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# A HETEROSCEDASTICITY-CONSISTENT COVARIANCE MATRIX ESTIMATOR FOR TIME SERIES REGRESSIONS

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This paper provides a covariance matrix estimator for the ordinary least squares coefficients of a linear time series model which is consistent even when the disturbances are heteroscedastic. This estimator does not require a formal model of the heteroscedasticity. One can also obtain a direct test of heteroscedasticity, although Monte Carlo experiments show that it may have low power.

## 1. Introduction

White (1980) provided a covariance matrix estimator of the ordinary least squares coefficients which is consistent even when the error term is conditionally or unconditionally heteroscedastic. However, White's proofs are valid only for those cases when the regressors are independent over observations. This paper shows that the same covariance matrix estimator can be used in a time series model, provided certain conditions are met.

In fact, this result is an extension of a special case considered in Hansen (1982), who assumes that the error terms are unconditionally homoscedastic but may be conditionally heteroscedastic. Here, we replace the stationary assumption with moment bounds. This allows the errors to be conditionally or unconditionally heteroscedastic.[1]

The next section provides the conditions needed to extend the results in Hansen (1982) and White (1980). Section 3 modifies White's tests of heteroscedasticity to allow for heterokurtic errors. Monte Carlo experiments are performed in section 4. Some concluding remarks are offered in section 5.

## 2. Extension of Hansen's and White's estimator

The proofs make use of two standard results in martingale theory. The first is the weak law of large numbers:

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[1]At the first draft of this paper, the author was unaware of the work by Domowitz and White (1981). They prove the existence of a heteroscedasticity-consistent covariance estimator for non-linear regressions. Although this paper only deals with linear regressions, which is a special case of Domowitz and White (1981), the conditions are slightly weaker because of linearity (see footnote 2). In addition, some interesting Monte Carlo results are presented here.

*Theorem 1.* *Let $Z_t$ be a (univariate) stochastic process, such that:*

(*i*) $\mathrm{E}[Z_t \mid Z_{t-1},\ldots]=0$, *with probability one (w.p. 1) for all t;*

(*ii*) *there exists $M>0$ such that $\mathrm{E}[|Z_t|^2]\leqq M$ for all t.*

*Then*

$$\sum_{t=1}^{T} Z_t/T \to 0 \quad \textit{in probability} \quad T\to\infty.$$

Theorem 1 is a straightforward application of Doob (1953, p. 155, theorem 4.1). The second is a martingale central limit theorem:

*Theorem 2.* *Let $Z_t$ be a stochastic process such that:*

(*iii*) $\mathrm{E}[Z_t \mid Z_{t-1},\ldots]=0$, *w.p. 1, for all t;*

(*iv*) *there exists $\delta>0$, $M>0$, such that $\mathrm{E}[|Z_t|^{2+\delta}]\leqq M$ for all t.*

*Denote*

$$S_T=\sum_{t=1}^{T} Z_t \quad \textit{and} \quad s_T^2=\sum_{t=1}^{T} \mathrm{E}[Z_t^2].$$

*Assume $s_T^2/T\to s^2>0$, as $T\to\infty$, and*

$$\sum_{t=1}^{T} \mathrm{E}[|\mathrm{E}[Z_t^2]-\mathrm{E}[Z_t^2 \mid Z_{t-1},\ldots]|]=\mathrm{O}(s_T^2).$$

*Then $S_T/s_T\to\Phi$, as $T\to\infty$ where $\Phi$ is a standard normal distribution.*

The proofs of Theorem 2 and subsequent theorems are provided in a mathematical appendix. Note that Theorems 1 and 2 also hold for the multivariate stochastic process $Z_t=[Z_{1t},\ldots,Z_{kt}]'$ when stated in terms of all linear combinations $\alpha' Z_t$, where $\alpha$ is any vector in $R^K$.

Now we turn to the standard linear model,

$$y_t=x_t'\beta+e_t \quad \text{for} \quad t=1,2,\ldots,T, \tag{1}$$

where $y_t$ and $e_t$ are scalars, and $x_t$ and $\beta$ are vectors of length $K$.

We make the following assumptions:

A.1. $\mathrm{E}[e_t \mid x_t,\ldots,e_{t-1},\ldots]=0$, w.p. 1, for all $t$.
A.2. There exists $M>0$, such that $\mathrm{E}[|e_t x_{nt}|^2]\leqq M$ for all $n$ and $t$.
A.3. $(X'X/T)\to A>0$ in probability as $T\to\infty$, where $X$ is the matrix formed by stacking the row vectors $x_t'$, $t=1,\ldots,T$.

(The notation '$A>0$' means '$A$ is a positive definite matrix'.)

*Theorem 3 (Consistency of OLS). Under A.1–A.3, $b_T=(X'X)^{-1}(X'y)$ is a consistent estimator of $\beta$.*

Condition A.1 assures that $x_t$ and $e_t$ are not correlated contemporaneously, and that $e_t$ is not correlated with past $x_t$ and $e_t$. This is often assumed in time series models, especially autoregressions. Under this condition, we also know that $e_t x_t$ is a multivariate martingale difference. In order to show consistency, we make use of the law of large numbers, and so we put second moment bounds on $e_t x_t$ (i.e., A.2).

Condition A.3 is stronger than required for this theorem. It can be relaxed as follows. Let $A_T=(X'X/T)$. If the determinant of $A_T$ is bounded away from zero for sufficiently large $T$, then Theorem 3 is still true. We do not state the theorem in this way, because (later) $A_T$ will appear as a component of the estimate of the asymptotic covariance of OLS, which is sensible only when $A_T$ converges.

To prove the asymptotic normality of OLS, we need to make some further assumptions:

A.4. There exists $\delta>0$, $M>0$, such that $\mathrm{E}[|e_t^2 x_{mt} x_{nt}|^{1+\delta}]\leqq M$ for all $m$, $n$, and $t$.

A.5. $B_T=\sum_{t=1}^{T}\mathrm{E}[e_t^2 x_t x_t']/T\to B>0$ as $T\to\infty$.

A.6. $$\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{t=1}^{T}\mathrm{E}[|\mathrm{E}[e_t^2 x_{it}' x_{jt}\alpha_{ij}]-\mathrm{E}[e_t^2 x_{it} x_{jt}\alpha_{ij}\,|\,x_{t-1},\ldots,e_{t-1},\ldots]|]$$
$$=\mathrm{O}\left(\sum_{i=1}^{K}\sum_{j=1}^{K}(B_T)_{ij}\alpha_{ij}\right),$$ for any sequence of real numbers $\alpha_{ij}$.

*Theorem 4 (Asymptotic normality of OLS). Under A.1–A.6, $T^{\frac{1}{2}}(b_T-\beta)$ is asymptotically distributed as a normal variate with mean O and covariance $A^{-1}B(A^{-1})'$.*

Condition A.4 allows us to apply Liapunov's central limit theorem when $(x_t, e_t)$ are independent. Theorem 4 requires more stringent conditions, i.e., A.5 and A.6, because independence is not assumed. It is likely that A.5 can be relaxed in a manner analogous to that of A.3. However, since (later) $B_T$ will appear as a component of the estimate of the asymptotic covariance of OLS, this is sensible when $B_T$ converges. Condition A.6 requires that the conditional and unconditional expectations of $e_t^2 x_t x_t'$ be 'close' to each other. This is a standard assumption for martingale central limit theorems.[2]

[2]This condition is weaker than the mixing conditions required by Domowitz and White (1981), because of the linearity assumption.

In order to obtain an estimate of the covariance of OLS, we need one further assumption:

A.7. There exists $M>0$ such that

$$\mathrm{E}[|e_t x_{it} x_{jt} x_{kt}|^2] \leqq M \quad \text{for all} \quad i, j, k, \text{ and } t,$$

$$\mathrm{E}[|x_{it} x_{jt} x_{kt} x_{mt}|^2] \leqq M \quad \text{for all} \quad i, j, k, m, \text{ and } t.$$

*Theorem 5* (*Consistency of the covariance estimator*). *Under A.1–A.7, $C_T = \sum_{t=1}^{T} u_t^2 x_t x_t'/T \to B$ in probability as $T \to \infty$, where $u_t = y_t - x_t' b_T$.*

As a result, the matrix

$$V_{HC} = (X'X/T)^{-1}\left(\sum_{t=1}^{T} u_t^2 x_t x_t'/T\right)(X'X/T)^{-1} \tag{2}$$

is a consistent estimate of $A^{-1}B(A^{-1})'$. This shall be referred to as the *HC* matrix. Note that there is no need to specify the structure of heteroscedasticity of $e_t$, either conditioned or unconditioned on the $x_t$'s.

If the errors are conditionally homoscedastic, i.e., $\mathrm{E}[e_t^2 \mid x_t, \ldots, e_{t-1}, \ldots] = \sigma^2 < \infty$, w.p. 1 for all $t$, then they are unconditionally homoscedastic, and the asymptotic covariance of $b_T$ is $\sigma^2 A^{-1}$, which can be estimated by the usual covariance matrix estimator,

$$V_{OC} = \left(\sum_{t=1}^{T} u_t^2/T\right)(X'X/T)^{-1}, \tag{3}$$

and shall be referred to as the *OC* matrix.

Note that our results have been stated for the single equation model. But they easily extend to the several equations context, if we also take into account the contemporaneous correlation between the disturbances across equations.

## 3. Test of heteroscedasticity

In comparing *HC* and *OC*, White (1980) pointed out that the difference between these two estimators tends to zero if and only if the $e_t^2$ is uncorrelated with $[x_t x_t']$. To test this, White runs the following regression:

$$u_t^2 = \alpha + \psi_t' \gamma + \eta_t, \tag{4}$$

where $\alpha$ is a constant, and $\psi_t$ is the vector of distinct elements of $[x_t x_t']$, excluding any constant term. He tests if $\gamma=0$. His test statistic requires that the error terms $\eta_t$ be homoscedastic, i.e., $e_t$ is homokurtic.[3]

In order to relax this requirement and allow $e_t$ to be heterokurtic (since, after all, $e_t$ is allowed to be heteroscedastic), we should run OLS on (4), compute the $HC$ covariance for $\gamma$, and test if $\gamma=0$. This procedure is justified if $e_t$ and $x_t$ satisfies the following conditions:

A.8. There exists $M>0$ such that $\mathrm{E}[|e_t^2\psi_{it}\psi_{jt}\psi_{kt}|^2]\leqq M$ for all $i$, $j$, $k$ and $t$, and $\mathrm{E}[|\psi_{it}\psi_{jt}\psi_{kt}\psi_{lt}|^2]\leqq \mathrm{M}$ for all $i$, $j$, $k$, $l$ and $t$.

A.9. $$\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{t=1}^{T}\mathrm{E}[\mathrm{E}[e_t^4\psi_{it}\psi_{jt}\alpha_{ij}]-\mathrm{E}[e_t^4\psi_{it}\psi_{it}\alpha_{ij}\,|\,x_{t-1},\dots,e_{t-1}]]$$
$$=\mathrm{O}\left(\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{t=1}^{T}\mathrm{E}[e_t^4\psi_{it}\psi_{ij}\alpha_{ij}]\right).$$

In other words, eq. (4) must satisfy A.1–A.7.

## 4. Monte Carlo results

To study the properties of the $HC$ covariance and the test of heteroscedasticity, we use the first-order Markov process, $y_t=\beta y_{t-1}+e_t$, where $\beta=0.05$, and $e_t$ is a serially uncorrelated error term, normally distributed with mean and variance $\sigma_t^2$. We shall consider the cases when $\sigma_t^2$ is bounded away from zero and infinity, $0<\underline{\sigma}^2\leqq\sigma_t^2\leqq\bar{\sigma}^2<\infty$. Then it is easily verified that the assumptions A.1 through A.9 are satisfied. Hence the OLS estimate of $\beta$ is consistent, and the heteroscedastic-consistent covariance matrix estimator can be used to test restrictions on $\beta$.

We performed some Monte Carlo runs of this first-order Markov process. A total of 500 draws of 200 observations were used. The following types of heteroscedasticities were considered:

V.1. $\sigma_t^2=1$;
V.2. $\sigma_t$ normally distributed, with mean 0.1 and variance 10;
V.3. $\sigma_t^2=\sin(t)^2$;
V.4. $\sigma_t^2$ increased from 0.005 to 10 in steps of 0.005;
V.5. $\sigma_t^2$ equaled 100 for $t=1,\dots,100$, and equaled 1 for $1=101,\dots,200$.

Two sets of results are presented. Table 1 summarizes the tests of the hypothesis $\beta=0.5$, using the various covariance estimators. The first column reports the average OLS coefficients for the 500 draws. The second and third columns give the average standard errors of the estimate computed under $V_{OC}$ and $V_{HC}$. The last two columns list the percent of the 500 draws in which the hypothesis $\beta=0.5$ was rejected at the 5-percent significance level.

[3]This is similar to Glejser (1969) and Goldfeld and Quandt (1972).

Table 1

Comparison of $OC$ and $HC$ in Monte Carlo experiments. Regression: $y_t = \beta y_{t-1} + e_t$.[a]

| | Average estimated coefficient | Average standard error | | Percent of draws rejecting $\beta=0.5$[b] | |
|---|---|---|---|---|---|
| | | $OC$[c] | $HC$[d] | $OC$[c] | $HC$[d] |
| V.1 | 0.495709 | 0.061391 | 0.060635 | 4.6 | 4.8 |
| V.2 | 0.498803 | 0.061216 | 0.059902 | 6.2 | 6.8 |
| V.3 | 0.495486 | 0.061685 | 0.048180 | 2.0 | 6.2 |
| V.4 | 0.487716 | 0.062284 | 0.079632 | 13.8 | 6.0 |
| V.5 | 0.493473 | 0.061116 | 0.084245 | 15.0 | 6.4 |

[a]These are averages for 500 draws of 200 observations.
[b]At the 5-percent asymptotic significance level.
[c]Using the usual covariance matrix ($V_{OC}$), given in (3).
[d]Using the heteroscedasticity-consistent covariance matrix ($V_{HC}$), given in (2).

Table 2

Tests of heteroscedasticity in Monte Carlo experiments. Regression: $u_t^2 = \alpha + \gamma y_{t-1}^2 + \eta_t$.[a]

| | Average estimate of $\gamma$ | Percent of draws rejecting $\gamma=0$[b] | |
|---|---|---|---|
| | | $OC$[c] | $HC$[d] |
| V.1 | −0.004155 | 1.69 | 6.60 |
| V.2 | −0.01051 | 1.40 | 20.00 |
| V.3 | 0.08863 | 47.00 | 23.40 |
| V.4 | −0.05483 | 0.60 | 52.80 |
| V.5 | 0.11965 | 67.20 | 65.20 |

[a]These are averages for 500 draws of 200 observations.
[b]At the 5-percent asymptotic significance level.
[c]Using the statistic proposed by White (1980).
[d]Using the alternative statistic proposed in section 3.

In both V.1 and V.2, the rejection rates for $\beta=0.5$ do not differ substantially between the two covariance estimators, and both are reasonably close to the asymptotic rejection rate of 5%. In V.3, $OC$ tends to overestimate the true covariance, rejecting $\beta=0.5$ only 2% of the time. On the other hand, $HC$ is closer to the true covariance since it rejects $\beta=0.5$ at 6.2%. In V.3 and V.4, $OC$ underestimates the true covariance, rejecting $\beta=0.5$ at 13.8% and 15.0%, respectively, while $HC$ is again closer to the true covariance, rejecting $\beta=0.5$ at 6.0% and 6.4%, respectively.

This first set of results shows that there may be little loss of power by using $HC$ rather than $OC$ to conduct tests on the estimated coefficients. In addition, $HC$ leads to a consistent test, while $OC$ may not. Since there is little theory to guide us in making assumptions about the variances of the error terms, it is more prudent to use $HC$.

Table 2 gives the results of the tests of heteroscedasticity using various test statistics. Since the errors are homokurtic, White's (1980) test as well as the one proposed in section 3 are both consistent. (However, if the errors are not homokurtic, then only the second test is consistent.) The results are very disappointing. The rate of rejecting homoscedasticity is very low, even in the cases where there is heteroscedasticity: V.3, V.4, and V.5. An obvious suggestion is to lower the requirements for rejection, i.e., set the significance level to be 10, 20, or even 30 percent. Another suggestion is to use the $HC$ covariance in all circumstances, regardless of whether homoscedasticity is rejected or not.

## 5. Concluding remarks

We have shown that the heteroscedasticity-consistent covariance matrix estimator for OLS in Hansen (1982) and White (1980) can be extended to include time series regression. This estimator can be used without the need to specify the structure of the heteroscedasticity. Also we have extended White's test of heteroscedasticity to allow for heterokurtic disturbances.

Some Monte Carlo runs were performed to study the properties of this new covariance estimator and the test of heteroscedasticity. At least in the context of the first-order Markov process, the results indicate that there is little loss of power in using the $HC$ covariance even if the errors were homoscedastic. But there is a gain in consistency if we are testing restrictions on the coefficients of the model. Furthermore, the tests of heteroscedasticity seem to have little power to discriminate between homoscedasticity and heteroscedasticity, at least at the 5-percent significance level. One way to counteract this is to use larger significance levels. Another way is to abandon the heteroscedasticity test, and to use always the $HC$ covariance, because an error of the second kind in the heteroscedasticity test (i.e., accepting homoscedasticity even though the error term is truly heteroscedastic) is much more serious than an error of the first kind (i.e., rejecting homoscedasticity while the error term is truly homoscedastic).

## Mathematical appendix

### *Proof of Theorem 2*

We note that the moment bound condition implies that $Z_t$ satisfies the Lindberg condition. Hence we can apply Theorem 1 in Chow and Teicher (1978, p. 313).

*Proof of Theorem 3*

$(b_T-\beta)=(X'X/T)^{-1}(X'e/T)$. We have $(X'X/T)\to A$ in probability, where $A>0$. We want to show $(X'e/T)\to 0$ in probability. Now take any $\alpha$, a vector in $R^K$. Define $Z_t=\alpha' x_t e_t$. Then,

$$\mathrm{E}[Z_t \mid Z_{t-1},\ldots]=0,$$

$$\mathrm{E}[|Z_t|^2]=\mathrm{E}[|\alpha' x_t e_t^2|]\leqq \sum_{k=1}^{K}[\alpha_k^2 \mid \mathrm{E}[|x_{kt}e_t^2|]]\leqq M\sum_{k=1}^{K}\alpha_k^2.$$

Hence $\sum_{t=1}^{T} Z_t/T\to 0$ in probability as $T\to\infty$. Q.E.D.

*Proof of Theorem 4*

$T^{\frac{1}{2}}(b_T-\beta)=(X'X/T)^{-1}(X'e/T^{\frac{1}{2}})$. We want to show $(X'e/T^{\frac{1}{2}})$ converges to a normal distribution. Pick any $\alpha$ in $R^K$. Define $Z_t=\alpha' x_t e_t$. Then,

$$\mathrm{E}[Z_t \mid Z_{t-1},\ldots]=0,$$

$$\mathrm{E}[|Z_t|^{2+\delta}]=\mathrm{E}[|\alpha' x_t e_t|^{2+\delta}]\leqq \sum_{k=1}^{K}[|\alpha_k|\mathrm{E}[|x_{kt}e_t|^{2+\delta}]^{1/(2+\delta)}]^{2+\delta}$$

$$\leqq\left[\sum_{k=1}^{K}|\alpha_k|^{2+\delta}\right]M.$$

Hence apply Theorem 2, getting $\sum_{t=1}^{T} Z_t/T^{\frac{1}{2}}$ converges in distribution to $\mathrm{N}(0,\alpha' B\alpha)$. Q.E.D.

*Proof of Theorem 5*

To prove Theorem 5, we need the following lemma:

*Lemma 1. Let $X_t$, $Y_t$ be two stochastic processes, such that*

(*a*) *$Y_t\to 0$ in probability as $T\to\infty$.*

(*b*) *There exists $\delta>0$, $M>0$ such that*

$$\mathrm{E}[|X_t|^{1+\delta}]\leqq M \quad \textit{for all} \quad t.$$

*Then $X_t Y_t\to 0$ in probability as $T\to\infty$.*

*Proof.* Pick any $\varepsilon > 0$.

$$\Pr\{|X_t Y_t| > \varepsilon\} \leqq \Pr\{|X_t Y_t| > \varepsilon \text{ and } |X_t| < \varepsilon/k\}$$

$$+ \Pr\{|X_t 1/t| > \varepsilon \text{ and } |X_t| \geqq \varepsilon/k\} \quad \text{for any} \quad k > 0,$$

$$\leqq \Pr\{|Y_t| > k\} + \Pr\{|Y_t| \geqq \varepsilon/k\}$$

$$\leqq \Pr\{|Y_t| > k\} + \mathrm{E}[|X_t|^{1+\delta}]/|\varepsilon/k|^{1+\delta}$$

$$\leqq \Pr\{|Y_t| > k\} + Mk^{1+\delta}/\varepsilon^{1+\delta}.$$

We can pick $k$ s.t.

$$Mk^{1+\delta}/\varepsilon^{1+\delta} < \varepsilon/2.$$

Since $Y_t \to 0$ in probability, we can find $t$ large enough so that $\Pr\{|Y_t| > k\} < \varepsilon/2$. Hence

$$\Pr\{|\mathrm{X}_t \mathrm{Y}_t| > \varepsilon\} < \varepsilon. \quad \text{Q.E.D.}$$

To prove Theorem 5, we proceed as follows: First, we show that the following expression tends to zero in probability:

$$\sum_{t=1}^{T} (u_t^2 x_t x_t' - e_t^2 x_t x_t')/T = 2 \sum_{t=1}^{T} e_t x_t'(\beta - b_T) x_t x_t'/T$$

$$+ \sum_{t=1}^{T} [x_t'(\beta - b_T)^2] x_t x_t'/T.$$

Now take any sequence of each number $\alpha_{ij}$, $i = 1, \ldots, k$, $j = 1, \ldots, k$. We want to show that

(a) $$\sum_{k=1}^{K} \left[ \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} e_t x_{kt} \alpha_{ij} x_{it} x_{jt}/T \right] [\beta_k - b_{kT}] \to 0$$

in probability as $T \to \infty$,

(b) $$\sum_{k=1}^{K} \sum_{m=1}^{K} \left[ \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} \alpha_{ij} x_{kt} x_{mt} x_{it} x_{jt}/T \right] [\beta_k - b_{kT}][\beta_m - b_{mT}] \to 0$$

in probability as $T \to \infty$.

We know for each $k$, $(\beta_k - b_{kT}) \to 0$ in probability as $T \to \infty$.

Now

$$\mathrm{E}\left[\left|\sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{t=1}^{t} e_t x_{jt} x_{it} \alpha_{ij}/T\right|^{1+\delta}\right]$$

$$\leqq \left[\sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{t=1}^{t} [\mathrm{E}|e_t x_{jt} x_{it} x_{jt} \alpha_{ij}|^{1/(1+\delta)}]^{1+\delta}\right]^{1+\delta} \Big/ T^{1+\delta}$$

$$\leqq \left[\sum_{i=1}^{k}\sum_{j=1}^{k} |\alpha_{ij}| M\right]^{1+\delta}.$$

Apply Lemma 1 to establish (a). Similarly, (b) is established.

Hence

$$\sum_{t=1}^{T} (u_t^2 - e_t^2) x_t x_t'/T \to 0$$

in probability as $T \to \infty$. Now

$$\sum_{t=1}^{T} e_t^2 x_t x_t'/T \to B$$

in probability as $T \to \infty$ by A.5 and A.6. Q.E.D.

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