Spurious Inference in Reduced-Rank Asset-Pricing Models

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Abstract

This note studies some seemingly anomalous results that arise in possibly misspecified, reduced-rank linear asset-pricing models estimated by the continuously-updated generalized method of moments. When a spurious factor (that is, a factor that is uncorrelated with the returns on the test assets) is present, the test for correct model specification has asymptotic power that is equal to the nominal size. In other words, applied researchers will erroneously conclude that the model is correctly specified even when the degree of misspecification is arbitrarily large. The rejection probability of the test for overidentifying restrictions typically decreases further in underidentified models where the dimension of the null space is larger than one.

Keywords: Asset pricing; Spurious risk factors; Reduced-rank models; Model misspecification; Continuously-updated GMM; Rank test; Test for overidentifying restrictions.

JEL classification numbers: C12; C13; G12.

1 Introduction

This note characterizes the limiting behavior of the specification test based on the continuously-updated generalized method of moments (CU-GMM) estimator in linear asset-pricing models when the derivative matrix of the moment conditions is rank deficient. For example, this could arise when the model includes spurious factors; that is, factors that are uncorrelated with the returns on the test assets.

In a recent paper, Gospodinov, Kan, and Robotti (2014) analyze the detrimental effects of lack of identification on estimation, testing, and evaluation of asset-pricing models using the Hansen and Jagannathan (1997) distance. Concerns about the reliability of goodness-of-fit measures for noninvariant estimators in beta-pricing models and stochastic discount factor (SDF) models with excess returns have also been raised by Kan and Zhang (1999), Kleibergen and Zhan (2015), and Burnside (2016). In this study, we show that the use of CU-GMM[1] does not alleviate these inference problems and, somewhat surprisingly, makes them substantially worse when spurious factors are included in the model.

In particular, we demonstrate that, in the presence of a spurious factor, the power of the specification test is equal to its size. As a consequence, an applied researcher would conclude with high probability that the model is correctly specified and proceed with constructing standard errors and test statistics that assume correct model specification. Since these statistics would not take into account the extra uncertainty arising from potential model misspecification, the inference on the model parameters would be distorted and would manifest itself in highly significant estimates for factors that do not contribute to improved pricing. In addition, we derive explicitly the limiting distribution of the specification test when the dimension of the null space of the model exceeds one. In this case, the power of the test under the alternative of a misspecified model is below the size of the test that uses critical values from the standard chi-squared approximation.

The rest of the note is organized as follows. Section 2 derives the limiting distribution of the model specification test in correctly specified and misspecified models with possible underidentification. This section also presents Monte Carlo simulation results. Section 3 concludes.

We adopt the following notation throughout the note: $E[y_t]$ and $\text{Var}[y_t]$ denote the expected value and the variance of a random variable y_t, respectively; 1_N is an $N \times 1$ vector of ones; 0_N

[1] The CU-GMM estimator is invariant to data scaling, reparameterizations and normalizations, curvature-altering and stationarity-inducing transformations, etc. (Hall, 2005). Peñaranda and Sentana (2015) demonstrate convincingly the appeal of the CU-GMM estimator by showing the numerical equality of prices of risk, overidentifying restrictions tests, and pricing errors in alternative representations of asset-pricing models estimated by CU-GMM. Also, note that the invariance property is not special to CU-GMM and is a feature of the class of generalized empirical likelihood estimators.

is an $N \times 1$ vector of zeros; I_N is the identity matrix of dimension $N \times N$; rank(A) denotes the column rank of a matrix A; vec(A) signifies column vectorization of a matrix A; $\otimes$ denotes the Kronecker product; $\stackrel{p}{\to}$ and $\stackrel{d}{\to}$ stand for "convergence in probability" and "convergence in distribution", respectively; $\sim$ stands for "distributed as"; $\mathcal{N}(\cdot)$ denotes the normal distribution and χ^2_m denotes the chi-squared distribution with m degrees of freedom.

2 Limiting Behavior Under Rank Deficiency

2.1 Model and Assumptions

Let $x_t'\lambda$ be a candidate SDF at time t, where $x_t = [1,\ f_t']'$, f_t is a $(K-1)$-vector of systematic risk factors, and $\lambda = [\lambda_0,\ \lambda_1']'$ is a K-vector of SDF parameters.[2] Also, let R_t denote the gross returns on N $(N > K)$ test assets and $e_t(\lambda) = D_t\lambda - 1_N$, where $D_t = R_t x_t'$.[3] The CU-GMM estimator of λ is defined as the solution to (Hansen, Heaton, and Yaron, 1996)

$$\mathcal{J} = T \min_{\lambda} \bar{e}(\lambda)'\hat{V}_e(\lambda)^{-1}\bar{e}(\lambda), \tag{1}$$

where $\bar{e}(\lambda) = T^{-1}\sum_{t=1}^{T} e_t(\lambda)$ and $\hat{V}_e(\lambda)$ is a consistent estimator of the long-run variance matrix of the sample pricing errors $V_e(\lambda) = \sum_{j=-\infty}^{\infty} E[(e_t(\lambda) - \bar{e}(\lambda))(e_{t+j}(\lambda) - \bar{e}(\lambda))']$.[4] This is the $\mathcal{J}$ test of the validity of the asset-pricing model restriction $D\lambda = 1_N$, where $D = E[D_t]$. The model is said to be misspecified if $D\lambda \neq 1_N$ for all λ.

Let $Y_t = [f_t',\ R_t']'$ with $E[Y_t] \equiv \begin{bmatrix} \mu_f \\ \mu_R \end{bmatrix}$ and $\mathrm{Var}[Y_t] \equiv V = \begin{bmatrix} V_f & V_{fR} \\ V_{Rf} & V_R \end{bmatrix}$. We now formally define a spurious factor.

Definition (Spurious Factor). *A spurious factor $f_{t,i}$ is defined such that $E[R_t f_{t,i}] = \mu_R \mu_{f,i}$, where $\mu_{f,i} \equiv E[f_{t,i}]$.*

It follows from this definition that the presence of a spurious factor renders the D matrix rank deficient. For example, by writing D more explicitly as

$$D = \left[E[R_t],\ E[R_t f_{t,1}], \ldots,\ E[R_t f_{t,i}], \ldots,\ E[R_t f_{t,K-1}]\right], \tag{2}$$

[2]The SDF is typically defined in terms of conditional expectations (Hansen and Richard, 1987). Our specification of the SDF remains valid if conditioning information is incorporated through scaled factors and returns (see, for example, Section 8.1 in Cochrane, 2005). We should note that, given the widely documented weak predictive ability of conditioning variables for future returns, this approach could further exacerbate the spurious factor problem discussed in this note.

[3]When R_t is a vector of payoffs with initial cost $q \neq 0_N$, we just need to replace 1_N with q. In addition, the analysis in the note can be adapted to handle the case of excess returns with $q = 0_N$.

[4]In the case of *iid* data, Newey and Smith (2004, footnote 2) and Antoine, Bonnal, and Renault (2007) establish the equivalence of this CU-GMM estimator and the CU-GMM estimator based on the uncentered optimal weighting matrix. This equivalency continues to hold for time series data when the weighting matrix is of general form.

it is easy to see that $E[R_t f_{t,i}] = \mu_R \mu_{f,i}$ and $E[R_t] = \mu_R$ are collinear and D has one degree of rank deficiency.

To gain some intuition for the results to follow, consider the $N \times (K+1)$ matrix $H \equiv [1_N,\ D]$ and note that the asset-pricing model restriction $D\lambda = 1_N$ can be rewritten as $Hv = 0_N$, where $v = [1,\ -\lambda']'$. Hence, the specification test is essentially testing that matrix H is of reduced rank since, if the asset-pricing model holds, the vector 1_N is in the column space of the matrix D. This implies that there exists a nonzero vector v that solves $Hv = 0_N$. When the model is correctly specified ($D\lambda = 1_N$) and well identified (matrix D is of full column rank), $v = [1,\ -\lambda']'$ and there is a member of the null space of H for which the first entry of v is equal to, or can be normalized to, one. However, there are also cases where H is of reduced rank but the first entry of v is zero. For instance, when the model is misspecified ($D\lambda \neq 1_N$) with a spurious factor (D is rank deficient of degree one), the vector v that solves $Hv = 0_N$ is proportional to $[0,\ -\mu_{f,K-1},\ 0'_{K-2},\ 1]'$, where, for convenience, the spurious factor is ordered last. Any attempt to normalize the first element of v to one in a given sample when its population value is actually zero will be approximately offset by making the rest of the coefficients large in magnitude. Despite these mathematical differences, a test that is insensitive to the scaling of a vector in the null space will fail to make a meaningful distinction in these two cases. In summary, the invariant $\mathcal{J}$ test cannot distinguish whether the reduced rank of H arises because the vector 1_N lies in the column space of D (correctly specified model) or because the vector 1_N is not in the column space of D but D is of reduced column rank $K-1$ (for example, a misspecified model with a spurious factor).

From this discussion, it proves useful to rewrite the objective function $\mathcal{J}(\lambda) \equiv T\bar{e}(\lambda)'\hat{V}_e(\lambda)^{-1}\bar{e}(\lambda)$ in (1) in a slightly different form. Let $H_t = [1_N,\ D_t]$, and $\hat{V}_d$ and $\hat{V}_h$ denote consistent estimators of the long-run variance matrices $V_d = \lim_{T\to\infty} \mathrm{Var}[T^{-\frac{1}{2}} \sum_{t=1}^T \mathrm{vec}(D_t)]$ and $V_h = \lim_{T\to\infty} \mathrm{Var}[T^{-\frac{1}{2}} \sum_{t=1}^T \mathrm{vec}(H_t)]$, respectively. Then, we have

$$e_t(\lambda) = (-v' \otimes I_N)\mathrm{vec}(H_t), \tag{3}$$

$$\hat{V}_e(\lambda) = (-v' \otimes I_N)\hat{V}_h(-v' \otimes I_N)' = (\lambda' \otimes I_N)\hat{V}_d(\lambda \otimes I_N), \tag{4}$$

and

$$\mathcal{J}(\lambda) = T(\hat{D}\lambda - 1_N)'[(\lambda' \otimes I_N)\hat{V}_d(\lambda \otimes I_N)]^{-1}(\hat{D}\lambda - 1_N), \tag{5}$$

where $\hat{D} = \frac{1}{T}\sum_{t=1}^T D_t$. The expression (5) for the CU-GMM objective function is convenient because it requires a consistent estimator of V_d which is only a function of the data. We show below that this form of the CU-GMM objective function is directly related to the objective function for testing the reduced rank of a matrix.

For our main results, we impose the following assumption.

Assumption 1. *Assume that Y_t is a jointly stationary and ergodic process with a finite fourth moment and V is a positive-definite matrix. In addition, assume that $\hat{V}_d \xrightarrow{p} V_d$, where V_d is a positive-definite matrix.*

Assumption 1 provides primitive conditions for the central limit theorem approximation of the product of returns and factors. It allows for general heteroskedasticity and serial correlation in the covariance matrix V_d. This assumption is sufficient for bounding the asymptotic distribution of the $\mathcal{J}$ test under possible underidentification. In order to obtain the explicit limiting distribution of the $\mathcal{J}$ test when the null space of the model is more than one-dimensional, we impose some further restrictions on the data and the model.

2.2 Asymptotic Distribution of the Specification Test

As mentioned above, the asset-pricing model restriction can be expressed as $Hv = 0_N$, where $H = [1_N,\ D]$ is of dimension $N \times (K+1)$. In this section, we study the limiting behavior of the model specification test when H is rank deficient of degree r ($r = 1, 2, \ldots, K$), where the rank deficiency arises either because the asset-pricing restrictions are satisfied or D itself is rank deficient, or both. In that sense, the limiting distribution of the specification test will depend only on degree of rank deficiency r and not on whether the model is correctly specified or misspecified. More specifically, suppose that the matrix H has a column rank $K + 1 - r$ ($r = 1, 2, \ldots, K$), that is, there exist r distinct linear combinations of the columns of H that are equal to zero vectors. Also, let P_1 be an $N \times (N-1)$ orthonormal matrix whose columns are orthogonal to 1_N such that $P_1' P_1 = I_{N-1}$ and

$$P_1 P_1' = I_N - 1_N (1_N' 1_N)^{-1} 1_N'. \tag{6}$$

Note that premultiplying by P_1' removes the column of ones from the matrix H. Thus, performing a rank test on the $(N-1) \times K$ matrix $P_1' D$ provides a convenient way of testing for rank deficiency of H. Under the null that $P_1' D$ is of reduced rank $K - 1$, there exists a nonzero K-vector c such that $P_1' D c = 0_{N-1}$ with the normalization $c'c = 1$.[5] As a result, for the purpose of testing whether the asset-pricing model is correctly specified, one could use the Cragg and Donald (1997) test[6] of $H_0 : \text{rank}(P_1' D) = K - 1$, which can be rewritten as an invariant test of the form

$$\mathcal{CD} = T \min_{c:c'c=1} (P_1' \hat{D} c)' [(c' \otimes P_1') \hat{V}_d (c \otimes P_1)]^{-1} (P_1' \hat{D} c). \tag{7}$$

[5]Solving $P_1' D c = 0_{N-1}$ requires some normalization since this condition only determines the direction of the vector c (up to a sign) but not its length (Hillier, 1990). While various normalizations are possible, here we employ the normalization $c'c = 1$. The unit norm also imposes compactness (see Hansen, 2012). The relationship between c and λ is made explicit in the proof of Theorem 1 in the Appendix.

[6]See Cragg and Donald (1997), Robin and Smith (2000), and Kleibergen and Paap (2006) for a detailed analysis of rank restriction tests.

Let r denote the dimension of the null space of H. We first establish the limiting behavior of the $\mathcal{J}$ test for $r \geq 1$ under the general conditions in Assumption 1. More specifically, The limiting behavior of the $\mathcal{J}$ test is obtained under a general structure of the V_d matrix for which a consistent, possibly heteroskedasticity and autocorrelation consistent (HAC), estimator is available.

THEOREM 1: *Suppose that the matrix H has a column rank $K+1-r$ for an integer $r \geq 1$, and Assumption 1 holds. Then, the limiting behavior of the $\mathcal{J}$ test for correct model specification can be characterized as follows: (a) when $r=1$, $\mathcal{J} \overset{d}{\to} \chi^2_{N-K}$, and (b) when $r \geq 2$, $\lim_{T\to\infty} \Pr[\mathcal{J} \leq a] \geq \Pr[x_{N-1} \leq a]$, where $x_{N-1} \sim \chi^2_{N-1}$.*

Proof: See Appendix.

The result in Theorem 1 shows that the limiting behavior of the $\mathcal{J}$ test is determined entirely by the dimension of the null space of H and not by whether the model is correctly specified or misspecified. This observation is key in understanding the lack of power of the $\mathcal{J}$ test when the model is misspecified. While the rank reduction in H in well-identified models arises only when the asset-pricing restrictions are satisfied, the presence of spurious factors leads to a rank reduction in H through a rank deficiency in D even when the asset-pricing restrictions do not hold. In the latter case, the test will have difficulties rejecting the null of correct model specification even if the degree of model misspecification is arbitrarily large. The proof of this result explores the numerical equality between the $\mathcal{J}$ and $\mathcal{CD}$ tests which has important implications for testing the validity of the asset-pricing model (see also Kleibergen and Mavroeidis, 2009, and Arellano, Hansen, and Sentana, 2012). Part (a) of Theorem 1 is concerned with the situation when the null space of H is one-dimensional ($r=1$) and the parameter vector c is uniquely identified, up to a sign. Part (a) embeds two cases: (i) the model is correctly specified and identified, and (ii) the model is misspecified with a spurious factor. In the first case, the asset-pricing restrictions hold ($D\lambda = 1_N$) and 1_N lies in the column space of D so that matrix $P_1'D$ is of reduced column rank with a one-dimensional null space. The asymptotic result for this case, $\mathcal{J} = \mathcal{CD} \overset{d}{\to} \chi^2_{N-K}$, is standard (Hansen, 1982; Hansen, Heaton, and Yaron, 1996; Cragg and Donald, 1997).

In the second case, 1_N is not in the column space of D (that is, $D\lambda \neq 1_N$), but D is of reduced column rank. Hence, $P_1'D$ is of reduced rank and the null space is again one-dimensional. Because D is rank deficient due to the presence of a spurious factor, we have $\text{rank}(P_1'D) = K-1$ and $\mathcal{CD} \overset{d}{\to} \chi^2_{N-K}$. From the numerical equality of the $\mathcal{J}$ and $\mathcal{CD}$ tests, we have that $\mathcal{J} \overset{d}{\to} \chi^2_{N-K}$ even when the model is misspecified and the $\mathcal{J}$ test does not exhibit power in rejecting the null hypothesis of correct model specification.[7]

[7] There are other cases when D is rank deficient with similar consequences on the power of the $\mathcal{J}$ test. For

When the null space of $P_1'D$ is more than one-dimensional, there is a multiplicity of solutions to $P_1'Dc = 0_{N-1}$ and the model is underidentified (see Arellano, Hansen, and Sentana, 2012; Manresa, Peñaranda, and Sentana, 2016). Part (b) in Theorem 1 establishes that the $\mathcal{J}$ test is asymptotically bounded by the χ^2_{N-1} distribution. Since the set of solutions of c for $r > 1$ is multi-dimensional, this underidentification implies that there are more degrees of freedom in selecting a vector c for solving $P_1'Dc = 0_{N-1}$. In this situation, the specification test will lack power when the model is indeed misspecified and the rejection rates of the test will be bounded by the nominal size of the χ^2_{N-1} distribution. In the context of asset-pricing models for equity returns, N is typically much larger than K and the χ^2_{N-1} asymptotic bound is not substantially more conservative than the asymptotic distribution in part (a) of Theorem 1.

Theorem 1 characterizes the limiting distribution of the $\mathcal{J}$ test under very general conditions but provides only a conservative upper bound when $r \geq 2$. Theorem 2 below presents the explicit asymptotic distribution for this underidentified case, where the dimension of the null space of H is $r \geq 2$, at the cost of some more restrictive assumptions. To introduce these assumptions, let $\Sigma = V_R - V_{Rf}V_f^{-1}V_{fR}$, $\beta = V_{Rf}V_f^{-1}$, $\alpha = \mu_R - \beta\mu_f$, $B = [\alpha,\ \beta]$, and $\hat{B}$ be the sample counterpart of B.

THEOREM 2: *Suppose that the matrix H has a column rank $K + 1 - r$ for an integer $r \geq 1$ and Assumption 1 holds. In addition, assume that $\sqrt{T}\text{vec}(\hat{B} - B) \xrightarrow{d} \mathcal{N}(0_{NK}, E[x_t x_t']^{-1} \otimes \Sigma)$. Then, we have*

$$\mathcal{J} \xrightarrow{d} w_r, \tag{8}$$

where w_r is the smallest eigenvalue of $W_r \sim \mathcal{W}_r(N - K - 1 + r, I_r)$, and $\mathcal{W}_r(N - K - 1 + r, I_r)$ denotes the Wishart distribution with $N - K - 1 + r$ degrees of freedom and a scaling matrix I_r. Furthermore, $\Pr[w_r \leq a] \geq \Pr[x_{N-K} \leq a]$, *where $x_{N-K} \sim \chi^2_{N-K}$.*

Proof: See Appendix.

Sufficient conditions for the high level assumption on $\text{vec}(\hat{B})$ in the beta representation of the model are contemporaneous conditional homoskedasticity and a martingale difference sequence requirement for the projection error of returns on factors. Note that these conditions are only sufficient and the result in Theorem 2 may continue to hold when more general features of the data are allowed. For example, under the assumption that $[f_t',\ R_t']'$ are jointly elliptically distributed, the returns R_t can exhibit conditional heteroskedasticity but Theorem 2 still holds (see the Online Appendix). Also, Theorem 2 imposes lack of serial correlation on the projection errors but not on

example, the full-rank condition on D may also be violated when the model includes two factors that are noisy versions of the same underlying factor.

the data. In a more general context of a multi-period conditional linear factor model (Hansen and Richard, 1987), the implied errors will have a zero conditional expectation under correct model specification but not necessarily under model misspecification with spurious factors. Even in this case, it may be better practice to impose the martingale structure (or approximate martingale structure as in Hansen, 1985) on the error term. While this would render the weighting matrix and the asymptotic limit in Theorem 2 invalid, our main point that model misspecification will be difficult to detect in the presence of spurious factors remains intact. For full generality, however, one should resort to the asymptotic bound result in Theorem 1.

The more restrictive conditions in Theorem 2 are imposed to ensure that the weighting matrix in (7) has an approximate Kronecker structure.[8] This allows us to express the objective function in (7) as a ratio of quadratic forms and an eigenvalue problem, which gives rise to the Wishart limiting distribution. As shown in part (a) of Theorem 1, the case $r = 1$, where the Wishart distribution specializes to the χ^2_{N-K} distribution, holds under general conditions but it is included in Theorem 2 for completeness. Theorem 2 also shows that the $\mathcal{J}$ test is asymptotically bounded by the χ^2_{N-K} distribution which is a sharper bound than the asymptotic bound in part (b) of Theorem 1. The following figure plots the limiting distribution of the $\mathcal{J}$ test for $r = 1, 2$, and 3 when $N - K = 7$.

Figure 1 about here

The lack of power of the specification tests in underidentified models suggests that the decision regarding the model specification should be augmented with additional diagnostics. For instance, the tests developed by Arellano, Hansen, and Sentana (2012), Peñaranda and Sentana (2015), and Manresa, Peñaranda, and Sentana (2016) can detect if the lack of rejection of the model specification tests is genuine or is due to the presence of a spurious factor.

2.3 Simulation Results

In this section, we undertake a small Monte Carlo simulation experiment to evaluate the empirical rejection rates of the specification test for the CU-GMM estimator. We consider four linear models: (i) a model with a constant term and a useful factor, (ii) a model with a constant term and a spurious factor, (iii) a model with a constant term, a useful, and a spurious factor, and (iv) a model with a constant term, three useful, and two spurious factors. For each of the four specifications, we consider separately the case of a correctly specified and a misspecified model. This allows us to

[8] See, for instance, Guggenberger, Kleibergen, Mavroeidis, and Chen (2012, p. 2655) for a discussion on the importance of the Kronecker product assumption for this type of result.

assess the properties of the specification test when the null space of H is of dimension $r = 0$, 1, 2, and 3.

The returns on the test assets and the useful factors are drawn from a multivariate normal distribution. In all simulation designs, the covariance matrix of the simulated test asset returns is set equal to the sample covariance matrix from the 1959:2–2012:12 sample of monthly gross returns on the 25 Fama-French size and book-to-market ranked portfolios. For misspecified models, the means of the simulated returns are set equal to the means of the actual returns. For correctly specified models, the means of the simulated returns are set such that the asset-pricing model restrictions are satisfied. The means and variances of the simulated useful factors are calibrated to the sample means and variances of the three Fama-French factors (see Fama and French, 1993).[9] The covariances between the useful factors and the returns are chosen based on the sample covariances estimated from the data. The spurious factors are generated as standard normal random variables which are independent of the returns and the useful factors. The time series sample size is $T = 200$, 600, and 1000, and all results are based on 100,000 Monte Carlo replications. We also report the limiting rejection probabilities (denoted by $T = \infty$) for the specification test based on our asymptotic results in Theorem 2 since our simulation setup satisfies the assumptions of Theorem 2.

Table I presents the probabilities of rejection of the model specification test at the 10%, 5%, and 1% nominal levels.

Table I about here

When the model contains only a useful factor (Panel A), the $\mathcal{J}$ test is correctly sized and consistent under the alternative as $T \to \infty$. Some size distortions occur for small T, but this is a well documented finding and is mainly due to the relatively large number of test assets used in our simulations.

Consistent with our theoretical results, the empirical rejection probabilities of the specification test are less than the nominal size when the model is correctly specified but it contains one or more spurious factors. In addition, the specification test does not exhibit any power in the presence of a spurious factor and the empirical rejection probabilities approach the nominal size under the alternative of a misspecified model (last three columns of Panels B and C). As a result, when a spurious factor is included in the model, the researcher will erroneously conclude (with probability one minus the nominal size of the test) that the model is correctly specified even when the misspecification of the model is arbitrarily large. Finally, the last three columns of Panel D show that, when $r = 2$,

[9] When the model contains only one useful factor, the mean and variance of the simulated useful factor is calibrated to the sample mean and variance of the value-weighted market excess return from Fama and French (1993).

the power of the CU-GMM test under the alternative of a misspecified model is below the size of the test that uses critical values from the standard chi-squared approximation. These spurious results should serve as a warning signal in applied work where many macroeconomic factors are only weakly correlated with the returns on the test assets.

3 Concluding Remarks

In this note, we establish the limiting properties of the CU-GMM specification test of asset-pricing models, and show that the inference based on this test can be misleading when spurious factors are present. It is important to stress that this is not an isolated problem limited to a particular sample, test assets, and asset-pricing models.

While the results in this note are developed in the context of linear factor models, we conjecture that similar results characterize the limiting behavior of specification tests in a more general setup. For example, Cragg and Donald (1996) establish the inconsistency of the Anderson-Rubin test for overidentifying restrictions in underidentified linear instrumental variable models while Dovonon and Renault (2013) derive the asymptotic distribution of the specification test under lack of first-order identification. Furthermore, Guggenberger, Kleibergen, Mavroeidis, and Chen (2012) characterize the asymptotic behavior of the upper bound of the subset Anderson-Rubin statistic in linear instrumental variables regression models with potentially weak identification. Extending the results to the class of generalized empirical likelihood estimators (Newey and Smith, 2004) is also a promising direction for future research.

Appendix: Proofs

A.1 Proof of Theorem 1

We start by showing the numerical equality between the $\mathcal{J}$ and $\mathcal{CD}$ tests. Note that this is an algebraic equality and does not depend on statistical assumptions. Let $P = [1_N/\sqrt{N},\ P_1]$, where P_1 is the orthonormal matrix defined in the text. Then, we can write

$$\begin{aligned}
\mathcal{J}(\lambda) &= T(\hat{D}\lambda - 1_N)'[(\lambda' \otimes I_N)\hat{V}_d(\lambda \otimes I_N)]^{-1}(\hat{D}\lambda - 1_N) \\
&= T(\hat{D}\lambda - 1_N)'P(P'\hat{V}_e(\lambda)P)^{-1}P'(\hat{D}\lambda - 1_N) \\
&= T \begin{bmatrix} \frac{1_N'(\hat{D}\lambda - 1_N)}{\sqrt{N}} \\ P_1'\hat{D}\lambda \end{bmatrix}' \begin{bmatrix} \frac{1_N'\hat{V}_e(\lambda)1_N}{N} & \frac{1_N'\hat{V}_e(\lambda)P_1}{\sqrt{N}} \\ \frac{P_1'\hat{V}_e(\lambda)1_N}{\sqrt{N}} & P_1'\hat{V}_e(\lambda)P_1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1_N'(\hat{D}\lambda - 1_N)}{\sqrt{N}} \\ P_1'\hat{D}\lambda \end{bmatrix}.
\end{aligned} \tag{A.1}$$

Denote the matrix in the middle as

$$A \equiv \begin{bmatrix} \frac{1_N'\hat{V}_e(\lambda)1_N}{N} & \frac{1_N'\hat{V}_e(\lambda)P_1}{\sqrt{N}} \\ \frac{P_1'\hat{V}_e(\lambda)1_N}{\sqrt{N}} & P_1'\hat{V}_e(\lambda)P_1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{A.2}$$

Using the following formula for the inverse of a partitioned matrix

$$A^{-1} = \begin{bmatrix} 0 & 0_{N-1}' \\ 0_{N-1} & A_{22}^{-1} \end{bmatrix} + \frac{1}{A_{11} - A_{12}A_{22}^{-1}A_{21}} \begin{bmatrix} -1 \\ A_{22}^{-1}A_{21} \end{bmatrix} \begin{bmatrix} -1 \\ A_{22}^{-1}A_{21} \end{bmatrix}', \tag{A.3}$$

we obtain

$$\mathcal{J}(\lambda) = \mathcal{CD}(\lambda) + \frac{T}{N(A_{11} - A_{12}A_{22}^{-1}A_{21})}[1_N'\hat{V}_e(\lambda)P_1(P_1'\hat{V}_e(\lambda)P_1)^{-1}P_1'\hat{D}\lambda - 1_N'(\hat{D}\lambda - 1_N)]^2, \tag{A.4}$$

where $\mathcal{CD}(\lambda) = T\lambda'\hat{D}'P_1[(\lambda' \otimes P_1')\hat{V}_d(\lambda \otimes P_1)]^{-1}P_1'\hat{D}\lambda$.

Note first that

$$\begin{aligned} N(A_{11} - A_{12}A_{22}^{-1}A_{21}) &= 1_N'\hat{V}_e(\lambda)1_N - 1_N'\hat{V}_e(\lambda)P_1(P_1'\hat{V}_e(\lambda)P_1)^{-1}P_1'\hat{V}_e(\lambda)1_N = \\ &= 1_N'\hat{V}_e(\lambda)[1_N - P_1(P_1'\hat{V}_e(\lambda)P_1)^{-1}P_1'\hat{V}_e(\lambda)1_N] \\ &= 1_N'\hat{V}_e(\lambda)\hat{V}_e(\lambda)^{-\frac{1}{2}}[I_N - \hat{V}_e(\lambda)^{\frac{1}{2}}P_1(P_1'\hat{V}_e(\lambda)P_1)^{-1}P_1'\hat{V}_e(\lambda)^{\frac{1}{2}}]\hat{V}_e(\lambda)^{\frac{1}{2}}1_N \\ &= 1_N'\hat{V}_e(\lambda)\hat{V}_e(\lambda)^{-\frac{1}{2}}[\hat{V}_e(\lambda)^{-\frac{1}{2}}1_N(1_N'\hat{V}_e(\lambda)^{-1}1_N)^{-1}1_N'\hat{V}_e(\lambda)^{-\frac{1}{2}}]\hat{V}_e(\lambda)^{\frac{1}{2}}1_N \\ &= \frac{N^2}{1_N'\hat{V}_e(\lambda)^{-1}1_N}. \end{aligned} \tag{A.5}$$

Similarly, rearranging the term in the square brackets gives

$$\begin{aligned} & 1_N'\hat{V}_e(\lambda)P_1(P_1'\hat{V}_e(\lambda)P_1)^{-1}P_1'\hat{D}\lambda - 1_N'\hat{D}\lambda + N \\ = & -[1_N - P_1(P_1'\hat{V}_e(\hat{c})P_1)^{-1}P_1'\hat{V}_e(\hat{c})1_N]'\hat{D}\lambda + N \\ = & -\frac{N1_N'\hat{V}_e(\lambda)^{-1}(\hat{D}\lambda - 1_N)}{1_N'\hat{V}_e(\lambda)^{-1}1_N}. \end{aligned} \tag{A.6}$$

Thus,

$$\frac{T}{N(A_{11} - A_{12}A_{22}^{-1}A_{21})}[1_N'\hat{V}_e(\lambda)P_1(P_1'\hat{V}_e(\lambda)P_1)^{-1}P_1'\hat{D}\lambda - 1_N'(\hat{D}\lambda - 1_N)]^2 = \frac{T(1_N'\hat{V}_e(\lambda)^{-1}(\hat{D}\lambda - 1_N))^2}{1_N'\hat{V}_e(\lambda)^{-1}1_N}. \tag{A.7}$$

For the numerical equality of the $\mathcal{J}$ and $\mathcal{CD}$ tests, we need to show that $1_N'\hat{V}_e(\lambda)^{-1}(\hat{D}\lambda - 1_N) = 0$ when evaluated at the minimizer of $\mathcal{J}(\lambda)$, $\hat{\lambda}$. The first-order conditions for the CU-GMM estimator are given by (see Antoine, Bonnal, and Renault, 2007)[10]

$$\left[\sum_{t=1}^{T} \pi_t \left(\frac{\partial e_t(\hat{\lambda})}{\partial \lambda'}\right)'\right] \hat{V}_e(\hat{\lambda})^{-1}\bar{e}(\hat{\lambda}) \equiv \hat{D}_\pi'\hat{V}_e(\hat{\lambda})^{-1}(\hat{D}\hat{\lambda} - 1_N) = 0_K, \tag{A.8}$$

[10]Note that $\hat{V}_e(\lambda)^{-1}(\hat{D}\lambda - 1_N)$ is, up to a sign, the vector of Lagrange multipliers associated with the N moment conditions (Antoine, Bonnal, and Renault, 2007).

where the weights $\pi_t = \left(1 - \bar{e}(\hat{\lambda})'\hat{V}_e(\hat{\lambda})^{-1}\left[e_t(\hat{\lambda}) - \bar{e}(\hat{\lambda})\right]\right)/T$ induce the moment conditions to be exactly satisfied, $\sum_{t=1}^{T} \pi_t e_t(\hat{\lambda}) = 0_N$. Since this forces the vector 1_N to be in the column span of $\hat{D}_\pi$, it follows that $1_N'\hat{V}_e(\hat{\lambda})^{-1}(\hat{D}\hat{\lambda} - 1_N) = 0$. Note that the $\mathcal{CD}$ test is invariant to scaling and normalizations of the parameter vector so that $\min_\lambda \mathcal{CD}(\lambda) = \min_{c:c'c=1}\mathcal{CD}(c) \equiv \mathcal{CD}$ and $\min_\lambda \mathcal{J}(\lambda) \equiv \mathcal{J} = \mathcal{CD}$.[11] Since under the null H_0 : rank$(P_1'D) = K - 1$ we have $\mathcal{CD} \overset{d}{\to} \chi^2_{N-K}$ (Cragg and Donald, 1997), it immediately follows by $\mathcal{J} = \mathcal{CD}$ that $\mathcal{J} \overset{d}{\to} \chi^2_{N-K}$. This completes the proof of part (a).

For part (b), consider the more general case when the true dimension of the null space of H is $r \geq 1$. Let c_* denote the parameter vector under the null H_0 : rank$(P_1'D) = K - 1$ that solves $P_1'Dc_* = 0_{N-1}$. Note that this restriction can hold for both correctly specified and misspecified models when $r \geq 1$. Furthermore, a non-zero vector c_* that solves $P_1'Dc_* = 0_{N-1}$ always exists although the set of solutions for c_* is r-dimensional. This is not the case for the parameter vector λ for which λ_* that solves $D\lambda_* - 1_N = 0_N$ cannot be defined when the model is misspecified and it contains spurious factors. Since $\hat{c}$ is the minimizer of $\min_{c:c'c=1}\mathcal{CD}(c) \equiv \mathcal{CD}$, we have that $\mathcal{CD}(c_*) \geq \mathcal{CD}$. Also, from Stock and Wright (2000), we have $\mathcal{CD}(c_*) \overset{d}{\to} \chi^2_{N-1}$. Then, it follows that the test $\mathcal{J} = \mathcal{CD}$ is asymptotically bounded by the χ^2_{N-1} distribution when $r \geq 2$. This completes the proof of part (b).

A.2 Proof of Theorem 2

First, we will rewrite the $\mathcal{CD}$ test in an asymptotically equivalent but simpler form. Let X be a $T \times K$ matrix with a typical row x_t' and note that

$$\hat{B} = \hat{D}\left(\frac{X'X}{T}\right)^{-1}, \tag{A.9}$$

Then, using that

$$\sqrt{T}\text{vec}(\hat{B} - B) \overset{d}{\to} \mathcal{N}(0_{NK}, E[x_t x_t']^{-1} \otimes \Sigma) \tag{A.10}$$

and the delta method, we have

$$\sqrt{T}\text{vec}(\hat{D} - D) \overset{d}{\to} \mathcal{N}\left(0_{NK}, E[x_t x_t'] \otimes \Sigma + (I_K \otimes B)V_x(I_K \otimes B')\right), \tag{A.11}$$

where V_x is the asymptotic variance of $\sqrt{T}\text{vec}\left((X'X)/T - E[x_t x_t']\right)$.[12] Therefore, for any nonzero vector c, we have

$$\sqrt{T}(P_1'\hat{D}c - P_1'Dc) \overset{d}{\to} \mathcal{N}\left(0_{N-1}, c'E[x_t x_t']cP_1'\Sigma P_1 + (c' \otimes P_1'B)V_x(c \otimes B'P_1)\right). \tag{A.12}$$

[11] If we let $\hat{\lambda} = \hat{a}\hat{c}$, the constant $\hat{a}$ that solves $1_N'[\hat{a}^2\hat{V}_e(\hat{c})]^{-1}(\hat{D}\hat{a}\hat{c} - 1_N) = 0$ is given by $\hat{a} = \frac{1_N'\hat{V}_e(\hat{c})^{-1}1_N}{1_N'\hat{V}_e(\hat{c})^{-1}\hat{D}\hat{c}}$.

[12] Note that V_x is singular because x_t has one as its first element.

Hence under the assumptions of Theorem 2, we can consistently estimate $(c' \otimes P_1')V_d(c \otimes P_1)$ using $\mathcal{A}(c) = \mathcal{A}_1(c) + \mathcal{A}_2(c)$, where $\mathcal{A}_1(c) = c'(X'X/T)cP_1'\hat{\Sigma}P_1$ and $\mathcal{A}_2(c) = (c' \otimes P_1'\hat{B})\hat{V}_x(c \otimes \hat{B}'P_1)$, with $\hat{\Sigma}$ and $\hat{V}_x$ being consistent estimators of Σ and V_x, respectively. When $P_1'D$ has a reduced rank, we have

$$\mathcal{J} = \mathcal{CD} = T \min_{c:c'c=1} (P_1'\hat{D}c)'[(c' \otimes P_1')\hat{V}_d(c \otimes P_1)]^{-1}(P_1'\hat{D}c) = \mathcal{J}_A + o_p(1), \tag{A.13}$$

where

$$\mathcal{J}_A = T \min_{c:c'c=1} (P_1'\hat{D}c)'\mathcal{A}(c)^{-1}(P_1'\hat{D}c). \tag{A.14}$$

Next, we will show that

$$\mathcal{J}_A = T \min_{c:c'c=1} (P_1'\hat{D}c)'\mathcal{A}_1(c)^{-1}(P_1'\hat{D}c) + o_p(1). \tag{A.15}$$

Let $\hat{c}$ be the optimal c in (A.14) and note that $\hat{c}$ is $O_p(1)$ since $\hat{c}'\hat{c} = 1$ by the adopted normalization. Since $\mathcal{A}_2(c)$ is a positive-definite matrix, it follows that $\mathcal{A}(c)^{-1} \preceq \mathcal{A}_1(c)^{-1}$ and

$$T(P_1'\hat{D}\hat{c})'\mathcal{A}(\hat{c})^{-1}(P_1'\hat{D}\hat{c}) \leq T \min_{c:c'c=1} (P_1'\hat{D}c)'\mathcal{A}_1(c)^{-1}(P_1'\hat{D}c) \leq T(P_1'\hat{D}\hat{c})'\mathcal{A}_1(\hat{c})^{-1}(P_1'\hat{D}\hat{c}). \tag{A.16}$$

Then, in order to establish the result in (A.15), it is sufficient to show that

$$T(P_1'\hat{D}\hat{c})'\mathcal{A}(\hat{c})^{-1}(P_1'\hat{D}\hat{c}) = T(P_1'\hat{D}\hat{c})'\mathcal{A}_1(\hat{c})^{-1}(P_1'\hat{D}\hat{c}) + o_p(1). \tag{A.17}$$

Using that

$$\mathcal{A}_1(\hat{c})^{-1} = \mathcal{A}(\hat{c})^{-1} + \mathcal{A}(\hat{c})^{-1}\mathcal{A}_2(\hat{c})[\mathcal{A}_2(\hat{c}) - \mathcal{A}_2(\hat{c})\mathcal{A}(\hat{c})^{-1}\mathcal{A}_2(\hat{c})]^{-1}\mathcal{A}_2(\hat{c})\mathcal{A}(\hat{c})^{-1}, \tag{A.18}$$

where $[\mathcal{A}_2(\hat{c}) - \mathcal{A}_2(\hat{c})\mathcal{A}(\hat{c})^{-1}\mathcal{A}_2(\hat{c})]^{-1}$ is a positive-definite matrix, we obtain

$$\begin{aligned} T(P_1'\hat{D}\hat{c})'\mathcal{A}_1(\hat{c})^{-1}(P_1'\hat{D}\hat{c}) &= T(P_1'\hat{D}\hat{c})'\mathcal{A}(\hat{c})^{-1}(P_1'\hat{D}\hat{c}) \\ &+ \sqrt{T}(P_1'\hat{D}\hat{c})'\mathcal{A}(\hat{c})^{-1}\mathcal{A}_2(\hat{c})[\mathcal{A}_2(\hat{c}) - \mathcal{A}_2(\hat{c})\mathcal{A}(\hat{c})^{-1}\mathcal{A}_2(\hat{c})]^{-1}\mathcal{A}_2(\hat{c})\mathcal{A}(\hat{c})^{-1}\sqrt{T}(P_1'\hat{D}\hat{c}). \end{aligned} \tag{A.19}$$

Since $\mathcal{A}(\hat{c})$, $\mathcal{A}_2(\hat{c})$, and $[\mathcal{A}_2(\hat{c}) - \mathcal{A}_2(\hat{c})\mathcal{A}(\hat{c})^{-1}\mathcal{A}_2(\hat{c})]^{-1}$ are $O_p(1)$, it suffices to show that

$$\mathcal{A}_2(\hat{c})\mathcal{A}(\hat{c})^{-1}\sqrt{T}P_1'\hat{D}\hat{c} = o_p(1). \tag{A.20}$$

From the first-order conditions of (A.14), we have

$$\sqrt{T}(P_1'\hat{D}\hat{c})'\mathcal{A}(\hat{c})^{-1}P_1'\hat{D} - \frac{1}{\sqrt{T}}[\sqrt{T}(P_1'\hat{D}\hat{c})'\mathcal{A}(\hat{c})^{-1}\mathcal{B}(\hat{c}) \otimes \sqrt{T}(P_1'\hat{D}\hat{c})'\mathcal{A}(\hat{c})^{-1}][I_K \otimes \operatorname{vec}(I_{N-1})] = 0_K', \tag{A.21}$$

where $\mathcal{B}(\hat{c}) = (\hat{c}' \otimes I_{N-1})[(X'X/T) \otimes P_1'\hat{\Sigma}P_1 + (I_K \otimes P_1'\hat{B})\hat{V}_x(I_K \otimes \hat{B}'P_1)]$. In deriving the limiting distribution below, we show that the middle term in (A.16), $T \min_{c:c'c=1}(P_1'\hat{D}c)'\mathcal{A}_1(c)^{-1}(P_1'\hat{D}c)$, is

$O_p(1)$. The first inequality in (A.16) then implies that a quadratic form in $\sqrt{T}(P_1'\hat{D}\hat{c})'\mathcal{A}(\hat{c})^{-\frac{1}{2}}$ is bounded by an $O_p(1)$ random variable. Since $\mathcal{B}(\hat{c})$ and $\mathcal{A}(\hat{c})$ are $O_p(1)$, then

$$\frac{1}{\sqrt{T}}[\sqrt{T}(P_1'\hat{D}\hat{c})'\mathcal{A}(\hat{c})^{-1}\mathcal{B}(\hat{c})\otimes\sqrt{T}(P_1'\hat{D}\hat{c})'\mathcal{A}(\hat{c})^{-1}][I_K\otimes\mathrm{vec}(I_{N-1})]=o_p(1) \tag{A.22}$$

and

$$\hat{D}'P_1\mathcal{A}(\hat{c})^{-1}\sqrt{T}P_1'\hat{D}\hat{c}=o_p(1). \tag{A.23}$$

Furthermore, substituting for $\hat{D}=\hat{B}(X'X/T)$ and pre-multiplying both sides by the $O_p(1)$ matrix $(X'X/T)^{-1}$, we obtain

$$\hat{B}'P_1\mathcal{A}(\hat{c})^{-1}\sqrt{T}P_1'\hat{D}\hat{c}=o_p(1). \tag{A.24}$$

Finally, substituting for $\mathcal{A}_2(\hat{c})$, we have

$$\mathcal{A}_2(\hat{c})\mathcal{A}(\hat{c})^{-1}\sqrt{T}P_1'\hat{D}\hat{c}=(\hat{c}'\otimes P_1'\hat{B})\hat{V}_x(\hat{c}\otimes\hat{B}'P_1\mathcal{A}(\hat{c})^{-1}\sqrt{T}P_1'\hat{D}\hat{c})=o_p(1), \tag{A.25}$$

where the last equality follows from (A.24). Thus,

$$T(P_1'\hat{D}\hat{c})'\mathcal{A}(\hat{c})^{-1}(P_1'\hat{D}\hat{c})=T(P_1'\hat{D}\hat{c})'\mathcal{A}_1(\hat{c})^{-1}(P_1'\hat{D}\hat{c})+o_p(1) \tag{A.26}$$

and

$$\begin{aligned}\mathcal{J}&=T\min_{c:c'c=1}\frac{c'\hat{D}'P_1(P_1'\hat{\Sigma}P_1)^{-1}P_1'\hat{D}c}{c'(X'X/T)c}+o_p(1)\\&=T\min_{\tilde{c}:\tilde{c}'\tilde{c}=1}\frac{\tilde{c}'\hat{B}'P_1(P_1'\hat{\Sigma}P_1)^{-1}P_1'\hat{B}\tilde{c}}{\tilde{c}'[(X'X/T)^{-1}]\tilde{c}}+o_p(1),\end{aligned} \tag{A.27}$$

where $\tilde{c}$ is proportional to $(X'X/T)c$. Using (A.27) and the invariance property of the estimator, it then follows that the $\mathcal{J}$ test is asymptotically distributed as T times the smallest eigenvalue of (Anderson, 1951; Sargan, 1958)

$$\tilde{\Omega}=(X'X/T)\hat{B}'P_1(P_1'\hat{\Sigma}P_1)^{-1}P_1'\hat{B}. \tag{A.28}$$

Let L_f be a lower triangular matrix such that $L_fL_f'=V_f$ and define

$$L=\begin{bmatrix}1 & 0_{K-1}'\\ \mu_f & L_f\end{bmatrix}. \tag{A.29}$$

Using that $(X'X)/T\xrightarrow{p}LL'$ and $\hat{\Sigma}\xrightarrow{p}\Sigma$, the $\mathcal{J}$ test has the same distribution as the smallest eigenvalue of

$$W=TL'\hat{B}'P_1(P_1'\Sigma P_1)^{-1}P_1'\hat{B}L. \tag{A.30}$$

Define $Z=(P_1'\Sigma P_1)^{-\frac{1}{2}}P_1'\hat{B}L$ and $M=(P_1'\Sigma P_1)^{-\frac{1}{2}}P_1'BL$. We have

$$\sqrt{T}\mathrm{vec}(Z-M)\xrightarrow{d}\mathcal{N}(0_{(N-1)K},I_{(N-1)K}). \tag{A.31}$$

Since $P_1'BL$ has rank $K-r$, there exists a $K\times r$ orthonormal matrix C_1 such that $MC_1 = 0_{(N-1)\times r}$. Let $C = [C_1,\ C_2]$ be a $K \times K$ orthonormal matrix, and define $\tilde{Z} = [\tilde{Z}_1,\ \tilde{Z}_2] = [ZC_1,\ ZC_2]$. Using (A.10) and $MC_1 = 0_{(N-1)\times r}$, we have

$$\sqrt{T}\begin{bmatrix} \mathrm{vec}(\tilde{Z}_1) \\ \mathrm{vec}(\tilde{Z}_2 - M_2) \end{bmatrix} \xrightarrow{d} \mathcal{N}\left(\begin{bmatrix} 0_{(N-1)r} \\ 0_{(N-1)(K-r)} \end{bmatrix}, \begin{bmatrix} I_{(N-1)r} & 0_{(N-1)r\times(N-1)(K-r)} \\ 0_{(N-1)(K-r)\times(N-1)r} & I_{(N-1)(K-r)} \end{bmatrix}\right), \tag{A.32}$$

where $M_2 = MC_2$ and all columns of M_2 are nonzero vectors. From the fact that $W = T(Z'Z)$ and $\tilde{W} = T(\tilde{Z}'\tilde{Z})$ share the same eigenvalues, it is sufficient to obtain the limiting distribution of the smallest eigenvalue of $\tilde{W}$ which is equal to the reciprocal of the largest eigenvalue of

$$\tilde{W}^{-1} = \begin{bmatrix} \tilde{W}^{11} & \tilde{W}^{12} \\ \tilde{W}^{21} & \tilde{W}^{22} \end{bmatrix}. \tag{A.33}$$

Using the formula for the inverse of a partitioned matrix, we have

$$\tilde{W}^{11} = \left(\sqrt{T}\tilde{Z}_1'[I_{N-1} - \tilde{Z}_2(\tilde{Z}_2'\tilde{Z}_2)^{-1}\tilde{Z}_2']\sqrt{T}\tilde{Z}_1\right)^{-1} \xrightarrow{d} \mathcal{W}_r(N-K-1+r, I_r)^{-1}, \tag{A.34}$$

$$\tilde{W}^{12} = -\tilde{W}^{11}\tilde{Z}_1'\tilde{Z}_2(\tilde{Z}_2'\tilde{Z}_2)^{-1} = O_p(T^{-\frac{1}{2}}), \tag{A.35}$$

$$\tilde{W}^{22} = (T\tilde{Z}_2'\tilde{Z}_2)^{-1} + (\tilde{Z}_2'\tilde{Z}_2)^{-1}(\tilde{Z}_2'\tilde{Z}_1)\tilde{W}^{11}(\tilde{Z}_1'\tilde{Z}_2)(\tilde{Z}_2'\tilde{Z}_2)^{-1} = O_p(T^{-1}), \tag{A.36}$$

where $\mathcal{W}_r(N-K-1+r, I_r)$ denotes the Wishart distribution with $N-K-1+r$ degrees of freedom and a scaling matrix I_r. Therefore, the limiting distribution of the largest eigenvalue of $\tilde{W}^{-1}$ is the same as the limiting distribution of the largest eigenvalue of $\tilde{W}^{11}$. Equivalently, the smallest eigenvalue of $T\tilde{\Omega}$ has the same limiting distribution as w_r, the smallest eigenvalue of $W_r \sim \mathcal{W}_r(N-K-1+r, I_r)$, where W_r denotes the limit of the inverse of $\tilde{W}^{11}$.

We now show that $\Pr[w_r \le a] \ge \Pr[x_{N-K} \le a]$, where $x_{N-K} \sim \chi^2_{N-K}$. When $r = 1$, $\mathcal{J} \xrightarrow{d} w_1 \sim \chi^2_{N-K}$. When $r \ge 2$, by the Bartlett decomposition of a Wishart matrix, we can write

$$W_r = \begin{bmatrix} W_{r-1} & W_{r-1}^{\frac{1}{2}}z \\ z'W_{r-1}^{\frac{1}{2}} & x_{N-K} + z'z \end{bmatrix}, \tag{A.37}$$

where $W_{r-1} \sim \mathcal{W}_{r-1}(N-K-2+r, I_{r-1})$, $z \sim \mathcal{N}(0_{r-1}, I_{r-1})$, and they are independent of each other and x_{N-K}. Using the fact that the eigenvalues of W_r are the same as the reciprocal of the eigenvalues of W_r^{-1}, it follows that

$$w_r = \min_{\omega:\omega'\omega=1} \omega' W_r \omega = \left(\max_{\omega:\omega'\omega=1} \omega' W_r^{-1}\omega\right)^{-1} \le ([0_{r-1}',\ 1]W_r^{-1}[0_{r-1}',\ 1]')^{-1} = x_{N-K} \sim \chi^2_{N-K}. \tag{A.38}$$

This completes the proof.

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Table I
Rejection Rates of the Specification Test

The table presents the rejection rates of Hansen, Heaton, and Yaron's (1996) test for overidentifying restrictions ($\mathcal{J}$) under correctly specified and misspecified models. The results are reported for different levels of significance (10%, 5%, and 1%) and for different values of the number of time series observations (T) using 100,000 simulations, assuming that the returns are generated from a multivariate normal distribution with means and covariance matrix calibrated to the 25 size and book-to-market Fama-French portfolio returns for the period 1959:2–2012:12. The $\mathcal{J}$ test statistic is compared with the critical values from a χ^2_{N-K} distribution. The rejection rates for the limiting case ($T=\infty$) in Panels B, C, and D are based on the results in Theorem 2.

	Size			Power		
T	10%	5%	1%	10%	5%	1%
Panel A: Model with 1 Useful Factor Only						
200	0.211	0.128	0.040	0.900	0.831	0.636
600	0.134	0.073	0.018	1.000	1.000	0.999
1000	0.121	0.065	0.014	1.000	1.000	1.000
∞	0.100	0.050	0.010	1.000	1.000	1.000
Panel B: Model with 1 Spurious Factor Only						
200	0.022	0.007	0.000	0.127	0.060	0.010
600	0.008	0.002	0.000	0.114	0.057	0.011
1000	0.007	0.002	0.000	0.108	0.054	0.011
∞	0.005	0.001	0.000	0.100	0.050	0.010
Panel C: Model with 1 Useful and 1 Spurious Factor						
200	0.017	0.005	0.000	0.102	0.044	0.006
600	0.008	0.002	0.000	0.109	0.055	0.010
1000	0.007	0.002	0.000	0.108	0.054	0.010
∞	0.005	0.001	0.000	0.100	0.050	0.010
Panel D: Model with 3 Useful and 2 Spurious Factors						
200	0.000	0.000	0.000	0.003	0.000	0.000
600	0.000	0.000	0.000	0.005	0.001	0.000
1000	0.000	0.000	0.000	0.006	0.001	0.000
∞	0.000	0.000	0.000	0.006	0.001	0.000

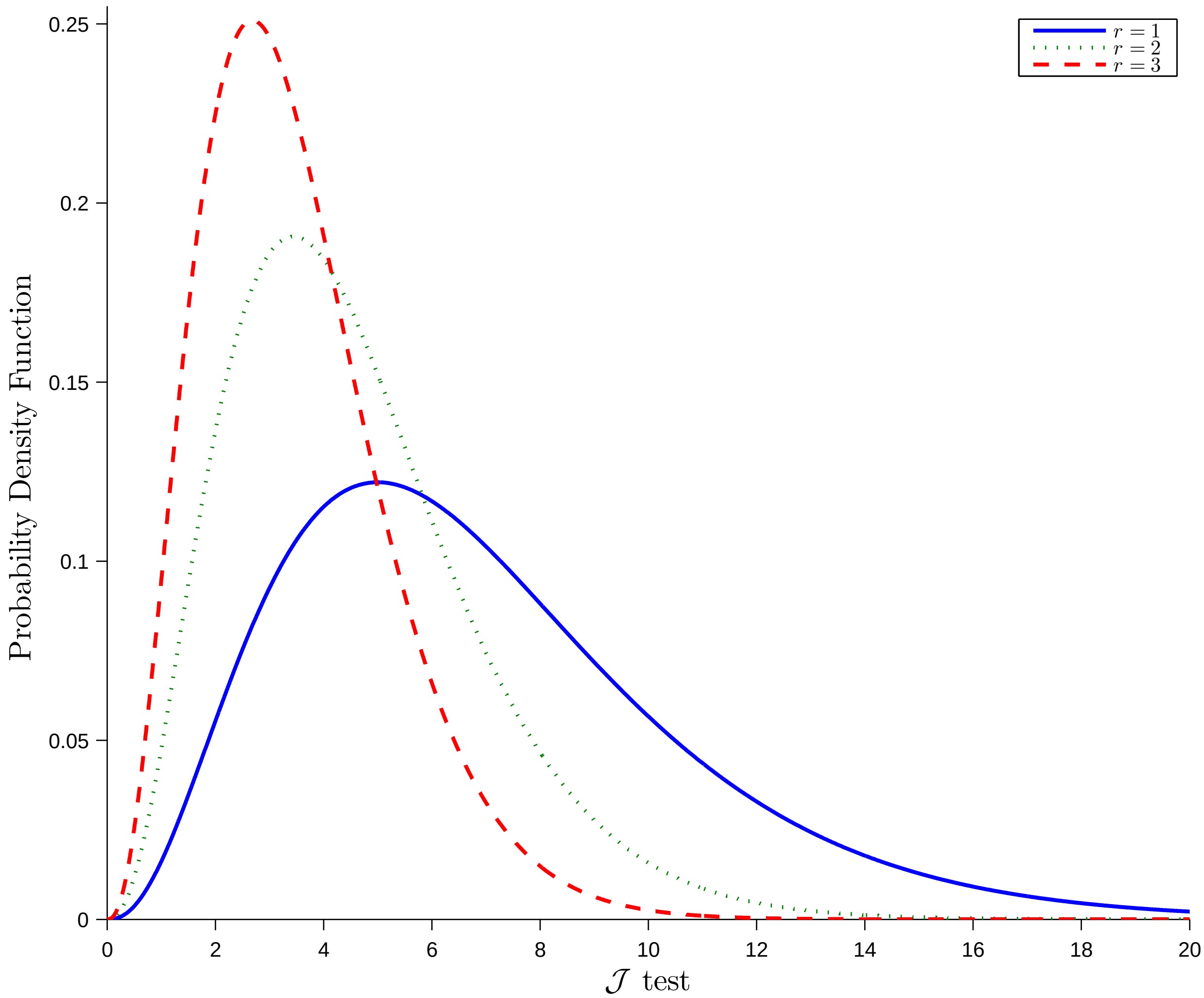

Figure 1. Limiting Distribution of the Specification Test $\mathcal{J}$. The figure plots the asymptotic distributions of $\mathcal{J}$ presented in Theorem 2 for $r = 1$, 2, and 3 (for $N - K = 7$).