond's maturity, and (5) the option's maturity) is the following:

a. the approximation approach always overprices in-the-money options and underprices out-of-the-money options;

b. the overpricing is less than 1% for all cases;

c. the underpricing is less than 3% for all coupon rates, all bond maturities, and all option maturities, and the absolute errors are within one penny for most cases; and

d. in the case of underpricing, the largest percentage errors are associated with the mean-reversion rate, and to a lesser extent, the interest rate volatility; however, in terms of absolute size, almost all errors are within one penny.

Therefore, it can be concluded that the proposed approximation works in all conceivable interest rate environments and for all conceivable option specifications. If cautions are to be exercised, they should be directed at the (low) mean-reversion rate and the interest rate volatility.

Two additional points should be made. First, the proposed approximation scheme will undoubtedly save calculation time. Aside from bypassing the iterations to solve for the critical interest rate, the approach requires calculation of only one option price. In Jamshidian's approach, the number of options to be evaluated is equal to the number of coupons. Thus, the proposed approach will cut the calculation time by at least n times if n is the number of coupons. Second, perhaps more importantly, the proposed approach should be appraised for other benefits such as the ease of hedging and the potential to bring additional insights into pricing which are not available otherwise. For instance, this approach indicates that an option's value will be either near-zero or the positive difference between the current value of the portfolio and the present value of the exercise price if the underlying portfolio's duration is close to the maturity of the option. This is true because a discount bond with a face value equal to the exercise price and maturity equal to the option's maturity will immunize the bond portfolio.

SUMMARY AND CONCLUSIONS

This study presents a simple approach to pricing European options on bond portfolios. While the traditional approach is to decompose the option into a portfolio of options on the component discount bonds in the portfolio, the proposed approach requires only one option value to be calculated. In addition, no iterative calculations are necessary in the proposed approach, at least for Vasicek and CIR models. Because the underlying bond portfolio is approximated by a single discount bond, hedging is also simplified. The proposed approach also applies when the short rate is non-Markovian, which is not possible with the Jamshidian's approach.

The essence of the simple approach is to match the volatility of the bond portfolio with that of a single discount bond. Since the approach precisely matches only the current value of the volatilities, while leaving the dynamics of the volatilities only roughly matched, the approach is only an approximation. Experiments with Vasicek and CIR models in a variety of parameter settings indicate that the accuracy of the approximation is quite impressive.

Besides providing a useful methodology for pricing coupon bond options with a non-Markovian spot rate, the proposed approach can be used also to examine bond option pricing from an alternative angle: impacts of duration and convexity on bond option prices. This should be valuable to both academics and practitioners who are concerned with the effects of bond characteristics on the bond option prices.

The proposed approach is not limited to Vasicek or CIR models. In principle, it is valid for all one-factor interest rate models. As long as closed form formulas exist for options on discount bonds, the approximation can always be used to price options on any portfolio of discount bonds (with positive weights). The approach can be generalized also to multi-factor settings, albeit less elegantly. Essentially, in an n-factor model, one needs n discount bonds to approximate a portfolio of more than n discount bonds.

APPENDIX: AN EXAMPLE OF A VOLATILITY STRUCTURE IMPLYING A NON-MARKOV SPOT RATE

An examination is conducted on a deterministic bond volatility structure which admits closed-form pricing formulas for discount bond options but does not lead to a Markov spot rate. In this case, the Jamshidian's approach cannot be applied. The approximation scheme proposed in the text is used.

A volatility structure which permits the short-term and long-term forward rate volatilities to converge to two different nonzero levels is examined. (In contrast, in the framework of Vasicek or CIR, the long-term forward rate volatility is simply zero.) Specifically, suppose the forward rate volatility takes the following form:[18]

$$\sigma_f(t, T) = \sigma_2 + (\sigma_1 - \sigma_2)\, e^{-k(T-t)}$$

where k is the mean-reversion speed and σ_1 and σ_2 are constant parameters. It can be verified that when $(T - t)$ approaches zero (infinity), the forward rate volatility approaches σ_1 (σ_2). Now, a discount bond's volatility can be written as:

$$\sigma_B(t, T) = \int_t^T \sigma_f(t, u)\, du = \frac{\sigma_1 - \sigma_2}{k}\left(1 - e^{-k(T-t)}\right) + (T - t)\sigma_2$$

As shown by Heath, Jarrow, and Morton (1992) and Carverhill (1994b), given the above volatility structure, a call option with maturity T_1 on a discount bond with maturity T_2 can be priced with the following closed-form formula:

$$C = B(t, T_2)N(d_1) - XB(t, T_1)N(d_2)$$

where

$$\begin{aligned}
d_1 &= \frac{\ln \dfrac{B(t, T_2)}{XB(t, T_1)} + \dfrac{\sigma^2}{2}(T_1 - t)}{\sigma\sqrt{T_1 - t}}, \; d_2 = d_1 - \sigma\sqrt{T_1 - t},\\
\sigma^2 &= \frac{1}{T_1 - t}\int_t^{T_1} [\sigma_B(u, T_2) - \sigma_B(u, T_1)]^2\, du\\
&= \sigma_2^2 (T_2 - T_1)^2 + \frac{(\sigma_1 - \sigma_2)^2}{2k^2(T_1 - t)}[(1 - a)^2 - (b - c)^2]\\
&\quad + \frac{2(\sigma_1 - \sigma_2)\sigma_2(T_2 - T_1)}{k^2(T_1 - t)}[1 - a - b + c],\\
a &= e^{-k(T_2 - T_1)}, \quad b = e^{-k(T_1 - t)}, \quad c = e^{-k(T_2 - t)}
\end{aligned}$$

Since the volatility of a discount bond is deterministic, a coupon bond

[18]This volatility structure was discussed in Carverhill (1994b).

can be approximated by a single discount bond whose volatility is equal to the weighted average of the individual volatilities of the discount bonds, as illustrated in the text. Thus, an option on a coupon bond can be priced as an option on this proxy discount bond.

It is shown below that the proposed volatility structure implies a non-Markov spot rate, and, as a result, the Jamshidian's approach cannot be applied.

According to Jeffrey (1995), the necessary and sufficient condition for a volatility structure to imply a Markov spot rate can be summarized as the following:

A forward rate volatility structure, σ_f(r, t, T), will lead to a Markov spot rate if and only if there exists a pair of functions, θ(r, t) and h(t, T), so that

$$\sigma_f(r, t, T) \int_t^T \sigma_f(r, t, v)dv = \frac{\sigma_f(r, t, T)}{\sigma_f(r, t, t)}\,\theta(r, t) + \frac{\partial}{\partial t}\left[\int_0^r \frac{\sigma_f(R, t, T)}{\sigma_f(R, t, t)}\,dR\right] + h(t, T) + \frac{1}{2}\,\sigma_f^2(r, t, t)\,\frac{\partial}{\partial t}\left[\frac{\sigma_f(r, t, T)}{\sigma_f(r, t, t)}\right].$$

In other words, for the spot rate to be Markov, one must be able to identify a pair of functions, $\theta(r, t)$ and $h(t, T)$, which satisfy the above equation. Notice here that $\theta(r, t)$ must be independent of the maturity date, T, and $h(t, T)$ must be independent of the spot rate, r. Now, substituting the expression for the forward rate volatility proposed at the beginning of this appendix into the above equation, one obtains:

$$\theta(r, t) = -\frac{\dfrac{\sigma_1 - \sigma_2}{k\sigma_1}\, e^{-k(T-t)} r}{\dfrac{\sigma_2}{\sigma_1} + \left(1 - \dfrac{\sigma_2}{\sigma_1}\right) e^{-k(T-t)}} + \frac{\sigma_f(t, T)\,\sigma_B(t, T) - \dfrac{(\sigma_1 - \sigma_2)\sigma_1}{2k}\, e^{-k(T-t)} - h(t, T)}{\dfrac{\sigma_2}{\sigma_1} + \left(1 - \dfrac{\sigma_2}{\sigma_1}\right) e^{-k(T-t)}}$$

It can be seen that no matter how the function $h(t, T)$ is chosen, $\theta(r, t)$ will always depend on the maturity date, T. In other words, it is impossible to find a pair of $h(t, T)$ and $\theta(r, t)$ which satisfies the necessary and sufficient condition. Thus, the volatility structure proposed here does not admit a Markov spot rate. In this case, a bond price cannot be written simply as a function of the spot rate. It will also depend on the whole path that the spot rate has taken. This will make the Jamshidian's approach impossible to apply.

To further illustrate the above point, let $f(0, \tau)$ represent the initial forward rate curve for the period, $0 \leq \tau \leq T$. The corresponding discount bond price is denoted by $B(0, \tau)$. The single underlying factor under the equivalent martingale measure is a standard Wiener process denoted by $w(t)$. Then, according to Heath, Jarrow, and Morton (1992), the drift of the forward rate process is

$$u_f(t, T) = \sigma_f(t, T) \int_t^T \sigma_f(t, s)\, ds$$

which is a deterministic function under the forward rate volatility structure. With the above specifications and the identity definitions in (2), (4), and (5) of Heath, Jarrow, and Morton (1992), it can be shown that the time t price of a bond maturing at time T takes the following form:

$$B(t, T) = \frac{B(0, T)}{B(0, t)} \exp\left\{ -\int_t^T ds \int_0^t \mu_f(u, s)\, du + \frac{f(0, t) + \int_0^t \mu_f(u, t) du}{k} \right. \\ \left. - \frac{r(t)}{k} + \frac{\sigma_1}{k} w(t) + \int_0^t [(t - u)\, \sigma_2 - \sigma_B(u, T)]\, dw(u) \right\}$$

where $r(t)$ is the spot interest rate at time t.[19] It can be seen that the bond price $B(t, T)$ not only depends on the spot rate $r(t)$, but also depends on the path of the interest rate which is captured by the last term in the exponential, the stochastic integral. This path-dependency renders the solving of a critical interest rate impossible, because the interest rate itself does not solely capture all the randomness. Consequently, the Jamshidian's approach is not applicable.

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[19]Readers who are interested in the detailed derivations are welcome to contact the author.

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