On the Moments of Ratios of Quadratic Forms in Normal Random Variables

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Abstract

In this paper, we present both integral and infinite series expressions of $\mu_q^p \equiv E[(x'Ax)^p/(x'Bx)^q]$ when $x \sim N(\mu, I_n)$, where p, q are nonnegative real numbers, A is a symmetric matrix, and B is a positive semi-definite matrix. We also present efficient numerical methods for computing μ_q^p under each approach.

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1. Introduction

Let $x = [x_1, \ldots, x_n]' \sim N(\mu, I_n)$ be a normal random vector, A be a symmetric nonzero matrix, and B be a positive semi-definite matrix. For nonnegative real numbers p and q, we are interested in obtaining computationally efficient expressions of

$$\mu_q^p \equiv E\left[\frac{(x'Ax)^p}{(x'Bx)^q}\right].$$

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When p is not an integer, we assume that A is also a positive semi-definite matrix in order for $(x'Ax)^p$ to be well defined.[1]

Since many statistical estimators can be written as ratios of quadratic forms, the computation of μ_q^p has been of great interest to statisticians and econometricians, especially for $p = q$. Broadly speaking, there are two different approaches for evaluating μ_q^p. The first approach is an integration approach that starts with Sawa [12], who provides an integral formula that allows us to compute μ_q^p when p is an integer. This method is by far the most popular one in the literature. Magnus [7] provides a nice discussion, in particular the numerical aspect, of this method, and Meng [9] provides an excellent review of this literature. When p is not an integer, the integral formula for μ_q^p is also available (see, for example, Mathai and Provost [8] (Section 4.5d) and Meng [9] (Lemma 2)), but this would typically involve a double integral. In addition, the integrand is often difficult to evaluate, so this formula has seen little use in practice.

The second approach is to rely on some infinite series expansion of μ_q^p. Smith [13] is the first one to provide a triple infinite series (or double infinite series if p is an integer) expansion of μ_q^p in terms of top-order invariant polynomials. Due to the difficulty of computing top-order invariant polynomials and the complexity of this approach, his formula of μ_q^p appears to be only of academic interest. Recently, Hillier, Kan, and Wang [4] provide an efficient method for computing top-order invariant polynomials. Nevertheless, they are able to implement this formula only for integral p and $\mu = 0_n$. Note that in deriving the infinite series expansion of μ_q^p, Smith [13] needs to assume that both A and B are positive definite. Therefore, it appears that his formula is not applicable for the case when A or B is positive semi-definite.

The objective of this paper is to improve both approaches for evaluating μ_q^p. For the integration approach, we provide an explicit expression of the integrand when p is not an integer. In addition, we provide a numerically efficient method for computing the integrand. This improvement is particularly important in light of the fact that this approach calls for a double integral. With our numerical method, it becomes practical to use the double

[1]Our results can be easily adapted to deal with the case when $x \sim N(\mu, \Sigma)$, where Σ is a positive definite matrix. This is because we can write $x'Ax = \tilde{x}'(\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}})\tilde{x}$ and $x'Bx = \tilde{x}'(\Sigma^{\frac{1}{2}}B\Sigma^{\frac{1}{2}})\tilde{x}$, where $\tilde{x} = \Sigma^{-\frac{1}{2}}x \sim N(\Sigma^{-\frac{1}{2}}\mu, I_n)$. In addition, we can also deal with the case that B is a negative semi-definite matrix if q is an integer.

integral formula for evaluating μ_q^p. For the infinite series approach, we show that Smith's triple infinite series formula actually holds even when A or B is positive semi-definite, but a different and more elaborate proof is required. In view of the numerical difficulty of using Smith's formula to compute μ_q^p, we provide a different infinite series expansion of μ_q^p. This alternative infinite series expansion of μ_q^p has already been introduced by Hillier, Kan, and Wang [5] for the case when p is an integer, but we extend it to allow for non-integral p. With our new infinite series expansion, we are able to reduce the triple infinite series expansion of μ_q^p to a double infinite series expansion. More importantly, we introduce a fast recurrence algorithm to compute the coefficients in our double infinite series expansion, which makes our new infinite series approach very competitive against the integration approach.

The rest of the paper is organized as follows. Section 2 presents the necessary and sufficient conditions for the existence of μ_q^p. Section 3 presents an integral expression of μ_q^p and discusses a numerically efficient approach of evaluating the integrand. Section 4 shows that Smith's triple infinite series expression of μ_q^p in terms of top-order invariant polynomials continues to hold even when A or B is positive semi-definite. In Section 5, we develop a new double infinite series expression of μ_q^p and provide a numerically efficient approach for evaluating the coefficients in the double infinite series. Section 6 concludes the paper. Proofs of all the propositions are collected in the Appendix.

2. Necessary and sufficient conditions for the existence of μ_q^p

Before we present different expressions of μ_q^p, we need to first understand the conditions for the existence of μ_q^p. Suppose the rank of B is $m \leq n$. Let $P_1 D_{1b} P_1' = B$, where $D_{1b} = \text{Diag}(b_1, \ldots, b_m)$ with $b_1 \geq \cdots \geq b_m > 0$ are the positive eigenvalues of B, and P_1 is an $n \times m$ matrix of the eigenvectors associated with the positive eigenvalues of B. Denoting $P = [P_1,\ P_2]$, where P_2 is an $n \times (n-m)$ matrix of the eigenvectors associated with the zero eigenvalues of B, we can write

$$\frac{(x'Ax)^p}{(x'Bx)^q} = \frac{(x'PP'APP'x)^p}{(x'P_1 D_{1b} P_1' x)^q} = \frac{(z'\tilde{A}z)^p}{(z_1' D_{1b} z_1)^q},$$

where

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} P_1'x \\ P_2'x \end{bmatrix} \sim N(P'\mu, I_n),$$

and

$$\tilde{A} = P'AP = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix},$$

with $\tilde{A}_{ij} = P_i'AP_j$. The following proposition presents the necessary and sufficient conditions for the existence of μ_q^p.

Proposition 1. *(1) Suppose B is a positive definite matrix. μ_q^p exists if and only if $\frac{n}{2} + p > q$.*

(2) Suppose B is a positive semi-definite matrix with rank $m < n$. We have three cases to consider. (i) When $\tilde{A}_{12} = 0_{m\times(n-m)}$ and $\tilde{A}_{22} = 0_{(n-m)\times(n-m)}$, μ_q^p exists if and only if $\frac{m}{2} + p > q$. (ii) When $\tilde{A}_{12} \neq 0_{m\times(n-m)}$ and $\tilde{A}_{22} = 0_{(n-m)\times(n-m)}$, μ_q^p exists if and only if $\frac{m+p}{2} > q$. (iii) When $\tilde{A}_{22} \neq 0_{(n-m)\times(n-m)}$, μ_q^p exists if and only if $\frac{m}{2} > q$.

Roberts [11] (Sections 3.1 and 7.2.2) has already shown that the conditions in Proposition 1 are sufficient for the existence of μ_q^p. He also shows that when p is an integer, the conditions in Proposition 1 are both necessary and sufficient. However, our Proposition 1 provides a stronger result by showing that the conditions are necessary and sufficient even when p is not an integer. Note that when p is not an integer, we need to assume A is positive semi-definite. It can be easily shown that when A is positive semi-definite, $\tilde{A}_{22} = 0_{(n-m)\times(n-m)}$ implies $\tilde{A}_{12} = 0_{m\times(n-m)}$,[2] so we can eliminate case (ii) from consideration when A is positive semi-definite.

3. Integral expression of μ_q^p

When p is an integer, the most popular method for numerical evaluation of μ_q^p is to use the results of Sawa [12] and Cressie, Davis, Folks, and Policello [3] to write

$$\mu_q^p = \frac{1}{\Gamma(q)} \int_0^\infty t^{q-1} \left. \frac{\partial^p}{\partial t_1^p} \phi(t_1, t_2) \right|_{t_1=0, t_2=-t} \mathrm{d}t, \tag{1}$$

[2] Since A is positive semi-definite, $\tilde{A}$ is also positive semi-definite. For a positive semi-definite matrix $\tilde{A} = (\tilde{a}_{ij})$, we have $\tilde{a}_{ii}\tilde{a}_{jj} \geq \tilde{a}_{ij}^2$ for $j \neq i$. Therefore, $\tilde{a}_{ii} = 0$ implies $\tilde{a}_{ij} = 0$ for $j \neq i$. This suggests that if a diagonal element of a positive semi-definite matrix is zero, then the entire row and column to which it belongs must also be zero.

where

$$\phi(t_1,t_2)=|I_n-2t_1A-2t_2B|^{-\frac{1}{2}}\exp\left(\frac{\mu'(I_n-2t_1A-2t_2B)^{-1}\mu}{2}-\frac{\mu'\mu}{2}\right)$$

is the joint moment generating function of $x'Ax$ and $x'Bx$. Meng [9] provides an excellent review of this literature.

When p is not an integer, Mathai and Provost [8] (Eq.4.5d.8) show that an integration formula is also available for μ_q^p (see also Lemma 2 of Meng [9]), and it is given by the following double integral expression:

$$\mu_q^p=\frac{1}{\Gamma(\langle p\rangle)\Gamma(q)}\int_0^\infty\int_0^\infty s^{\langle p\rangle-1}t^{q-1}\left.\frac{\partial^{\lceil p\rceil}}{\partial t_1^{\lceil p\rceil}}\phi(t_1,t_2)\right|_{t_1=-s,t_2=-t}\mathrm{d}s\mathrm{d}t, \quad (2)$$

where $\lceil p\rceil$ is the smallest integer that is greater than or equal to p, and $\langle p\rangle=\lceil p\rceil-p$.

In order for us to use (1) or (2) to evaluate μ_q^p, we need an explicit expression for the partial derivatives of $\phi(t_1,t_2)$. Magnus [7] presents a simple expression of this for the case when p is an integer. In the following proposition, we present a slightly different expression of his result as well as a corresponding expression for the case when p is not an integer.

Proposition 2. *Suppose μ_q^p exists. When p is an integer, we have*

$$\mu_q^p=\frac{1}{\Gamma(q)}\int_0^\infty t^{q-1}\phi(0,-t)E[(w'Rw)^p]\mathrm{d}t, \quad (3)$$

where $R=L'AL$, $w\sim N(\tilde{\mu},I_n)$ with $\tilde{\mu}=L'\mu$, and L is an $n\times n$ matrix such that $LL'=(I_n+2tB)^{-1}$. When p is not an integer, we have

$$\mu_q^p=\frac{1}{\Gamma(\langle p\rangle)\Gamma(q)}\int_0^\infty\int_0^\infty s^{\langle p\rangle-1}t^{q-1}\phi(-s,-t)E[(w'Rw)^{\lceil p\rceil}]\mathrm{d}s\mathrm{d}t, \quad (4)$$

but L is now defined as an $n\times n$ matrix such that $LL'=(I_n+2sA+2tB)^{-1}$.

There are two computational issues for us to tackle when using (4) to compute μ_q^p. The first issue is the computation of $E[(w'Rw)^k]$. When k is small, a simple explicit formula of $E[(w'Rw)^k]$ is available. For example, when $k=1$, we have $E[w'Rw]=\mathrm{tr}(R)+\tilde{\mu}'R\tilde{\mu}$. However, when k is large, it is very tedious and computationally expensive to use the explicit formula (see Magnus [7] for an explicit formula of $E[(w'Rw)^k]$). In the following, we provide a fast recurrence algorithm for computing $E[(w'Rw)^k]$.

Let

$$\tilde{D}(t) = |I_n - tR|^{-\frac{1}{2}} \exp\left(\frac{\tilde{\mu}'(I_n - tR)^{-1}\tilde{\mu} - \tilde{\mu}'\tilde{\mu}}{2}\right) = \sum_{k=0}^{\infty} \tilde{d}_k t^k.$$

It can be readily shown that $\tilde{d}_k = E[(w'Rw)^k]/(2^k k!)$ and we have the following recurrence relation (see, for example, Ruben [10] and Mathai and Provost [8] (Eq.3.2b.8))

$$\tilde{d}_k = \frac{1}{2k}\sum_{j=1}^{k} p_j \tilde{d}_{k-j}, \tag{5}$$

where

$$p_j = \mathrm{tr}(R^j) + j\tilde{\mu}'R^j\tilde{\mu}.$$

Let $Q\Lambda Q' = R$, where $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix of the eigenvalues of R, and $Q = [q_1, \ldots, q_n]$ is a matrix of the corresponding eigenvectors. We can then write

$$p_j = \sum_{i=1}^{n} (\lambda_i^j + j\delta_i\lambda_i^j), \tag{6}$$

where $\delta_i = (q_i'\tilde{\mu})^2$. Using the initial condition $\tilde{d}_0 = 1$, we can use (5) successively to obtain $\tilde{d}_k$. While (5) is far more efficient than the explicit formula for computing $\tilde{d}_k$, it has a shortcoming in that the length of recursion increases with k, so it can be inefficient when k is large.

Using a method suggested by Brown [1], it is possible to update $\tilde{d}_k$ from $\tilde{d}_{k-1}$ without the need of using all the $\tilde{d}_i$ for $0 \le i \le k-1$. In order to obtain this short recurrence relation, we substitute (6) into (5) and exchange the order of summation to obtain the following expression

$$\tilde{d}_k = \frac{1}{2k}\sum_{i=1}^{n}\sum_{j=1}^{k}(\lambda_i^j \tilde{d}_{k-j} + j\delta_i\lambda_i^j\tilde{d}_{k-j}) = \frac{1}{2k}\sum_{i=1}^{n}(u_{i,k} + v_{i,k}), \tag{7}$$

where[3]

$$u_{i,k} = \sum_{j=1}^{k} \lambda_i^j \tilde{d}_{k-j}, \qquad v_{i,k} = \delta_i \sum_{j=1}^{k} j\lambda_i^j \tilde{d}_{k-j}. \tag{8}$$

It can be easily verified that $u_{i,k}$ and $v_{i,k}$ can be updated using the following recurrence relation:

$$u_{i,k} = \lambda_i(u_{i,k-1} + \tilde{d}_{k-1}), \qquad v_{i,k} = \delta_i u_{i,k} + \lambda_i v_{i,k-1}, \tag{9}$$

with the initial conditions $u_{i,0} = 0$ and $v_{i,0} = 0$. This method is extremely efficient because we update $u_{i,k}$ and $v_{i,k}$ using just $u_{i,k-1}$, $v_{i,k-1}$ and $\tilde{d}_{k-1}$. As a result, the memory requirement for this recurrence relation is only $2n + 1$ elements (n elements for $u_{i,k}$, n elements for $v_{i,k}$ and one element for $\tilde{d}_k$) regardless of k, and the computation time for updating $\tilde{d}_k$ does not increase with k.

The second issue is the computational efficiency of the integrand. In general, R depends on both s and t, and since we need to obtain the eigenvalues and eigenvectors of R in order to compute the moments of $w'Rw$, this can be extremely time consuming. In order to speed up the computation of the integrand, we propose a method that frees us from performing eigenvalue decomposition of R. Under this method, eigenvalue decomposition is only performed at the outer integral, and there is no need to perform eigenvalue decomposition of R as we change the value of s in the inner integral.

Our method hinges on choosing an appropriate L such that $R = L'AL$ is a diagonal matrix. Let $PD_bP' = B$, where $D_b = \mathrm{Diag}(b_1, \ldots, b_n)$ is a diagonal matrix of the eigenvalues of B (some of the b_i's can be zero), and P is an $n \times n$ matrix of the corresponding eigenvectors. Denoting $\tilde{A} = P'AP$, we can then write

$$I_n + 2sA + 2tB = P(I_n + 2s\tilde{A} + 2tD_b)P' = P\Delta^{-1}(I_n + 2s\breve{A})\Delta^{-1}P',$$

where $\Delta = (I_n + 2tD_b)^{-\frac{1}{2}}$ and $\breve{A} = \Delta\tilde{A}\Delta$. Let $HD_{\breve{a}}H' = \breve{A}$, where $D_{\breve{a}} = \mathrm{Diag}(\breve{a}_1, \ldots, \breve{a}_n)$ is a diagonal matrix of the eigenvalues of $\breve{A}$, and H is an $n \times n$ matrix of the corresponding eigenvectors. With these transformations, we can write

$$(I_n + 2sA + 2tB)^{-1} = P\Delta H(I_n + 2sD_{\breve{a}})^{-1}H'\Delta P',$$

[3]When $\mu = 0_n$, we can drop the $v_{i,k}$ terms because they are equal to zero.

and choose $L = P\Delta H(I_n + 2sD_{\breve{a}})^{-\frac{1}{2}}$. Under this choice of L, we have

$$\begin{aligned} R &= L'AL \\ &= (I_n + 2sD_{\breve{a}})^{-\frac{1}{2}} H'\Delta P'AP\Delta H(I_n + 2sD_{\breve{a}})^{-\frac{1}{2}} \\ &= (I_n + 2sD_{\breve{a}})^{-\frac{1}{2}} H'\breve{A}H(I_n + 2sD_{\breve{a}})^{-\frac{1}{2}} \\ &= (I_n + 2sD_{\breve{a}})^{-\frac{1}{2}} D_{\breve{a}}(I_n + 2sD_{\breve{a}})^{-\frac{1}{2}}, \end{aligned}$$

which is a diagonal matrix with diagonal elements $\lambda_i = \breve{a}_i/(1 + 2s\breve{a}_i)$. In addition, we have $\delta_i = \tilde{\mu}_i^2$, where

$$\tilde{\mu} = L'\mu = (I_n + 2sD_{\breve{a}})^{-\frac{1}{2}} H'\Delta P'\mu.$$

The important point to note here is that we can obtain λ_i and δ_i by using H and $D_{\breve{a}}$, both of which depend on t but not on s, so we no longer need to perform eigenvalue decomposition of R in the inner integral. In addition, we have

$$|I_n + 2sA + 2tB|^{-\frac{1}{2}} = |\Delta||I_n + 2sD_{\breve{a}}|^{-\frac{1}{2}} = \prod_{i=1}^n \frac{1}{(1 + 2tb_i)^{\frac{1}{2}}} \prod_{i=1}^n \frac{1}{(1 + 2s\breve{a}_i)^{\frac{1}{2}}}.$$

With all these expressions, we can rewrite (4) as

$$\mu_q^p = \frac{e^{-\frac{\mu'\mu}{2}} 2^{\lceil p \rceil} \lceil p \rceil!}{\Gamma(\langle p \rangle)\Gamma(q)} \int_0^\infty \frac{t^{q-1}}{\prod_{i=1}^n (1 + 2tb_i)^{\frac{1}{2}}} \left[\int_0^\infty \frac{s^{\langle p \rangle - 1} e^{\frac{\tilde{\mu}'\tilde{\mu}}{2}} \tilde{d}_{\lceil p \rceil}}{\prod_{i=1}^n (1 + 2s\breve{a}_i)^{\frac{1}{2}}} \mathrm{d}s \right] \mathrm{d}t,$$

and this is our preferred integral expression for evaluating μ_q^p when p is not an integer.[4]

4. Infinite series expression of μ_q^p in terms of top-order invariant polynomials

Besides the integral expression of μ_q^p, there is an infinite series expression of μ_q^p given by Smith [13]. Before we present his expression, we first introduce the normalized top-order invariant polynomials. For the rest of the paper,

[4] A set of Matlab programs for computing μ_q^p using this integration method and other methods discussed later in the paper is available at `http://www.rotman.utoronto.ca/~kan/research.htm`.

we shall adopt the following notation: $\boldsymbol{t} = (t_1, \ldots, t_r)$, $\boldsymbol{\kappa} = (k_1, \ldots, k_r)$, the k_i being nonnegative integers, $|\boldsymbol{\kappa}|$ will denote the sum of the parts of $\boldsymbol{\kappa}$, i.e., $|\boldsymbol{\kappa}| = \sum_{i=1}^r k_i$, and $\boldsymbol{t}^{\boldsymbol{\kappa}} = \prod_{i=1}^r t_i^{k_i}$.

Throughout the paper, we use the notation in Wilf [14] for coefficients in a generating function: the expression $[\boldsymbol{t}^{\boldsymbol{\kappa}}]D(\boldsymbol{t})$ denotes the coefficient of $\boldsymbol{t}^{\boldsymbol{\kappa}}$ in the power series expansion of the function

$$D(\boldsymbol{t}) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\kappa}|=k} d_{\boldsymbol{\kappa}} \boldsymbol{t}^{\boldsymbol{\kappa}}.$$

Using Wilf's notation, we can write $d_{\boldsymbol{\kappa}}$ as

$$d_{\boldsymbol{\kappa}} = [\boldsymbol{t}^{\boldsymbol{\kappa}}]D(\boldsymbol{t}).$$

For $n \times n$ symmetric matrices A_1 to A_r, we define the following generating function

$$D(\boldsymbol{t}) = |I_n - t_1 A_1 - \cdots - t_r A_r|^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\kappa}|=k} d_{\boldsymbol{\kappa}}(A_1, \cdots, A_r),$$

where $d_{\boldsymbol{\kappa}}(A_1, \ldots, A_r)$ is the normalized top-order invariant polynomial (see Chikuse [2] and Hillier, Kan, and Wang [4]). Assuming both A and B are positive definite and $\frac{n}{2} + p > q$, Smith [13] provides a triple infinite series expansion of μ_q^p in terms of the top-order invariant polynomials. In terms of our notation, his expression is

$$\mu_q^p = \frac{2^{p-q}\beta^q e^{-\frac{\mu'\mu}{2}}}{\alpha^p} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-p)_i (q)_j \Gamma\left(\frac{n}{2} + p - q + k\right)}{2^k \left(\frac{1}{2}\right)_k \Gamma\left(\frac{n}{2} + i + j + k\right)} d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu'), \tag{10}$$

where $\hat{A} = I_n - \alpha A$, $\hat{B} = I_n - \beta B$ with $0 < \alpha < 2/a_{\max}$, $0 < \beta < 2/b_{\max}$, $a_{\max}$ and $b_{\max}$ are the maximum eigenvalues of A and B, respectively, and $(a)_k = a(a+1)\cdots(a+k-1)$ is the usual Pochhammer symbol.

When p is an integer, (10) can be simplified to a double infinite series:

$$\mu_q^p = 2^{p-q}\beta^q e^{-\frac{\mu'\mu}{2}} p! \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q)_j \Gamma\left(\frac{n}{2} + p - q + k\right)}{2^k \left(\frac{1}{2}\right)_k \Gamma\left(\frac{n}{2} + p + j + k\right)} d_{p,j,k}(A, \hat{B}, \mu\mu'). \tag{11}$$

In addition, (11) also holds when A is a general symmetric matrix.

Since top-order invariant polynomials are in general difficult to evaluate, these infinite series expressions of μ_q^p are rarely used by researchers in practice. Recently, Hillier, Kan, and Wang [4] have made some progress in the numerical computation of top-order invariant polynomials. For the special case when p is an integer and $\mu = 0_n$, (11) simplifies to a single infinite series, and Hillier, Kan, and Wang [4] provide an algorithm for numerical evaluation of μ_q^p. However, for the more general case, these infinite series expressions remain only of academic interest.

When proving (10), Smith relies on the assumption that both A and B are positive definite matrices and his proof does not go through when A or B is positive semi-definite. He attempts to deal with some special cases when B is positive semi-definite, but leaves the general case as an open question. To the best of our knowledge, general infinite series expression of μ_q^p is currently unavailable when A or B is positive semi-definite. Surprisingly, our next proposition shows that as long as μ_q^p exists, Smith's formula continues to hold even when A or B is positive semi-definite. Therefore, with the moment existence conditions in Proposition 1 satisfied, we can safely use (10) and (11).

Proposition 3. *Suppose μ_q^p exists. μ_q^p is given by (10) when p is not an integer, and by (11) when p is an integer.*

Our proof of Proposition 3 relies on the integral formula of μ_q^p given in Section 3 because that formula works also for the case when A or B is positive semi-definite. In addition, our proof provides a unified framework that allows us to reconcile the two different approaches (integral vs. infinite series) of evaluating μ_q^p.

5. New infinite series expression of μ_q^p

While we establish in the last section that Smith's expression of μ_q^p holds even when A or B is positive semi-definite, his expression is not computationally friendly because it involves a triple infinite series. In order to overcome this problem, we present in this section a new infinite series expansion of μ_q^p in terms of a double infinite series. For the special case when p is an integer, μ_q^p reduces to a single infinite series. It should be noted that for the case that p is an integer, this new infinite series expansion has already been proposed by Hillier, Kan, and Wang [5]. However, their proof assumes B is positive definite, which somewhat restricts the generality of their result.

In order to present our results, we first define two generating functions

$$
\begin{aligned}
H(t_1,t_2) &= |I_n - t_1A_1 - t_2A_2|^{-\frac{1}{2}} \exp\left(\frac{(1-t_1-t_2)\mu'(I_n - t_1A_1 - t_2A_2)^{-1}\mu}{2} - \frac{\mu'\mu}{2}\right) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} h_{i,j}(A_1,A_2)t_1^i t_2^j, \\
\tilde{H}(t_1,t_2) &= |I_n - t_1A_1 - t_2A_2|^{-\frac{1}{2}} \exp\left(\frac{(1-t_2)\mu'(I_n - t_1A_1 - t_2A_2)^{-1}\mu}{2} - \frac{\mu'\mu}{2}\right) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \tilde{h}_{i,j}(A_1,A_2)t_1^i t_2^j,
\end{aligned}
$$

where A_1 and A_2 are two symmetric matrices. With $h_{i,j}$ and $\tilde{h}_{i,j}$ defined, the following proposition presents our new infinite series expression of μ_q^p.

Proposition 4. *Suppose μ_q^p exists. When p is not an integer, we have*

$$
\begin{aligned}
\mu_q^p &= \frac{2^{p-q}\beta^q\Gamma\left(\frac{n}{2}+p-q\right)}{\alpha^p}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-p)_i(q)_j}{\Gamma\left(\frac{n}{2}+i+j\right)}h_{i,j}(\hat{A},\hat{B}) \\
&= \frac{2^{p-q}\beta^q\Gamma\left(\frac{n}{2}+p-q\right)}{\alpha^p}\sum_{k=0}^{\infty}\frac{w_k}{\Gamma\left(\frac{n}{2}+k\right)}, \qquad (12)
\end{aligned}
$$

where $w_k = \sum_{i=0}^{k}(-p)_i(q)_{k-i}h_{i,k-i}(\hat{A},\hat{B})$, $\hat{A} = I_n - \alpha A$, $\hat{B} = I_n - \beta B$ with $0 < \alpha < 2/a_{\max}$, $0 < \beta < 2/b_{\max}$, and $a_{\max}$ and $b_{\max}$ are the maximum eigenvalues of A and B, respectively. When p is an integer, we have

$$
\mu_q^p = \frac{2^{p-q}\beta^q p!\Gamma\left(\frac{n}{2}+p-q\right)}{\Gamma\left(\frac{n}{2}+p\right)}\sum_{j=0}^{\infty}\frac{(q)_j}{\Gamma\left(\frac{n}{2}+j\right)}\tilde{h}_{p,j}(A,\hat{B}). \qquad (13)
$$

Note that when $\mu = 0_n$, $h_{i,j}(\hat{A},\hat{B}) = d_{i,j}(\hat{A},\hat{B})$ and $\tilde{h}_{i,j}(A,\hat{B}) = d_{i,j}(A,\hat{B})$. In order to use (12) and (13), we need to have an algorithm to compute $h_{i,j}$ and $\tilde{h}_{i,j}$. The following proposition shows that both $h_{i,j}$ and $\tilde{h}_{i,j}$ can be obtained recursively, with the initial conditions $h_{0,0} = 1$ and $\tilde{h}_{0,0} = 1$.

Proposition 5. *Defining $\tau_{i,j} = [t_1^i t_2^j]\mathrm{tr}((t_1A_1+t_2A_2)^{i+j})$ and $\eta_{i,j} = [t_1^i t_2^j]\mu'(t_1A_1+$*

$t_2A_2)^{i+j}\mu$, *we have*

$$\begin{aligned} h_{i,j} &= \frac{1}{2(i+j)}\sum_{\nu_1=0}^{i}\sum_{\substack{\nu_2=0\\ \nu_1+\nu_2>0}}^{j} p_{\nu_1,\nu_2}h_{i-\nu_1,j-\nu_2}, \\ \tilde{h}_{i,j} &= \frac{1}{2(i+j)}\sum_{\nu_1=0}^{i}\sum_{\substack{\nu_2=0\\ \nu_1+\nu_2>0}}^{j} \tilde{p}_{\nu_1,\nu_2}\tilde{h}_{i-\nu_1,j-\nu_2} \end{aligned}$$

where

$$\begin{aligned} p_{\nu_1,\nu_2} &= \tau_{\nu_1,\nu_2}+(\nu_1+\nu_2)(\eta_{\nu_1,\nu_2}-\eta_{\nu_1-1,\nu_2}-\eta_{\nu_1,\nu_2-1}), \\ \tilde{p}_{\nu_1,\nu_2} &= \tau_{\nu_1,\nu_2}+(\nu_1+\nu_2)(\eta_{\nu_1,\nu_2}-\eta_{\nu_1,\nu_2-1}), \end{aligned}$$

with the convention that $\eta_{i,j}=0$ *when* i *or* j *is negative.*

There are two major problems with using the recursions in Proposition 5. First, it is computationally expensive. For instance, $h_{i,j}$ is expressed as a linear combination of all the h_{ν_1,ν_2}'s with $\nu_1 \le i$ and $\nu_2 \le j$, so this requires a lot of storage space, and is computationally expensive when i and j are large. Second, $\tau_{i,j}$ and $\eta_{i,j}$ are not easy to compute because a binomial expansion on $(t_1A_1+t_2A_2)^{i+j}$ can lead to many terms when $i+j$ is large.

In order to overcome these two problems, we present an efficient algorithm for computing $h_{i,j}$ and $\tilde{h}_{i,j}$, which can be regarded as a multivariate generalization of Brown's [1] algorithm, and is introduced in Hillier, Kan, and Wang [5]. To present this algorithm, we let $A(\boldsymbol{t}) = t_1A_1+t_2A_2$ and define matrix $G_{i,k-i}$ and vector $g_{i,k-i}$ as follows:

$$\begin{aligned} G_{i,k-i} &= \sum_{\nu_1=0}^{i}\sum_{\substack{\nu_2=0\\ \nu_1+\nu_2>0}}^{k-i} [t_1^{\nu_1}t_2^{\nu_2}]A(\boldsymbol{t})^{\nu_1+\nu_2}h_{i-\nu_1,k-i-\nu_2}, \\ g_{i,k-i} &= \sum_{\nu_1=0}^{i}\sum_{\substack{\nu_2=0\\ \nu_1+\nu_2>0}}^{k-i} (\nu_1+\nu_2)\left([t_1^{\nu_1}t_2^{\nu_2}]A(\boldsymbol{t})^{\nu_1+\nu_2}-[t_1^{\nu_1}t_2^{\nu_2-1}]A(\boldsymbol{t})^{\nu_1+\nu_2-1}\right. \\ &\quad \left. -[t_1^{\nu_1-1}t_2^{\nu_2}]A(\boldsymbol{t})^{\nu_1+\nu_2-1}\right)\mu h_{i-\nu_1,k-i-\nu_2}. \end{aligned}$$

With $G_{i,k-i}$ and $g_{i,k-i}$ available, we can then obtain $h_{i,k-i}$ using

$$h_{i,k-i} = \frac{1}{2k}\sum_{\nu_1=0}^{i}\sum_{\substack{\nu_2=0\\ \nu_1+\nu_2>0}}^{k-i} p_{\nu_1,\nu_2}h_{i-\nu_1,k-i-\nu_2} = \frac{\operatorname{tr}(G_{i,k-i})+\mu' g_{i,k-i}}{2k}.$$

As it turns out, both $G_{i,k-i}$ and $g_{i,k-i}$ can be obtained by short recursions, which are given by

$$\begin{aligned} G_{i,k-i} &= A_1(h_{i-1,k-i}I_n + G_{i-1,k-i}) + A_2(h_{i,k-1-i}I_n + G_{i,k-1-i}), \\ g_{i,k-i} &= (G_{i,k-i} - G_{i,k-1-i} - G_{i-1,k-i})\mu - (h_{i,k-1-i} + h_{i-1,k-i})\mu \\ &\quad + A_1 g_{i-1,k-i} + A_2 g_{i,k-1-i}. \end{aligned}$$

Here, we adopt the convention that $G_{i,j} = 0_{n\times n}$, $g_{i,j} = 0_n$, $h_{i,j} = 0$ when either i or j is negative.

Together with the initial conditions $h_{0,0} = 0$, $g_{0,0} = 0_n$, and $G_{0,0} = 0_{n\times n}$, the above two recursions allow us to efficiently compute $h_{i,k-i}$ for $i = 0,\dots,k$ using only $h_{i,k-1-i}$, $G_{i,k-1-i}$ and $g_{i,k-1-i}$ for $i = 0,\dots,k-1$. This stands in contrast to the long recursion in Proposition 5. More importantly, the updating time of $h_{i,k-i}$ is the same regardless of k, and this is particularly desirable for the computation of μ_p^q that may require summing up $h_{i,k-i}$ for very large values of k.

With our new infinite series representation of μ_q^p and the corresponding fast updating algorithm for $h_{i,k-i}$, the infinite series approach becomes very competitive against the integration approach for numerical evaluation of μ_q^p. For example, with $n = 120$, we find that numerical calculation of μ_q^p based on the new infinite series representation with the fast algorithm is on average ten times faster than using the integration formula in Section 3.

An important advantage of (12) over the integration approach is that $h_{i,j}(\hat{A},\hat{B})$ in (12) does not depend on p and q. Therefore, if one needs to compute a table of μ_q^p for different values of p and q, then one only needs to compute $h_{i,j}$'s once and they can be used to update the series approximation of μ_q^p for all values of p and q. In contrast, the integration approach needs to perform a separate integration for each single p and q, and it can be much more time consuming.

For the sake of completeness, we also present the fast updating algorithm for $\tilde{h}_{i,k-i}$. Defining matrix $\tilde{G}_{i,k-i}$ and vector $\tilde{g}_{i,k-i}$ as follows:

$$\begin{aligned} \tilde{G}_{i,k-i} &= \sum_{\nu_1=0}^{i} \sum_{\substack{\nu_2=0 \\ \nu_1+\nu_2>0}}^{k-i} [t_1^{\nu_1} t_2^{\nu_2}] A(\boldsymbol{t})^{\nu_1+\nu_2} \tilde{h}_{i-\nu_1,k-i-\nu_2}, \\ \tilde{g}_{i,k-i} &= \sum_{\nu_1=0}^{i} \sum_{\substack{\nu_2=0 \\ \nu_1+\nu_2>0}}^{k-i} (\nu_1+\nu_2)\left([t_1^{\nu_1} t_2^{\nu_2}] A(\boldsymbol{t})^{\nu_1+\nu_2} - [t_1^{\nu_1} t_2^{\nu_2-1}] A(\boldsymbol{t})^{\nu_1+\nu_2-1}\right) \mu \tilde{h}_{i-\nu_1,k-i-\nu_2}, \end{aligned}$$

then we have

$$\tilde{h}_{i,k-i} = \frac{\mathrm{tr}(\tilde{G}_{i,k-i}) + \mu'\tilde{g}_{i,k-i}}{2k}.$$

It can be easily verified that $\tilde{G}_{i,k-i}$ and $\tilde{g}_{i,k-i}$ have the following recurrence relations:

$$\begin{aligned}
\tilde{G}_{i,k-i} &= A_1(\tilde{h}_{i-1,k-i}I_n + \tilde{G}_{i-1,k-i}) + A_2(\tilde{h}_{i,k-1-i}I_n + \tilde{G}_{i,k-1-i}), \\
\tilde{g}_{i,k-i} &= (\tilde{G}_{i,k-i} - \tilde{G}_{i,k-1-i})\mu - \tilde{h}_{i,k-1-i}\mu + A_1\tilde{g}_{i-1,k-i} + A_2\tilde{g}_{i,k-1-i},
\end{aligned}$$

with the initial conditions $\tilde{h}_{0,0} = 0$, $\tilde{g}_{0,0} = 0_n$, and $\tilde{G}_{0,0} = 0_{n\times n}$.

6. Conclusion

In this paper, we present new results related to the mixed moments of ratios of quadratic forms in normal random variables. We first provide the necessary and sufficient conditions for the existence of mixed moments. We then present the integration and infinite series approaches for evaluating the mixed moments. In addition, we introduce a new and more efficient infinite series expansion of the mixed moments. For both the integration and infinite series approaches, we develop efficient numerical algorithms that are vastly superior than the existing ones in the literature.

Our results can be extended to deal with other problems. For example, it is rather straightforward to extend our results when the numerator is defined as $(x'Ax + b'x + c)^p$. For future research, it is of interest to generalize the integration and infinite series approaches to deal with ratios of quadratic forms in non-normal random variables.

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Appendix

Proof of Proposition 1: Since the proposition is known to be true for integer p, we only need to prove it for the case when p is not an integer, and we assume A is positive semi-definite in our proof. In addition, we only need to prove the necessary conditions for the existence of μ_q^p since the sufficient conditions in Proposition 1 have already been established by Roberts [11].

When B is positive definite, we let $\lambda_1 > 0$ be the largest eigenvalue of $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ and we have

$$\frac{x'Ax}{x'Bx} \leq \lambda_1 \Rightarrow \frac{x'Bx}{x'Ax} \geq \frac{1}{\lambda_1}.$$

Let $\lceil p \rceil$ be the smallest integer that is larger than p. Since

$$\frac{(x'Ax)^p}{(x'Bx)^q} = \left(\frac{x'Bx}{x'Ax}\right)^{\lceil p \rceil - p} \frac{(x'Ax)^{\lceil p \rceil}}{(x'Bx)^{q+\lceil p \rceil - p}} \geq \frac{1}{\lambda_1^{\lceil p \rceil - p}} \frac{(x'Ax)^{\lceil p \rceil}}{(x'Bx)^{q+\lceil p \rceil - p}},$$

we have

$$\mu_q^p \geq \frac{1}{\lambda_1^{\lceil p \rceil - p}} \mu_{q+\lceil p \rceil - p}^{\lceil p \rceil}.$$

As $\lceil p \rceil$ is an integer, $\mu_{q+\lceil p \rceil - p}^{\lceil p \rceil}$ exists if and only $\frac{n}{2} + \lceil p \rceil > q + \lceil p \rceil - p$, or equivalently $\frac{n}{2} + p > q$. Therefore, when $\frac{n}{2} + p \leq q$, μ_q^p does not exist.

We now consider the case when B is positive semi-definite. For case (i) with $\tilde{A}_{12} = 0_{m \times (n-m)}$ and $\tilde{A}_{22} = 0_{(n-m)\times(n-m)}$, we have

$$\frac{(x'Ax)^p}{(x'Bx)^q} = \frac{(z'\tilde{A}z)^p}{(z_1'D_{1b}z_1)^q} = \frac{(z_1'\tilde{A}_{11}z_1)^p}{(z_1'D_{1b}z_1)^q}.$$

Since D_{1b} is a positive definite matrix, we can use the result from the positive definite B case to show that $\frac{m}{2} + p > q$ is both necessary and sufficient for μ_q^p to exist.

For case (iii) with $\tilde{A}_{22} \neq 0_{(n-m)\times(n-m)}$, we let $QD_aQ' = \tilde{A}$, where $D_a = \text{Diag}(a_1, \ldots, a_r)$ with $a_1 \geq a_2 \geq \cdots \geq a_r > 0$ being the $r \leq n$ nonzero eigenvalues of $\tilde{A}$, and Q is an $n \times r$ matrix of the corresponding eigenvectors. Let $\tilde{z} = Q'z$; then we have

$$z'\tilde{A}z = \tilde{z}'D_a\tilde{z} = \sum_{i=1}^{r} a_i\tilde{z}_i^2.$$

Note that $\tilde{z}_i = q_i'z = q_{i,1}{}'z_1 + q_{i,2}{}'z_2$, where $q_i = [q_{i,1}',\ q_{i,2}']'$ is the i-th column of Q. Since $\tilde{A}_{22}$ is a nonzero matrix, at least one of the $q_{i,2}$'s is not a zero vector. Suppose $q_{j,2}$ is not a zero vector. Conditional on z_1, $\tilde{z}_j^2 \sim \sigma^2\chi_1^2(\lambda)$, where $\lambda = (q_{j,1}'z_1 + q_{j,2}{}'\mu_2)^2/\sigma^2$ and $\sigma^2 = q_{j,2}'q_{j,2}$.

For a noncentral chi-squared random variable with ν degrees of freedom and a noncentrality parameter of λ, its p-th moment exists if and only if $p > -\nu/2$ and it is given by (see, for example, Krishnan [6]),

$$E[\chi^2_\nu(\lambda)^p] = \frac{2^p e^{-\frac{\lambda}{2}}\Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}\right)} {}_1F_1\left(\frac{2p+\nu}{2}; \frac{1}{2}; \frac{\lambda}{2}\right),$$

where ${}_1F_1(\cdot;\cdot;\cdot)$ is the confluent hypergeometric function. Using this result and the fact that $\tilde{z}'D_a\tilde{z} \geq a_j\tilde{z}_j^2$, we have

$$\begin{aligned}
E\left[\frac{(\tilde{z}'D_a\tilde{z})^p}{(z_1'D_{1b}z_1)^q}\right] &\geq a_j^p E\left[\frac{\tilde{z}_j^{2p}}{(z_1'D_{1b}z_1)^q}\right] \\
&= \frac{(2a_j\sigma^2)^p\Gamma\left(\frac{1}{2}+p\right)}{\Gamma\left(\frac{1}{2}\right)} E\left[\frac{e^{-\frac{\lambda}{2}}}{(z_1'D_{1b}z_1)^q} {}_1F_1\left(\frac{2p+1}{2}; \frac{1}{2}; \frac{\lambda}{2}\right)\right] \\
&> \frac{(2a_j\sigma^2)^p\Gamma\left(\frac{1}{2}+p\right)}{\Gamma\left(\frac{1}{2}\right)} E\left[\frac{e^{-\frac{\lambda}{2}}}{(z_1'D_{1b}z_1)^q} {}_1F_1\left(\frac{1}{2}; \frac{1}{2}; \frac{\lambda}{2}\right)\right] \\
&= \frac{(2a_j\sigma^2)^p\Gamma\left(\frac{1}{2}+p\right)}{\Gamma\left(\frac{1}{2}\right)} E\left[\frac{1}{(z_1'D_{1b}z_1)^q}\right].
\end{aligned}$$

As $b_m(z_1'z_1) \leq z_1'D_{1b}z_1 \leq b_1(z_1'z_1)$ and $z_1'z_1 \sim \chi^2_m(\mu'P_1P_1'\mu)$, the expectation on the right hand side exists if and only if $m/2 > q$. It follows that μ_q^p does not exist when $m/2 \leq q$. □

Proof of Proposition 2: Note that

$$\begin{aligned}
&\frac{\partial^k\phi(t_1,t_2)}{\partial t_1^k} \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}(x'Ax)^k\exp(t_1x'Ax+t_2x'Bx)\exp\left(-\frac{(x-\mu)'(x-\mu)}{2}\right)(\mathrm{d}x) \\
&= \exp\left(-\frac{\mu'\mu}{2}\right)\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}(x'Ax)^k\exp\left(-\frac{x'(I_n-2t_1A-2t_2B)x}{2}+x'\mu\right)(\mathrm{d}x).
\end{aligned}$$

Making a transformation of $w = L^{-1}x$, where L is an $n \times n$ matrix such that $(I_n - 2t_1A - 2t_2B)^{-1} = LL'$, we have $x'Ax = w'(L'AL)w = w'Rw$. Then denoting $\tilde{\mu} = L'\mu$, we obtain

$$\begin{aligned}\frac{\partial^k \phi(t_1,t_2)}{\partial t_1^k} &= |L| \exp\left(\frac{\tilde{\mu}'\tilde{\mu}}{2} - \frac{\mu'\mu}{2}\right) \\ &\quad \times \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} (w'Rw)^k \exp\left(-\frac{(w-\tilde{\mu})'(w-\tilde{\mu})}{2}\right)(\mathrm{d}w) \\ &= \phi(t_1,t_2) E[(w'Rw)^k], \qquad (14)\end{aligned}$$

where we make use of the identities $x'\mu = w'\tilde{\mu}$ and $|L| = |I_n - 2t_1A - 2t_2B|^{-\frac{1}{2}}$, and $w \sim N(\tilde{\mu}, I_n)$ in the second equality. Setting $t_1 = 0$ and $t_2 = -t$ in (14) yields (3) and setting $t_1 = -s$ and $t_2 = -t$ yields (4). □

Proof of Proposition 3: We provide only the derivation of (10) here (the derivation of (11) is similar). When μ_q^p exists, we use (2) to obtain

$$\begin{aligned}\mu_q^p &= \frac{\beta^q}{\alpha^p} E\left[\frac{(x'(\alpha A)x)^p}{(x'(\beta B)x)^q}\right] \\ &= \frac{(-1)^{\lceil p\rceil}\beta^q e^{-\frac{\mu'\mu}{2}}}{\alpha^p \Gamma(\langle p\rangle)\Gamma(q)} \int_0^\infty \int_0^\infty s^{\langle p\rangle -1} t^{q-1} \frac{\partial^{\lceil p\rceil}}{\partial s^{\lceil p\rceil}} |I_n + 2s\alpha A + 2t\beta B|^{-\frac{1}{2}} \\ &\quad \times \exp\left(\frac{\mu'(I_n + 2s\alpha A + 2t\beta B)^{-1}\mu}{2}\right) \mathrm{d}s\mathrm{d}t,\end{aligned}$$

where the extra $(-1)^{\lceil p\rceil}$ term is introduced because we write the integrand as derivatives of $\phi(-s,-t)$. Using the fact

$$I_n + 2s\alpha A + 2t\beta B = (1+2s+2t)\left(I_n - \frac{2s}{1+2s+2t}\hat{A} - \frac{2t}{1+2s+2t}\hat{B}\right),$$

where $\hat{A} = I_n - \alpha A$ and $\hat{B} = I_n - \beta B$, we can write

$$\begin{aligned}\mu_q^p &= \frac{(-1)^{\lceil p\rceil}\beta^q e^{-\frac{\mu'\mu}{2}}}{\alpha^p \Gamma(\langle p\rangle)\Gamma(q)} \int_0^\infty \int_0^\infty s^{\langle p\rangle -1} t^{q-1} \frac{\partial^{\lceil p\rceil}}{\partial s^{\lceil p\rceil}} (1+2s+2t)^{-\frac{n}{2}} \\ &\quad \times |I_n - t_1\hat{A} - t_2\hat{B}|^{-\frac{1}{2}} \exp\left(\frac{\mu'(I_n - t_1\hat{A} - t_2\hat{B})^{-1}\mu}{2(1+2s+2t)}\right) \mathrm{d}s\mathrm{d}t, \qquad (15)\end{aligned}$$

where $t_1 = 2s/(1+2s+2t)$ and $t_2 = 2t/(1+2s+2t)$.

Let $c = 2(1+2s+2t)$ and note that

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu')}{c^k \left(\frac{1}{2}\right)_k} \\
= &\sum_{k=0}^{\infty} \frac{1}{c^k \left(\frac{1}{2}\right)_k} [t_1^i t_2^j t_3^k] |I_n - t_1\hat{A} - t_2\hat{B} - t_3\mu\mu'|^{-\frac{1}{2}} \\
= &\sum_{k=0}^{\infty} \frac{1}{c^k \left(\frac{1}{2}\right)_k} [t_1^i t_2^j t_3^k] |I_n - t_1\hat{A} - t_2\hat{B}|^{-\frac{1}{2}} [1 - t_3\mu'(I_n - t_1\hat{A} - t_2\hat{B})^{-1}\mu]^{-\frac{1}{2}} \\
= &\sum_{k=0}^{\infty} \frac{1}{c^k \left(\frac{1}{2}\right)_k} [t_1^i t_2^j] |I_n - t_1\hat{A} - t_2\hat{B}|^{-\frac{1}{2}} \frac{\left(\frac{1}{2}\right)_k [\mu'(I_n - t_1\hat{A} - t_2\hat{B})^{-1}\mu]^k}{k!} \\
= &[t_1^i t_2^j] |I_n - t_1\hat{A} - t_2\hat{B}|^{-\frac{1}{2}} \exp\left(\frac{\mu'(I_n - t_1\hat{A} - t_2\hat{B})^{-1}\mu}{c}\right), \qquad (16)
\end{aligned}$$

so we can now write (15) as

$$\begin{aligned}
\mu_q^p = &\frac{(-1)^{\lceil p \rceil} \beta^q e^{-\frac{\mu'\mu}{2}}}{\alpha^p \Gamma(\langle p \rangle) \Gamma(q)} \int_0^{\infty} \int_0^{\infty} s^{\langle p \rangle - 1} t^{q-1} \frac{\partial^{\lceil p \rceil}}{\partial s^{\lceil p \rceil}} (1+2s+2t)^{-\frac{n}{2}} \\
&\times \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu') t_1^i t_2^j}{2^k (1+2s+2t)^k \left(\frac{1}{2}\right)_k} \mathrm{d}s\mathrm{d}t \\
= &\frac{(-1)^{\lceil p \rceil} \beta^q e^{-\frac{\mu'\mu}{2}}}{\alpha^p \Gamma(\langle p \rangle) \Gamma(q)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu')}{2^k \left(\frac{1}{2}\right)_k} \int_0^{\infty} t^{q-1} (2t)^j \\
&\times \int_0^{\infty} s^{\langle p \rangle - 1} \frac{\partial^{\lceil p \rceil}}{\partial s^{\lceil p \rceil}} (2s)^i (1+2s+2t)^{-\frac{n}{2}-i-j-k} \mathrm{d}s\mathrm{d}t.
\end{aligned}$$

In order to prove (10), we need the following lemma.

Lemma 1. *Suppose $\frac{n}{2} + p > q$ and i, j, k are nonnegative integers. We have*

$$\begin{aligned}
&\frac{(-1)^{\lceil p \rceil}}{\Gamma(\langle p \rangle) \Gamma(q)} \int_0^{\infty} t^{q-1} (2t)^j \int_0^{\infty} s^{\langle p \rangle - 1} \frac{\partial^{\lceil p \rceil}}{\partial s^{\lceil p \rceil}} (2s)^i (1+2s+2t)^{-\frac{n}{2}-i-j-k} \mathrm{d}s\mathrm{d}t \\
= &\frac{2^{p-q} (-p)_i (q)_j \Gamma\left(\frac{n}{2} + p - q + k\right)}{\Gamma\left(\frac{n}{2} + i + j + k\right)}.
\end{aligned}$$

Proof: Using the Leibniz rule, we have

$$\begin{aligned}
&\frac{\partial^{\lceil p\rceil}}{\partial s^{\lceil p\rceil}}(2s)^i(1+2s+2t)^{-\frac{n}{2}-i-j-k}\\
=\ &\sum_{l=0}^{\min[i,\lceil p\rceil]}\binom{\lceil p\rceil}{l}(-1)^l(-i)_l 2^l(2s)^{i-l}\\
&\times(-1)^{\lceil p\rceil-l}\left(\frac{n}{2}+i+j+k\right)_{\lceil p\rceil-l}2^{\lceil p\rceil-l}(1+2s+2t)^{-\frac{n}{2}-i-j-k-\lceil p\rceil+l}\\
=\ &\frac{2^{\lceil p\rceil}(-1)^{\lceil p\rceil}}{\Gamma\left(\frac{n}{2}+i+j+k\right)}\sum_{l=0}^{\min[i,\lceil p\rceil]}\binom{\lceil p\rceil}{l}(-i)_l\Gamma\left(\frac{n}{2}+i+j+k+\lceil p\rceil-l\right)\\
&\times(2s)^{i-l}(1+2s+2t)^{-\frac{n}{2}-i-j-k-\lceil p\rceil+l}.
\end{aligned}$$

When $i+\langle p\rangle>l$ (which is satisfied here because $l\le\min[i,\lceil p\rceil]$), we have

$$\begin{aligned}
&\int_0^\infty s^{\langle p\rangle-1}(2s)^{i-l}(1+2s+2t)^{-\frac{n}{2}-i-j-k-\lceil p\rceil+l}\mathrm{d}s\\
=\ &\frac{2^{-\langle p\rangle}(1+2t)^{-\frac{n}{2}-p-j-k}\Gamma\left(\frac{n}{2}+p+j+k\right)\Gamma(i-l+\langle p\rangle)}{\Gamma\left(\frac{n}{2}+i+j+k+\lceil p\rceil-l\right)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_0^\infty s^{\langle p\rangle-1}\frac{\partial^{\lceil p\rceil}}{\partial s^{\lceil p\rceil}}(2s)^i(1+2s+2t)^{-\frac{n}{2}-i-j-k}\mathrm{d}s\\
=\ &\frac{2^p(-1)^{\lceil p\rceil}\Gamma(i-p)(1+2t)^{-\frac{n}{2}-p-j-k}\Gamma\left(\frac{n}{2}+p+j+k\right)}{\Gamma\left(\frac{n}{2}+i+j+k\right)}\sum_{l=0}^{\min[\lceil p\rceil,i]}\binom{\lceil p\rceil}{l}(-i)_l(i-p)_{\lceil p\rceil-l}\\
=\ &\frac{2^p(-1)^{\lceil p\rceil}\Gamma(i-p)(1+2t)^{-\frac{n}{2}-p-j-k}\Gamma\left(\frac{n}{2}+p+j+k\right)}{\Gamma\left(\frac{n}{2}+i+j+k\right)}\sum_{l=0}^{\lceil p\rceil}\binom{\lceil p\rceil}{l}(-i)_l(i-p)_{\lceil p\rceil-l}\\
=\ &\frac{2^p(-1)^{\lceil p\rceil}\Gamma(i-p)(-p)_{\lceil p\rceil}\Gamma\left(\frac{n}{2}+p+j+k\right)}{\Gamma\left(\frac{n}{2}+i+j+k\right)}(1+2t)^{-\frac{n}{2}-p-j-k},
\end{aligned}$$

where the last equality follows from the Chu-Vandermonde identity. When $\frac{n}{2}+p>q$, we have

$$\int_0^\infty t^{q-1}(2t)^j(1+2t)^{-\frac{n}{2}-p-j-k}\mathrm{d}t=\frac{2^{-q}\Gamma\left(\frac{n}{2}+p-q+k\right)\Gamma(q+j)}{\Gamma\left(\frac{n}{2}+p+j+k\right)}.$$

Using this result, we obtain

$$\begin{aligned}
&\frac{(-1)^{\lceil p\rceil}}{\Gamma(\langle p\rangle)\Gamma(q)}\int_0^\infty t^{q-1}(2t)^j\int_0^\infty s^{\langle p\rangle-1}\frac{\partial^{\lceil p\rceil}}{\partial s^{\lceil p\rceil}}(2s)^i(1+2s+2t)^{-\frac{n}{2}-i-j-k}\mathrm{d}s\mathrm{d}t\\
= \;& \frac{(-1)^{\lceil p\rceil}}{\Gamma(\langle p\rangle)\Gamma(q)}\frac{2^p(-1)^{\lceil p\rceil}\Gamma(i-p)(-p)_{\lceil p\rceil}}{\Gamma\left(\frac{n}{2}+i+j+k\right)}2^{-q}\Gamma\left(\frac{n}{2}+p-q+k\right)\Gamma(q+j)\\
= \;& \frac{2^{p-q}(-p)_i(q)_j\Gamma\left(\frac{n}{2}+p-q+k\right)}{\Gamma\left(\frac{n}{2}+i+j+k\right)}.
\end{aligned}$$

This completes the proof of Lemma 1. □

Note that since μ_q^p exists, the condition $\frac{n}{2}+p>q$ must be satisfied, and we can apply Lemma 1 in (16) to obtain (10). However, in the above derivation, we perform integration term by term, so we need to justify this operation by showing that when μ_q^p exists, the infinite series is uniformly absolute convergent under the assumptions of $0<\alpha<2/a_{\max}$ and $0<\beta<2/b_{\max}$.

We first show that when B is positive definite, then the triple infinite series in (10) is uniformly absolute convergent when $\frac{n}{2}+p>q$ (i.e., μ_q^p exists) even under the weaker assumption that A is positive semi-definite. Let $\hat{a}$ be the smallest absolute eigenvalue of $\hat{A}$ and $\hat{b}$ be the largest absolute eigenvalue of $\hat{B}$. Under the assumption $0<\alpha<2/a_{\max}$, all the eigenvalues of $\hat{A}$ have absolute values less than or equal to one. Since A is not a zero matrix, we must have $0\le\hat{a}<1$. In addition, under the assumption $0<\beta<2/b_{\max}$ and the fact that B is positive definite, we have $0\le\hat{b}<1$.

Let $u=[u_1,\ u_2']'\sim N(0_n,I_n)$, where u_1 is the first element of u; then it is easy to show that

$$\begin{aligned}
|u'\hat{A}u| &\le \hat{a}q_1+q_2,\\
|u'\hat{B}u| &\le \hat{b}(q_1+q_2),\\
|u'\mu\mu'u| &\le (\mu'\mu)(u'u)=\delta(q_1+q_2),
\end{aligned}$$

where $q_1=u_1'u_1\sim\chi_1^2$, $q_2=u_2'u_2\sim\chi_{n-1}^2$, independent of each other, and $\delta=\mu'\mu$. With these inequalities, we derive an upper bound on $|d_{p,j,k}(\hat{A},\hat{B},\mu\mu')|$ for this case. Using the relation between top-order invariant polynomials and product moments of ratios of quadratic forms (see, for example, Eq.(48) of

Hillier, Kan, and Wang [4]), we have

$$\begin{aligned}
& |d_{i,j,k}(\hat{A},\hat{B},\mu\mu')| \\
= & \left|\frac{E[(u'\hat{A}u)^i(u'\hat{B}u)^j(u'\mu\mu'u)^k]}{2^{i+j+k}i!j!k!}\right| \\
\leq & \frac{\hat{b}^j\delta^k E[(\hat{a}q_1+q_2)^i(q_1+q_2)^{j+k}]}{2^{i+j+k}i!j!k!} \\
= & \frac{\hat{b}^j\delta^k}{2^{i+j+k}i!j!k!}\sum_{s=0}^{i}\sum_{t=0}^{j+k}\binom{i}{s}\binom{j+k}{t}E[q_1^{s+t}]E[q_2^{i+j+k-s-t}]\hat{a}^s \\
= & \frac{\hat{b}^j\delta^k}{i!j!k!}\sum_{s=0}^{i}\binom{i}{s}\hat{a}^s\sum_{t=0}^{j+k}\binom{j+k}{t}\left(\frac{1}{2}\right)_{s+t}\left(\frac{n-1}{2}\right)_{i+j+k-s-t} \\
= & \frac{\hat{b}^j\delta^k}{i!j!k!}\sum_{s=0}^{i}\binom{i}{s}\left(\frac{1}{2}\right)_s\left(\frac{n-1}{2}\right)_{i-s}\hat{a}^s\sum_{t=0}^{j+k}\left(\frac{1}{2}+s\right)_t\left(\frac{n-1}{2}+i-s\right)_{j+k-t} \\
= & \left(\frac{n}{2}+i\right)_{j+k}\frac{\hat{b}^j\delta^k}{j!k!}\sum_{s=0}^{i}\frac{\left(\frac{1}{2}\right)_s\left(\frac{n-1}{2}\right)_{i-s}\hat{a}^s}{s!(i-s)!},
\end{aligned}$$

where the third equality is obtained by making use of the fact that $E[Q^r] = 2^r(\nu/2)_r$ for $Q \sim \chi^2_\nu$, and the last equality follows from the Chu-Vandermonde identity. Let

$$g_i = i!\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q)_j\Gamma\left(\frac{n}{2}+p-q+k\right)}{2^k\left(\frac{1}{2}\right)_k\Gamma\left(\frac{n}{2}+i+j+k\right)}d_{i,j,k}(\hat{A},\hat{B},\mu\mu'), \tag{17}$$

then we can write (10) as

$$\mu_q^p = \frac{2^{p-q}\beta^q e^{-\frac{\delta}{2}}}{\alpha^p}\sum_{i=0}^{\infty}\frac{(-p)_i}{i!}g_i = \frac{2^{p-q}\beta^q e^{-\frac{\delta}{2}}}{\alpha^p}\left[\sum_{i=0}^{\lceil p\rceil-1}\frac{(-p)_i}{i!}g_i + \sum_{i=\lceil p\rceil}^{\infty}\frac{(-p)_i}{i!}g_i\right]. \tag{18}$$

With the bound on $|d_{i,j,k}(\hat{A},\hat{B},\hat{\mu}\mu')|$ and under the assumption that $\frac{n}{2}+p > q$, we can obtain a bound on $|g_i|$ as follows:

$$\begin{aligned}
|g_i| &\leq i!\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q)_j\Gamma\left(\frac{n}{2}+p-q+k\right)}{2^k\left(\frac{1}{2}\right)_k\Gamma\left(\frac{n}{2}+i+j+k\right)}\left(\frac{n}{2}+i\right)_{j+k}\frac{\hat{b}^j\delta^k}{j!k!}\sum_{s=0}^{i}\frac{\left(\frac{1}{2}\right)_s\left(\frac{n-1}{2}\right)_{i-s}\hat{a}^s}{s!(i-s)!} \\
&= \frac{i!\Gamma\left(\frac{n}{2}+p-q\right)}{\Gamma\left(\frac{n}{2}+i\right)}{}_1F_1\left(\frac{n}{2}+p-q;\frac{1}{2};\frac{\delta}{2}\right)(1-\hat{b})^{-q}\sum_{s=0}^{i}\frac{\left(\frac{1}{2}\right)_s\left(\frac{n-1}{2}\right)_{i-s}\hat{a}^s}{s!(i-s)!} \\
&= \frac{i!c}{\Gamma\left(\frac{n}{2}+i\right)}\sum_{s=0}^{i}\frac{\left(\frac{1}{2}\right)_s\left(\frac{n-1}{2}\right)_{i-s}\hat{a}^s}{s!(i-s)!},
\end{aligned}$$

where $c=\Gamma\left(\frac{n}{2}+p-q\right){}_1F_1\left(\frac{n}{2}+p-q;\frac{1}{2};\frac{\delta}{2}\right)(1-\hat{b})^{-q}$ is finite because $\hat{b}<1$. Note that the first sum in (18) is finite because it has only finite number of terms, and each term is uniformly absolute convergent. For the second sum in (18), we know $(-p)_i$'s have the same sign when $i\geq\lceil p\rceil$, so we have

$$\sum_{i=\lceil p\rceil}^{\infty}\left|\frac{(-p)_i}{i!}g_i\right|\leq\left|\sum_{i=\lceil p\rceil}^{\infty}\frac{(-p)_i}{i!}|g_i|\right|=\left|\sum_{i=0}^{\infty}\frac{(-p)_i}{i!}|g_i|-\sum_{i=0}^{\lceil p\rceil-1}\frac{(-p)_i}{i!}|g_i|\right|,$$

and it suffices to show that the first infinite sum on the right hand side is uniformly absolute convergent. Letting $r=i-s$ and exchanging the order of summation, we obtain

$$\begin{aligned}
\sum_{i=0}^{\infty}\frac{(-p)_i}{i!}|g_i| &\leq c\sum_{i=0}^{\infty}\frac{(-p)_i}{\Gamma\left(\frac{n}{2}+i\right)}\sum_{s=0}^{i}\frac{\left(\frac{1}{2}\right)_s\left(\frac{n-1}{2}\right)_{i-s}\hat{a}^s}{s!(i-s)!} \\
&= c\sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-p)_{r+s}\hat{a}^s}{\Gamma\left(\frac{n}{2}+r+s\right)}\frac{\left(\frac{1}{2}\right)_s\left(\frac{n-1}{2}\right)_r}{r!s!} \\
&= c\sum_{s=0}^{\infty}\frac{(-p)_s\left(\frac{1}{2}\right)_s\hat{a}^s}{\Gamma\left(\frac{n}{2}+s\right)s!}\sum_{r=0}^{\infty}\frac{(-p+s)_r\left(\frac{n-1}{2}\right)_r}{\left(\frac{n}{2}+s\right)_r r!} \\
&= c\sum_{s=0}^{\infty}\frac{(-p)_s\left(\frac{1}{2}\right)_s\hat{a}^s}{\Gamma\left(\frac{n}{2}+s\right)s!}\frac{\Gamma\left(\frac{n}{2}+s\right)\Gamma\left(\frac{1}{2}+p\right)}{\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(\frac{n}{2}+p\right)} \\
&= \frac{c\Gamma\left(\frac{1}{2}+p\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}+p\right)}\sum_{s=0}^{\infty}\frac{(-p)_s\hat{a}^s}{s!} \\
&= \frac{c\Gamma\left(\frac{1}{2}+p\right)(1-\hat{a})^p}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}+p\right)},
\end{aligned}$$

where the third equality follows from the Gauss hypergeometric theorem, which states that

$$\sum_{i=0}^{\infty} \frac{(a)_i (b)_i}{(c)_i i!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

when $c > a + b$. Therefore, we establish the uniform absolute convergence of (10) when B is positive definite.

We now turn our attention to the cases that B is positive semi-definite. Suppose the rank of B is $0 < m < n$. Without loss of generality (see the proof of Proposition 1), we assume $B = \mathrm{Diag}(b_1, \ldots, b_m, 0_{n-m}')$, where $b_1 \geq b_2 \geq \cdots \geq b_m > 0$. There are two cases to consider. The first case is $A_{22} \neq 0_{(n-m)\times(n-m)}$. For this case, we show that when $\frac{m}{2} > q$ (i.e., μ_q^p exists), (10) is uniformly absolute convergent.

Let $\hat{a}$ be the largest absolute eigenvalue of $\hat{A}$ and $\hat{b} = \max_{i=1}^m |1 - \beta b_i|$. Under the assumptions $0 < \alpha < 2/a_{\max}$ and $0 < \beta < 2/b_{\max}$, we have $0 < \hat{a} \leq 1$ and $0 < \hat{b} < 1$. Defining $u = [u_1',\ u_2']' \sim N(0_n, I_n)$, where u_1 is the first m elements of u, we have $q_1 = u_1'u_1 \sim \chi_m^2$, $q_2 = u_2'u_2 \sim \chi_{n-m}^2$, and they are independent of each other. It is easy to show that

$$\begin{aligned} |u'\hat{A}u| &\leq \hat{a}(u'u) \leq q_1 + q_2, \\ |u'\hat{B}u| &\leq \hat{b}q_1 + q_2, \\ |u'\mu\mu'u| &\leq \delta(q_1 + q_2). \end{aligned}$$

With these inequalities, we can derive an upper bound of $|d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu')|$ as follows:

$$\begin{aligned} |d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu')| &\leq \frac{\delta^k E[(q_1+q_2)^i(\hat{b}q_1+q_2)^j(q_1+q_2)^k]}{2^{i+j+k} i! j! k!} \\ &= \frac{2^{i+j+k}\delta^k}{i!j!k!} \sum_{r=0}^{i+k} \sum_{s=0}^{j} \binom{i+k}{r} \binom{j}{s} \hat{b}^s E[q_1^{r+s}] E[q_2^{i+k-r+j-s}] \\ &= \frac{\delta^k \left(\frac{n}{2}+j\right)_{i+k}}{i!k!} \sum_{s=0}^{j} \frac{\left(\frac{m}{2}\right)_s \left(\frac{n-m}{2}\right)_{j-s} \hat{b}^s}{s!(j-s)!}. \end{aligned}$$

It follows that an upper bound for $|g_i|$ in (17) is:

$$\begin{aligned}
|g_i| &\leq \sum_{j=0}^{\infty}\frac{(q)_j}{\Gamma\left(\frac{n}{2}+j\right)}\sum_{k=0}^{\infty}\frac{\left(\frac{\delta}{2}\right)^k\Gamma\left(\frac{n}{2}+p-q+k\right)}{k!\left(\frac{1}{2}\right)_k}\sum_{s=0}^{j}\frac{\left(\frac{m}{2}\right)_s\left(\frac{n-m}{2}\right)_{j-s}\hat{b}^s}{s!(j-s)!} \\
&= \Gamma\left(\frac{n}{2}+p-q\right){}_1F_1\left(\frac{n}{2}+p-q;\frac{1}{2};\frac{\delta}{2}\right)\sum_{j=0}^{\infty}\frac{(q)_j}{\Gamma\left(\frac{n}{2}+j\right)}\sum_{s=0}^{j}\frac{\left(\frac{m}{2}\right)_s\left(\frac{n-m}{2}\right)_{j-s}\hat{b}^s}{s!(j-s)!} \\
&= \Gamma\left(\frac{n}{2}+p-q\right){}_1F_1\left(\frac{n}{2}+p-q;\frac{1}{2};\frac{\delta}{2}\right)\sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\frac{(q)_{r+s}}{\Gamma\left(\frac{n}{2}+r+s\right)}\frac{\left(\frac{m}{2}\right)_s\left(\frac{n-m}{2}\right)_r\hat{b}^s}{s!r!} \\
&= \Gamma\left(\frac{n}{2}+p-q\right){}_1F_1\left(\frac{n}{2}+p-q;\frac{1}{2};\frac{\delta}{2}\right)\sum_{s=0}^{\infty}\frac{(q)_s\hat{b}^s}{\Gamma\left(\frac{n}{2}+s\right)\left(\frac{m}{2}\right)_s s!}\sum_{r=0}^{\infty}\frac{(q+s)_r}{\left(\frac{n}{2}+s\right)_r}\frac{\left(\frac{n-m}{2}\right)_r}{r!} \\
&= \Gamma\left(\frac{n}{2}+p-q\right){}_1F_1\left(\frac{n}{2}+p-q;\frac{1}{2};\frac{\delta}{2}\right)\sum_{s=0}^{\infty}\frac{(q)_s\hat{b}^s}{\Gamma\left(\frac{n}{2}+s\right)\left(\frac{m}{2}\right)_s s!}\frac{\Gamma\left(\frac{n}{2}+s\right)\Gamma\left(\frac{m}{2}-q\right)}{\Gamma\left(\frac{m}{2}+s\right)\Gamma\left(\frac{n}{2}-q\right)} \\
&= \frac{\Gamma\left(\frac{n}{2}+p-q\right)\Gamma\left(\frac{m}{2}-q\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}-q\right)}{}_1F_1\left(\frac{n}{2}+p-q;\frac{1}{2};\frac{\delta}{2}\right)(1-\hat{b})^{-q}\equiv d,
\end{aligned}$$

where the second last equality is obtained by using the Gauss hypergeometric theorem when $\frac{m}{2}>q$. Since the confluent hypergeometric function ${}_1F_1$ converges uniformly for all values of its arguments, g_i is uniformly absolute convergent when $m/2>q$.

As before, it suffices to prove that the second sum in (18) is uniformly absolute convergent. The second sum is finite because it can be written as

$$\begin{aligned}
\sum_{i=\lceil p\rceil}^{\infty}\frac{(-p)_i}{i!} &= \sum_{j=0}^{\infty}\frac{(-p)_{\lceil p\rceil+j}}{\Gamma(\lceil p\rceil+j+1)} \\
&= \frac{(-p)_{\lceil p\rceil}}{\Gamma(\lceil p\rceil+1)}\sum_{j=0}^{\infty}\frac{(\langle p\rangle)_j}{(\lceil p\rceil+1)_j} \\
&= \frac{(-p)_{\lceil p\rceil}}{\Gamma(\lceil p\rceil+1)}\frac{\lceil p\rceil}{p} \\
&= \frac{(-p)_{\lceil p\rceil}}{p\Gamma(\lceil p\rceil)},
\end{aligned}$$

where the third equality follows from the Gauss hypergeometric theorem. Therefore, the second infinite sum is uniformly absolute convergent when $m/2>q$.

The last case that we need to consider is $A_{12} = 0_{m\times(n-m)}$ and $A_{22} = 0_{(n-m)\times(n-m)}$. For this case, we need to prove the following identity. Suppose $A_{12} = 0_{m\times(n-m)}$, $A_{22} = 0_{(n-m)\times(n-m)}$, $B_{12} = 0_{m\times(n-m)}$, and $B_{22} = 0_{(n-m)\times(n-m)}$. Let $\delta_1 = \mu_1'\mu_1$ and $\delta_2 = \mu_2'\mu_2$, where μ_1 is the first m elements of μ and μ_2 is the last $n-m$ elements of μ. We have

$$\begin{aligned} & \frac{2^{p-q}\beta^q e^{-\frac{\delta}{2}}}{\alpha^p}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-p)_i(q)_j\Gamma\left(\frac{n}{2}+p-q+k\right)}{2^k\left(\frac{1}{2}\right)_k\Gamma\left(\frac{n}{2}+i+j+k\right)}d_{i,j,k}(\hat{A},\hat{B},\mu\mu') \\ = & \frac{2^{p-q}\beta^q e^{-\frac{\delta_1}{2}}}{\alpha^p}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-p)_i(q)_j\Gamma\left(\frac{m}{2}+p-q+t\right)}{2^t\left(\frac{1}{2}\right)_t\Gamma\left(\frac{m}{2}+r+s+t\right)}d_{i,j,k}(\hat{A}_{11},\hat{B}_{11},\mu_1\mu_1'). \end{aligned}$$

Since B_{11} is positive definite, once we prove this identity, we can then apply the earlier result for positive definite B to show that the infinite series is uniformly absolute convergent when $\frac{m}{2}+p > q$.

We need two identities in our proof. The first identity is given in the following lemma.

Lemma 2. *Suppose $d > a+b+c$. We have*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a)_i(b)_j(c)_{i+j}}{(d)_{i+j}i!j!} = \frac{\Gamma(d)\Gamma(d-a-b-c)}{\Gamma(d-a-b)\Gamma(d-c)}. \tag{19}$$

Proof: Letting $k = i+j$, we have

$$\begin{aligned} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a)_i(b)_j(c)_{i+j}}{(d)_{i+j}i!j!} &= \sum_{k=0}^{\infty}\frac{(c)_k}{(d)_k k!}\sum_{i=0}^{k}\binom{k}{i}(a)_i(b)_{k-i} \\ &= \sum_{k=0}^{\infty}\frac{(c)_k(a+b)_k}{(d)_k k!} \\ &= \frac{\Gamma(d)\Gamma(d-a-b-c)}{\Gamma(d-c)\Gamma(d-a-b)}, \end{aligned}$$

where the second equality follows from the Chu-Vandermonde identity, and the last equality follows from the Gauss hypergeometric theorem when $d > a+b+c$. This completes the proof of Lemma 2. □

The second identity relates $d_{i,j,k}(\hat{A},\hat{B},\mu\mu')$ to $d_{i,j,k}(\hat{A}_{11},\hat{B}_{11},\mu_1\mu_1')$ and it is given in the following lemma.

Lemma 3. *Suppose* $A_{12} = 0_{m\times(n-m)}$*,* $A_{22} = 0_{(n-m)\times(n-m)}$*,* $B_{12} = 0_{m\times(n-m)}$*, and* $B_{22} = 0_{(n-m)\times(n-m)}$*. We have*

$$\begin{aligned} d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu') &= \sum_{r=0}^{i}\sum_{s=0}^{j}\sum_{t=0}^{k} \frac{d_{r,s,t}(\hat{A}_{11}, \hat{B}_{11}, \mu_1\mu_1')}{(i-r)!(j-s)!(k-t)!} \\ &\quad \times \frac{\left(\frac{1}{2}\right)_k \delta_2^{k-t}}{\left(\frac{1}{2}\right)_t} \left(\frac{n-m}{2} + k - t\right)_{i-r+j-s}. \end{aligned} \tag{20}$$

Proof: In order to prove this identity, we first note that when $A_{12} = 0_{m\times(n-m)}$, $A_{22} = 0_{(n-m)\times(n-m)}$, $B_{12} = 0_{m\times(n-m)}$, and $B_{22} = 0_{(n-m)\times(n-m)}$, we have

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & I_{n-m} \end{bmatrix}, \qquad \hat{B} = \begin{bmatrix} \hat{B}_{11} & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & I_{n-m} \end{bmatrix},$$

where $\hat{A}_{11} = I_m - \alpha A_{11}$ and $\hat{B}_{11} = I_m - \beta B_{11}$. Let $u = [u_1',\ u_2']' \sim N(0_n, I_n)$, where u_1 is the first m elements of u. We have $q_2 = u_2'u_2 \sim \chi^2_{n-m}$ and it is independent of u_1. Applying binomial expansions to $(u'\hat{A}u)^i = (u_1'\hat{A}_{11}u_1 + q_2)^i$, $(u'\hat{B}u)^j = (u_1'\hat{B}_{11}u_1 + q_2)^j$, $(u'\mu\mu'u)^k = (\mu_1'u_1 + \mu_2'u_2)^{2k}$, and noting that $E[(u_1'\hat{A}_{11}u_1)^r(u_1'\hat{B}_{11}u_1)^s(u_1'\mu_1)^t] = 0$ when t is odd, we have

$$\begin{aligned} & E[(u'\hat{A}u)^i(u'\hat{B}u)^j(u'\mu\mu'u)^k] \\ = & \sum_{r=0}^{i}\sum_{s=0}^{j}\sum_{t=0}^{k} \binom{i}{r}\binom{j}{s}\binom{2k}{2t} E[(u_1'\hat{A}_{11}u_1)^r(u_1'\hat{B}_{11}u_1)^s(u_1'\mu_1)^{2t}] E[q_2^{i-r+j-s}(\mu_2'u_2)^{2k-2t}]. \end{aligned}$$

Using the fact that $(2k)! = 2^{2k}k!\left(\frac{1}{2}\right)_k$, we can write

$$\binom{2k}{2t} = \binom{k}{t} \frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_t \left(\frac{1}{2}\right)_{k-t}}.$$

For the second expectation, we note that

$$X = \frac{(\mu_2'u_2)^2}{(\mu_2'\mu_2)(u_2'u_2)} \sim \text{Beta}\left(\frac{1}{2}, \frac{n-m}{2}\right)$$

is independent of $q_2 = u_2'u_2$ and its $(k-t)$-th moment is given by

$$E[X^{k-t}] = \frac{\left(\frac{1}{2}\right)_{k-t}}{\left(\frac{n-m}{2}\right)_{k-t}}.$$

It follows that

$$\begin{aligned}
E[q_2^{i-r+j-s}(\mu_2' u_2)^{2k-2t}] &= (\mu_2'\mu_2)^{k-t} E[q_2^{i-r+j-s+k-t} X^{k-t}] \\
&= \delta_2^{k-t} 2^{i-r+j-s+k-t} \left(\frac{n-m}{2}\right)_{i-r+j-s+k-t} \frac{\left(\frac{1}{2}\right)_{k-t}}{\left(\frac{n-m}{2}\right)_{k-t}} \\
&= \delta_2^{k-t} 2^{i-r+j-s+k-t} \left(\frac{1}{2}\right)_{k-t} \left(\frac{n-m}{2} + k - t\right)_{i-r+j-s}.
\end{aligned}$$

With these results and the fact that

$$\begin{aligned}
d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu') &= \frac{E[(u'\hat{A}u)^i (u'\hat{B}u)^j (u'\mu\mu' u)^k]}{2^{i+j+k} i! j! k!}, \\
d_{r,s,t}(\hat{A}_{11}, \hat{B}_{11}, \mu_1\mu_1') &= \frac{E[(u_1'\hat{A}_{11}u)^r (u_1'\hat{B}_{11}u_1)^s (u_1'\mu_1\mu_1' u_1)^t]}{2^{r+s+t} r! s! t!},
\end{aligned}$$

we can easily prove (20). This completes the proof of Lemma 3. □

With these two identities, we now prove our main result. Substituting (20) into (10), we obtain

$$\begin{aligned}
\mu_q^p &= \frac{2^{p-q}\beta^q e^{-\frac{\delta}{2}}}{\alpha^p} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-p)_i (q)_j \Gamma\left(\frac{n}{2}+p-q+k\right)}{2^k \left(\frac{1}{2}\right)_k \Gamma\left(\frac{n}{2}+i+j+k\right)} \\
&\quad \times \sum_{r=0}^{i}\sum_{s=0}^{j}\sum_{t=0}^{k} \frac{d_{r,s,t}(\hat{A}_{11}, \hat{B}_{11}, \mu_1\mu_1')}{(i-r)!(j-s)!(k-t)!} \frac{\left(\frac{1}{2}\right)_k \delta_2^{k-t} \left(\frac{n-m}{2}+k-t\right)_{i-r+j-v}}{\left(\frac{1}{2}\right)_t}.
\end{aligned}$$

Letting $u = i-r$, $v = j-s$, $w = k-t$ and exchanging the order of summation, we obtain

$$\begin{aligned}
\mu_q^p &= \frac{2^{p-q}\beta^q e^{-\frac{\delta}{2}}}{\alpha^p} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{(-p)_r (q)_s}{2^t \left(\frac{1}{2}\right)_t} d_{r,s,t}(\hat{A}_{11}, \hat{B}_{11}, \mu_1\mu_1') \\
&\quad \times \sum_{w=0}^{\infty} \frac{\left(\frac{\delta_2}{2}\right)^w \Gamma\left(\frac{n}{2}+p-q+t+w\right)}{w!\Gamma\left(\frac{n}{2}+r+s+t+w\right)} \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{(-p+r)_u (q+s)_v \left(\frac{n-m}{2}+w\right)_{u+v}}{\left(\frac{n}{2}+r+s+t+w\right)_{u+v} u! v!} \\
&= \frac{2^{p-q}\beta^q e^{-\frac{\delta}{2}}}{\alpha^p} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{(-p)_r (q)_s}{2^t \left(\frac{1}{2}\right)_t} d_{r,s,t}(\hat{A}_{11}, \hat{B}_{11}, \mu_1\mu_1') \\
&\quad \times \sum_{w=0}^{\infty} \frac{\left(\frac{\delta_2}{2}\right)^w \Gamma\left(\frac{n}{2}+p-q+t+w\right)}{w!\Gamma\left(\frac{n}{2}+r+s+t+w\right)} \frac{\Gamma\left(\frac{n}{2}+r+s+t+w\right)\Gamma\left(\frac{m}{2}+p-q+t\right)}{\Gamma\left(\frac{n}{2}+p-q+t+w\right)\Gamma\left(\frac{m}{2}+r+s+t\right)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2^{p-q}\beta^q e^{-\frac{\delta}{2}}}{\alpha^p}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-p)_r(q)_s\Gamma\left(\frac{m}{2}+p-q+t\right)}{2^t\left(\frac{1}{2}\right)_t\Gamma\left(\frac{m}{2}+r+s+t\right)}d_{r,s,t}(\hat{A}_{11},\hat{B}_{11},\mu_1\mu_1')\sum_{w=0}^{\infty}\frac{\left(\frac{\delta_2}{2}\right)^w}{w!} \\
&= \frac{2^{p-q}\beta^q e^{-\frac{\delta_1}{2}}}{\alpha^p}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-p)_r(q)_s\Gamma\left(\frac{m}{2}+p-q+t\right)}{2^t\left(\frac{1}{2}\right)_t\Gamma\left(\frac{m}{2}+r+s+t\right)}d_{r,s,t}(\hat{A}_{11},\hat{B}_{11},\mu_1\mu_1'),
\end{aligned}
$$

where the second equality is obtained by using (19). □

Proof of Proposition 4: We can provide a direct proof of (12), but we choose to present a transformation to obtain (12) from (10). By doing so, we no longer need to prove the uniform absolute convergence for (12) because this new infinite series expansion is just a rearrangement of Smith's infinite series expansion, and its convergence was established in Proposition 3.

We first build a connection between $h_{i,j}(\hat{A},\hat{B})$ and $d_{i,j,k}(\hat{A},\hat{B},\mu\mu')$. Writing $H(t_1,t_2)$ as

$$
\begin{aligned}
H(t_1,t_2) &= e^{-\frac{\mu'\mu}{2}}|I_n-t_1\hat{A}-t_2\hat{B}|^{-\frac{1}{2}}\sum_{m=0}^{\infty}\frac{(1-t_1-t_2)^m}{2^m m!}(\mu'(I_n-t_1A_1-t_2A_2)^{-1}\mu)^m \\
&= e^{-\frac{\mu'\mu}{2}}|I_n-t_1\hat{A}-t_2\hat{B}|^{-\frac{1}{2}}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^l(t_1+t_2)^l}{2^{m+l}l!m!}(\mu'(I_n-t_1\hat{A}-t_2\hat{B})^{-1}\mu)^{m+l},
\end{aligned}
$$

we have

$$
\begin{aligned}
h_{i,j} &= [t_1^i t_2^j]H(t_1,t_2) \\
&= e^{-\frac{\mu'\mu}{2}}\sum_{m=0}^{\infty}\frac{1}{2^m m!} \\
&\quad\times [t_1^i t_2^j]\sum_{l=0}^{i+j}\frac{(-1)^l(t_1+t_2)^l}{2^l l!}|I_n-t_1\hat{A}-t_2\hat{B}|^{-\frac{1}{2}}(\mu'(I_n-t_1\hat{A}-t_2\hat{B})^{-1}\mu)^{m+l} \\
&= e^{-\frac{\mu'\mu}{2}}\sum_{m=0}^{\infty}\frac{1}{2^m m!}\sum_{u=0}^{i}\sum_{v=0}^{j}\frac{(-1)^{u+v}}{2^{u+v}u!v!} \\
&\quad\times [t_1^{i-u}t_2^{j-v}]|I_n-t_1\hat{A}-t_2\hat{B}|^{-\frac{1}{2}}(\mu'(I_n-t_1\hat{A}-t_2\hat{B})^{-1}\mu)^{m+u+v} \\
&= e^{-\frac{\mu'\mu}{2}}\sum_{m=0}^{\infty}\sum_{u=0}^{i}\sum_{v=0}^{j}\frac{(-1)^{u+v}}{2^{m+u+v}m!u!v!}\frac{(m+u+v)!}{\left(\frac{1}{2}\right)_{m+u+v}}d_{i-u,j-v,m+u+v}(\hat{A},\hat{B},\mu\mu'),
\end{aligned}
$$

where the third equality is obtained by writing $l = u+v$, and the last equality follows from (16). Using this expression of $h_{i,j}$, we can obtain (12) from (10) as follows

$$\begin{aligned}
\mu_q^p &= \frac{2^{p-q}\beta^q e^{-\frac{\mu'\mu}{2}}}{\alpha^p}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-p)_i(q)_j\Gamma\left(\frac{n}{2}+p-q+k\right)}{2^k\left(\frac{1}{2}\right)_k\Gamma\left(\frac{n}{2}+i+j+k\right)}d_{i,j,k}(\hat{A},\hat{B},\mu\mu') \\
&= \frac{2^{p-q}\beta^q e^{-\frac{\mu'\mu}{2}}}{\alpha^p}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-p)_i(q)_j\Gamma\left(\frac{n}{2}+p-q\right)}{2^k\left(\frac{1}{2}\right)_k\Gamma\left(\frac{n}{2}+i+j\right)}d_{i,j,k}(\hat{A},\hat{B},\mu\mu') \\
&\quad\times\sum_{u=0}^{k}\sum_{v=0}^{k-u}\frac{(-k)_{u+v}(-p+i)_u(q+j)_v}{\left(\frac{n}{2}+i+j\right)_{u+v}u!v!} \\
&= \frac{2^{p-q}\beta^q e^{-\frac{\mu'\mu}{2}}}{\alpha^p}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{u=0}^{k}\sum_{v=0}^{k-u}\frac{(-p)_{i+u}(q)_{j+v}\Gamma\left(\frac{n}{2}+p-q\right)}{2^k\left(\frac{1}{2}\right)_k\Gamma\left(\frac{n}{2}+i+u+j+v\right)} \\
&\quad\times\frac{k!(-1)^{u+v}(-p+i)_u(q+j)_v}{(k-u-v)!u!v!}d_{i,j,k}(\hat{A},\hat{B},\mu\mu') \\
&= \frac{2^{p-q}\beta^q\Gamma\left(\frac{n}{2}+p-q\right)e^{-\frac{\mu'\mu}{2}}}{\alpha^p}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-p)_r(q)_s}{\Gamma\left(\frac{n}{2}+r+s\right)} \\
&\quad\times\sum_{m=0}^{\infty}\sum_{u=0}^{r}\sum_{v=0}^{s}\frac{(-1)^{u+v}(m+u+v)!}{2^{m+u+v}m!u!v!\left(\frac{1}{2}\right)_{m+u+v}}d_{r-u,s-v,m+u+v}(\hat{A},\hat{B},\mu\mu') \\
&= \frac{2^{p-q}\beta^q\Gamma\left(\frac{n}{2}+p-q\right)}{\alpha^p}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-p)_r(q)_s}{\Gamma\left(\frac{n}{2}+r+s\right)}h_{r,s}(\hat{A},\hat{B}),
\end{aligned}$$

where the second last equality follows by setting $r = i+u$, $s = j+v$, $m = k-u-v$, and the first equality is obtained by using Lemma 2 with $a = -p+i$, $b = q+j$, $c = -k$, and $d = \frac{n}{2}+i+j$,

$$\begin{aligned}
\sum_{u=0}^{k}\sum_{v=0}^{k-u}\frac{(-p+i)_u(q+j)_v(-k)_{u+v}}{\left(\frac{n}{2}+i+j\right)_{u+v}u!v!} &= \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\frac{(-p+i)_u(q+j)_v(-k)_{u+v}}{\left(\frac{n}{2}+i+j\right)_{u+v}u!v!} \\
&= \frac{\left(\frac{n}{2}+p-q\right)_k}{\left(\frac{n}{2}+i+j\right)_k}.
\end{aligned}$$

□

Proof of Proposition 5: We only prove the recurrence relation for $h_{i,j}$

because the proof of the recurrence relation for $\tilde{h}_{i,j}$ is almost identical. Let

$$\begin{aligned} P(t_1,t_2) &= 2t_1 \frac{\partial \ln H(t_1,t_2)}{\partial t_1} + 2t_2 \frac{\partial \ln H(t_1,t_2)}{\partial t_2} \\ &= \operatorname{tr}(A(\boldsymbol{t})(I_n - A(\boldsymbol{t}))^{-1}) + (1 - t_1 - t_2)\mu'(I_n - A(\boldsymbol{t}))^{-2}\mu \\ &\quad - \mu'(I_n - A(\boldsymbol{t}))^{-1}\mu \\ &= \sum_{k=1}^{\infty} \operatorname{tr}(A(\boldsymbol{t})^k) + (1 - t_1 - t_2)\sum_{k=0}^{\infty}(k+1)\mu' A(\boldsymbol{t})^k \mu - \sum_{k=1}^{\infty} \mu' A(\boldsymbol{t})^k \mu \\ &= \sum_{k=1}^{\infty} \operatorname{tr}(A(\boldsymbol{t})^k) + \sum_{k=1}^{\infty} k\mu' A(\boldsymbol{t})^k \mu - (t_1 + t_2)\sum_{k=0}^{\infty}(k+1)\mu' A(\boldsymbol{t})^k \mu; \end{aligned}$$

then it is easy to see that

$$p_{i,j} = [t_1^i t_2^j]P(t_1,t_2) = \tau_{i,j} + (i+j)(\eta_{i,j} - \eta_{i-1,j} - \eta_{i,j-1}).$$

Since

$$P(t_1,t_2)H(t_1,t_2) = 2\left[t_1 \frac{\partial H(t_1,t_2)}{\partial t_1} + t_2 \frac{\partial H(t_1,t_2)}{\partial t_2}\right],$$

we can compare the coefficients of $t_1^i t_2^j$ on both sides to obtain the recurrence relation:

$$\sum_{\nu_1=0}^{i} \sum_{\substack{\nu_2=0 \\ \nu_1+\nu_2>0}}^{j} p_{\nu_1,\nu_2} h_{i-\nu_1,j-\nu_2} = 2(i+j)h_{i,j}.$$

□

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