

# An Explicit Expression for the Expectation of a Product of Quadratic Forms in Multivariate Normal Random Variables

Raymond Kan<sup>a,\*</sup>, Jiening Pan<sup>b</sup>

<sup>a</sup>Rotman School of Management, University of Toronto, 105 St. George Street, Ontario, Canada, M5S 3E6  
Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong  
<sup>b</sup>Nankai University, 38 Tongyan Road, Jinnan District, Tianjin, Tianjin 300350, China

---

## Abstract

We present an explicit formula for the expectation of a product of quadratic forms in multivariate real normal random variables. The proposed expression is optimized for zero redundancy, containing the fewest possible terms without repetition. Moreover, the coefficients of all the terms are given in closed-form, and we provide an efficient algorithm to enumerate the contributing terms. Our results are further extended to Hermitian quadratic forms in multivariate complex normal random variables. As an application, we show that our formula can be used to obtain a simple expression for the product moments of central and noncentral Wishart distributions. Finally, we derive a general expression that allows for repeated quadratic forms in the product.

*Keywords:* Expectation of product, Quadratic form, Multivariate normal distribution, Wishart distribution  
*2020 MSC:* Primary 60E10, Secondary 62H10

---

## 1. Introduction

Consider a real random vector  $\mathbf{z} = [z_1, \dots, z_m]^T \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ , where  $\boldsymbol{\mu} = [\mu_1, \dots, \mu_m]^T$  is the mean vector and  $\boldsymbol{\Sigma}$  is a positive semidefinite covariance matrix, and  $T$  denotes the transpose of a vector or a matrix. Given  $k$  symmetric matrices  $\mathbf{A}_1$  to  $\mathbf{A}_k$ , we seek an explicit expression for  $E[\prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z}]$ . In the central case where  $\boldsymbol{\mu} = \mathbf{0}$ , explicit expressions for  $E[\prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z}]$  have been derived in earlier work by Kumar [12], Magnus [15], Magnus [16], and Don [3]. However, for the more general noncentral case ( $\boldsymbol{\mu} \neq \mathbf{0}$ ), no closed-form expression is available in the literature, except for small values of  $k$ . This paper aims to fill this gap.

In the literature, there exists some recurrence relations on  $E[\prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z}]$ . For example, Bao and Ullah [1] presented a recurrence relation on  $E[\prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z}]$ . Hillier, Kan, and Wang [8] presented three other recurrence relations on  $E[\prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z}]$ . Kan [9] established an identity that expresses  $E[\prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z}]$  as linear combinations of  $k$ -th moment of various quadratic forms in  $\mathbf{z}$ . These  $k$ -th moments can, in turn, be computed efficiently via a simple recurrence relation. All these methods allow us to numerically evaluate  $E[\prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z}]$ . After expansion, some of these methods can also deliver analytical expressions of  $E[\prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z}]$ . However, the expressions obtained from these recurrence methods generally contain repeated terms, and the coefficient associated with each term must be determined recursively. Although such expressions can be manually simplified when  $k$  is small, an explicit formula for  $E[\prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z}]$  for general  $k$  remains unknown when  $\boldsymbol{\mu} \neq \mathbf{0}$ . In this paper, we address this issue by presenting a simple and explicit expression for  $E[\prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z}]$ . Our formula contains no repeated terms and is therefore in the simplest possible form. Moreover, the coefficient of each term is given in closed-form.

The remainder of the paper is organized as follows. Section 2 reviews existing explicit expression for  $E[\prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z}]$  in the case that  $\boldsymbol{\mu} = \mathbf{0}$ . Section 3 extends the result to the case that  $\boldsymbol{\mu} \neq \mathbf{0}$ . However, enumerating the terms in this expression is computationally expensive. To address this issue, Section 4 presents an alternative explicit expression for  $E[\prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z}]$  that is significant more efficient to compute. Section 5 extends our results to expectation of a product

---

\*Corresponding author. Email address: Raymond.Kan@rotman.utoronto.ca

of Hermitian quadratic forms in complex normal random variables. We also demonstrate that our results yield simple expressions for the product moments of real and complex Wishart random matrices. Section 6 further generalizes our results by providing an explicit expression for  $E[\prod_{i=1}^k (z^T A_i z)^{s_i}]$  where  $s_i$  are positive integers. Section 7 concludes the paper.

## 2. Explicit Expression for $E[\prod_{i=1}^k z^T A_i z]$ when $z \sim \mathcal{N}(\mathbf{0}, \Sigma)$

In a series of papers, Kumar [12], Magnus [15], Magnus [16], and Don [3] developed an explicit formula for  $E[\prod_{i=1}^k z^T A_i z]$  when  $z \sim \mathcal{N}(\mathbf{0}, \Sigma)$ . We summarize their results as follows. Let  $\nu = (\nu_1, \dots, \nu_p)$  be a vector of positive integers, and define

$$\tau_\nu = \text{tr}(A_{\nu_1} \Sigma A_{\nu_2} \Sigma \cdots \Sigma A_{\nu_p} \Sigma). \quad (1)$$

For a positive integer  $k$ , we let  $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_r) \vdash k$  be a partition of integer  $k$  with  $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_r > 0$ , and  $r = \ell(\kappa)$  denotes the number of nonzero parts of  $\kappa$ . We may also write  $\kappa$  in frequency representation as  $\kappa = 1^{f_1} 2^{f_2} \dots k^{f_k}$ , where  $f_i$  denotes the frequency of  $i$  in  $\kappa$ .

When  $z \sim \mathcal{N}(\mathbf{0}, \Sigma)$ , Don [3] showed that

$$E\left[\prod_{i=1}^k z^T A_i z\right] = \sum_{\kappa \vdash k} 2^{k-f_1-f_2} \sum_{(\pi_1, \dots, \pi_r) \in \mathcal{S}_\kappa} \tau_{\pi_1} \cdots \tau_{\pi_r}, \quad (2)$$

where  $\pi = (\pi_1, \dots, \pi_r)$  is a permutation of  $\{1, \dots, k\}$ , with the length of  $\pi_i$  is equal to  $\kappa_{r+1-i}$ ,  $i = 1, \dots, r$ . For our subsequent notational convenience, we arrange the  $r$  blocks of  $\pi$  in a different order from Don [3]. Instead of setting  $\ell(\pi_i) = \kappa_i$  for  $i = 1, \dots, r$ , we reverse the order: the first  $f_1$  blocks of  $\pi$  contain one element each, the next  $f_2$  blocks of  $\pi$  contain two elements each, and so on, up to the last  $f_k$  blocks containing  $k$  elements each. Finally,  $\mathcal{S}_\kappa$  is defined as a subset of the permutation group of  $\{1, \dots, k\}$  that satisfies the following three conditions:

1. for every  $\pi_i$  with more than one element, its first element must be the smallest element,
2. if  $\pi_i$  has more than two elements, then its second element must be smaller than its last element,
3. if  $\pi_i$  and  $\pi_{i+1}$  have the same number of elements, then the first element of  $\pi_i$  must be smaller than the first element of  $\pi_{i+1}$ .

Essentially, the first two conditions require  $\pi_i$  to be a bracelet — that is, the lexicographically smallest element in an equivalence class of strings under rotation and reversal. For example  $\pi_i = (1, 2, 4)$  is a bracelet whereas  $\pi_i = (1, 4, 2), (2, 1, 4), (2, 4, 1), (4, 1, 2), (4, 2, 1)$  are not. This requirement arises because the trace of a product of matrices is invariant under both rotation and reversal of the list of matrices. Consequently, the first two conditions identify a unique representative (the bracelet) among all  $\pi_i$ 's that yield the same value of  $\tau_{\pi_i}$ . The third condition requires the blocks of  $\pi$  with the same length to be ordered in ascending order of their first elements. This is necessary because  $\tau_{\pi_i} \tau_{\pi_{i+1}} = \tau_{\pi_{i+1}} \tau_{\pi_i}$ , and without this ordering rule, duplicate  $\tau_{\pi_i} \tau_{\pi_{i+1}}$  would appear, preventing the terms on the right hand side of (2) from being unique.

Don [3] also derives the number of elements in  $\mathcal{S}_\kappa$ , given by

$$\varphi(\kappa) = \frac{k! 2^{f_1+f_2}}{\prod_{i=1}^k f_i! (2i)^{f_i}}. \quad (3)$$

Consequently, the total number of terms on the right hand side of (2) is

$$N(k) = \sum_{\kappa \vdash k} \varphi(\kappa). \quad (4)$$

In Table 1, we present the values of  $N(k)$  for  $k = 1$  to 10.

Table 1: Number of unique terms in the expansion of  $E[\prod_{i=1}^k z^T A_i z]$  when  $z \sim \mathcal{N}(\mathbf{0}, \Sigma)$

$k$	1	2	3	4	5	6	7	8	9	10
$N(k)$	1	2	5	17	73	388	2,461	18,155	152,531	1,436,714

It turns out that  $N(k)$  can also be interpreted as the number of ways to cover  $k$  labeled vertices by disjoint undirected cycles. According to <https://oeis.org/A002135> from the on-line encyclopedia of integer sequences,  $N(k)$  satisfies the recurrence:

$$N(k+1) = (k+1)N(k) - \binom{k}{2}N(k-2),$$

with the boundary conditions  $N(0) = N(1) = 1$  and  $N(2) = 2$ .

While (2) provides a compact expression for  $E[\prod_{i=1}^k z^T A_i z]$ , it is not computationally efficient for enumerating the terms. The inefficiency arises because, for each  $\kappa \vdash k$ , one must examine all  $k!$  permutations of  $\{1, \dots, k\}$  to check which satisfy the three conditions defining  $\mathcal{S}_\kappa$ . In Section 4, we present an alternative explicit expression that is significantly more efficient to compute.

### 3. Explicit Expression for $E[\prod_{i=1}^k z^T A_i z]$ when $z \sim \mathcal{N}(\mu, \Sigma)$

We now consider the general case where  $z \sim \mathcal{N}(\mu, \Sigma)$  with  $\mu \neq \mathbf{0}$ . Before deriving an explicit expression for  $E[\prod_{i=1}^k z^T A_i z]$ , we first present an explicit expression for  $E[(z^T A z)^k]$ , corresponding to the special case where  $A_1 = \dots = A_k = A$ .

Using the recurrence relation between cumulants and moments, Lemma 3 of Magnus [17] gives the following explicit expression of  $E[(z^T A z)^k]$ :

$$E[(z^T A z)^k] = \sum_{\kappa \vdash k} \frac{k! 2^k}{\prod_{i=1}^k f_i! (2i)^{f_i}} \prod_{i=1}^k (i\tilde{\theta}_i + \tilde{\tau}_i)^{f_i}, \quad (5)$$

where  $\tilde{\tau}_i = \text{tr}((A\Sigma)^i)$  and  $\tilde{\theta}_i = \mu'(A\Sigma)^{i-1} A \mu$ . Although Magnus [17] assumes  $\Sigma$  is positive definite, this expression remains valid for positive semidefinite  $\Sigma$ . This is because the cumulants of  $z^T A z$  have the same expression regardless of whether  $\Sigma$  is positive definite or positive semidefinite (see p.55 of Mathai and Provost [18]).

However, the right hand side of (5) contains repeated terms. Our first task is to determine the number of unique terms on the right hand side of (5). Applying the binomial expansion, we have

$$(i\tilde{\theta}_i + \tilde{\tau}_i)^{f_i} = \sum_{g_i=0}^{f_i} \binom{f_i}{g_i} i^{g_i} \tilde{\theta}_i^{g_i} \tilde{\tau}_i^{f_i-g_i}, \quad (6)$$

and thus there are  $1 + f_i$  unique terms on its right hand side of (6). This implies that, for a given  $\kappa \vdash k$ , the number of unique terms in (5) is  $\prod_{i=1}^k (1 + f_i)$ . Together, the total number of unique terms in (5) is

$$m(k) = \sum_{\kappa \vdash k} \prod_{i=1}^k (1 + f_i). \quad (7)$$

In Table 2, we present the values of  $m(k)$  for  $k = 1$  to 10.

Table 2: Number of unique terms in the expansion of  $E[(z^T A z)^k]$  when  $z \sim \mathcal{N}(\mu, \Sigma)$

$k$	1	2	3	4	5	6	7	8	9	10
$m(k)$	2	5	10	20	36	65	110	185	300	481

Using the binomial expansion of (6), we obtain the following explicit expression for  $E[(z^T A z)^k]$ :

$$E[(z^T A z)^k] = \sum_{\kappa \vdash k} \sum_{\mathbf{0} \leq \mathbf{g} \leq \mathbf{f}} c_{\mathbf{f}, \mathbf{g}} \prod_{i=1}^k \tilde{\theta}_i^{g_i} \tilde{\tau}_i^{f_i-g_i}, \quad (8)$$

where  $\mathbf{f} = (f_1, \dots, f_k)$ ,  $\mathbf{g} = (g_1, \dots, g_k)$ ,  $\sum_{\mathbf{0} \leq \mathbf{g} \leq \mathbf{f}}$  is a short-hand notation for  $\sum_{g_1=0}^{f_1} \dots \sum_{g_k=0}^{f_k}$ , and

$$c_{\mathbf{f}, \mathbf{g}} = \frac{k! 2^k}{\prod_{i=1}^k f_i! (2i)^{f_i}} \prod_{i=1}^k \binom{f_i}{g_i} i^{g_i}.$$

Unlike (5), every term in (8) is unique.

We now present an explicit expression for

$$\alpha(\mathbf{A}_1, \dots, \mathbf{A}_k) := \mathbb{E} \left[ \prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z} \right]. \quad (9)$$

It turns out that (8), together with the polarization argument suggested by Don [3], allows us to obtain an explicit expression for  $\alpha(\mathbf{A}_1, \dots, \mathbf{A}_k)$ . For the noncentral case, the expression involves both  $\tau_{\mathbf{v}}$  as defined in (1), and

$$\theta_{\mathbf{v}} = \boldsymbol{\mu}^T \mathbf{A}_{v_1} \boldsymbol{\Sigma} \mathbf{A}_{v_2} \boldsymbol{\Sigma} \dots \boldsymbol{\Sigma} \mathbf{A}_{v_p} \boldsymbol{\mu}. \quad (10)$$

For a given  $\boldsymbol{\kappa}$  (i.e.,  $\mathbf{f}$ ), we split  $\boldsymbol{\pi} \in \mathcal{S}_k$  into  $r = \ell(\boldsymbol{\kappa})$  blocks as  $\boldsymbol{\pi} = (\boldsymbol{\pi}_1, \dots, \boldsymbol{\pi}_r)$ , where  $\ell(\boldsymbol{\pi}_i) = \kappa_{r+1-i}$ . Then for a given  $\mathbf{f}$  and  $\mathbf{g}$ , we define

$$\eta_{\mathbf{f}, \mathbf{g}, \boldsymbol{\pi}} = \prod_{i=1}^k \left( \prod_{j=1}^{g_i} \theta_{\boldsymbol{\pi}_{s_i+j}} \right) \left( \prod_{j=g_i+1}^{f_i} \tau_{\boldsymbol{\pi}_{s_i+j}} \right), \quad (11)$$

where  $s_i = \sum_{l=1}^{i-1} f_l$ . While the definition of  $\eta_{\mathbf{f}, \mathbf{g}, \boldsymbol{\pi}}$  seems somewhat complicated, it is just a product of various  $\theta$  and  $\tau$  terms. Specifically, the first  $f_1$  factors correspond to  $g_1$   $\theta$  terms and  $f_1 - g_1$   $\tau$  terms with  $\ell(\mathbf{v}) = 1$ ; the next  $f_2$  factors correspond to  $g_2$   $\theta$  terms and  $f_2 - g_2$   $\tau$  terms with  $\ell(\mathbf{v}) = 2$ , and so on, up to the last  $f_k$  factors correspond to  $g_k$   $\theta$  terms and  $f_k - g_k$   $\tau$  terms with  $\ell(\mathbf{v}) = k$ . The subscripts of these terms are taken from  $\boldsymbol{\pi}_1$  to  $\boldsymbol{\pi}_r$ .

**Example 1.** For  $\boldsymbol{\kappa} = (2, 2, 1)$ , we have  $\mathbf{f} = (1, 2, 0)$ . If  $\mathbf{g} = (0, 1, 0)$ , we have

$$\eta_{\mathbf{f}, (0,1,0), \boldsymbol{\pi}} = \tau_{\boldsymbol{\pi}_1} \theta_{\boldsymbol{\pi}_2} \tau_{\boldsymbol{\pi}_3},$$

where  $\ell(\boldsymbol{\pi}_1) = 1$ ,  $\ell(\boldsymbol{\pi}_2) = \ell(\boldsymbol{\pi}_3) = 2$ . Similarly if  $\mathbf{g} = (1, 0, 0)$ , then

$$\eta_{\mathbf{f}, (1,2,0), \boldsymbol{\pi}} = \theta_{\boldsymbol{\pi}_1} \tau_{\boldsymbol{\pi}_2} \tau_{\boldsymbol{\pi}_3}.$$

With this notation, we can now write down a simple explicit expression of  $\alpha(\mathbf{A}_1, \dots, \mathbf{A}_k)$ :

$$\alpha(\mathbf{A}_1, \dots, \mathbf{A}_k) = \sum_{\boldsymbol{\kappa} \vdash k} \sum_{\mathbf{0} \leq \mathbf{g} \leq \mathbf{f}} \frac{c_{\mathbf{f}, \mathbf{g}}}{k!} \sum_{\boldsymbol{\pi} \in \mathcal{S}_k} \eta_{\mathbf{f}, \mathbf{g}, \boldsymbol{\pi}}, \quad (12)$$

where  $\mathcal{S}_k$  stands for the permutation group of  $\{1, \dots, k\}$ . That is, each term in (8) is expanded into  $k!$  terms, with subscripts determined by the permutations in  $\mathcal{S}_k$ .

**Example 2.** For  $k = 2$ , (8) gives

$$\mathbb{E}[(\mathbf{z}^T \mathbf{A} \mathbf{z})^2] = 2\tilde{\tau}_2 + 4\tilde{\theta}_2 + \tilde{\tau}_1^2 + 2\tilde{\theta}_1 \tilde{\tau}_1 + \tilde{\theta}_1^2.$$

Then since  $\mathcal{S}_2 = \{(1, 2), (2, 1)\}$ , (12) suggests

$$\alpha(\mathbf{A}_1, \mathbf{A}_2) = \frac{1}{2} [2\tau_{(1,2)} + 2\tau_{(2,1)} + 4\theta_{(1,2)} + 4\theta_{(2,1)} + \tau_1 \tau_2 + \tau_2 \tau_1 + 2\theta_1 \tau_2 + 2\tau_1 \theta_2 + \theta_1 \theta_2 + \theta_2 \theta_1]. \quad (13)$$

The correctness of (12) follows from the polarization argument used by Don [3]. Namely,  $\alpha(\mathbf{A}_1, \dots, \mathbf{A}_k)$  must be symmetric, i.e., invariant under permutations of the matrix indices, and (12) indeed satisfies this property. Moreover, both  $\mathbb{E}[\prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z}]$  and (12) are multilinear in  $\mathbf{A}_1, \dots, \mathbf{A}_k$ . Since two symmetric multilinear form in  $k$  variables are equal if and only if their associated homogeneous polynomial forms of total degree  $k$  are equal, it suffices to verify that  $\alpha(\mathbf{A}, \dots, \mathbf{A})$  reduces to (8), which is certainly true by construction.

While (12) gives a correct expression for  $[\prod_{i=1}^k z^T A_i z]$ , it contains many repeated terms. For example in (13), the terms in the first, second, third, and the fifth pairs are identical. Consequently, (12) does not yield the simplest explicit expression for  $[\prod_{i=1}^k z^T A_i z]$ . To obtain an explicit expression for  $\alpha(A_1, \dots, A_k)$  without repeated terms, we identify the subset  $\mathcal{S}_{f,g} \subset \mathcal{S}_k$ , consisting of permutations  $\pi$  that produce distinct (in general) values of  $\eta_{f,g,\pi}$ .

We define  $\mathcal{S}_{f,g}$  as follows. For  $\pi = (\pi_1, \dots, \pi_r)$  with  $\ell(\pi_i) = \kappa_{r+1-i}$  to belong to  $\mathcal{S}_{f,g}$ , it has to satisfy the following conditions:

1. For the  $g_i$  blocks of  $\pi$  with  $i$  elements that are designated as subscripts for  $\theta$  terms:
  - If  $i \geq 2$ , the first element in each block must be smaller than the last element.
  - Across all  $g_i$  such blocks, the first elements must be in ascending order.
2. For the  $f_i - g_i$  blocks of  $\pi$  with length  $i$  that are designated as subscripts for  $\tau$  terms:
  - If  $i \geq 2$ , the first element must be the smallest in the block.
  - If  $i \geq 3$ , the second element must be smaller than the last element.
  - Across all  $f_i - g_i$  such blocks, the first elements must be in ascending order.

The first condition ensures that the product of  $\theta$  terms is unique. Since  $\theta$  is invariant under reversal of its subscript (e.g.,  $\theta_{(1,2,3)} = \theta_{(3,2,1)}$ ), we require the first element to be less than the last. Furthermore, when multiple  $\theta$  terms in the product have the same subscript length, we sort them in ascending order of their first elements to avoid duplicates. The second condition mirrors that of  $\mathcal{S}_k$ , ensuring the subscripts of the  $\tau$  terms have a bracelet representation, and that for  $\tau$  terms with equal subscript length, their first elements are sorted in ascending order — thus ensuring the uniqueness in the product of  $\tau$  terms. We call any  $\pi$  satisfying these two conditions as admissible.

**Example 3.** For  $\kappa = (2, 1)$ , i.e.,  $f = (1, 1, 0)$  and  $g = (0, 1, 0)$ , the admissible permutations are  $\pi = (1)(2, 3)$ ,  $(2)(1, 3)$ , and  $(3)(1, 2)$ . The other three permutations  $(1)(3, 2)$ ,  $(2)(3, 1)$ , and  $(3)(2, 1)$  are inadmissible. For  $\kappa = (1, 1, 1)$ , i.e.,  $f = (3, 0, 0)$ , then for any  $g \leq f$ , the only admissible permutation is  $\pi = (1)(2)(3)$ .

With  $\mathcal{S}_{f,g}$  defined, we can now present our explicit expression of  $\alpha(A_1, \dots, A_k)$ :

$$\alpha(A_1, \dots, A_k) = \sum_{\kappa \vdash k} \sum_{0 \leq g \leq f} \frac{c_{f,g}}{h_{f,g}} \sum_{\pi \in \mathcal{S}_{f,g}} \eta_{f,g,\pi}, \quad (14)$$

where  $h_{f,g}$  is the number of elements in  $\mathcal{S}_{f,g}$ .

In contrast to (12), which contains repeated terms, every term on the right hand side of (14) is unique. We first determine the number of unique terms in (14), denoted by  $M(k)$ . For a given  $\kappa \vdash k$ , we place the integers 1 to  $k$  into  $r = \ell(\kappa)$  bins, with bin sizes  $\kappa_1$  to  $\kappa_r$ . The number of ways to do that is simply the multinomial coefficient  $k! / \prod_{i=1}^r \kappa_i!$ . Equivalently, since there are  $f_i$  bins containing  $i$  elements,  $i = 1, \dots, k$ , the multinomial coefficient can also be written as  $k! / \prod_{i=1}^k (i!)^{f_i}$ . If we further require that bins of the same size to be ordered in ascending order of their first elements, then the total number of ways of distributing  $\{1, \dots, k\}$  into these  $r$  bins is

$$q(\kappa) = \frac{k!}{\prod_{i=1}^k (i!)^{f_i}}. \quad (15)$$

For each bin, its contents can be assigned as subscripts for  $\theta$  and  $\tau$ . We now determine the number of distinct  $\theta$  and  $\tau$  terms that can be formed from the elements of a bin. For  $\theta$ , it is unique up to the reflection of its subscripts, so for a bin with  $i$  distinct integers, the number of unique  $\theta$  that can be created is

$$n_\theta(i) = \begin{cases} 1 & i \leq 1, \\ \frac{i!}{2} & i > 1. \end{cases} \quad (16)$$

For  $\tau$ , it is unique up to the bracelet representation of its subscripts. So for a bin with  $i$  distinct integers, the number of unique  $\tau$  that can be created is

$$n_\tau(i) = \begin{cases} 1 & i \leq 2, \\ \frac{(i-1)!}{2} & i > 2. \end{cases} \quad (17)$$

This follows because when  $i \leq 2$ , there is only one unique  $\tau$ . When  $i > 2$ , admissibility requires that the first subscript be the smallest element, leaving  $i - 1$  positions to be filled for the second through last elements, yielding  $(i - 1)!$  possibilities. Since the second element must be smaller than the last, we need to divide the number of possibilities by 2.

As a result, we can generate  $n_\theta(i) + n_\tau(i)$  distinct terms from a bin containing  $i$  distinct integers. Therefore, the total number of unique terms in (14) is

$$M(k) = \sum_{\kappa \vdash k} q(\kappa) \prod_{i=1}^k [n_\theta(i) + n_\tau(i)]^{f_i}. \quad (18)$$

In Table 3, we present the values of  $M(k)$  for  $k = 1$  to 10.

Table 3: Number of unique terms in the expansion of  $E\left[\prod_{i=1}^k z^T A_i z\right]$  when  $z \sim \mathcal{N}(\mu, \Sigma)$

$k$	1	2	3	4	5	6	7	8	9	10
$M(k)$	2	6	24	123	774	5,778	49,824	486,531	5,300,262	63,673,506

We now derive a formula for  $h_{f,g}$ , the number of permutations in  $\mathcal{S}_{f,g}$ . For a given  $\kappa \vdash k$  (equivalently, given  $f$ ), we know there are  $q(\kappa)$  ways to partition  $\{1, \dots, k\}$  into  $r$  blocks, with  $f_i$  blocks of length  $i$ ,  $i = 1, \dots, k$ , and for blocks with equal length, they are arranged in ascending order of their first elements. For the  $f_i$  blocks of length  $i$ , we choose  $g_i$  blocks to serve as subscripts for the  $\theta$  terms, with the remaining  $f_i - g_i$  blocks serving as subscripts for the  $\tau$  terms. For the blocks assigned to  $\theta$ , the number of distinct terms is  $n_\theta(i)$ ; for the blocks assigned to  $\tau$ , it is  $n_\tau(i)$ . Hence,

$$h_{f,g} = q(\kappa) \prod_{i=1}^k \binom{f_i}{g_i} n_\theta(i)^{g_i} n_\tau(i)^{f_i - g_i}.$$

Furthermore,

$$\begin{aligned} \frac{c_{f,g}}{h_{f,g}} &= 2^k \prod_{i=1}^k \frac{(i!)^{f_i} i^{g_i}}{(2i)^{f_i} n_\theta(i)^{g_i} n_\tau(i)^{f_i - g_i}} \\ &= 2^{k - f_1 - f_2 + g_2} \prod_{i=1}^k \frac{(i!)^{f_i} i^{g_i}}{(2i)^{f_i} \left(\frac{i!}{2}\right)^{g_i} \left[\frac{(i-1)!}{2}\right]^{f_i - g_i}} \\ &= 2^{k - f_1 - f_2 + g_2}, \end{aligned}$$

where the second equality follows from

$$\begin{aligned} \prod_{i=1}^k n_\theta(i)^{g_i} &= 2^{g_1} \prod_{i=1}^k \left(\frac{i!}{2}\right)^{g_i}, \\ \prod_{i=1}^k n_\tau(i)^{f_i - g_i} &= 2^{f_1 - g_1 + f_2 - g_2} \prod_{i=1}^k \left[\frac{(i-1)!}{2}\right]^{f_i - g_i}. \end{aligned}$$

This leads to the following generalization of Don [3] to the case where  $\mu \neq \mathbf{0}$ .

**Proposition 1.** *Suppose  $z \sim \mathcal{N}(\mu, \Sigma)$  and  $A_i$ ,  $i = 1, \dots, k$  are symmetric matrices. Then*

$$E\left[\prod_{i=1}^k z^T A_i z\right] = \sum_{\kappa \vdash k} \sum_{\mathbf{0} \leq g \leq f} 2^{k - f_1 - f_2 + g_2} \sum_{\pi \in \mathcal{S}_{f,g}} \eta_{f,g,\pi}, \quad (19)$$

where each  $\eta_{f,g,\pi}$  denotes a unique product of  $r = \ell(\kappa)$  terms of  $\theta$  and  $\tau$ , as defined in (11).

This expression accounts for all valid monomials in the expectation, and is both combinatorially minimal and fully constructive. In the case where  $\boldsymbol{\mu} = \mathbf{0}$ , the only nonzero terms in the above expression correspond to  $\mathbf{g} = \mathbf{0}$ . Using the fact that  $\mathcal{S}_{f,\mathbf{0}} = \mathcal{S}_\kappa$ , the expression in (19) reduces to (2).

We illustrate the use of (19) with the case  $k = 3$ . Table 4 enumerates all the contributing terms in (19). The number of unique terms matches the known count of  $M(3) = 24$  in Table 3. Based on the entries in Table 4, we obtain

$$\begin{aligned} \mathbb{E} \left[ \prod_{i=1}^3 \mathbf{z}^T \mathbf{A}_i \mathbf{z} \right] &= 8\tau_{(1,2,3)} + 8\theta_{(1,2,3)} + 8\theta_{(1,3,2)} + 8\theta_{(2,1,3)} \\ &\quad + 2\tau_1\tau_{(2,3)} + 2\tau_2\tau_{(1,3)} + 2\tau_3\tau_{(1,2)} + 4\tau_1\theta_{(2,3)} + 4\tau_2\theta_{(1,3)} + 4\tau_3\theta_{(1,2)} \\ &\quad + 2\theta_1\tau_{(2,3)} + 2\theta_2\tau_{(1,3)} + 2\theta_3\tau_{(1,2)} + 4\theta_1\theta_{(2,3)} + 4\theta_2\theta_{(1,3)} + 4\theta_3\theta_{(1,2)} \\ &\quad + \tau_1\tau_2\tau_3 + \theta_1\tau_2\tau_3 + \theta_2\tau_1\tau_3 + \theta_3\tau_1\tau_2 + \theta_1\theta_2\tau_3 + \theta_1\theta_3\tau_2 + \theta_2\theta_3\tau_1 + \theta_1\theta_2\theta_3. \end{aligned} \quad (20)$$

Table 4: Construction of an explicit expression of  $\mathbb{E} \left[ \prod_{i=1}^3 \mathbf{z}^T \mathbf{A}_i \mathbf{z} \right]$  when  $\mathbf{z} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ , using (19)

$\mathbf{g}$	$2^{k-f_1-f_2+g_2}$	Admissible $\boldsymbol{\pi}$	term
$\boldsymbol{\kappa} = (3), \mathbf{f} = (0, 0, 1)$			
(0,0,0)	8	(1,2,3)	$\tau_{(1,2,3)}$
(0,0,1)	8	(1,2,3)	$\theta_{(1,2,3)}$
		(1,3,2)	$\theta_{(1,3,2)}$
		(2,1,3)	$\theta_{(2,1,3)}$
$\boldsymbol{\kappa} = (2, 1), \mathbf{f} = (1, 1, 0)$			
(0,0,0)	2	(1)(2,3)	$\tau_1\tau_{(2,3)}$
		(2)(1,3)	$\tau_2\tau_{(1,3)}$
		(3)(1,2)	$\tau_3\tau_{(1,2)}$
(0,1,0)	4	(1)(2,3)	$\tau_1\theta_{(2,3)}$
		(2)(1,3)	$\tau_2\theta_{(1,3)}$
		(3)(1,2)	$\tau_3\theta_{(1,2)}$
(1,0,0)	2	(1)(2,3)	$\theta_1\tau_{(2,3)}$
		(2)(1,3)	$\theta_2\tau_{(1,3)}$
		(3)(1,2)	$\theta_3\tau_{(2,3)}$
(1,1,0)	4	(1)(2,3)	$\theta_1\theta_{(2,3)}$
		(2)(1,3)	$\theta_2\theta_{(1,3)}$
		(3)(1,2)	$\theta_3\theta_{(1,2)}$
$\boldsymbol{\kappa} = (1, 1, 1), \mathbf{f} = (3, 0, 0)$			
(0,0,0)	1	(1)(2)(3)	$\tau_1\tau_2\tau_3$
(1,0,0)	1	(1)(2)(3)	$\theta_1\tau_2\tau_3$
		(2)(1)(3)	$\theta_2\tau_1\tau_3$
		(3)(1)(2)	$\theta_3\tau_1\tau_2$
(2,0,0)	1	(1)(2)(3)	$\theta_1\theta_2\tau_3$
		(1)(3)(2)	$\theta_1\theta_3\tau_2$
		(2)(3)(1)	$\theta_2\theta_3\tau_1$
(3,0,0)	1	(1)(2)(3)	$\theta_1\theta_2\theta_3$

#### 4. A Computationally Efficient Expression for $\mathbb{E} \left[ \prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z} \right]$

Although (2) and (19) present compact expressions for  $\mathbb{E} \left[ \prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z} \right]$  in the central and noncentral normal case, respectively, these forms are not computationally efficient. The main limitation is that (2) requires enumerating  $\mathcal{S}_\kappa$  for every  $\boldsymbol{\kappa} \vdash k$  and (19) requires enumerating  $\mathcal{S}_{f,\mathbf{g}}$  for every  $\mathbf{f}$  and  $\mathbf{g}$ . This generally involves checking all  $k!$  permutations

to determine admissibility, which becomes prohibitively expensive as  $k$  increases. To address this issue, we seek an alternative expression for  $E[\prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z}]$  that is computationally more efficient. We focus on the noncentral case ( $\boldsymbol{\mu} \neq \mathbf{0}$ ) as the central case follows in a similar fashion.

We first introduce the concept of a set partition. Let  $\mathcal{B}_k$  denote the set of all partitions of  $\{1, \dots, k\}$  into nonempty subsets (blocks). For each partition in  $\mathcal{B}_k$ , there is no significance attached to the ordering of its blocks, so without loss of generality, we arrange them in ascending order of block size. For example, when  $k = 3$ ,  $\mathcal{B}_3$  contains the following five partitions: (i) (1)(2)(3), (ii) (1)(2, 3), (iii) (2)(1, 3), (iv) (3)(1, 2), (v) (1, 2, 3). The number of partitions in  $\mathcal{B}_k$  is called the Bell number,  $B_k$ , and an efficient algorithm for generating the partitions is given by Nijenhuis and Wilf [19].

For a given  $\boldsymbol{\pi} = (\boldsymbol{\pi}_1, \dots, \boldsymbol{\pi}_r) \in \mathcal{B}_k$  with  $r$  blocks, let  $f_i$  denote the number of blocks containing  $i$  elements. Each block can be assigned to a  $\theta$  term or a  $\tau$  term. A block of size  $i$  can yield  $n_\theta(i) + n_\tau(i)$  different terms. Consequently, the total number of unique terms generated from all  $\boldsymbol{\pi} \in \mathcal{B}_k$  is

$$M(k) = \sum_{\boldsymbol{\pi} \in \mathcal{B}_k} \prod_{i=1}^k [n_\theta(i) + n_\tau(i)]^{f_i}, \quad (21)$$

which provides an alternative expression of  $M(k)$  given in (18).

We will introduce additional notation before proceeding. For a vector  $\boldsymbol{v}$  containing  $i$  distinct integers, let  $\boldsymbol{p}_j \equiv \boldsymbol{p}_j(\boldsymbol{v})$  for  $j = 1, \dots, n_\theta(i)$  denote the different rearrangements of the elements of  $\boldsymbol{v}$  that produce unique  $\theta$  terms, and let  $\boldsymbol{q}_j \equiv \boldsymbol{q}_j(\boldsymbol{v})$  for  $j = 1, \dots, n_\tau(i)$  denote different rearrangements of the elements of  $\boldsymbol{v}$  yielding unique  $\tau$  terms.<sup>1</sup>

**Example 4.** Suppose  $\boldsymbol{v} = (3, 7, 9)$ , which has three elements. We have  $n_\theta(3) = 3$  and we can create three different  $\tau$  terms, with subscripts  $\boldsymbol{p}_1 = (3, 7, 9)$ ,  $\boldsymbol{p}_2 = (3, 9, 7)$ , or  $\boldsymbol{p}_3 = (7, 3, 9)$ . On the other hand, we have  $n_\tau(3) = 1$  and there is only one possible  $\tau$  term, with subscripts  $\boldsymbol{q}_1 = (3, 7, 9)$ .

Using  $\boldsymbol{p}_j(\boldsymbol{v})$  and  $\boldsymbol{q}_j(\boldsymbol{v})$ , we define the sum of all possible distinct terms that can be constructed from  $\boldsymbol{v}$  as

$$\lambda_{\boldsymbol{v}} = 2^{I_{\ell(\boldsymbol{v})=2}} \sum_{j=1}^{n_\theta(\ell(\boldsymbol{v}))} \theta_{\boldsymbol{p}_j(\boldsymbol{v})} + \sum_{j=1}^{n_\tau(\ell(\boldsymbol{v}))} \tau_{\boldsymbol{q}_j(\boldsymbol{v})}, \quad (22)$$

where  $\ell(\boldsymbol{v})$  is the number of elements in  $\boldsymbol{v}$ . The factor  $2^{I_{\ell(\boldsymbol{v})=2}}$  for the  $\theta$  terms accounts for the  $2^{g_2}$  coefficient in the expansion of  $E[\prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z}]$  when the block has exactly two elements.

With  $\lambda_{\boldsymbol{v}}$  defined, we now present our computationally efficient expression.

**Proposition 2.** Suppose  $\mathbf{z} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  and  $\mathbf{A}_i$ ,  $i = 1, \dots, k$  are symmetric matrices. Then

$$E \left[ \prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z} \right] = \sum_{\boldsymbol{\pi} \in \mathcal{B}_k} 2^{k-f_1-f_2} \prod_{i=1}^{\ell(\boldsymbol{\pi})} \lambda_{\boldsymbol{\pi}_i}, \quad (23)$$

where  $f_1$  and  $f_2$  are the number of blocks of  $\boldsymbol{\pi}$  that have one and two elements, respectively, and  $\lambda_{\boldsymbol{\pi}_i}$  is defined in (22).

For the central case ( $\boldsymbol{\mu} = \mathbf{0}$ ), the  $\theta$  terms in  $\lambda_{\boldsymbol{\pi}_i}$  vanish, so  $\lambda_{\boldsymbol{\pi}_i}$  reduces to the sum of possible  $\tau$  terms that be generated from  $\boldsymbol{\pi}_i$ .

(23) is preferred over (19) because (19) requires examining all  $k!$  permutations to determine admissibility for each  $(\boldsymbol{f}, \boldsymbol{g})$ , whereas (23) only requires enumerating  $\boldsymbol{\pi} \in \mathcal{B}_k$  once and generating  $\lambda_{\boldsymbol{\pi}_i}$  from the blocks of  $\boldsymbol{\pi}$ . Since the bell number  $B_k$  is much smaller than  $k!$  (for instance,  $B_{10} = 115,975$  while  $10! = 3,628,800$ ), the efficiency gain from using (23) is substantial, especially for large  $k$ .

<sup>1</sup>An efficient algorithm for generating bracelets with fixed contents is given by Karim, Sawada, Alamgir, and Husnine [10]. Their algorithm can be adapted to generate the indices for unique  $\theta_{\boldsymbol{v}}$ .

We revisit the case  $k = 3$  with the new formula (23). Table 5 lists all the terms that are necessary to construct  $E[\prod_{i=1}^3 \mathbf{z}^T \mathbf{A}_i \mathbf{z}]$ . From this, we obtain

$$\begin{aligned} E\left[\prod_{i=1}^3 \mathbf{z}^T \mathbf{A}_i \mathbf{z}\right] &= (\theta_1 + \tau_1)(\theta_2 + \tau_2)(\theta_3 + \tau_3) + 2(\theta_1 + \tau_1)[2\theta_{(2,3)} + \tau_{(2,3)}] \\ &\quad + 2(\theta_2 + \tau_2)[2\theta_{(1,3)} + \tau_{(1,3)}] + 2(\theta_3 + \tau_3)[2\theta_{(1,2)} + \tau_{(1,2)}] \\ &\quad + 8[\theta_{(1,2,3)} + \theta_{(1,3,2)} + \theta_{(2,1,3)} + \tau_{(1,2,3)}]. \end{aligned}$$

Expanding this yields an expression that is identical to (20), but obtained through a more efficient procedure. For large  $k$ , the computational savings from using (23) over (19) are dramatic.<sup>2</sup>

Table 5: Construction of an explicit expression of  $E[\prod_{i=1}^3 \mathbf{z}^T \mathbf{A}_i \mathbf{z}]$  when  $\mathbf{z} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ , using (23)

$\boldsymbol{\pi} \in \mathcal{B}_3$	$2^{k-f_1-f_2}$	$\lambda_{\boldsymbol{\pi}_1}$	$\lambda_{\boldsymbol{\pi}_2}$	$\lambda_{\boldsymbol{\pi}_3}$
(1)(2)(3)	1	$\theta_1 + \tau_1$	$\theta_2 + \tau_2$	$\theta_3 + \tau_3$
(1)(2,3)	2	$\theta_1 + \tau_1$	$2\theta_{(2,3)} + \tau_{(2,3)}$	—
(2)(1,3)	2	$\theta_2 + \tau_2$	$2\theta_{(1,3)} + \tau_{(1,3)}$	—
(3)(1,2)	2	$\theta_3 + \tau_3$	$2\theta_{(1,2)} + \tau_{(1,2)}$	—
(1,2,3)	8	$\theta_{(1,2,3)} + \theta_{(1,3,2)} + \theta_{(2,1,3)} + \tau_{(1,2,3)}$	—	—

## 5. Extension and Application

In this section, we show that our results can also be applied to obtain explicit expressions for the product moments of real and complex central and noncentral Wishart distributions. Before doing so, we first extend the results from previous sections to obtain an explicit expression for the expectation of a product of Hermitian quadratic forms in multivariate complex normal random variables.

The multivariate complex normal random distribution was introduced by Wooding [23]. Let  $\mathbf{z} = \mathbf{z}_r + i\mathbf{z}_c$  be a complex  $m$ -vector, where  $\mathbf{z}_r = \Re(\mathbf{z})$ ,  $\mathbf{z}_c = \Im(\mathbf{z})$ , and  $i = \sqrt{-1}$ . Define  $\tilde{\boldsymbol{\mu}} = [\boldsymbol{\mu}_r^T, \boldsymbol{\mu}_c^T]^T$  as a real vector and

$$\tilde{\boldsymbol{\Sigma}} = \frac{1}{2} \begin{bmatrix} \boldsymbol{\Sigma}_r & -\boldsymbol{\Sigma}_c \\ \boldsymbol{\Sigma}_c & \boldsymbol{\Sigma}_r \end{bmatrix},$$

where  $\boldsymbol{\Sigma}_r$  is a real symmetric matrix and  $\boldsymbol{\Sigma}_c$  is a real skew-symmetric matrix. It follows that  $\tilde{\boldsymbol{\Sigma}}$  is symmetric and we assume it is positive semidefinite. If  $\tilde{\mathbf{z}} = [\mathbf{z}_r^T, \mathbf{z}_c^T]^T \sim \mathcal{N}(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}})$ , then we say  $\mathbf{z} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  follows a (circular) multivariate complex normal distribution with mean  $\boldsymbol{\mu} = \boldsymbol{\mu}_r + i\boldsymbol{\mu}_c$  and covariance matrix  $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_r + i\boldsymbol{\Sigma}_c$ . For a Hermitian matrix  $\mathbf{A} = \mathbf{A}_r + i\mathbf{A}_c$ , where  $\mathbf{A}_r$  is a real symmetric matrix and  $\mathbf{A}_c$  is a real skew-symmetric matrix, we define a Hermitian quadratic form in  $\mathbf{z}$  as  $\mathbf{z}^H \mathbf{A} \mathbf{z}$ , where  $H$  denotes the conjugate transpose. We seek an explicit expression for  $E[\prod_{i=1}^k \mathbf{z}^H \mathbf{A}_i \mathbf{z}]$ , where  $\mathbf{A}_i$ ,  $i = 1, \dots, k$  are Hermitian matrices. This allows us to extend our results for the real case in previous sections to the complex case.

It is straightforward to show that  $\mathbf{z}^H \mathbf{A}_i \mathbf{z}$  is real, and in fact, equals  $\tilde{\mathbf{z}}^T \tilde{\mathbf{A}}_i \tilde{\mathbf{z}}$ , where

$$\tilde{\mathbf{A}}_i = \begin{bmatrix} \mathbf{A}_{ir} & -\mathbf{A}_{ic} \\ \mathbf{A}_{ic} & \mathbf{A}_{ir} \end{bmatrix},$$

is a real symmetric matrix with  $\mathbf{A}_{ir} = \Re(\mathbf{A}_i)$  and  $\mathbf{A}_{ic} = \Im(\mathbf{A}_i)$ . This implies that

$$E\left[\prod_{i=1}^k \mathbf{z}^H \mathbf{A}_i \mathbf{z}\right] = E\left[\prod_{i=1}^k \tilde{\mathbf{z}}^T \tilde{\mathbf{A}}_i \tilde{\mathbf{z}}\right]$$

<sup>2</sup>A set of Matlab programs were written to compute  $E[\prod_{i=1}^k \mathbf{z}^T \mathbf{A}_i \mathbf{z}]$  by using various algorithms in this paper. These programs can be downloaded at <https://www-2.rotman.utoronto.ca/~kan/research.htm>.

is simply an expectation of a product of quadratic forms in real multivariate normal random variables. Therefore, the formulae in (19) or (23) can be applied directly. The only modification is that  $\theta_v$  and  $\tau_v$  must be replaced by

$$\tilde{\theta}_v = \tilde{\boldsymbol{\mu}}^\top \tilde{\mathbf{A}}_{v_1} \tilde{\boldsymbol{\Sigma}} \tilde{\mathbf{A}}_{v_2} \tilde{\boldsymbol{\Sigma}} \cdots \tilde{\boldsymbol{\Sigma}} \tilde{\mathbf{A}}_{v_r} \tilde{\boldsymbol{\mu}}, \quad (24)$$

$$\tilde{\tau}_v = \text{tr}(\tilde{\mathbf{A}}_{v_1} \tilde{\boldsymbol{\Sigma}} \tilde{\mathbf{A}}_{v_2} \tilde{\boldsymbol{\Sigma}} \cdots \tilde{\mathbf{A}}_{v_r} \tilde{\boldsymbol{\Sigma}}). \quad (25)$$

Nevertheless, one may wish to express  $E[\prod_{i=1}^k z^H \mathbf{A}_i z]$  directly in terms of  $\boldsymbol{\mu}$ ,  $\boldsymbol{\Sigma}$  and  $\mathbf{A}_i$  without explicitly forming the augmented real representation. The following lemma provides the necessary information for obtaining such an expression.

**Lemma 1.** *Let  $\mathbf{v} = [v_1, \dots, v_p]$  be a vector of  $p$  positive integers, and define*

$$\theta_v = \Re(\boldsymbol{\mu}^H \mathbf{A}_{v_1} \boldsymbol{\Sigma} \mathbf{A}_{v_2} \boldsymbol{\Sigma} \cdots \boldsymbol{\Sigma} \mathbf{A}_{v_p} \boldsymbol{\mu}), \quad (26)$$

$$\tau_v = \Re(\text{tr}(\mathbf{A}_{v_1} \boldsymbol{\Sigma} \mathbf{A}_{v_2} \boldsymbol{\Sigma} \cdots \boldsymbol{\Sigma} \mathbf{A}_{v_p} \boldsymbol{\Sigma})), \quad (27)$$

then

$$\tilde{\theta}_v = \frac{\theta_v}{2^{p-1}}, \quad (28)$$

$$\tilde{\tau}_v = \frac{\tau_v}{2^{p-1}}. \quad (29)$$

*Proof.* For two complex square matrices  $\mathbf{A} = \mathbf{A}_r + i\mathbf{A}_c$  and  $\mathbf{B} = \mathbf{B}_r + i\mathbf{B}_c$ , the real and imaginary parts of  $\mathbf{C} = \mathbf{A}\mathbf{B}$  are

$$\mathbf{C}_r = \Re(\mathbf{C}) = \mathbf{A}_r \mathbf{B}_r - \mathbf{A}_c \mathbf{B}_c,$$

$$\mathbf{C}_c = \Im(\mathbf{C}) = \mathbf{A}_r \mathbf{B}_c + \mathbf{A}_c \mathbf{B}_r.$$

The augmented real representations of  $\mathbf{A}$  and  $\mathbf{B}$  are

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_r & -\mathbf{A}_c \\ \mathbf{A}_c & \mathbf{A}_r \end{bmatrix},$$

$$\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_r & -\mathbf{B}_c \\ \mathbf{B}_c & \mathbf{B}_r \end{bmatrix}.$$

It is straightforward to verify that

$$\tilde{\mathbf{C}} = \tilde{\mathbf{A}}\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{C}_r & -\mathbf{C}_c \\ \mathbf{C}_c & \mathbf{C}_r \end{bmatrix}.$$

By induction, if

$$\mathbf{C} = \mathbf{A}_{v_1} \boldsymbol{\Sigma} \mathbf{A}_{v_2} \cdots \boldsymbol{\Sigma} \mathbf{A}_{v_r} \equiv \mathbf{C}_r + i\mathbf{C}_c,$$

with  $\mathbf{C}_r = \Re(\mathbf{C})$  and  $\mathbf{C}_c = \Im(\mathbf{C})$ , then

$$\tilde{\mathbf{A}}_{v_1} \tilde{\boldsymbol{\Sigma}} \tilde{\mathbf{A}}_{v_2} \cdots \tilde{\boldsymbol{\Sigma}} \tilde{\mathbf{A}}_{v_r} = \frac{1}{2^r} \begin{bmatrix} \mathbf{C}_r & -\mathbf{C}_c \\ \mathbf{C}_c & \mathbf{C}_r \end{bmatrix}.$$

Consequently,

$$\tilde{\tau}_v = \text{tr}(\tilde{\mathbf{A}}_{v_1} \tilde{\boldsymbol{\Sigma}} \tilde{\mathbf{A}}_{v_2} \cdots \tilde{\boldsymbol{\Sigma}} \tilde{\mathbf{A}}_{v_r}) = \frac{2\text{tr}(\mathbf{C}_r)}{2^r} = \frac{\Re(\text{tr}(\mathbf{C}_r + i\mathbf{C}_c))}{2^{r-1}} = \frac{\tau_v}{2^{r-1}}.$$

Using the fact that

$$\boldsymbol{\mu}^H \mathbf{A}_{v_1} \boldsymbol{\Sigma} \cdots \boldsymbol{\Sigma} \mathbf{A}_{v_r} \boldsymbol{\mu} = \text{tr}(\mathbf{A}_{v_1} \boldsymbol{\Sigma} \cdots \boldsymbol{\Sigma} \mathbf{A}_{v_r} \boldsymbol{\mu} \boldsymbol{\mu}^H),$$

and applying the same argument yields (28). □

With the use of Lemma 1, we obtain the following proposition.

**Proposition 3.** Suppose  $\mathbf{z} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  and  $\mathbf{A}_i$ ,  $i = 1, \dots, k$  are Hermitian matrices. Then

$$\mathbb{E} \left[ \prod_{i=1}^k \mathbf{z}^H \mathbf{A}_i \mathbf{z} \right] = \sum_{\kappa=k} \sum_{0 \leq g \leq f} 2^{\ell(\kappa) - f_1 - f_2 + g_2} \sum_{\pi \in S_{f,g}} \eta_{f,g,\pi}, \quad (30)$$

where each  $\eta_{f,g,\pi}$  denotes a unique product of  $\ell(\kappa)$  terms of  $\theta$  and  $\tau$  as defined in (11). Alternatively, a computationally efficient form is

$$\mathbb{E} \left[ \prod_{i=1}^k \mathbf{z}^H \mathbf{A}_i \mathbf{z} \right] = \sum_{\pi \in \mathcal{B}_k} 2^{\ell(\pi) - f_1 - f_2} \prod_{i=1}^{\ell(\pi)} \lambda_{\pi_i}, \quad (31)$$

where  $\ell(\pi)$  is the number of blocks in  $\pi$  and  $\lambda_{\pi_i}$  is defined in (22).

There are two modifications required when extending the results from the real case to the complex case. The first is to replace  $2^k$  with  $2^{\ell(\kappa)}$  or  $2^{\ell(\pi)}$ , depending on whether (19) or (23) is being used. The second modification is more subtle: in defining  $\theta_\nu$  and  $\lambda_\nu$ , one must take the real part when  $\mathbf{z} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ . Failing to do so would yield an expression with a nonzero imaginary component, and thus an incorrect result. For instance, Sultan and Tracy [22, Eq. 5.3] claimed that when  $k = 3$  and  $\mathbf{z} \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{\Sigma})$ ,<sup>3</sup>

$$\begin{aligned} \mathbb{E} \left[ \prod_{i=1}^3 (\mathbf{z}^H \mathbf{A}_i \mathbf{z}) \right] &= 2\text{tr}(\mathbf{A}_1 \boldsymbol{\Sigma} \mathbf{A}_2 \boldsymbol{\Sigma} \mathbf{A}_3 \boldsymbol{\Sigma}) + \text{tr}(\mathbf{A}_1 \boldsymbol{\Sigma}) \text{tr}(\mathbf{A}_2 \boldsymbol{\Sigma} \mathbf{A}_3 \boldsymbol{\Sigma}) + \text{tr}(\mathbf{A}_2 \boldsymbol{\Sigma}) \text{tr}(\mathbf{A}_3 \boldsymbol{\Sigma} \mathbf{A}_1 \boldsymbol{\Sigma}) \\ &\quad + \text{tr}(\mathbf{A}_3 \boldsymbol{\Sigma}) \text{tr}(\mathbf{A}_1 \boldsymbol{\Sigma} \mathbf{A}_2 \boldsymbol{\Sigma}) + \text{tr}(\mathbf{A}_1 \boldsymbol{\Sigma}) \text{tr}(\mathbf{A}_2 \boldsymbol{\Sigma}) \text{tr}(\mathbf{A}_3 \boldsymbol{\Sigma}). \end{aligned} \quad (32)$$

For our expression, it is necessary to take the real part of all trace terms. For trace terms involving only one or two  $\mathbf{A}$  matrices, the imaginary part vanishes, so taking the real part is inconsequential in those cases.<sup>4</sup> However, for the term

$$\text{tr}(\mathbf{A}_1 \boldsymbol{\Sigma} \mathbf{A}_2 \boldsymbol{\Sigma} \mathbf{A}_3 \boldsymbol{\Sigma}),$$

the imaginary part is generally nonzero, and thus (32) does not yield the correct result.<sup>5</sup> An alternative expression can be obtained from Theorem 1 of Graczyk, Letac, and Massam [5], by setting  $p = 1$  in their expression, which gives

$$\begin{aligned} \mathbb{E} \left[ \prod_{i=1}^3 (\mathbf{z}^H \mathbf{A}_i \mathbf{z}) \right] &= \text{tr}(\mathbf{A}_1 \boldsymbol{\Sigma} \mathbf{A}_2 \boldsymbol{\Sigma} \mathbf{A}_3 \boldsymbol{\Sigma}) + \text{tr}(\mathbf{A}_1 \boldsymbol{\Sigma} \mathbf{A}_3 \boldsymbol{\Sigma} \mathbf{A}_2 \boldsymbol{\Sigma}) + \text{tr}(\mathbf{A}_1 \boldsymbol{\Sigma}) \text{tr}(\mathbf{A}_2 \boldsymbol{\Sigma} \mathbf{A}_3 \boldsymbol{\Sigma}) \\ &\quad + \text{tr}(\mathbf{A}_2 \boldsymbol{\Sigma}) \text{tr}(\mathbf{A}_1 \boldsymbol{\Sigma} \mathbf{A}_3 \boldsymbol{\Sigma}) + \text{tr}(\mathbf{A}_3 \boldsymbol{\Sigma}) \text{tr}(\mathbf{A}_1 \boldsymbol{\Sigma} \mathbf{A}_2 \boldsymbol{\Sigma}) + \text{tr}(\mathbf{A}_1 \boldsymbol{\Sigma}) \text{tr}(\mathbf{A}_2 \boldsymbol{\Sigma}) \text{tr}(\mathbf{A}_3 \boldsymbol{\Sigma}). \end{aligned} \quad (33)$$

Here, the imaginary parts of the first two terms cancel, so no explicit real-part operation is required, and (33) provides the correct answer.

Nevertheless, (33) involves calculating one more trace term than our approach. More importantly, for general  $k$ , the formula of Graczyk, Letac, and Massam [5] contains  $k!$  terms whereas our expression yields only  $N(k)$  terms in the central case — where  $N(k)$ , as listed in Table 1, is significantly smaller than  $k!$ , especially for large  $k$ .

With our main results extended to the complex case, we turn to an important application. Let  $\mathbf{z}_t \sim \mathcal{N}(\boldsymbol{\mu}_t, \boldsymbol{\Sigma})$  for  $t = 1, \dots, n$ , and they are independent across  $t$ . We say

$$\mathbf{W} = \sum_{t=1}^n \mathbf{z}_t \mathbf{z}_t^T \sim \mathcal{W}_m(n, \boldsymbol{\Sigma}, \boldsymbol{\Lambda})$$

follows a noncentral real Wishart distribution with  $n$  degrees of freedom, covariance matrix of  $\boldsymbol{\Sigma}$  and noncentrality matrix of  $\boldsymbol{\Lambda} = \sum_{t=1}^n \boldsymbol{\mu}_t \boldsymbol{\mu}_t^T$ . In standard notation, the noncentrality matrix is  $\boldsymbol{\Omega} = \boldsymbol{\Sigma}^{-1} \boldsymbol{\Lambda}$ . We prefer to use the triplet

<sup>3</sup>Their Eq.(5.2) for  $\mathbb{E}[(\mathbf{z}^H \mathbf{A}_1 \mathbf{z})(\mathbf{z}^H \mathbf{A}_2 \mathbf{z})]$  has an obvious typo. The coefficient for the first term ought to be 1 rather than 2.

<sup>4</sup>For the case when  $\boldsymbol{\mu} \neq \mathbf{0}$ , the term  $\boldsymbol{\mu}^H \mathbf{A}_1 \boldsymbol{\Sigma} \mathbf{A}_2 \boldsymbol{\mu}$  in general has nonzero imaginary part.

<sup>5</sup>When  $k \geq 4$  for the noncentral case, or  $k \geq 6$  for the central case, even the real part of the expression is incorrect if we do not take the real parts of  $\theta$  and  $\tau$  as in (26) and (27).

$(n, \Sigma, \Lambda)$  to characterize the noncentral Wishart distribution because it allows for the case in which  $\Sigma$  is positive semidefinite and  $\Sigma^{-1}$  may not exist. When  $\mu_t = \mathbf{0}$ ,  $\mathbf{W}$  is said to have a central real Wishart distribution.

The definition for complex noncentral Wishart is similar. Suppose  $z_t \sim \mathcal{CN}(\mu, \Sigma)$  for  $t = 1, \dots, n$ , and they are independent across  $t$ . We say

$$\mathbf{W} = \sum_{t=1}^n z_t z_t^H \sim \mathcal{CW}_m(n, \Sigma, \Lambda)$$

follows a noncentral complex Wishart distribution with  $n$  degrees of freedom, covariance matrix of  $\Sigma$  and noncentrality matrix of  $\Lambda = \sum_{t=1}^n \mu_t \mu_t^H$ .

Our goal is to obtain an explicit expression for

$$\mathbb{E} \left[ \prod_{i=1}^k \text{tr}(\mathbf{W} \mathbf{A}_i) \right], \quad (34)$$

where  $\mathbf{W}$  follows a noncentral real or complex Wishart distribution, and  $\mathbf{A}_i$ ,  $i = 1, \dots, k$  are symmetric matrices in the real case and Hermitian matrices in the complex case. With an appropriate choice of  $\mathbf{A}$ , i.e.,  $\mathbf{A} = (\mathbf{e}_i \mathbf{e}_j^T + \mathbf{e}_j \mathbf{e}_i^T)/2$ , where  $\mathbf{e}_i$  is the  $i$ -th column of  $\mathbf{I}_m$ , we have  $\text{tr}(\mathbf{W} \mathbf{A}) = w_{i,j}$ , which is the  $(i, j)$ -th element of  $\mathbf{W}$ . Therefore, having the expression for (34) will allow us to directly obtain an explicit expression for the product moments of the elements of  $\mathbf{W}$ .

Our key observation is that  $\text{tr}(\mathbf{W} \mathbf{A}_i)$  can be written as a quadratic form in multivariate normal random variables. We demonstrate this for the real case. Let

$$\mathbf{y} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \quad \mathbf{m} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}.$$

Then  $\mathbf{y} \sim \mathcal{N}(\mathbf{m}, \mathbf{I}_n \otimes \Sigma)$ , where  $\otimes$  denotes the Kronecker product. We have

$$\text{tr}(\mathbf{W} \mathbf{A}_i) = \text{tr} \left( \sum_{t=1}^n z_t z_t^T \mathbf{W} \right) = \sum_{t=1}^n \text{tr}(z_t z_t^T \mathbf{W}) = \sum_{t=1}^n z_t^T \mathbf{A}_i z_t = \mathbf{y}^T (\mathbf{I}_n \otimes \mathbf{A}_i) \mathbf{y},$$

which a quadratic form in  $\mathbf{y}$ . Therefore, we can apply either (19) or (23) to obtain an explicit expression for (34), with the following adjustments to  $\theta_v$  and  $\tau_v$ . For  $\tau_v$ ,

$$\begin{aligned} \tau_v &= \text{tr}((\mathbf{I}_n \otimes \mathbf{A}_{v_1})(\mathbf{I}_n \otimes \Sigma) \cdots (\mathbf{I}_n \otimes \mathbf{A}_{v_r})(\mathbf{I}_n \otimes \Sigma)) \\ &= \text{tr}(\mathbf{I}_n \otimes \mathbf{A}_{v_1} \Sigma \cdots \mathbf{A}_{v_r} \Sigma) \\ &= n \text{tr}(\mathbf{A}_{v_1} \Sigma \cdots \mathbf{A}_{v_r} \Sigma), \end{aligned} \quad (35)$$

so  $\tau_v$  is multiplied by  $n$  compared to the original definition. For  $\theta_v$ , let  $\mathbf{m} = \text{vec}(\mathbf{M})$ , where  $\mathbf{M} = [\mu_1, \dots, \mu_n]$ . Then

$$\begin{aligned} \theta_v &= \mathbf{m}^T (\mathbf{I}_n \otimes \mathbf{A}_{v_1})(\mathbf{I}_n \otimes \Sigma) \cdots (\mathbf{I}_n \otimes \Sigma)(\mathbf{I}_n \otimes \mathbf{A}_{v_r}) \mathbf{m} \\ &= \text{vec}(\mathbf{M})^T (\mathbf{I}_n \otimes \mathbf{A}_{v_1} \Sigma \cdots \Sigma \mathbf{A}_{v_r}) \text{vec}(\mathbf{M}) \\ &= \text{vec}(\mathbf{M})^T \text{vec}(\mathbf{A}_{v_1} \Sigma \cdots \Sigma \mathbf{A}_{v_r} \mathbf{M} \mathbf{I}_n) \\ &= \text{tr}(\mathbf{A}_{v_1} \Sigma \cdots \Sigma \mathbf{A}_{v_r} \mathbf{M} \mathbf{M}^T) \\ &= \text{tr}(\mathbf{A}_{v_1} \Sigma \cdots \Sigma \mathbf{A}_{v_r} \Lambda). \end{aligned} \quad (36)$$

For the complex Wishart case, an explicit expression for (34) can be computed analogously using (30) or (31), with  $\tau_v$  and  $\theta_v$  defined as the real parts in (35) and (36).

For the central Wishart distribution, explicit formulae for (34) are given in Graczyk, Letac, and Massam [5] for the complex case, and in Graczyk, Letac, and Massam [6] for the real case. For the noncentral Wishart distribution, Letac and Massam [13] describe a method for obtaining an analytical expression for (34).<sup>6</sup> They demonstrate the use of their method for  $k = 1, 2$ , and  $3$ . However, their reported expression for  $k = 3$  (in p.1405) is incorrect and has six missing terms. This indicates the difficulty of using their method to obtain an explicit expression of (34), especially for large  $k$ . Di Nardo [2] suggested a symbolic method to obtain an explicit expression for (34) in the complex case. A Maple program was provided to generate the analytical expression. Unfortunately, the program fails to produce the correct answer. As a result, the explicit expression for  $E[\text{tr}(\mathbf{W}\mathbf{A}_1)\text{tr}(\mathbf{W}\mathbf{A}_2)^2]$  as given in her example 3 is incorrect. Using our framework for expectations of products of quadratic forms in multivariate normal random variables, we can easily obtain explicit expressions for (34) in both the real and the complex cases. Moreover, our expression is more compact. For the central case, the formulae in Graczyk, Letac, and Massam [5] and Graczyk, Letac, and Massam [6] have  $k!$  terms whereas ours require  $N(k)$  terms. For the noncentral case, the method of Letac and Massam [13] produces 34 terms for  $k = 3$  and 209 terms for  $k = 4$ , while ours requires  $M(k)$  terms, with  $M(3) = 24$  and  $M(4) = 123$ .

## 6. Explicit Expression for $E\left[\prod_{i=1}^k (z^T \mathbf{A}_i z)^{s_i}\right]$ when $z \sim \mathcal{N}(\mu, \Sigma)$

In this section, we provide an explicit expression for  $E[\prod_{i=1}^k (z^T \mathbf{A}_i z)^{s_i}]$ , where  $s_i, i = 1, \dots, k$  are positive integers. Consider a multiset containing  $s_1$  copies of 1,  $s_2$  copies of 2,  $\dots$ , and  $s_k$  copies of  $k$ . We define the multiplicity of this multiset as  $\mathbf{s} = [s_1, \dots, s_k]$  and its size as  $|\mathbf{s}| = \sum_{i=1}^k s_i$ . A multiset partition is a way of partitioning this multiset into nonempty subsets (blocks). We denote by  $\mathcal{B}_{\mathbf{s}}$  the collection of all such multiset partitions for a given  $\mathbf{s}$ . For example, when  $\mathbf{s} = [2, 2]$ , i.e., the multiset has two 1's and two 2's, there are nine possible multiset partitions: (i) (1)(1)(2)(2) (ii) (1)(1)(2,2), (iii) (1)(2)(1,2), (iv) (2)(2)(1,1), (v) (1,1)(2,2), (vi) (1,2)(1,2), (vii) (1)(1,2,2), (viii) (2)(1,1,2), (ix) (1,1,2,2). For a given  $\mathbf{s}$ , let  $p(s_1, \dots, s_k)$  denote the number of possible multiset partitions. Explicit and recurrence formulae for  $p(s_1, \dots, s_k)$  can be found in MacMahon [14] and Gutpa [7], respectively. Note that for the special case  $s_1 = \dots = s_k = 1$ , we have  $\mathcal{B}_{\mathbf{s}} = \mathcal{B}_k$  and  $p(1, \dots, 1) = B_k$  (the Bell number).

For our purpose, it is essential to have an algorithm to enumerate all multiset partitions in  $\mathcal{B}_{\mathbf{s}}$ . Fortunately, Knuth [11] (Algorithm M, pp.429–430) provides an efficient algorithm for generating  $\mathcal{B}_{\mathbf{s}}$ . For  $\pi \in \mathcal{B}_{\mathbf{s}}$  with  $r$  blocks, we write  $\pi = (\pi_1, \dots, \pi_r)$ . Unlike the partitions in  $\mathcal{B}_k$  where all blocks are distinct, some blocks of  $\pi \in \mathcal{B}_{\mathbf{s}}$  may be repeated. For example, in the above case of  $\mathbf{s} = [2, 2]$ , the multiset partitions (i), (ii), (iv) and (vi) contain repeated blocks. To account for repetitions, we define  $\tilde{\pi}_i, i = 1, \dots, r'$  as the set of unique blocks in  $\pi$  and  $h_i, i = 1, \dots, r'$  as their corresponding frequencies.

We take the following steps to obtain an explicit expression for  $E[\prod_{i=1}^k (z^T \mathbf{A}_i z)^{s_i}]$ :

1. Expansion of the number of matrices. Expand the list  $\mathbf{A}_i, i = 1, \dots, k$  into  $\tilde{\mathbf{A}}_i$  for  $i = 1 \dots, |\mathbf{s}|$ , where

$$\tilde{\mathbf{A}}_1 = \dots = \tilde{\mathbf{A}}_{s_1} = \mathbf{A}_1, \quad \tilde{\mathbf{A}}_{s_1+1} = \dots = \tilde{\mathbf{A}}_{s_1+s_2} = \mathbf{A}_2, \quad \dots, \quad \tilde{\mathbf{A}}_{\sum_{j=1}^{k-1} s_j+1} = \dots = \tilde{\mathbf{A}}_{|\mathbf{s}|} = \mathbf{A}_k.$$

2. Application of (23). Using (23), we write

$$E\left[\prod_{i=1}^k (z^T \mathbf{A}_i z)^{s_i}\right] = E\left[\prod_{i=1}^{|\mathbf{s}|} z^T \tilde{\mathbf{A}}_i z\right] = \sum_{\pi' \in \mathcal{B}_{|\mathbf{s}|}} 2^{|\mathbf{s}| - f_1 - f_2} \prod_{i=1}^{\ell(\pi')} \tilde{\lambda}_{\pi'_i}, \quad (37)$$

where  $\tilde{\lambda}_{\pi'_i}$  represents the sum of possible  $\theta$  and  $\tau$  terms generated from block  $\pi'_i$ , and  $\theta$  terms associated with a block of exactly two elements are multiplied by 2. The notation  $\tilde{\lambda}$  emphasizes that the  $\theta$  and  $\tau$  terms in the sum are constructed from  $\tilde{\mathbf{A}}_i$ .

3. Avoiding duplicates. In (37), not every  $\pi' \in \mathcal{B}_{|\mathbf{s}|}$  yields a distinct term, since some of the  $\tilde{\mathbf{A}}_i$ 's are repeated. To avoid duplicated terms, we generate  $\pi$  from  $\mathcal{B}_{\mathbf{s}}$  instead of from  $\mathcal{B}_{|\mathbf{s}|}$ . However, we must determine, for each  $\pi \in \mathcal{B}_{\mathbf{s}}$ , how many  $\pi' \in \mathcal{B}_{|\mathbf{s}|}$  have the same block membership as  $\pi$ .

<sup>6</sup>Although Letac and Massam [13] considered only the real Wishart case, their results are equally applicable for the complex Wishart case. In their notation, the result for the real case is obtained by setting  $p = n/2$  and  $\sigma = 2\Sigma$ , and the result for the complex case is obtained by setting  $p = n$  and  $\sigma = \Sigma$ .

**Example 5.** Suppose  $s_1 = s_2 = 2$ . Table 6 illustrates the mapping between  $\pi \in \mathcal{B}_{(2,2)}$  and  $\pi' \in \mathcal{B}_4$ . We observe that  $\mathcal{B}_4$  contains 15 set partitions, whereas  $\mathcal{B}_{(2,2)}$  contains only 9 multiset partitions. Consequently, some multiset partitions in  $\mathcal{B}_{(2,2)}$  correspond to multiple set partitions in  $\mathcal{B}_4$ . For instance, consider

$$\pi = (1)(2)(1, 2),$$

which has  $A_1$  in the first block,  $A_2$  in the second block, and both  $A_1$  and  $A_2$  in the third block. This  $\pi$  corresponds to four distinct  $\pi' \in \mathcal{B}_4$ . The first is

$$(1)(3)(2, 4),$$

where  $\tilde{A}_1$  is in the first block,  $\tilde{A}_3$  in the second, and  $\tilde{A}_2$  and  $\tilde{A}_4$  in the third. Since  $\tilde{A}_1 = \tilde{A}_2 = A_1$  and  $\tilde{A}_3 = \tilde{A}_4 = A_2$ , the block contents of  $\pi' = (1)(3)(2, 4)$  are identical to those of  $\pi = (1)(2)(1, 2)$ . Similarly, the other three set partitions

$$(1)(4)(2, 3), \quad (2)(3)(1, 4), \quad (2)(4)(1, 3)$$

also have the same block membership as  $\pi = (1)(2)(1, 2)$ .

Table 6: Mapping between  $\mathcal{B}_{(2,2)}$  and  $\mathcal{B}_4$

$\pi \in \mathcal{B}_{(2,2)}$	$\pi' \in \mathcal{B}_4$
(1)(1)(2)(2)	(1)(2)(3)(4)
(1)(1)(2,2)	(1)(2)(3,4)
(1)(2)(1,2)	(1)(3)(2,4)
	(1)(4)(2,3)
	(2)(3)(1,4)
	(2)(4)(1,3)
(2)(2)(1,1)	(3)(4)(1,2)
(1,1)(2,2)	(1,2)(3,4)
(1,2)(1,2)	(1,3)(2,4)
	(1,4)(2,3)
(1)(1,2,2)	(1)(2,3,4)
	(2)(1,3,4)
(2)(1,1,2)	(3)(1,2,4)
	(4)(1,2,3)
(1,1,2,2)	(1,2,3,4)

In the following lemma, we give an explicit expression of the multiplicity of a multiset partition in  $\mathcal{B}_{|s|}$ .

**Lemma 2.** For a given multiset partition  $\pi = (\pi_1, \dots, \pi_r) \in \mathcal{B}_s$ , let  $c_s(\pi)$  denote the number of set partitions in  $\mathcal{B}_{|s|}$  that have the same block contents as  $\pi$ . An explicit expression of  $c_s(\pi)$  is given by

$$c_s(\pi) = \frac{1}{\prod_{i=1}^{r'} h_i!} \prod_{j=1}^k \frac{s_j!}{\prod_{i=1}^{r'} b_{i,j}!}, \quad (38)$$

where  $b_{i,j}$  is the number of occurrences of integer  $j$  in block  $\pi_i$ , and  $h_i$ ,  $i = 1, \dots, r'$  are the multiplicities of the  $r'$  unique blocks in  $\pi$ .

*Proof.* Note that in  $\pi$ , the integer  $j$  appears exactly  $s_j$  times, with the restriction that  $b_{i,j}$  of them are placed in the  $i$ -th block of  $\pi$ ,  $i = 1, \dots, r'$ . If these  $s_j$  instances of  $j$  were distinct elements, then the number of ways to arrange them into blocks with the prescribed multiplicities would be

$$\frac{s_j!}{\prod_{i=1}^{r'} b_{i,j}!}.$$

Repeating this for all  $k$  integers, the number of set partitions  $\pi' \in \mathcal{B}_{|S|}$  with the same block contents as  $\pi$  would be

$$\prod_{j=1}^k \frac{s_j!}{\prod_{i=1}^{r'} b_{i,j}!}.$$

However, if  $\pi$  contains repeated blocks, some of these set partitions will be identical. To account for over-counting, we need to divide the above expression by  $\prod_{i=1}^{r'} h_i!$ , where  $r'$  is the number of distinct blocks in  $\pi$ , and  $h_i$ 's are multiplicities of the unique blocks of  $\pi$ . This yields (38).  $\square$

**Example 6.** We illustrate the calculation of  $c_s(\pi)$  using the example in Table 6. For the multiset partition  $\pi = (1)(1)(2)(2)$ , there are two unique blocks (i.e.,  $r' = 2$ ) and each appearing twice (i.e.,  $h_1 = h_2 = 1$ ). Thus

$$c_s(\pi) = \frac{1}{2!2!} \prod_{j=1}^2 \frac{s_j!}{\prod_{i=1}^2 b_{i,j}!} = \frac{2!2!}{2!2!} = 1$$

since all  $b_{i,j}$  are either 0 or 1.

For the multiset partition  $\pi = (1)(2)(1, 2)$ , there are 3 distinct blocks (i.e.,  $r' = 3$ , and  $h_1 = h_2 = h_3 = 1$ ). In this case,

$$c_s(\pi) = \frac{1}{1!1!1!} \prod_{j=1}^2 \frac{s_j!}{\prod_{i=1}^3 b_{i,j}!} = 2!2! = 4,$$

again because all  $b_{i,j}$  are either 0 or 1. This shows that the multiset partition  $\pi = (1)(2)(1, 2)$  corresponds to four different set partitions in  $\mathcal{B}_4$ .

With  $c_s(\pi)$  defined, we now generate  $\pi$  from  $\mathcal{B}_S$  instead of  $\pi'$  from  $\mathcal{B}_{|S|}$  as in (37). This allows us to write

$$\mathbb{E} \left[ \prod_{i=1}^k (\mathbf{z}^T \mathbf{A}_i \mathbf{z})^{s_i} \right] = \sum_{\pi \in \mathcal{B}_S} 2^{|\mathcal{S}| - f_1 - f_2} c_s(\pi) \prod_{i=1}^{\ell(\pi)} \tilde{\lambda}_{\pi'_i}. \quad (39)$$

In the above expression, all the duplicated set partitions in  $\mathcal{B}_{|S|}$  have been removed. However, the last term, i.e.,  $\tilde{\lambda}_{\pi'_i}$ , is still defined in terms of  $\tilde{\mathbf{A}}_i$  and  $\pi'_i$  ( $\pi'$  can be any set partition to which  $\pi$  maps). Our next step is to reexpress  $\tilde{\lambda}_{\pi'_i}$  in terms of  $\mathbf{A}_i$  and  $\pi_i$ .

Before doing so, we must first determine the number of unique  $\theta$  and  $\tau$  terms that can be generated from  $\pi_i$ , which may contain repeated elements. Let  $\mathbf{v}$  be a vector of positive integers, and  $v_1, \dots, v_p$  be the distinct elements of  $\mathbf{v}$ , with multiplicities  $n_1, \dots, n_p$ , respectively. The total number of elements in  $\mathbf{v}$  is  $n \equiv \sum_{i=1}^p n_i$ . We define  $n_\theta(\mathbf{v})$  as the number of distinct  $\theta$  terms that can be constructed from the elements of  $\mathbf{v}$ , and  $n_\tau(\mathbf{v})$  as the number of distinct  $\tau$  terms that can be constructed from the elements of  $\mathbf{v}$ . Explicit expressions of  $n_\theta(\mathbf{v})$  and  $n_\tau(\mathbf{v})$ , which rely on combinatorial arguments, are provided in the Appendix.

With  $n_\theta(\mathbf{v})$  and  $n_\tau(\mathbf{v})$  defined, the total number of possible  $\theta$  and  $\tau$  terms that can be generated from the  $i$ -th block of  $\pi$  is  $n(\pi_i) \equiv n_\theta(\pi_i) + n_\tau(\pi_i)$ . Since  $\pi$  may contain repeated blocks, the total number of distinct products of  $\theta$  and  $\tau$  generated from  $\pi$  is not equal to

$$\prod_{i=1}^r n(\pi_i).$$

Instead, it is given by

$$\prod_{i=1}^{r'} \binom{n(\tilde{\pi}_i) + h_i - 1}{h_i},$$

where  $\tilde{\pi}_i$  for  $i = 1, \dots, r'$  are the distinct blocks of  $\pi$  and  $h_i, i = 1, \dots, r'$  are their multiplicities. This follows because from the fact that for the  $h_i$  blocks with identical contents as  $\tilde{\pi}_i$ , the number of distinct products is the same as the

number of possible ways of choosing  $h_i$  objects from  $n(\tilde{\pi}_i)$  distinct objects with replacement. Consequently, the total number of unique terms in  $E[\prod_{i=1}^k (\mathbf{z}^T \mathbf{A}_i \mathbf{z})^{s_i}]$  is

$$M(\mathbf{s}) = \sum_{\pi \in \mathcal{B}_k} \prod_{i=1}^{r'} \binom{n(\tilde{\pi}_i) + h_i - 1}{h_i}. \quad (40)$$

In Table 7, we present  $M(\mathbf{s})$  for various choices of  $\mathbf{s}$  with  $|\mathbf{s}| = 10$ .<sup>7</sup> The results show that when some matrices in the product of quadratic forms are repeated, the number of distinct terms in the expression can be significantly reduced. Consequently, for computational efficiency, it is valuable to obtain an explicit expression for  $E[\prod_{i=1}^k (\mathbf{z}^T \mathbf{A}_i \mathbf{z})^{s_i}]$  that contains only the unique terms.

Table 7: Number of Unique Terms in the Expansion of  $E[\prod_{i=1}^k (\mathbf{z}^T \mathbf{A}_i \mathbf{z})^{s_i}]$  when  $\mathbf{z} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$\mathbf{s}$	$\boldsymbol{\mu} = \mathbf{0}$	$\boldsymbol{\mu} \neq \mathbf{0}$
(10)	42	481
(5,5)	554	15,310
(3,3,4)	3,895	143,795
(2,2,3,3)	16,561	673,364
(2,2,2,2,2)	60,775	2,572,407
(1,1,1,1,1,1,1,1,1,1)	1,436,714	63,673,506

For a given vector of positive integers  $\boldsymbol{\nu}$ , we define  $\mathbf{p}_j \equiv \mathbf{p}_j(\boldsymbol{\nu})$  for  $j = 1, \dots, n_\theta(\boldsymbol{\nu})$  as the permutations of  $\boldsymbol{\nu}$  that generate unique  $\theta$  terms. Similarly, we define  $\mathbf{q}_j \equiv \mathbf{q}_j(\boldsymbol{\nu})$  for  $j = 1, \dots, n_\tau(\boldsymbol{\nu})$  as the permutations of  $\boldsymbol{\nu}$  that generate unique  $\tau$  terms. Generating  $\mathbf{p}_j$  is straightforward: one enumerates all multiset permutations of  $\boldsymbol{\nu}$  and retains only those that do not lexicographically dominate their reversal. Generating  $\mathbf{q}_j$  requires enumerating all bracelets from the elements of  $\boldsymbol{\nu}$ . An efficient algorithm for this task (known as generating bracelets with fixed content) is given by Karim, Sawada, Alamgir, and Husnine [10].

**Example 7.** When  $\boldsymbol{\nu} = (1, 1, 2, 2)$ , we have  $n_\theta(\boldsymbol{\nu}) = 4$  unique  $\theta$  terms, with subscripts  $\mathbf{p}_1 = (1, 1, 2, 2)$ ,  $\mathbf{p}_2 = (1, 2, 1, 2)$ ,  $\mathbf{p}_3 = (1, 2, 2, 1)$ , and  $\mathbf{p}_4 = (2, 1, 1, 2)$ . For the unique  $\tau$  terms, we have  $n_\tau(\boldsymbol{\nu}) = 2$ , with subscripts  $\mathbf{q}_1 = (1, 1, 2, 2)$  and  $\mathbf{q}_2 = (1, 2, 1, 2)$ .

With  $\mathbf{p}_j$  and  $\mathbf{q}_j$  defined, we can replace  $\tilde{\lambda}_{\pi'_i}$  in (39) with  $\lambda_{\pi_i}$ , defined as

$$\lambda_{\pi_i} = 2^{I(\ell(\pi_i)=2)} \sum_{j=1}^{n_\theta(\pi_i)} c_{i,j} \theta_{\mathbf{p}_j(\pi_i)} + \sum_{j=1}^{n_\tau(\pi_i)} d_{i,j} \tau_{\mathbf{q}_j(\pi_i)}, \quad (41)$$

where  $c_{i,j}$  is the multiplicity of  $\theta_{\mathbf{p}_j(\pi_i)}$  in  $\tilde{\lambda}_{\pi'_i}$ , and  $d_{i,j}$  is the multiplicity of  $\tau_{\mathbf{q}_j(\pi_i)}$  in  $\tilde{\lambda}_{\pi'_i}$ . These multiplicities arise because  $\tilde{\lambda}_{\pi'_i}$  contains  $n_\theta(\ell(\pi'_i)) + n_\tau(\ell(\pi'_i))$  terms, where  $n_\theta(i)$  and  $n_\tau(i)$  are defined in (16) and (17). However, since  $\tilde{\mathbf{A}}_i$  contains repeated elements, some of the terms in  $\lambda_{\pi_i}$  are not unique. When using (41) — which retains only unique terms — we must keep track each term's multiplicity. The values of  $c_{i,j}$  and  $d_{i,j}$  can be directly obtained from  $\mathbf{p}_j(\pi_i)$  and  $\mathbf{q}_j(\pi_i)$ , respectively.

Let  $b_{i,l}$  for  $l = 1, \dots, k$ , denote the frequency of integer  $l$  in  $\pi_i$ . In the Appendix, we show that

$$c_{i,j} = \begin{cases} \frac{1}{2} \prod_{l=1}^k b_{i,l}! & \text{if } \mathbf{p}_j(\pi_i) \text{ is equal to its reverse,} \\ \prod_{l=1}^k b_{i,l}! & \text{otherwise,} \end{cases} \quad (42)$$

$$d_{i,j} = \begin{cases} \frac{\rho_{i,j}}{2\ell(\pi_i)} \prod_{l=1}^k b_{i,l}! & \text{if } \mathbf{q}_j(\pi_i) \text{ maps to one necklace,} \\ \frac{\rho_{i,j}}{\ell(\pi_i)} \prod_{l=1}^k b_{i,l}! & \text{otherwise,} \end{cases} \quad (43)$$

<sup>7</sup>The result for  $\boldsymbol{\mu} = \mathbf{0}$  is obtained by setting  $n(\tilde{\pi}_i) = n_\tau(\tilde{\pi}_i)$ .

where  $\rho_{i,j}$  denotes the primitive period of  $\mathbf{q}_j(\boldsymbol{\pi}_i)$ .<sup>8</sup>

We say that a bracelet  $\mathbf{q}_j(\boldsymbol{\pi}_i)$  maps to one necklace if, after reversal (and possible rotations), it coincides with the original bracelet. For example, reversing the bracelet  $(1,2,1,2,1,2)$  yields  $(2,1,2,1,2,1)$ , which, after a single rotation returns to  $(1,2,1,2,1,2)$ . Thus,  $(1,2,1,2,1,2)$  maps to one necklace. In contrast, reversing the bracelet  $(1,1,2,1,2,2)$  yields  $(2,2,1,2,1,1)$ , but it will not return to the original bracelet after rotations. Thus  $(1,1,2,1,2,2)$  maps to two necklaces. In the above example,  $\mathbf{q}_1 = (1, 1, 2, 2)$  has a primitive period of 4 and it maps to one necklace, and  $\mathbf{q}_2 = (1, 2, 1, 2)$  has a primitive period of 2 and it also maps to just one necklace.

After establishing the notation, we present our final explicit expression for  $E[\prod_{i=1}^k (\mathbf{z}^T \mathbf{A}_i \mathbf{z})^{s_i}]$  in the following proposition.

**Proposition 4.** *Suppose  $\mathbf{z} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  and  $\mathbf{A}_i$ ,  $i = 1, \dots, k$  are symmetric matrices. For positive integers  $s_1, \dots, s_k$ , we have*

$$E \left[ \prod_{i=1}^k (\mathbf{z}^T \mathbf{A}_i \mathbf{z})^{s_i} \right] = \sum_{\boldsymbol{\pi} \in \mathcal{B}_s} 2^{|\mathbf{s}| - f_1 - f_2} c_s(\boldsymbol{\pi}) \prod_{i=1}^{\ell(\boldsymbol{\pi})} \lambda_{\boldsymbol{\pi}_i}, \quad (44)$$

where  $f_1$  and  $f_2$  denote the number of blocks in  $\boldsymbol{\pi}$  containing one and two elements, respectively,  $|\mathbf{s}| = s_1 + \dots + s_k$ ,  $c_s(\boldsymbol{\pi})$  is defined in Lemma 2, and  $\lambda_{\boldsymbol{\pi}_i}$  is defined in (41).

We illustrate the use of (44) with an example that  $s_1 = s_2 = 2$ . Table 8 provides all quantities required to compute  $E[(\mathbf{z}^T \mathbf{A}_1 \mathbf{z})^2 (\mathbf{z}^T \mathbf{A}_2 \mathbf{z})^2]$ . The column  $c_s(\boldsymbol{\pi})$  in Table 8 is obtained from Table 6 and these coefficients can be calculated by using Lemma 2. The coefficients of the  $\theta$  and  $\tau$  terms in  $\lambda_{\boldsymbol{\pi}_i}$  are derived from (42) and (43). For example, when  $\boldsymbol{\pi} = (1, 1, 2, 2)$ , there four distinct  $\theta$  terms and two distinct  $\tau$  terms. The first two  $\theta$  indices, i.e.,  $(1,1,2,2)$  and  $(1,2,1,2)$ , are not equal to their reversals, and hence have coefficients  $2!2! = 4$ . The other two, i.e.,  $(1,2,2,1)$  and  $(2,1,1,2)$ , are equal to their reversals and their coefficients are halved. For the  $\tau$  terms, the possible indices are:  $(1,1,2,2)$  and  $(1,2,1,2)$ . Both map to one necklace, and their primitive periods are 4 and 2, respectively, yielding coefficient  $4 \times 2!2! / (2 \times 4) = 2$  for  $\tau_{(1,1,2,2)}$  and  $2 \times 2!2! / (2 \times 4) = 1$  for  $\tau_{(1,2,1,2)}$ .

From Table 8, we obtain

$$\begin{aligned} E \left[ (\mathbf{z}^T \mathbf{A}_1 \mathbf{z})^2 (\mathbf{z}^T \mathbf{A}_2 \mathbf{z})^2 \right] &= (\theta_1 + \tau_1)^2 (\theta_2 + \tau_2)^2 + 2(\theta_1 + \tau_1)^2 [2\theta_{(2,2)} + \tau_{(2,2)}] \\ &\quad + 8(\theta_1 + \tau_1)(\theta_2 + \tau_2) [2\theta_{(1,2)} + \tau_{(1,2)}] + 2(\theta_2 + \tau_2)^2 [2\theta_{(1,1)} + \tau_{(1,1)}] \\ &\quad + 4[2\theta_{(1,1)} + \tau_{(1,1)}] [2\theta_{(2,2)} + \tau_{(2,2)}] + 8[2\theta_{(1,2)} + \tau_{(1,2)}]^2 \\ &\quad + 16(\theta_1 + \tau_1) [2\theta_{(1,2,2)} + \theta_{(2,1,2)} + \tau_{(1,2,2)}] + 16(\theta_2 + \tau_2) [2\theta_{(1,1,2)} + \theta_{(1,2,1)} + \tau_{(1,1,2)}] \\ &\quad + 16[4\theta_{(1,1,2,2)} + 4\theta_{(1,2,1,2)} + 2\theta_{(1,2,2,1)} + 2\theta_{(2,1,1,2)} + 2\tau_{(1,1,2,2)} + \tau_{(1,2,1,2)}]. \end{aligned}$$

Expanding the above expression yields 54 unique terms—significantly fewer than  $M(4) = 123$  unique terms when all four quadratic forms involve distinct matrices.

The generalization of Proposition 4 to  $\mathbf{z} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  is straightforward: replace  $2^{|\mathbf{s}| - f_1 - f_2}$  with  $2^{\ell(\boldsymbol{\pi}) - f_1 - f_2}$  in (44), and take the real parts of all  $\theta$  and  $\tau$  terms as defined in Lemma 1.

<sup>8</sup>A sequence  $\{a_i\}$  is of period  $n$  if  $a_i = a_{i+n}$  for all  $i$ . Primitive period of the sequence is the shortest period that one can find in a sequence. For example, the sequence  $(1,2,1,2,1,2,1,2)$  has periods of 2, 4, and 8, but its primitive period is 2.

Table 8: Construction of an explicit expression of  $E[(z^T A_1 z)^2 (z^T A_2 z)^2]$  when  $z \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ , using (44)

$\boldsymbol{\pi} \in \mathcal{B}_{(2,2)}$	$2^{ \mathbf{s} -f_1-f_2}$	$c_s(\boldsymbol{\pi})$	$\lambda_{\boldsymbol{\pi}_1}$	$\lambda_{\boldsymbol{\pi}_2}$	$\lambda_{\boldsymbol{\pi}_3}$	$\lambda_{\boldsymbol{\pi}_4}$
(1)(1)(2)(2)	1	1	$\theta_1 + \tau_1$	$\theta_1 + \tau_1$	$\theta_2 + \tau_2$	$\theta_2 + \tau_2$
(1)(1)(2,2)	2	1	$\theta_1 + \tau_1$	$\theta_1 + \tau_1$	$2\theta_{(2,2)} + \tau_{(2,2)}$	–
(1)(2)(1,2)	2	4	$\theta_1 + \tau_1$	$\theta_2 + \tau_2$	$2\theta_{(1,2)} + \tau_{(1,2)}$	–
(2)(2)(1,1)	2	1	$\theta_2 + \tau_2$	$\theta_2 + \tau_2$	$2\theta_{(1,1)} + \tau_{(1,1)}$	–
(1,1),(2,2)	4	1	$2\theta_{(1,1)} + \tau_{(1,1)}$	$2\theta_{(2,2)} + \tau_{(2,2)}$	–	–
(1,2),(1,2)	4	2	$2\theta_{(1,2)} + \tau_{(1,2)}$	$2\theta_{(1,2)} + \tau_{(1,2)}$	–	–
(1)(1,2,2)	8	2	$\theta_1 + \tau_1$	$2\theta_{(1,2,2)} + \theta_{(2,1,2)} + \tau_{(1,2,2)}$	–	–
(2)(1,1,2)	8	2	$\theta_2 + \tau_2$	$2\theta_{(1,1,2)} + \theta_{(1,2,1)} + \tau_{(1,1,2)}$	–	–
(1,1,2,2)	16	1	$4\theta_{(1,1,2,2)} + 4\theta_{(1,2,1,2)} +$ $2\theta_{(1,2,2,1)} + 2\theta_{(2,1,1,2)} +$ $2\tau_{(1,1,2,2)} + \tau_{(1,2,1,2)}$	–	–	–

## 7. Conclusion

In this paper, we present an explicit expression for the expectation of a product of quadratic forms in multivariate normal random variables for both the real and the complex case. The proposed expression is novel and contains the minimum possible number of terms. Furthermore, we develop a computationally efficient algorithm for enumerating these terms. We also demonstrate that our expression yields simple formulae for the product moments of central and noncentral Wishart distributions. Finally, we present an explicit formulation for the case in which some matrices in the product of quadratic forms are repeated, leading to a further reduction in the number of terms.

## Appendix

In this Appendix, we consider the following problem. Let  $\boldsymbol{\nu}$  be a vector of positive integers containing  $p$  distinct elements  $\nu_1, \dots, \nu_p$ , with corresponding multiplicities of  $n_1, \dots, n_p$ . The total number of elements in  $\boldsymbol{\nu}$  is equal to  $n \equiv \sum_{i=1}^p n_i$ , and the possible number of multiset permutations that we can obtain from  $\boldsymbol{\nu}$  is given by the multinomial coefficient

$$N = \frac{n!}{\prod_{i=1}^p n_i!}. \quad (45)$$

However, when multiplicities are present, not all these multiset permutations yield distinct  $\theta$  and  $\tau$  terms. Our goal is to determine how many unique  $\theta$  and  $\tau$  terms can be generated by using permutations of  $\boldsymbol{\nu}$  as subscripts.

### Counting Unique $\theta$ Terms

A multiset permutation of  $\boldsymbol{\nu}$ , say  $\boldsymbol{\lambda}$ , is considered an index for a unique  $\theta$  if it does not lexicographically dominate its reverse,  $\boldsymbol{\lambda}^R$ .

**Example 8.** If  $\boldsymbol{\nu} = (1, 1, 2, 2)$ , then the permutations

$$(1, 1, 2, 2), (1, 2, 1, 2), (1, 2, 2, 1), (2, 1, 1, 2)$$

are admissible indices for unique  $\theta$  terms. However,

$$(2, 1, 2, 1), (2, 2, 1, 1)$$

are not. Out of the six multiset permutations of  $(1, 1, 2, 2)$ , only four yield unique  $\theta$ .

We are interested in counting, among the  $N$  multiset permutations of  $\boldsymbol{\nu}$ , those that do not dominate their reverse. Let  $P$  be the number of *palindromic* permutations (those equal to their own reverse, i.e.,  $\boldsymbol{\lambda} = \boldsymbol{\lambda}^R$ ), then:

- Every non-palindromic permutation,  $\lambda$  must either dominate  $\lambda^R$ , or be dominated by it, since lexicographical order is a total order.
- Thus,  $(N - P)/2$  permutations dominate their reverse, and  $N - (N - P)/2 = (N + P)/2$  permutations do not dominate their reverse. Therefore, the number of unique  $\theta$  terms is

$$n_\theta(\mathbf{v}) = \frac{N + P}{2}. \quad (46)$$

It remains to determine  $P$ . Note that a palindromic permutation must have at most one element with odd multiplicity because all elements except possibly one must pair symmetrically around the center. Thus,

- If more than one  $n_i$  is odd, then no palindromic permutations exist, i.e.,  $P = 0$ .
- If at most one  $n_i$  is odd, define  $n'_i = \lfloor n_i/2 \rfloor$  and  $n' = \sum_{i=1}^p n'_i$ , where  $\lfloor x \rfloor$  stands for the greatest integer less than or equal to  $x$ . Then to the left of the center of a palindromic permutation, it contains  $n'_i$  copies of  $v_i$  for  $i = 1, \dots, p$ , and the part to the right of the center mirrors it. Hence, the number of palindromic permutations is

$$P = \frac{n'!}{\prod_{i=1}^p n'_i!} = \frac{\lfloor n/2 \rfloor!}{\prod_{i=1}^p \lfloor n_i/2 \rfloor!}. \quad (47)$$

#### Mapping to Non-unique $\theta$ Terms

If multiplicities are ignored, there are  $n!/2$  possible  $\theta$  terms, though many are duplicates. We now describe how the  $n_\theta(\mathbf{v})$  unique  $\theta$  terms map into this larger set.

- Among the  $(N - P)/2$  non-palindromic permutations, each is dominated by its reverse. Every such permutation corresponds to  $\prod_{i=1}^p n_i!$  non-unique  $\theta$  terms, arising from relabeling identical elements.
- Among the  $P$  palindromic permutations, it only corresponds to  $\frac{1}{2} \prod_{i=1}^p n_i!$  non-unique  $\theta$  terms, since reversal produces no new terms.

Combining those contributions, the total mapping is

$$\frac{N - P}{2} \times \prod_{i=1}^p n_i! + P \times \frac{1}{2} \prod_{i=1}^p n_i! = \frac{N}{2} \times \prod_{i=1}^p n_i! = \frac{n!}{2}.$$

This observation explains the origin of the coefficient  $c_{i,j}$  in (42).

#### Counting Unique $\tau$ Terms

To determine the number of unique  $\tau$  terms, we introduce the concepts of *necklaces* and *bracelets*. For a sequence of integers, say  $\mathbf{v}$ , we can connect the two ends of the sequence to obtain a necklace. Two necklaces are considered to be different if we cannot obtain one by rotating the other. The total number of necklaces that can be constructed from  $\mathbf{v}$  (known as the number of necklaces with fixed content) is well known (see Gilbert and Riordan [4]), and is given by

$$N(n_1, \dots, n_p) = \frac{1}{n} \sum_{j|\gcd(n_1, \dots, n_p)} \phi(j) \frac{(n/j)!}{\prod_{i=1}^p (n_i/j)!}, \quad (48)$$

where  $j|h$  means that  $j$  is the set of integer divisors of  $h$ ,  $\gcd(n_1, \dots, n_p)$  denotes the greatest common divisor of  $n_1, \dots, n_p$ , and  $\phi(j)$  is the Euler totient function, which is the number of positive integers less than or equal to  $j$  that are relative prime to  $j$ .

If we consider two necklaces to be equivalent when we can obtain one by reversing the other, we get the set of bracelets. Although Karim, Sawada, Alamgir, and Husnine [10] provided an algorithm to enumerate the set of bracelets that can be generated from  $\mathbf{v}$ , the total number of bracelets with fixed content has not been explicitly established. In the following, we present a closed-form expression for the total number of bracelets that can be constructed from  $\mathbf{v}$ .

**Lemma 3.** Let  $\nu$  be a vector with  $p$  distinct elements, where the frequency of the  $i$ -th element is  $n_i$ ,  $i = 1, \dots, p$ . Let  $j$  denote the number of  $n_i$  that are odd. Then the number of bracelets that can be constructed from  $\nu$  is given by

$$B(n_1, \dots, n_p) = \begin{cases} \frac{1}{2}N(n_1, \dots, n_p) + \frac{((n-j)/2)!}{2 \prod_{i=1}^p \lfloor n_i/2 \rfloor!} & \text{if } j \leq 2, \\ \frac{1}{2}N(n_1, \dots, n_p) & \text{if } j > 2, \end{cases} \quad (49)$$

where  $n = \sum_{i=1}^p n_i$ .

*Proof.* We use Pólya's enumeration theorem to prove the lemma.<sup>9</sup> The cycle index for the cyclic group  $C_n$  is

$$Z(C_n) = \frac{1}{n} \sum_{d|n} \phi(d) a_d^{\frac{n}{d}},$$

where the generating function of the object is  $a_d = f(x_1^d, \dots, x_p^d)$ . For the necklace and bracelet problems with  $p$  distinct elements, we have

$$a_d = x_1^d + \dots + x_p^d. \quad (50)$$

The number of necklaces that can be constructed from  $\nu$  is given by the coefficient of  $x_1^{n_1} \dots x_p^{n_p}$  in  $Z(C_n)$ , which can be readily shown to be  $N(n_1, \dots, n_p)$  as defined in (48).

For the bracelets, we need to consider the cycle index for the dihedral group  $D_n$ , which is<sup>10</sup>

$$Z(D_n) = \begin{cases} \frac{Z(C_n)}{2} + \frac{a_1 a_2^{\frac{n-1}{2}}}{2} & \text{if } n \text{ is odd,} \\ \frac{Z(C_n)}{2} + \frac{a_1^2 a_2^{\frac{n-2}{4}} + a_2^{\frac{n}{2}}}{4} & \text{if } n \text{ is even.} \end{cases}$$

The number of bracelets that can be constructed from  $\nu$  is given by the coefficient of  $x_1^{n_1} \dots x_p^{n_p}$  in  $Z(D_n)$ . The contribution coming from the first term is simply  $N(n_1, \dots, n_p)/2$ . The additional contribution coming from the second term depends on whether  $n$  is odd or even.

- **$n$  is odd:** The additional contribution comes from the coefficient of  $x_1^{n_1} \dots x_p^{n_p}$  in the polynomial

$$\frac{1}{2}(x_1 + \dots + x_p)(x_1^2 + \dots + x_p^2)^{\frac{n-1}{2}}.$$

This is nonzero only when exactly one  $n_i$ 's is odd (i.e.,  $j = 1$ ). The corresponding  $x_i$  must be picked from  $(x_1 + \dots + x_p)$ . The number of ways to choose the other  $x_i$ 's from the expression  $(x_1^2 + \dots + x_p^2)^{\frac{n-1}{2}}$  is

$$\frac{((n-1)/2)!}{\prod_{i=1}^p \lfloor n_i/2 \rfloor!}.$$

Hence the additional contribution coming from the second term is

$$\frac{((n-j)/2)!}{2 \prod_{i=1}^p \lfloor n_i/2 \rfloor!}.$$

- **$n$  is even:** The additional contribution comes from the coefficient of  $x_1^{n_1} \dots x_p^{n_p}$  in the polynomial

$$\frac{1}{4}(x_1 + \dots + x_p)^2 (x_1^2 + \dots + x_p^2)^{\frac{n-2}{2}} + \frac{1}{4}(x_1^2 + \dots + x_p^2)^{\frac{n}{2}}. \quad (51)$$

<sup>9</sup>See Read [20] for a good review of Pólya's enumeration theorem.

<sup>10</sup>See Riordan [21] (p.150) for a derivation of the cycle index for the dihedral group.

Since this polynomial contains at most two  $x_i$ 's with odd powers, the coefficient is zero if more than two  $n_i$  are odd (i.e.,  $j > 2$ ). If all  $n_i$  are even (i.e.,  $j = 0$ ), the additional contribution coming from (51) is

$$\frac{(n/2)!}{4 \prod_{i=1}^r (n_i/2)!} + \frac{(n/2)!}{4 \prod_{i=1}^p (n_i/2)!} = \frac{(n/2)!}{2 \prod_{i=1}^p (n_i/2)!}.$$

When two  $n_i$ 's are odd (i.e.,  $j = 2$ ), we can only obtain the two  $x_i$ 's with odd power from the first term of (51), and there are two different ways to do that. For the remaining  $x_i$ 's, there are

$$\frac{((n-2)/2)!}{\prod_{i=1}^p \lfloor n_i/2 \rfloor!} \quad (52)$$

different ways of picking them from the expression  $(x_1^2 + \dots + x_p^2)^{\frac{n-2}{2}}$ . Together, the additional contribution coming from (51) is

$$\frac{((n-j)/2)!}{2 \prod_{i=1}^p \lfloor n_i/2 \rfloor!} \quad (53)$$

This shows that the first expression of (49) holds for  $j = 0, 1$  and  $2$ .  $\square$

Since the trace of a product of matrices is invariant to rotation and reflection of the matrices in the product, the number of unique  $\tau$  terms that can be created from using the permutations of  $\nu$  as subscripts is simply the total number of bracelets that can be constructed from  $\nu$ . Therefore, we have

$$n_\tau(\nu) = B(n_1, \dots, n_p). \quad (54)$$

#### Mapping to Non-unique $\tau$

If multiplicities are ignored, there are  $(n-1)/2$  possible  $\tau$  terms, though many are duplicates. Our goal is to understand how these  $(n-1)/2$   $\tau$ 's correspond to the unique set of  $\tau$ 's that has  $n_\tau(\nu)$  elements.

Consider a multiset permutation of  $\nu$ , say  $\lambda$ , which is a bracelet. Replacing  $n_i$  occurrences of  $\nu_i$  in  $\lambda$  with  $n_i$  distinct integers give rise to  $\prod_{i=1}^p n_i!$  possible permutations. By applying suitable rotation and reversal, each of these permutations can be associated with a bracelet that is constructed from  $n$  distinct integers. However, not all of the  $\prod_{i=1}^p n_i!$  resulting bracelets are distinct, for two main reasons.

1. **Primitive period.** A permutation  $\lambda$  may have a primitive period shorter than  $n$ . Let  $\rho$  be its primitive period, then the first  $\rho$  elements in  $\nu$  repeat  $n/\rho$  times within  $\nu$ . For instance,  $\lambda = (1, 2, 1, 2, 1, 2)$  has a primitive period of 2, and the block (1,2) repeats three times within  $\lambda$ . Although  $n_1 = n_2 = 3$ , replacing the 1's and 2's in  $\lambda$  with (1,2,3) with (4,5,6), respectively will not give us  $3! \times 3! = 36$  different bracelets. Ignoring the duplication due to reversals, rotation alone suggests that the number of distinct bracelets should be reduced by a factor of  $n/\rho$  (i.e., 3), leading to only 12 possible distinct bracelets.
2. **Symmetry under reflection.** A bracelet could remain the same after reversal and appropriate rotation (which can occur when the number of odd  $n_i$  is less than or equal to two). In this case, we say the bracelet maps to one necklace. For the other cases, we say the bracelet maps to two necklaces. For instance,  $\lambda = (1, 2, 1, 2, 1, 2)$  maps to one necklace because after reversal, we get (2,1,2,1,2,1) and after one more rotation, we get back  $\lambda$ . In contrast,  $\lambda = (1, 1, 2, 1, 2, 2)$  maps to two necklaces because after reversal, we get (2,2,1,2,1,1), but applying rotations to it will not allow us to recover  $\lambda$ . When a  $\lambda$  maps to only one necklace, then reversals will give us duplicate bracelets, so we need to reduce the number of bracelets by a factor of two.

In summary, if  $\lambda$  maps to one necklace, it corresponds to  $\prod_{i=1}^p n_i! / (2n/\rho)$  non-unique  $\tau$  terms, If  $\lambda$  maps to two necklaces, it corresponds to  $\prod_{i=1}^p n_i! / (n/\rho)$  non-unique  $\tau$  terms. This observation explains the origin of the coefficient  $d_{i,j}$  in (43).

In the algorithm of Karim, Sawada, Alamgir, and Husnine [10] for generating bracelets with fixed content, the information about the primitive period of a generated bracelet as well as whether it maps to one or two necklaces is readily available. As a result, it is easy to compute the number of non-unique  $\tau$ 's that each unique  $\tau$  maps to.

**Example 9.** Consider the case with  $\nu = (1, 1, 1, 2, 2, 2)$ , i.e.,  $n_1 = n_2 = 3$ . We have  $n_\tau(\nu) = 3$ , so there are three possible  $\tau$ 's that can be generated from  $\nu$ , with subscripts  $q_1(\nu) = (1, 1, 1, 2, 2, 2)$ ,  $q_2(\nu) = (1, 1, 2, 1, 2, 1)$ , and  $q_3(\nu) = (1, 2, 1, 2, 1, 2)$ . Table 9 summarizes the structural information of these three multiset permutations, which allows us to determine the number of non-unique  $\tau$ 's each  $\lambda$  maps to. The total number is  $18 + 36 + 6 = 60$ , consistent with the fact that the total number of non-unique  $\tau$ 's is  $(6 - 1)!/2 = 60$ .

Table 9: Mapping between Unique and Non-unique  $\tau$ 's when  $n_1 = n_2 = 3$

$\lambda$	1 or 2 necklaces?	Length of primitive period	Number of primitive periods	Number of non-unique $\tau$ 's
(1,1,1,2,2,2)	1	6	1	18
(1,1,2,1,2,2)	2	6	1	36
(1,2,1,2,1,2)	1	2	3	6

## Acknowledgments

We thank Yong Bao for helpful comments.

## References

- [1] Y. Bao, and A. Ullah, Expectation of quadratic forms in normal and nonnormal variables with applications, *Journal of Statistical Planning and Inference* 140 (2010), 1193–1205.
- [2] E. Di Nardo, On a symbolic representation of non-central Wishart random matrices with applications, *Journal of Multivariate Analysis* 125 (2014) 121–135.
- [3] F. J. H. Don, The expectation of products of quadratic forms in normal variables, *Statistica Neerlandica* 33 (1979) 73–79.
- [4] E. N. Gilbert, and J. Riordan, Symmetric types of periodic sequences, *Illinois Journal of Mathematics* 5 (1961) 657–665.
- [5] P. Graczyk, G. Letac, and H. Massam, The complex Wishart distribution and the symmetric group, *Annals of Statistics* 31(1) (2003) 287–309.
- [6] P. Graczyk, G. Letac, and H. Massam, The hyperoctahedral group, symmetric group representations and the moments of the real Wishart distribution, *Journal of Theoretical Probability* 18(1) (2005) 1–42.
- [7] H. Gupta, On the partition of J-partite numbers, *Proceedings of the National Institute of Sciences of India, Part A* 27 (1961) 579–587.
- [8] G. Hillier, R. Kan, and X. Wang, Generating functions and short recursions, with applications to the moments of quadratic forms in noncentral normal vectors, *Econometric Theory* 30 (2014) 436–473.
- [9] R. Kan, From moments of sum to moments of product, *Journal of Multivariate Analysis* 99 (2008) 542–554.
- [10] S. Karim, J. Sawada, Z. Alamgir, and S. M. Husnine, Generating bracelets with fixed content, *Theoretical Computer Science* 475 (2013) 103–112.
- [11] D. E. Knuth, *The Art of Computer Programming, Combinatorial Algorithms, Volume 4A*, Pearson, Boston, 2011.
- [12] A. Kumar, Expectation of product of quadratic forms, *Sankhyā Series B* 35 (1973) 359–362.

- [13] G. Letac, and H. Massam, The noncentral Wishart as an exponential family, and its moments, *Journal of Multivariate Analysis* 99 (2008) 1393–1417.
- [14] P. A. Macmahon, Seventh memoir on the partition of numbers. A detailed study of the enumeration of the partitions of multipartite numbers, *Philosophical Transactions of the Royal Society of London. Series A* 27 (1918) 81–113.
- [15] J. R. Magnus, The moments of products of quadratic forms in normal variables, *Statistica Neerlandica* 32 (1978) 201–210.
- [16] J. R. Magnus, The expectation of products of quadratic forms in normal variables: the practice, *Statistica Neerlandica* 33 (1979) 131–136.
- [17] J. R. Magnus, The exact moments of a ratio of quadratic forms in normal variables, *Annales d'Économie et de Statistique* 4 (1986) 95–109.
- [18] A. M. Mathai, S. B. Provost, *Quadratic forms in random variables: theory and applications*, Marcel Dekker, New York, 1992.
- [19] A. Nijenhuis, H. S. Wilf, *Combinatorial algorithms for computers and calculators*, 2nd edition, Academic Press, London, 1978.
- [20] R. C. Read, Pólya's theorem and its progeny, *Mathematics Magazine* 60(5) (1987) 275–282.
- [21] J. Riordan, *An introduction to combinatorial analysis*, Princeton University Press, New Jersey, 1980.
- [22] S. A. Sultan, D. S. Tracy, Moments of the complex multivariate normal distribution, *Linear Algebra and its Applications* 237/238 (1996) 191–204.
- [23] R. A. Wooding, The multivariate distribution of complex normal variables, *Biometrika* 43 (1956) 212–215.