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ON CORRELATIONS AND INFERENCES ABOUT MEAN-VARIANCE EFFICIENCY *

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A framework is presented for investigating the mean-variance efficiency of an unobservable portfolio based on its correlation with a proxy portfolio. A sensitivity analysis derives the highest correlation between the proxy and a portfolio that reverses the inference of a test of Sharpe-Lintner tangency. For example, the maximum correlation between the value-weighted NYSE-AMEX portfolio and a portfolio inferred tangent ranges from 0.76 to 0.48. We also test whether the correlation between the proxy and the tangent portfolio exceeds a given level. This hypothesis is often rejected for the NYSE-AMEX proxy at a correlation of 0.7.

1. Introduction

Many asset pricing models imply the mean-variance efficiency of one or more benchmark portfolios. Models for which the benchmarks have been identified include the Capital Asset Pricing Model (CAPM), the intertemporal consumption-based model, and the Arbitrage Pricing Theory.[1] Researchers concerned with testing asset pricing models have faced at least two important questions in recent years. First, how does one test the mean-variance efficiency of a given portfolio in a finite sample? Second, how can one make inferences about the mean-variance efficiency of a benchmark when its exact rate of return is unobservable?

In addressing the problem of testing a given portfolio's efficiency, researchers have sought to apply methods that account for finite-sample variability. Various tests are developed and applied by Gibbons (1982), Stambaugh (1982), Jobson and Korkie (1982), and Shanken (1985). Ross (1983) and

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[1] The CAPM is due to Sharpe (1964), Lintner (1965), and Black (1972); the consumption-based intertemporal model is due to Breeden (1979); the Arbitrage Pricing Theory is due to Ross (1976). Benchmark portfolio efficiency is discussed by Fama (1976), Roll (1977), and Ross (1977) for the CAPM, by Breeden (1979) for the consumption model, and by Chamberlain (1983), Grinblatt and Titman (1987), and Huberman, Kandel and Stambaugh (1987) for the Arbitrage Pricing Theory.

Kandel (1984a) obtain analytical solutions for computing one such test, the likelihood ratio. The distribution of the tests in finite samples has been investigated analytically [Ross (1983), Shanken (1985), and Gibbons, Ross and Shanken (1985)] and through simulations [Gibbons (1980), Stambaugh (1981), Jobson and Korkie (1982), and MacKinlay (1985)].

Precise measurement of the relevant benchmark return can be difficult. Roll (1977) discusses how unobservability of the market portfolio's return presents problems in testing the CAPM. When the benchmark portfolio is measured imprecisely by a proxy, the researcher may wish to investigate the sensitivity of inferences to alternative specifications of the proxy. One approach is to repeat tests using various proxies [Stambaugh (1982)]. Another approach, pursued in this study, is to ask whether any portfolio in a class of portfolios would provide a different inference, where the class includes all portfolios satisfying some specified relation (e.g., correlation) with the original proxy.

We characterize an alternative proxy portfolio in terms of the correlation between its return and the return on the original proxy. This characterization allows us to define a class of alternative proxies as all portfolios having a sample correlation of at least, say, 0.9 with the original proxy. We then ask whether any portfolio in that class gives an inference about mean-variance efficiency that differs from the inference about the original proxy. This question is examined in a finite-sample context. We also test whether the ex ante correlation between the proxy and the Sharpe-Lintner tangent portfolio of the global asset universe exceeds a given value. Thus, our study integrates the problems of finite-sample tests and benchmark-portfolio measurement.

A brief example can illustrate the type of information provided by the approach developed here. Using weekly data from July 1969 through October 1975, the second of three subperiods, a likelihood ratio test can reject at the 0.05 significance level the hypothesis that the value-weighted portfolio of all New York and American Stock Exchange stocks is the Sharpe-Lintner tangent portfolio. We find that no alternative index portfolio whose sample (ex post) correlation with that original proxy exceeds 0.50 could have provided a different inference. In addition to conducting this ex post sensitivity analysis, we also reject in the same subperiod the hypothesis that the original proxy has an ex ante correlation of at least 0.70 with the Sharpe-Lintner tangent portfolio of the global universe of assets. If the market portfolio has a correlation of at least 0.70 with the NYSE-AMEX value-weighted proxy, then the latter result also rejects the CAPM.

The paper proceeds as follows. Section 2 first analyzes the above problem when all parameter values are given. This framework allows us to introduce the relevant mean-variance mathematics before turning to the complications of finite-sample variability. The starting point is Roll's (1977) well known example, in which he shows that the market proxy used by Black, Jensen and

Scholes (1972) is correlated 0.9 with the sample Sharpe–Lintner tangent portfolio. We show that the correlation between the tangent portfolio and the proxy is sensitive to how one constructs the efficient set. This applies for both the population and the sample. For example, if the set is constructed from 16 portfolios of stocks and bonds, the sample correlation between the tangency and the proxy of Black, Jensen and Scholes drops to 0.48. If still more assets are used to construct the set, the correlation can only decrease. The last statement follows from the result that the correlation between a proxy and the tangent portfolio is simply the ratio of their Sharpe measures.

Section 2 also discusses a more general question examined by Kandel and Stambaugh (1986): where, in mean–variance space, are the portfolios having correlations with a given proxy of at least, say, 0.9? Do such portfolios include points on the minimum-variance boundary? Do they include the Sharpe–Lintner tangency? Do they exist at all levels of mean return? A complete analytical characterization of this set of portfolios is provided in Kandel and Stambaugh (1986). In section 2, we illustrate graphically the properties of this set that are useful in deriving the sensitivity analysis and the tests in sections 3 and 4.

Section 3 expands the analysis to include finite-sample inference. We analyze the sensitivity of inferences based on the likelihood ratio test of tangency in the presence of a riskless asset. As Ross (1983) demonstrates, this test has a known finite-sample distribution and, conveniently for our purposes, can be constructed easily from parameters of the sample efficient set. The latter feature allows us to extend the analytical results in section 2 and to compute the highest sample correlation between the proxy and a portfolio that, in the same sample, reverses the inference about the proxy's tangency. Using stock and bond returns data, we compute that correlation for various proxies and sets of assets. In many cases, the correlation is quite high. In other cases, there are no portfolios that would reverse inferences, whatever the correlation. The latter cases occur when, for that sample, there is no rejection region for the likelihood ratio test for standard test sizes.

Section 4 presents a new test of the hypothesis that a given proxy is correlated ex ante at least ρ_0 with the ex ante tangent portfolio. The null hypothesis is, in fact, a joint hypothesis that some portfolio whose exact return is unobservable, e.g., the market portfolio, is both (i) the ex ante tangent portfolio of the global universe of all assets and (ii) correlated ex ante at least ρ_0 with the given observable proxy. The null hypothesis is rejected if the proxy itself is inferred non-tangent for the econometrician's observed asset universe by Ross's test and if the highest sample correlation between the proxy and a portfolio that is inferred tangent is too low. The latter sample statistic is derived in section 3, and it is used there for sensitivity analysis within the same sample. In section 4 the same statistic is used in a formal test whose significance level is bounded from above.

In a recent paper, Shanken (1987) suggests an approach for testing the hypothesis that the correlation between an observable proxy and the ex ante tangent portfolio of the econometrician's observed asset universe exceeds a given level. His approach gives a test statistic whose distribution depends on an unknown parameter. Given that parameter, which can be estimated, the approach provides exact significance levels for a test of this hypothesis. The second part of section 4 generalizes Shanken's approach slightly to consider instead the ex ante tangent portfolio of the global asset universe, and we obtain a test whose significance level is bounded from above, conditional on the unknown parameter. We then apply this test and obtain inferences similar to those produced by the first approach. Both approaches indicate that the above hypothesis is typically rejected at conventional significance levels for ρ_0 equal to 0.9, and often for ρ_0 equal to 0.8 or 0.7, when the market proxy is either the equally weighted or value-weighted portfolio of New York and American Stock Exchange stocks.

Section 5 reviews the paper's conclusions.

2. Correlations between a given portfolio and alternative portfolios

This section explores the relations that govern the correlation between the returns on a given proxy portfolio and the returns on (i) the Sharpe–Lintner tangent portfolio, (ii) other portfolios on the minimum variance boundary, and (iii) arbitrary feasible portfolios, given only the locations of the latter in mean–variance space. We also note that these relations give a simple implication about the relevant benchmark portfolio of a subset of assets. The relations in this section are stated in terms of ex ante values, but they hold (and are subsequently used) for both ex ante and sample values.

2.1. The correlation between the proxy and the tangent portfolio

Given a universe of n risky assets and a riskless asset, define

$\mu(p)$ = mean return on portfolio p,
$\sigma(p)$ = standard deviation of the return on portfolio p,
$\rho(p,q)$ = correlation between returns on portfolios p and q,
r = riskless return.

The Sharpe measure of a portfolio is the ratio of its mean excess return to its standard deviation of return. That is, the Sharpe measure of p is given by

$$S(p)=\frac{\mu(p)-r}{\sigma(p)}. \tag{1}$$

The Sharpe–Lintner *tangent* portfolio is the portfolio of the n risky assets with the maximum Sharpe measure.[2]

The term 'minimum-variance portfolio' will be used throughout the paper to denote a portfolio on the minimum-variance boundary of the risky assets. A circumflex will denote a sample value. The following notation is used to denote various portfolios of the n risky assets in the universe:

α = proxy portfolio,
γ = ex ante tangent portfolio,
$\hat{\gamma}$ = sample tangent portfolio,
p_z = minimum-variance portfolio whose return is uncorrelated with the minimum-variance portfolio p, i.e., p_z is a 'zero-beta' portfolio with respect to p,
p^*= minimum-variance portfolio with the same mean return as portfolio p.

As established in the following proposition, Sharpe measures provide a convenient way to compute the correlation between the proxy and the tangent portfolio.

Proposition 1.[3]

$$\rho(\alpha,\gamma) = S(\alpha)/S(\gamma). \tag{2}$$

Proof. The linear mean–beta relation implied by the tangency of γ gives

$$\mu(\alpha) - r = \frac{\operatorname{cov}(\alpha,\gamma)}{\sigma^2(\gamma)}[\mu(\gamma) - r] = \frac{\sigma(\alpha)}{\sigma(\gamma)}\rho(\alpha,\gamma)[\mu(\gamma) - r]. \tag{3}$$

Solving (3) for $\rho(\alpha,\gamma)$ gives

$$\rho(\alpha,\gamma) = \frac{[\mu(\alpha) - r]/\sigma(\alpha)}{[\mu(\gamma) - r]/\sigma(\gamma)} = S(\alpha)/S(\gamma). \quad \square \tag{4}$$

In his critique of tests of the CAPM, Roll (1977) constructs efficient sets from sample parameters and then computes the correlation between a market proxy and the sample tangent portfolio. In Roll's examples, the correlations are high, 0.9 or more, which leads Roll to suggest that rejections of the CAPM can be reversed easily with an alternative proxy that is highly correlated with the original. Proposition 1 is useful in analyzing such examples. As we

[2] We assume throughout that such a portfolio exists for both the population and the sample, which is equivalent to assuming that the riskless return is less than the mean return on the portfolio of risky assets having the smallest variance.

[3] See also Jensen (1969) and Long (1977) for related results.

demonstrate, the correlation between the proxy and the sample tangent portfolio is sensitive to how one constructs the efficient set.

We construct several sample minimum-variance boundaries using monthly returns on various assets for the overall period January 1926 through November 1978 and for two subperiods. All returns are in excess of the one-month Treasury Bill rate [from Ibbotson and Sinquefield (1982)]. Our market proxy is the equally weighted New York Stock Exchange (NYSE) index, following Black, Jensen and Scholes (1972), one of the studies from which Roll (1977) generates his examples. The first column of table 1 displays the correlation between the equally weighted NYSE and the tangent portfolio for three minimum-variance boundaries. (The tangent emanates from the origin, since we use excess returns.) The first minimum-variance boundary is essentially the same one constructed by Roll. That is, we form a zero-beta portfolio in the manner of Black, Jensen and Scholes and compute a series of monthly returns on that portfolio. The boundary is then generated as all possible combinations of the NYSE and the zero-beta portfolio, using the return series to estimate the necessary parameters. Consistent with Roll's finding, the correlations between the NYSE proxy and the sample tangent portfolio are 0.9 or more in this case.

The other two boundaries constructed here, unlike the first, are not assumed to include the equally weighted NYSE. Rather, that portfolio now lies somewhere inside the sample boundary. We construct the first of these alternative boundaries from a universe containing the equally weighted NYSE plus ten equally weighted common stock portfolios formed by ranking firms into deciles of market value of equity at the end of the previous year.[4] The correlations between the sample tangent portfolio of this boundary and the NYSE proxy are considerably lower than in the original example – 0.60 for the overall period.

The decline in the correlation from the first boundary to the second is easily understood given the following corollary:

Corollary 1. *For a given proxy α, $\rho(\alpha, \gamma)$ cannot increase as risky assets are added to the universe (and γ changes).*

Proof. Use Proposition 1 and the fact that $S(\gamma)$ cannot decrease as the universe expands. □

This corollary is illustrated further with the third boundary, which is constructed by adding the value-weighted NYSE and four bond portfolios to

[4] The ten size-ranked portfolios exclude firms for which market values cannot be computed at the end of the previous calendar year. Thus the equally weighted NYSE index, which includes all stocks on the Exchange in any month, is not a redundant asset here.

Table 1

Sample correlations between the equally weighted NYSE (EW) and portfolios on the minimum-variance boundary.

Assets used to construct the minimum-variance boundary: Number	Description	Correlation between EW and the Sharpe–Lintner tangent portfolio	Maximum correlation between EW and a portfolio on the boundary
		1/1926–11/1978	
2	EW plus the BJS zero-beta portfolio[a]	0.994	1.000
11	EW plus ten size-ranked portfolios of common stocks[b]	0.596	0.783
16	EW, the size-ranked portfolios, the value-weighted NYSE, and four bond portfolios[c]	0.484	0.484
		1/1926–12/1952	
2	EW plus the BJS zero-beta portfolio	0.920	1.000
11	EW plus ten size-ranked portfolios of common stocks	0.440	0.650
16	EW, the size-ranked portfolios, the value-weighted NYSE, and four bond portfolios	0.269	0.328
		1/1953–11/1978	
2	EW plus the BJS zero-beta portfolio	0.902	1.000
11	EW plus ten size-ranked portfolios of common stocks	0.528	0.883
16	EW, the size-ranked portfolios, the value-weighted NYSE, and four bond portfolios	0.438	0.550

[a] The zero-beta portfolio is formed in essentially the same manner as Black, Jensen and Scholes (1972).

[b] All firms on the NYSE are assigned to deciles based on the market value of equity at the end of the previous year. Portfolios are equally weighted.

[c] The four bond portfolios consist of long-term U.S. Government bonds, long-term high-grade corporate bonds, BAA-rated corporate bonds, and below-BAA-rated corporate bonds.

the previous universe of eleven stock portfolios. The bond portfolios consist of long-term U.S. Government bonds, long-term high-grade corporate bonds, BAA-rated corporate bonds, and below-BAA-rated corporate bonds.[5] The correlations between the sample tangent portfolio and the NYSE proxy are, as they must be, lower than in the previous case – 0.48 for the overall period. Adding more assets can only decrease the correlation still further.

[5] The first two series are from Ibbotson and Sinquefield (1982); the latter two are from Ibbotson (1979).

2.2. *The correlation between the proxy and portfolios on the minimum-variance boundary*

When a riskless asset exists, the above examples illustrate the difference, in terms of correlation, between the proxy and the portfolio that, ex post, supports the Sharpe–Lintner pricing theory. When no riskless asset exists, a similar analysis is possible. In the latter case, however, there is no longer a unique boundary portfolio of the risky assets. Rather, many portfolios on the positively sloped minimum-variance boundary could in principle support the more general Black (1972) version of the two-parameter model.[6]

The following propositions are useful in understanding the correlation between the proxy and portfolios on the minimum variance boundary.

Proposition 2. For any minimum-variance portfolio p,

$$\rho(\alpha, p) = \frac{[\mu(\alpha) - \mu(p_z)]/\sigma(\alpha)}{[\mu(p) - \mu(p_z)]/\sigma(p)}. \tag{5}$$

Proof. Replace r with $\mu(p_z)$ in the proof of Proposition 1. □

Proposition 3. The minimum-variance portfolio having the highest correlation with α is α^*, the minimum-variance portfolio having the same mean return as α. The maximum correlation is

$$\rho(\alpha, \alpha^*) = \sigma(\alpha^*)/\sigma(\alpha). \tag{6}$$

Proof. Consider an arbitrary minimum-variance portfolio p. Let (μ_a, σ_a) be the point in mean–standard-deviation space on the line tangent to the minimum-standard-deviation boundary at p with $\mu_a = \mu(\alpha)$. Then

$$\begin{aligned}\rho(\alpha, p) &= \frac{[\mu(\alpha) - \mu(p_z)]/\sigma(\alpha)}{[\mu(p) - \mu(p_z)]/\sigma(p)} \quad \text{(Proposition 2)}\\ &= \frac{[\mu(\alpha) - \mu(p_z)]/\sigma(\alpha)}{[\mu_a - \mu(p_z)]/\sigma_a} = \frac{\sigma_a}{\sigma(\alpha)}.\end{aligned}$$

Since, for all choices of p, $\sigma_a \leq \sigma(\alpha^*)$, $\rho(\alpha, p)$ is maximized at $\sigma_a = \sigma(\alpha^*)$, which is accomplished by choosing $p = \alpha^*$.[7] □

[6] We say 'many' rather than 'any' because, as Ehrbar (1984) notes, portfolios on the positively sloped boundary but below the point of tangency of a ray emanating from −100% are inefficient for investors not constrained to invest all their money.

[7] We are grateful to John Long for suggesting this method of proof.

If the question of primary interest is the sensitivity of inferences to choosing alternative proxies, then an important boundary portfolio is the one having the highest correlation with the original proxy. That portfolio is, in a sense, the one that most easily reverses a violation of the pricing theory. From Proposition 3, that boundary portfolio is the one with the same mean return as the original proxy.

The second column of table 1 displays the maximum sample correlation between the equally weighted NYSE and portfolios on the sample minimum variance boundaries described earlier. For the first boundary, the maximum correlation equals one, since the proxy lies on the boundary by construction. The maximum correlations for the other boundaries are, as in the previous examples, considerably lower. In fact, for the overall period and for the most inclusive boundary, the boundary portfolio having maximum correlation with the proxy happens to be the sample tangent portfolio. In other words, the sample tangent portfolio in that case has the same mean return as the NYSE proxy (by Proposition 3).

As in the previous examples, the decline in correlations as assets are added to the universe is easily understood given the following corollary:

Corollary 2. For a given proxy α, the maximum correlation between α and a portfolio on the minimum-variance boundary cannot increase as risky assets are added to the universe.

Proof. Use Proposition 3 and the fact that $\sigma(\alpha^*)$ cannot increase as the opportunity set expands. □

2.3. The correlation between the proxy and an arbitrary portfolio

We have examined the correlation between a proxy portfolio and portfolios on the minimum-variance boundary, but a more complete characterization is possible. Where, in mean–variance space, are the portfolios correlated at least ρ_0 with the proxy? Kandel and Stambaugh (1986) provide a complete analytical characterization, in mean–variance space, of the set of portfolios having correlation of at least ρ_0 with a given proxy. Here we illustrate graphically the properties of this set that are useful in developing the sensitivity analysis in section 3.

We restrict attention to a universe of risky assets having a non-singular variance–covariance matrix, and we assume that the proxy does not lie on the minimum-variance boundary. Consider the set of portfolios whose correlation with the proxy is at least ρ_0. For $\rho_0 = 1$, the set contains only the proxy itself. The set expands as ρ_0 declines. For sufficiently low values of ρ_0, the set contains portfolios at all levels of mean return. Roll (1980) shows, for example, that portfolios uncorrelated with an inefficient proxy exist at all levels of mean return. For intermediate values of ρ_0, the set contains portfolios only at certain levels of mean return.

Just as the set of portfolios correlated at least ρ_0 with the proxy expands as ρ_0 declines, the region of mean–variance space that this set of portfolios can occupy also expands as ρ_0 declines. Fig. 1 displays some examples of these regions for various values of ρ_0. We plot the regions in mean–standard-deviation space, given that most readers are probably more familiar with graphs in those dimensions. The minimum-standard-deviation boundary is constructed from the sixteen stock and bond portfolios used in the earlier examples, where parameters are estimated over the 1926–1978 period. We again use the equally weighted NYSE as the proxy. Thus, the graphs represent the same proxy and asset universe used to compute the third row of table 1. The parameter values are annualized, and the returns are stated in excess of the T-Bill rate.

All portfolios having a correlation of at least ρ_0 with the proxy lie in a convex region in mean–standard-deviation space, although that region can also contain portfolios whose correlation with the proxy is less than ρ_0. The region may or may not be bounded in various directions, depending on ρ_0. The four cases displayed in fig. 1 illustrate some of the possibilities. For a sufficiently high ρ_0, as in the first case where $\rho_0 = 0.999$, the region is bounded in all directions. When $\rho_0 = 0.9$, portfolios having the required minimum correlation exist at all mean returns greater than a critical level, but for a given mean, the variance of such portfolios is bounded. When $\rho_0 = 0.7$, the portfolios exist at all mean returns, and the variance has no upper bound. The same is true when $\rho_0 = 0.45$, except the region of portfolios then extends to include points on the minimum-standard-deviation boundary. Given the earlier discussion surrounding table 1, recall that when $\rho_0 = 0.48$, the region touches the boundary at one point – the point with the same mean return as the proxy.

2.4. *Benchmark portfolios for subsets*

Investigations of asset pricing models must use subsets of the global universe of assets. The above relations also provide a relevant benchmark portfolio formed from a subset of assets. Let b denote the benchmark portfolio of the global universe, which is identified by the pricing theory, and let p' be the portfolio from the subset of assets that is most highly correlated with b. The following corollaries, easily shown given Propositions 1 and 2, establish p' as a relevant benchmark for testing the pricing theory with the subset of assets.

Corollary 3. If b is the Sharpe–Lintner tangent portfolio for the global universe of assets, and p' is the portfolio from a subset of assets that is most highly correlated with b, then p' is the tangent portfolio for the subset of assets.

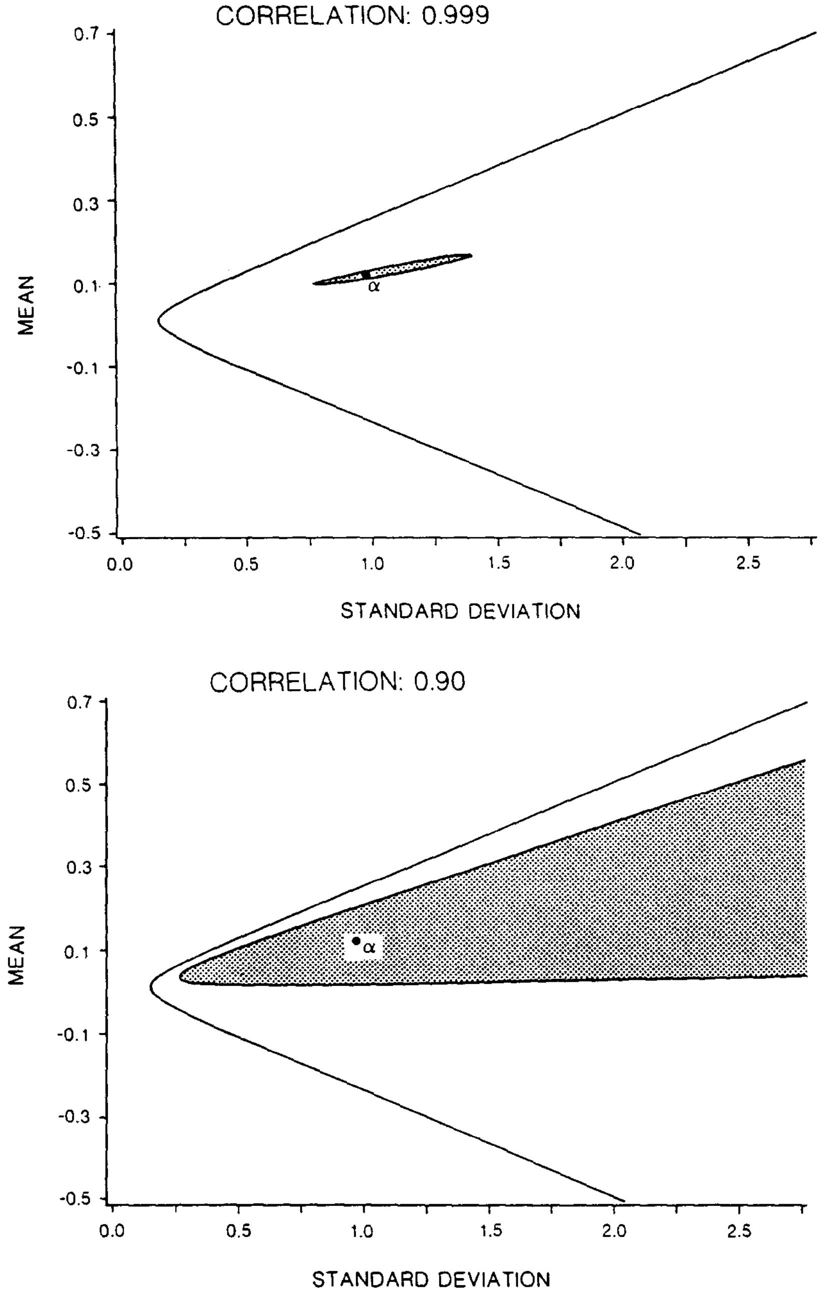

Fig. 1. Regions containing the portfolios having correlation with the proxy (α) greater than or equal to the value given.

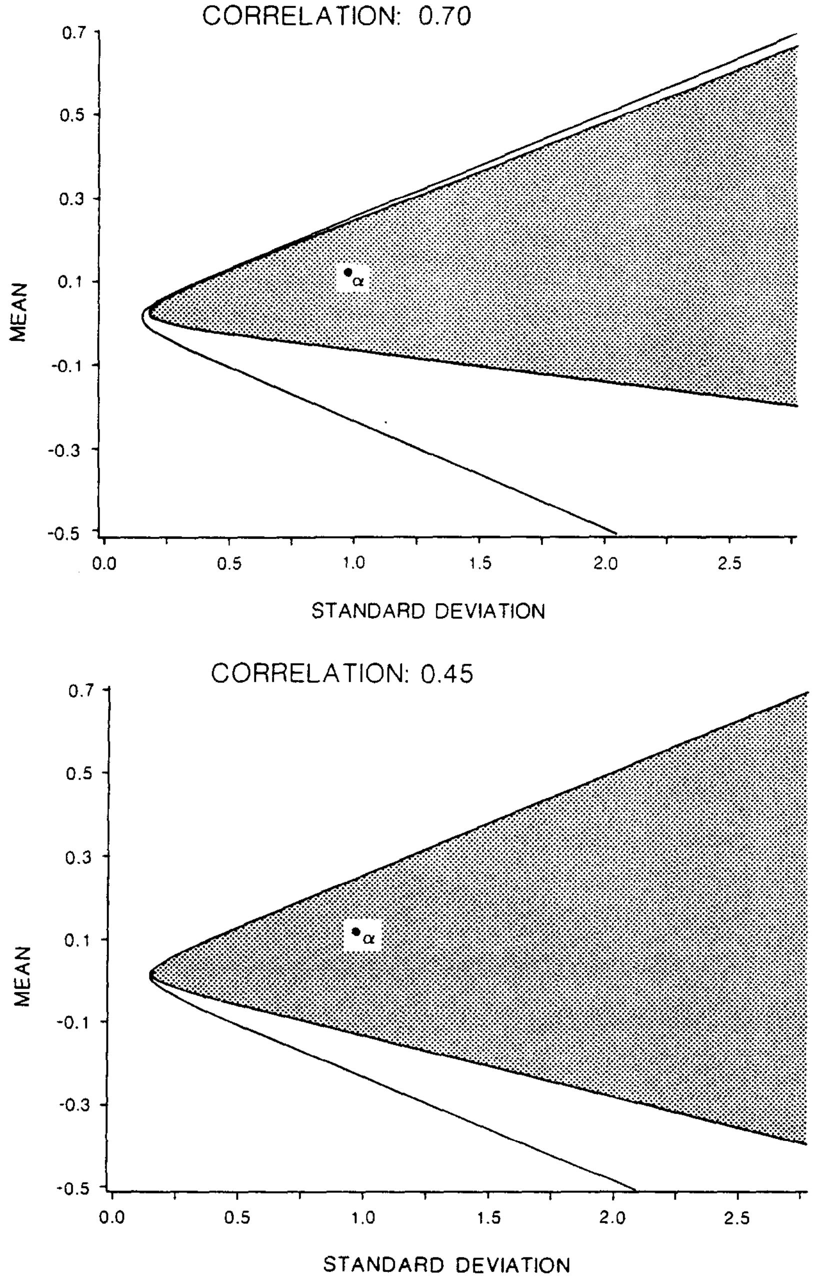

Fig. 1. (continued)

Proof. For any portfolio p of the subset, Proposition 1 gives $\rho(b, p) = S(p)/S(b)$. Since this correlation is maximized for $p = p'$, $S(p')$ is the maximum Sharpe measure for any portfolio of the subset, and thus p' is the tangent portfolio for the subset. □

Corollary 4. If b is on the minimum-variance boundary of all assets, and p' is the portfolio from a subset of assets that is most highly correlated with b, then p' is on the minimum-variance boundary of the subset of assets.

Proof. Identical to Corollary 3 except that, using Proposition 2, Sharpe measures are defined with respect to $\mu(b_z)$ instead of r. □

Corollary 3 implies, for example, that if the tangency of p' is rejected on a subset of assets, then the tangency of b for the global universe is also rejected. It is important to note that the exact construction of p' is not likely to be known by the researcher. In the discussions that follow, however, it is sufficient to view the observable portfolio α as a proxy for the unobservable p'.

3. The sensitivity of finite-sample inferences

The previous section addresses the question of how similar, in terms of correlation, is the proxy portfolio to a sample *efficient* portfolio that supports the pricing theory exactly. In a finite sample, however, there will also be sample *inefficient* portfolios for which the hypothesis of ex ante efficiency cannot be rejected. Those portfolios are not sufficiently 'far' from sample efficiency to rule out parameter estimation error as the cause of their sample inefficiency. The correlations in table 1 essentially provide information about inference sensitivity in infinite samples. We address in this section the issue of inference sensitivity in finite samples.

We coninue to pose the question raised originally by Roll (1977) and pursued in the previous section: how highly correlated with the original proxy can an alternative proxy be and still provide a different inference about ex ante mean–variance efficiency? In an infinite sample, this question is interesting only if the original proxy is inefficient – if the proxy happens to be efficient, there are clearly inefficient portfolios 'close by' whose correlations with the proxy are arbitrarily close to unity. In a finite sample, however, the question of inference reversal becomes interesting whether or not the proxy is inferred to be inefficient, since a *sample* inefficient portfolio is not necessarily inferred to be *ex ante* inefficient.

3.1. The statistical framework

The sensitivity of finite-sample inferences obviously depends, inter alia, on the type of test performed. We investigate here the sensitivity of the likelihood ratio test of whether a given portfolio is the Sharpe–Lintner tangent portfolio. A transformation of the test statistic has a finite-sample F distribution, as shown by Ross (1983). Another convenient feature of the test for our purposes is that it can be characterized completely in terms of the sample mean–standard-deviation space. A portfolio's tangency is accepted or rejected by comparing its estimated Sharpe measure to that of the sample tangent portfolio. If the difference in squared Sharpe measures is large enough, tangency is rejected. Specifically,

$$F = \frac{T-n}{n-1}\left[\frac{\hat{S}(\hat{\gamma})^2 - \hat{S}(p)^2}{1+\hat{S}(p)^2}\right] \tag{7}$$

has an F distribution with $n-1$ and $T-n$ degrees of freedom if p is the ex ante tangent portfolio, where n is the number of assets and T $(\geq n)$ is the number of time series observations [Ross (1983)].[8]

Our objective is to describe, in terms of sample correlation with the original proxy α, the portfolios that are inferred (i) tangent if α is inferred non-tangent or (ii) non-tangent if α is inferred tangent. The first step is to observe that, in some cases, no such reversal of inferences is possible for any correlation. For a given sample of assets, F has a maximum of

$$\bar{F} = \frac{T-n}{n-1}\hat{S}(\hat{\gamma})^2, \tag{8}$$

which is attained when $\hat{S}(p)=0$. Note that $\bar{F}$ could still be less than F_θ, the critical F value for significance level θ. In such a case, which is more likely to occur in samples where n is large relative to T, there are no feasible portfolios whose tangency is rejected at significance level θ.

When $\bar{F}$ in (8) exceeds F_θ, then the ex post rejection region is non-empty and can be characterized in terms of critical Sharpe measures. In such a case, the likelihood ratio test rejects tangency of p if $|\hat{S}(p)| < S_{\mathrm{CRIT}}$, where

$$S_{\mathrm{CRIT}} = \left[\frac{\hat{S}(\hat{\gamma})^2 - \nu F_\theta}{1+\nu F_\theta}\right]^{1/2}, \tag{9}$$

and $\nu = (n-1)/(T-n)$. [Set $F = F_\theta$ in (7) and solve for $\hat{S}(p)$.] Tangency is

[8] See also Gibbons, Ross and Shanken (1985) and MacKinlay (1985).

accepted for portfolios with sample Sharpe measures that are either high enough ($> S_{\mathrm{CRIT}}$) or low enough ($< -S_{\mathrm{CRIT}}$). Given the symmetric treatment of positive and negative Sharpe measures, the test's power clearly is greatest against a zero Sharpe measure and diminishes as the Sharpe measure moves either up or down. The test ignores the restriction that the mean excess return of the tangent portfolio is strictly positive.

Combined with our earlier analysis in section 2.3, the critical Sharpe measures provide a simple way to address the issue of inference sensitivity. For a given ρ_0 first construct the region, denoted Φ, in sample mean–standard-deviation space containing portfolios having sample correlation of at least ρ_0 with the proxy α. (Recall that examples of such a region are displayed in fig. 1.) Next construct the lines representing the critical Sharpe measures, S_{CRIT} and $-S_{\mathrm{CRIT}}$. If either line passes through Φ, then whatever the inference about α's tangency, some portfolio having a sample correlation of at least ρ_0 with α gives a different inference.

As described earlier, the region Φ expands as ρ_0 decreases. Fig. 2 illustrates two cases, where the universe contains the same sixteen risky assets used in the previous section, parameters are estimated for the 1926–1952 subperiod, and the proxy α is the equally weighted NYSE. The critical Sharpe measure (S_{CRIT}) reflects a 0.05 significance level, and α lies in the rejection region. In fig. 2A, Φ is constructed with $\rho_0 = 0.95$, and no points in Φ lie in the acceptance region. When ρ_0 is lowered to 0.70 in fig. 2B, Φ then crosses S_{CRIT}, so some portfolios in Φ then lie in the acceptance region. We compute the highest ρ_0 for which the region Φ is tangent to one of the critical Sharpe measures, that is, the highest ρ_0 for which a reversal of inferences is possible.

Proposition 4. Assume $\bar{F} > F_\theta$ (non-empty rejection region). The maximum correlation between α and a portfolio that is inferred (i) tangent if α is inferred non-tangent or (ii) non-tangent if α is inferred tangent at significance level θ, is given by

$$\bar{\rho}_\theta(\alpha) = \frac{S_{\mathrm{CRIT}} \cdot |\hat{S}(\alpha)| + c(\alpha)}{\hat{S}(\hat{\gamma})^2}, \tag{10}$$

where

$$c(\alpha) = \left\{\left[\hat{S}(\hat{\gamma})^2 - S_{\mathrm{CRIT}}^2\right]\left[\hat{S}(\hat{\gamma})^2 - \hat{S}(\alpha)^2\right]\right\}^{1/2}.$$

Proof. By Proposition 6 in Kandel and Stambaugh (1986), the maximum

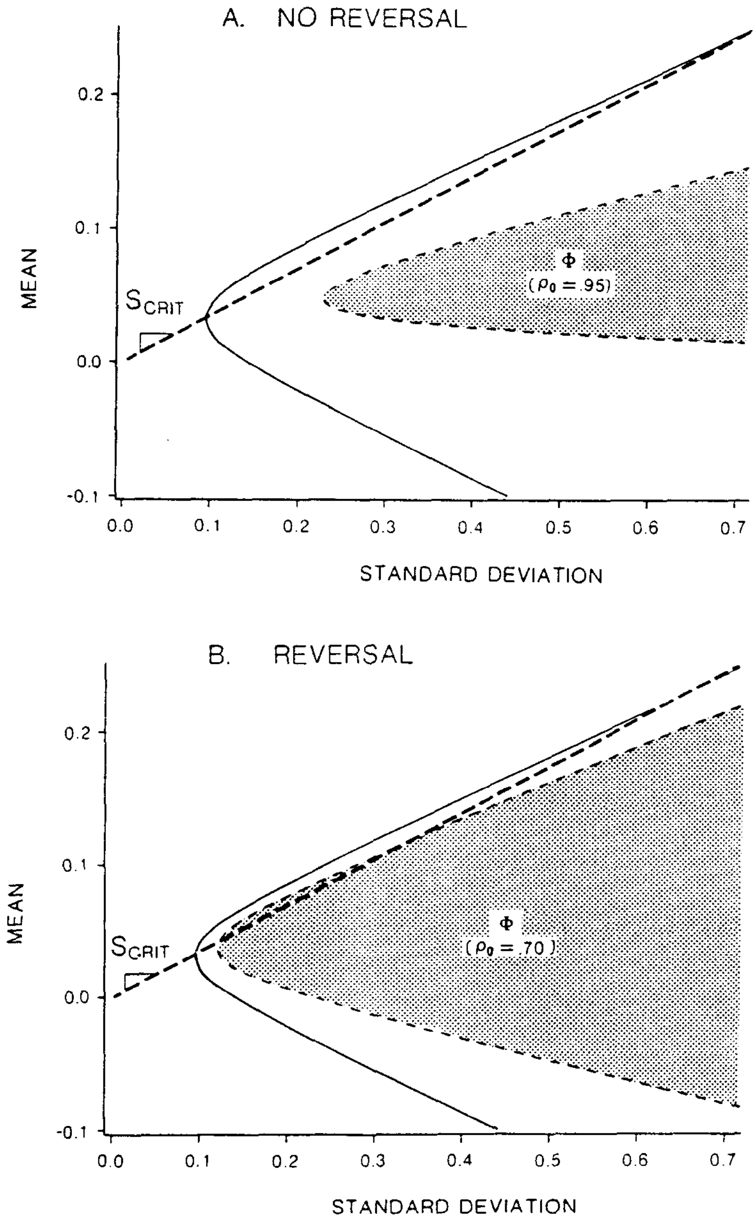

Fig. 2. Portfolios that plot above the dashed line whose slope is S_{CRIT} are inferred to be the ex ante tangent portfolio at a significance level of 0.05. Tangency is rejected for the original proxy as well as for all portfolios correlated at least 0.95 with the proxy (part A). Some portfolios correlated at least 0.70 with the proxy are inferred to be the ex ante tangent portfolio (part B).

sample correlation between α and a portfolio with sample Sharpe measure S is

$$\rho(S)=\frac{S\cdot\hat{S}(\alpha)+c(\alpha)}{\hat{S}(\hat{\gamma})^2}, \tag{11}$$

with $c(\alpha)$ defined as above except that S replaces S_{CRIT}. It is easily verified that $\rho'(S)<0$ (>0) if $\hat{S}(\alpha)<S$ $(>S)$, which implies that a portfolio satisfying the conditions of the proposition will lie on a critical Sharpe measure. (A portfolio beyond the boundary of the critical region cannot produce a higher correlation.) Therefore S can be either S_{CRIT} or $-S_{\mathrm{CRIT}}$, whichever produces the higher correlation. From (11), the higher correlation occurs for $S=S_{\mathrm{CRIT}}$ when $\hat{S}(\alpha)>0$ and for $S=-S_{\mathrm{CRIT}}$ when $\hat{S}(\alpha)<0$, and this choice is accomplished in (10). □

We note here that the alternative portfolios considered in Proposition 4 consist of different combinations of the original n assets used in the test. We do not investigate empirically the sensitivity of inferences when assets are added (n increases). With an infinite number of time-series observations, such sensitivity can be discussed easily. In that case, the only portfolio that will reverse an inference of non-tangency is the tangent portfolio itself. As discussed in section 2, the correlation between the original proxy α and the tangent portfolio cannot increase as more assets are added to the universe (and to the tangent portfolio). With a finite number of time-series observations, however, the problem of inference sensitivity when the number of assets is increased becomes less straightforward. In fact, it is difficult to pose a question in a way that could elicit an interesting answer.

The major problem in finite samples stems from how one handles the number of time-series observations (T). Holding T fixed presents one problem, but letting T increase presents another. For example, if T is held fixed as assets are added, the size of the rejection region can decrease. Unless $\hat{S}(\hat{\gamma})$, the maximum sample Sharpe measure, increases sufficiently, S_{CRIT} in (9) will decrease as the number of assets (n) increases. In fact, the ex post rejection region can disappear if $\bar{F}$ in (8) falls below the critical F value, F_{θ}. Therefore, an inference of non-tangency of the original proxy α could be reversed by a portfolio highly correlated with α, after adding enough assets to sufficiently reduce the rejection region of the test. On the other hand, if time-series observations are added in order to overcome the above problem, then a reversal of inferences could again occur for a portfolio highly correlated with the original proxy α, but the reversal could arise solely from the additional time-series observations. Indeed, one could in principle reverse the original inference by testing the *same* proxy using the longer time series. For these reasons, we confine our attention to alternative portfolios of the original n

Table 2

Sensitivity of the F test of Sharpe–Lintner tangency using monthly returns.[a]

Market proxy[b]	Sample Sharpe measure of proxy	Sample correlation of the proxy with the sample tangent portfolio	P-value of the test of tangency of the proxy	Maximum correlation between the proxy and a portfolio that gives the opposite inference at significance level 0.10	0.05	0.01
		1/1926–11/1978 ($T = 635$, $\hat{S}(\hat{\gamma}) = 0.262$)[c]				
EW	0.127	0.483	0.008	0.972	0.986	0.999
VW	0.109	0.416	0.004	0.952	0.970	0.995
		1/1926–12/1952 ($T = 324$, $\hat{S}(\hat{\gamma}) = 0.487$)				
EW	0.131	0.269	0.000	0.791	0.817	0.863
VW	0.110	0.226	0.000	0.764	0.791	0.840
		1/1953–11/1978 ($T = 311$, $\hat{S}(\hat{\gamma}) = 0.320$)				
EW	0.140	0.438	0.089	0.999	0.999	none[d]
VW	0.120	0.372	0.059	0.992	0.999	none

[a] The set of risky assets consists of sixteen portfolios: ten portfolios of common stocks of the New York Stock Exchange, the equally weighted and the value-weighted NYSE portfolios, and four bond portfolios. The ten stock portfolios are based on market value deciles at the end of the previous year and they are equally weighted portfolios.

[b] EW: equally weighted NYSE; VW: value-weighted NYSE.

[c] $\hat{S}(\hat{\gamma})$ is the sample's maximum Sharpe measure.

[d] 'None' indicates that there is no rejection region ex post.

assets. Given the preceding discussion of benchmarks for subsets of assets (section 2.4), such an investigation is relevant for testing asset pricing theories.

3.2. Results with the monthly data

Table 2 displays values of $\bar{\rho}_\theta(\alpha)$, the maximum sample correlation that allows reversal of inferences about tangency, in tests using the monthly returns data. The proxies are the equally weighted NYSE (EW) and the value-weighted NYSE (VW), and tangency is tested with respect to the sixteen risky assets examined earlier. In addition to the sample Sharpe measure $[\hat{S}(\alpha)]$ of the proxy, $\bar{\rho}_\theta(\alpha)$ depends on the maximum sample Sharpe measure $[\hat{S}(\hat{\gamma})]$, the number of assets (n), the number of observations (T), and the significance level of the test (θ). Thus, table 2 provides only a few examples of the analyses that could be conducted for various proxies and collections of assets. Nevertheless, some interesting observations emerge.

When reversals of inferences are possible, the correlations with alternative portfolios that allow such reversals are often higher than table 1 might lead one to suspect. For example, the equally weighted NYSE is correlated only

0.48 with the sample tangent portfolio in the overall period (table 1), but there exists a portfolio correlated as high as 0.99 with that proxy that would reverse the inference of ex ante non-tangency at the 0.05 significance level. In this case, the proxy lies very close to the critical Sharpe measure. Although such a result need not always occur, this example illustrates the often dramatic effect of allowing for finite-sample variability. A similar comparison can be made in the first subperiod, except that each of the correlations is lower than in the overall period (0.48 becomes 0.27; 0.99 becomes 0.82).

A rather different phenomenon occurs in the second subperiod. No portfolios would have been inferred to be non-tangent at significance levels of 0.01 or less. This illustrates the possibility discussed earlier, where the maximum of the test statistics [$\bar{F}$ in (8)] is less than the critical F value. Here again, however, observe the contrast between infinite and finite samples. In an infinite sample, all portfolios would be inferred non-tangent except one – the tangency. In the finite sample, no portfolios would be inferred non-tangent. Thus, any other portfolio would reverse an original inference of tangency in an infinite sample, whereas no portfolio could reverse such an inference in this finite sample.

3.3. Results with weekly data

Although the above discussion illustrates well a range of outcomes that can occur when investigating the sensitivity of inferences, the implied stationarity assumptions are fairly strong. The overall period of 53 years and the subperiods of 27 and 26 years are long by usual standards. Shorter subperiods, while relaxing the assumed stationarity, result in fairly small numbers of time-series observations when using monthly data, and the power of the test is thereby reduced (the rejection region is often empty ex post in such cases). In order to illustrate the above sensitivity analysis without imposing such strong stationarity assumptions, we conduct additional tests using weekly data.

Weekly returns are computed for ten value-weighted portfolios formed by ranking all firms on the New York and American Exchanges by market value at the end of the previous year. We use as two market proxies both the equally weighted and value-weighted portfolios of stocks on the NYSE–AMEX. The riskless rate is the return on a U.S. Treasury Bill with one week to maturity.[9] We test the tangency of each proxy with respect to a universe of twelve risky assets consisting of the ten size portfolios and both market proxies. Table 3 reports the results of the same sensitivity analysis conducted in table 2, except that now the analysis is performed on three periods, each about six years (324 weeks) in length. Thus, the number of time-series observations (T) is the same as in the first subperiod in table 2, but the period is only one fourth as long.

[9] We thank Richard Rogalski for providing the Treasury Bill data.

Table 3

Sensitivity of the F test of Sharpe–Lintner tangency using weekly returns.[a]

Market proxy[b]	Sample Sharpe measure of proxy	Sample correlation of the proxy with the sample tangency portfolio	P-value of the test of tangency of the proxy	Maximum correlation between the proxy and a portfolio that gives the opposite inference at significance level 0.10	0.05	0.01
		1/2/63–6/25/69 ($T = 324$, $\hat{S}(\hat{\gamma}) = 0.445$)[c]				
EW	0.200	0.449	0.000	0.876	0.897	0.934
VW	0.097	0.218	0.000	0.733	0.763	0.819
		7/2/69–10/1/75 ($T = 324$, $\hat{S}(\hat{\gamma}) = 0.681$)				
EW	0.018	0.026	0.000	0.433	0.459	0.510
VW	0.034	0.051	0.000	0.455	0.481	0.531
		10/8/75–12/23/81 ($T = 324$, $\hat{S}(\hat{\gamma}) = 0.608$)				
EW	0.178	0.292	0.000	0.686	0.709	0.753
VW	0.047	0.077	0.000	0.511	0.539	0.591

[a] The set of risky assets consists of twelve portfolios: ten portfolios of stocks of the New York and American Exchanges and their value-weighted and equally weighted portfolios. The ten portfolios are based on market value deciles at the end of the previous year and they are value-weighted portfolios.
[b] EW: equally weighted NYSE–AMEX; VW: value-weighted NYSE–AMEX.
[c] $\hat{S}(\hat{\gamma})$ is the sample's maximum Sharpe measure.

Tangency of both proxies is rejected strongly in all three subperiods. At a 0.05 significance level, the maximum sample correlation between the proxy and a portfolio inferred to be tangent, $\bar{\rho}_\theta(\alpha)$, ranges from 0.76 to 0.48 for the value-weighted NYSE–AMEX and from 0.90 to 0.46 for the equally weighted index.

If $\bar{\rho}_\theta(\alpha)$ is high, as occurs for some cases in tables 2 and 3, then the researcher knows there exists a portfolio highly correlated (in his sample) with the original proxy that will give a different inference about the pricing theory. This does not mean that any highly correlated portfolio will reverse inferences. For example, Stambaugh (1982) obtains the same inferences from several highly correlated market proxies. Rather, a high $\bar{\rho}_\theta(\theta)$ means that without specifying additional characteristics of reasonable alternative proxies, the researcher is unable to conclude that his inferences *cannot* be easily reversed by another portfolio. Additional characteristics could include, for example, the condition that the alternative proxies resemble portfolios of aggregate wealth or the condition that all asset weights be positive. Such conditions are not imposed in computing $\bar{\rho}_\theta(\alpha)$ in tables 2 and 3.

4. Testing the efficiency of an unobservable portfolio using partial information

The approach outlined in the previous section allows the researcher to investigate the ex post sensitivity of inferences to alternative specifications of the proxy portfolio. This approach is potentially useful if, for example, the researcher believes that the returns on an observable proxy have a sample correlation of at least ρ_0 with the unobservable portfolio of interest. Such partial information about the unobservable portfolio can also be included in the test itself, in the form of the ex ante correlation between the proxy and the unobservable portfolio.

Other studies discuss the value of partial information in evaluating the efficiency of an unobservable portfolio. Examples of such partial information include the non-negativity of market portfolio weights [Roll (1977) and Green (1986)] and upper bounds on the relative value and return variance of a missing asset [Kandel (1984b) and Shanken (1986)]. Shanken (1984) derives an inequality relation that contains the (multiple) correlation between the unobservable portfolio and a set of observable instruments, and he suggests that this relation could be useful in formulating tests of asset pricing theories.

We assume that the researcher summarizes his partial information about the unobservable portfolio by specifying a lower bound on the ex ante correlation between that portfolio and an observable proxy α. We also distinguish between the global universe of all assets and the *observed* universe consisting of the subset of n assets used by the econometrician. The unobservable portfolio can contain any assets in the global universe. Let γ^* denote the ex ante tangent portfolio of the global universe (as distinct from γ, the ex ante tangent portfolio for the observed universe of n assets). The null hypothesis is a joint hypothesis that the unobservable portfolio is (i) the ex ante tangent portfolio of the global universe and (ii) ex ante correlated at least ρ_0 with the proxy. In other words,

$$\mathrm{H}_0: \quad \rho(\alpha, \gamma^*) \geq \rho_0. \tag{12}$$

Special cases of H_0 include those where the global universe is identical to the observed universe and where $\rho_0 = 1$. Thus, H_0 is a generalization of the hypothesis tested in previous investigations of the CAPM, wherein the tangency of a given proxy was tested.

In this section we discuss two alternative approaches to testing H_0. Both approaches, in general, give tests whose significance levels can be bounded above. We first develop in section 4.1 an approach that uses the preceding sensitivity analysis. We then examine in section 4.2 an alternative approach, similar to that of Shanken (1987), which uses the distribution of the statistic in (7) under non-tangency of the tested portfolio (p). The latter distribution includes an unknown nuisance parameter, but, given that parameter, the approach gives an exact significance level when the global universe is identical to the observed universe.

4.1. A test based on the sensitivity analysis

Before proceeding to the formal development of the test, we first provide a brief informal description. Consider two sets of portfolios consisting of assets in the observed universe: (i) the portfolios inferred, at significance level θ, to be γ, the ex ante tangent portfolio of the observed universe, and (ii) the portfolios inferred, at significance level ψ, to have a correlation of at least ρ_0 with the proxy α. Note that both of these sets of portfolios can be observed by the researcher. The probability that γ lies outside the first set is θ, and, if H_0 is true, the probability that γ lies outside the second set is at most ψ [since H_0 implies that $\rho(\alpha, \gamma) \geq \rho_0$, by Corollary 1]. If H_0 is true, then the probability that the two sets are disjoint is at most $\theta + \psi$, and this provides us with the test developed below.

Let $S_{\mathrm{CRIT}}(\theta)$ be the critical Sharpe measure [in (9)] for testing the tangency of a given portfolio at significance level θ. The portfolio α is inferred non-tangent if $|\hat{S}(\alpha)| < S_{\mathrm{CRIT}}(\theta)$. In the previous section we derived $\bar{\rho}_\theta(\alpha)$, the maximum sample correlation between α and any portfolio that is inferred tangent at significance level θ when α is inferred non-tangent. Using the distribution of the sample correlation of bivariate normal random variables, $\hat{\rho}$, given the true population value, ρ, define the critical value $\rho_1(\rho_0, \psi)$ such that

$$\Pr\left[\hat{\rho} \leq \rho_1(\rho_0, \psi) | \rho = \rho_0\right] = \psi. \tag{13}$$

Reject H_0 if (i) $|\hat{S}(\alpha)| < S_{\mathrm{CRIT}}(\theta)$ and (ii) $\bar{\rho}_\theta(\alpha) < \rho_1(\rho_0, \psi)$. In other words, reject H_0 if the tangency of α is rejected at significance level θ and the maximum correlation between α and a portfolio inferred tangent is less than $\rho_1(\rho_0, \psi)$. As proved in the following proposition, the significance level (size) of this test is at most $\theta + \psi$.

Proposition 5.

$$\Pr\left[|\hat{S}(\alpha)| < S_{\mathrm{CRIT}}(\theta) \text{ and } \bar{\rho}_\theta(\alpha) < \rho_1(\rho_0, \psi) | \mathrm{H}_0\right] \leq \theta + \psi. \tag{14}$$

Proof. Consider the sample correlation between the proxy α and the ex ante tangent portfolio of the observed universe γ, denoted $\hat{\rho}(\alpha, \gamma)$. The weights in γ, and therefore the returns on γ, are not observed by the researcher, so $\hat{\rho}(\alpha, \gamma)$ cannot be computed. Nevertheless, this hypothetical sample correlation is useful in proving the proposition.

Define the events

(A$_1$) $|\hat{S}(\alpha)| < S_{\mathrm{CRIT}}(\theta)$,

(A$_2$) $\bar{\rho}_\theta(\alpha) < \rho_1(\rho_0, \psi)$,

(B) $\hat{\rho}(\alpha, \gamma) \geq \rho_1(\rho_0, \psi)$,

(C) $\hat{\rho}(\alpha, \gamma) \geq \bar{\rho}_\theta(\alpha)$,

(D) $|\hat{S}(\gamma)| < S_{\mathrm{CRIT}}(\theta)$.

First observe that $[\mathrm{A}_2 \cap \mathrm{B}] \Rightarrow \mathrm{C}$ by transitivity. Next observe that $[\mathrm{A}_1 \cap \mathrm{C}] \Rightarrow \mathrm{D}$, since the tangency of α is rejected (by A$_1$) and γ gives the same inference (by C). [Note that $\hat{S}(\gamma)$ is the hypothetical sample Sharpe measure of the ex ante tangent portfolio of the observed universe.] Therefore

$$\begin{aligned}\Pr[\mathrm{A}_1 \cap \mathrm{A}_2 \cap \mathrm{B} | \mathrm{H}_0] &\leq \Pr[\mathrm{A}_1 \cap \mathrm{C} | \mathrm{H}_0] \\ &\leq \Pr[\mathrm{D} | \mathrm{H}_0] = \Pr[\mathrm{D}] = \theta. \end{aligned} \tag{15}$$

Next observe that

$$\begin{aligned}\Pr[\mathrm{A}_1 \cap \mathrm{A}_2 \cap (\sim \mathrm{B}) | \mathrm{H}_0] &\leq \Pr[\sim \mathrm{B} | \mathrm{H}_0] \\ &\leq \Pr[\sim \mathrm{B} | \rho(\alpha, \gamma) \geq \rho_0] \leq \psi. \end{aligned} \tag{16}$$

The second inequality follows from Corollary 1 and the implication that H$_0$ implies $\rho(\alpha, \gamma) \geq \rho_0$. The third inequality follows from (13) and the fact that, for a fixed $\rho_1(\rho_0, \psi)$, the probability on the left-hand side of (13) is decreasing in ρ. The probability in (14), $\Pr[\mathrm{A}_1 \cap \mathrm{A}_2 | \mathrm{H}_0]$, can be written as

$$\Pr[\mathrm{A}_1 \cap \mathrm{A}_2 | \mathrm{H}_0] = \Pr[\mathrm{A}_1 \cap \mathrm{A}_2 \cap \mathrm{B} | \mathrm{H}_0] + \Pr[\mathrm{A}_1 \cap \mathrm{A}_2 \cap (\sim \mathrm{B}) | \mathrm{H}_0],$$

and combining this with (15) and (16) gives the desired result. □

The choice of θ and ψ in the above test is arbitrary. Together both parameters determine the maximum significance level of the test, but we do not know which combination of θ and ψ gives the highest power. In order to give a simple illustration of the test, we specify $\theta = \psi$. In addition, the exact sampling distribution of the correlation coefficient is rather complicated, so in computing $\rho_1(\rho_0, \psi)$ we use Fisher's z transformation, in which

$$z = \tfrac{1}{2}\log\frac{1+\hat{\rho}}{1-\hat{\rho}} \tag{17}$$

is distributed approximately Normal [see Kendall and Stewart (1977, eq. 16.77) for the moments of the distribution].

In table 4, we apply the test to the same weekly return data analyzed in table 3. Consider the first row of table 4, where the above hypothesis is tested in the first subperiod with ρ_0 equal to 0.9, a maximum significance level of 0.10 ($= \theta + \psi$), and the value-weighted NYSE–AMEX as the proxy. The test proceeds as follows. First, the tangency of the proxy itself is rejected at a significance level of 0.05 ($= \theta$). The maximum sample correlation between the proxy and a portfolio inferred tangent, $\bar{\rho}_\theta(\alpha)$, equals 0.763 (column 3). Up to this point, we have simply repeated the procedure in table 3. Next compute $\rho_1(\rho_0, \psi)$ for $\rho_0 = 0.9$ and $\psi = 0.05$, and this value is 0.881 (column 4). Since this value exceeds $\bar{\rho}_\theta(\alpha)$, we reject H_0. The same procedure is repeated in table 4 for maximum significance levels 0.05 and 0.01, for ρ_0 equal to 0.8 and 0.7, and with the equally weighted NYSE–AMEX as an alternative proxy.

For the value-weighted index, H_0 is rejected in the last two subperiods for $\rho_0 = 0.7$ at a significance level of at most 0.01. In the first subperiod, H_0 is rejected at the 0.01 level with $\rho_0 = 0.9$ and at the 0.10 level for $\rho_0 = 0.8$, but H_0 is not rejected for $\rho_0 = 0.7$ in that subperiod. The results for the equally weighted index are similar, the primary exceptions being that H_0 with $\rho_0 = 0.7$ is rejected only in the second subperiod and H_0 is not rejected at all in the first subperiod. In general, these results suggest that if the correlation between either of these proxies and the market portfolio exceeds 0.9, or even a lower value, then the CAPM is rejected.

4.2. A test based on the power function of Ross's statistic

As described in section 3, Ross's test statistic is distributed central F when the tested portfolio is the ex ante tangent portfolio of the observed universe. When the tested portfolio is not the tangent portfolio, the same test statistic is distributed, conditional on $\hat{S}(\alpha)$, as non-central F, with non-centrality parameter

$$\lambda = \frac{T}{1+\hat{S}(\alpha)^2}\left[S(\gamma)^2 - S(\alpha)^2\right], \tag{18}$$

as shown by Gibbons, Ross and Shanken (1985) [see also MacKinlay (1985)]. This result is useful for understanding the power function of Ross's test of

Table 4

Test of H_0: $\rho(\alpha, \gamma^*) \geq \rho_0$.[a]

Market proxy (α)[b]	$(\theta + \psi)$[c]	$\bar{\rho}_\theta(\alpha)$[d]	$\rho_0 = 0.9$ Critical value $\rho_1(\rho_0, \psi)$[e]	$\rho_0 = 0.9$ Inference about H_0	$\rho_0 = 0.8$ Critical value $\rho_1(\rho_0, \psi)$	$\rho_0 = 0.8$ Inference about H_0	$\rho_0 = 0.7$ Critical value $\rho_1(\rho_0, \psi)$	$\rho_0 = 0.7$ Inference about H_0
			1/2/63–6/25/69 ($T = 324$)					
VW	0.10	0.763	0.881	reject	0.765	reject	0.651	accept
VW	0.05	0.789	0.877	reject	0.758	accept	0.641	accept
VW	0.01	0.839	0.869	reject	0.743	accept	0.620	accept
EW	0.10	0.897	0.881	accept	0.765	accept	0.651	accept
EW	0.05	0.915	0.877	accept	0.758	accept	0.641	accept
EW	0.01	0.946	0.869	accept	0.743	accept	0.620	accept
			7/2/69–10/1/75 ($T = 324$)					
VW	0.10	0.481	0.881	reject	0.765	reject	0.651	reject
VW	0.05	0.504	0.877	reject	0.758	reject	0.641	reject
VW	0.01	0.549	0.869	reject	0.743	reject	0.620	reject
EW	0.10	0.459	0.881	reject	0.765	reject	0.651	reject
EW	0.05	0.483	0.877	reject	0.758	reject	0.641	reject
EW	0.01	0.528	0.869	reject	0.743	reject	0.620	reject
			10/8/75–12/23/81 ($T = 324$)					
VW	0.10	0.539	0.881	reject	0.765	reject	0.651	reject
VW	0.05	0.563	0.877	reject	0.758	reject	0.641	reject
VW	0.01	0.611	0.869	reject	0.743	reject	0.620	reject
EW	0.10	0.709	0.881	reject	0.765	reject	0.651	accept
EW	0.05	0.729	0.877	reject	0.758	reject	0.641	accept
EW	0.01	0.768	0.869	reject	0.743	accept	0.620	accept

[a]The null hypothesis (H_0) states that the proxy portfolio (α) has an ex ante correlation of at least ρ_0 with the Sharpe–Lintner tangent portfolio of the global universe (γ^*). The tests are based on weekly returns. The set of twelve risky assets consists of ten value-weighted portfolios based on market-value deciles at the end of the previous year, the value-weighted NYSE–AMEX portfolio and the equally weighted NYSE–AMEX portfolio.

[b]VW: value-weighted NYSE–AMEX; EW: equally weighted NYSE–AMEX.

[c]Upper bound on the significance level ($\theta = \psi$).

[d]Maximum correlation between α and a portfolio inferred tangent in the observed universe at significance level θ.

[e]The critical value $\rho_1(\rho_0, \psi)$ is chosen so that $\Pr[\hat{\rho} \leq \rho_1(\rho_0, \psi) \mid \rho = \rho_0] = \psi$, where $\hat{\rho}$ is the sample correlation between bivariate normal variables.

tangency against various alternatives, but, as Shanken (1987) demonstrates in a slightly different fashion, the same result can be used to test the hypothesis $\rho(\alpha, \gamma) \geq \rho_0$.

Using Proposition 1, we can rewrite (18) as

$$\lambda = \frac{T}{1 + \hat{S}(\alpha)^2} S(\gamma)^2 [1 - \rho^2(\alpha, \gamma)]. \tag{19}$$

In addition to T and $\hat{S}(\alpha)$, which are known to the researcher, the non-central-

Table 5

Test of H_0: $\rho(\alpha, \gamma^*) \geq \rho_0$, conditional on the maximum Sharpe measure $[S(\gamma)]$.[a]

	Upper bounds on the p-values for the test of H_0					
	Proxy (α): VW[b]			Proxy (α): EW[b]		
$S(\gamma)$	$\rho_0 = 0.9$	$\rho_0 = 0.8$	$\rho_0 = 0.7$	$\rho_0 = 0.9$	$\rho_0 = 0.8$	$\rho_0 = 0.7$
	1/2/63–6/25/69 ($T = 324$, $\hat{S}(\hat{\gamma}) = 0.445$)[c]					
0.3	0.000	0.001	0.004	0.001	0.006	0.022
0.4	0.001	0.011	0.056	0.005	0.053	0.181
0.5	0.004	0.087	0.328	0.026	0.246	0.586
0.6	0.025	0.345	0.763	0.099	0.604	0.914
0.7	0.100	0.719	0.970	0.272	0.890	0.994
0.8	0.281	0.941	0.999	0.533	0.986	0.999
	7/2/69–10/1/75 ($T = 324$, $\hat{S}(\hat{\gamma}) = 0.681$)					
0.3	0.000	0.000	0.000	0.000	0.000	0.000
0.4	0.000	0.000	0.000	0.000	0.000	0.000
0.5	0.000	0.000	0.000	0.000	0.000	0.000
0.6	0.000	0.000	0.000	0.000	0.000	0.000
0.7	0.000	0.000	0.014	0.000	0.000	0.014
0.8	0.000	0.006	0.143	0.000	0.006	0.140
	10/8/75–12/23/81 ($T = 324$, $\hat{S}(\hat{\gamma}) = 0.608$)					
0.3	0.000	0.000	0.000	0.000	0.000	0.000
0.4	0.000	0.000	0.000	0.000	0.000	0.000
0.5	0.000	0.000	0.000	0.000	0.000	0.002
0.6	0.000	0.001	0.015	0.000	0.002	0.041
0.7	0.000	0.011	0.151	0.000	0.031	0.272
0.8	0.000	0.088	0.546	0.001	0.178	0.703

[a]The null hypothesis (H_0) states that the proxy portfolio (α) has an ex ante correlation of at least ρ_0 with the Sharpe-Lintner tangent portfolio of the global universe (γ^*). $S(\gamma)$ is the maximum ex ante Sharpe measure of the observed universe of assets. The tests are based on weekly returns. The set of twelve risky assets consists of ten value-weighted portfolios based on market-value deciles at the end of the previous year, the value-weighted NYSE-AMEX portfolio, and the equally weighted NYSE-AMEX portfolio.

[b]VW: value-weighted NYSE-AMEX; EW: equally weighted NYSE-AMEX.

[c]$\hat{S}(\hat{\gamma})$ is the sample's maximum Sharpe measure.

ity parameter depends on the unknown parameters $\rho(\alpha,\gamma)$ and $S(\gamma)$ (the maximum ex ante Sharpe measure of the observed universe). Given a value of $S(\gamma)$, however, a test of H_0 is straightforward. Note that the non-centrality parameter λ is decreasing in $\rho(\alpha,\gamma)$, so that a test of the hypothesis $\rho(\alpha,\gamma) \geq \rho_0$ can be based on the non-central F distribution with λ evaluated at $\rho(\alpha,\gamma) = \rho_0$. The (exact) significance level of this test gives an upper bound for the significance level of a test of H_0, since H_0 implies that $\rho(\alpha,\gamma) \geq \rho_0$, given Corollary 1. The significance levels for both tests are identical if the global and observed universes coincide ($\gamma^* = \gamma$).

Table 5 displays results of the above test for values of ρ_0 equal to 0.9, 0.8 and 0.7, using the same subperiods and weekly return data as in tables 3 and 4. We show results for values of the maximum Sharpe measure of the observed universe, $S(\gamma)$, ranging from 0.3 to 0.8. The ex post values of $\hat{S}(\hat{\gamma})$, also shown, range from 0.445 to 0.681. The non-centrality parameter λ, and thus the p-value as well, increases with $S(\gamma)$, and the increases in the p-values can be large in table 5 as $S(\gamma)$ increases to 0.7 and 0.8. Such apparent sensitivity to the larger values of $S(\gamma)$ suggests caution in interpreting the results. Nevertheless, for values of $S(\gamma)$ near or slightly larger than the ex post values, the inferences provided by this method are similar to those obtained in table 4.

5. Summary and conclusions

This paper presents a framework for investigating the mean–variance efficiency of an unobservable portfolio based on its correlation with an observable proxy portfolio. We first analyze some useful mean–variance relations based on the correlation between a given proxy portfolio and other portfolios in both the observed and global universes. We then develop a sensitivity analysis that provides the highest sample correlation between the proxy and a portfolio that reverses the inference of a test of Sharpe–Lintner tangency. Extending that analysis, we formally test whether an observable proxy is ex ante highly correlated with the ex ante tangent portfolio.

We conclude that the correlation between the tangent portfolio and the market proxy is sensitive to how one constructs the efficient set, both ex ante and in the sample. In his critique of tests of the CAPM, Roll (1977) shows that the sample inefficient market proxy used by Black, Jensen and Scholes (1972) is correlated 0.9 with the estimated Sharpe–Lintner tangent portfolio. He concludes that inferences about the CAPM can be reversed easily with an alternative market proxy whose return is highly correlated with the return on the original proxy. Starting with the same minimum-variance boundary constructed by Roll, we show that the correlation between the sample tangent portfolio and the proxy of Black, Jensen and Scholes decreases as additional assets are added to the observed universe. For example, the correlation drops to 0.48 when the universe consists of sixteen portfolios of stocks and bonds. The decline in correlation is easily understood when one realizes that the correlation between the proxy and the tangent portfolio is the ratio of the Sharpe measures of the two portfolios.

The relation between correlations and Sharpe measures also implies that the mean–variance efficiency of a 'true' benchmark portfolio, possibly containing all assets, is rejected if one rejects efficiency of a particular alternative benchmark portfolio in an observed universe consisting of a subset of assets. The relevant alternative benchmark in the observed universe is the portfolio from the observed universe that is most highly correlated with the true benchmark.

In a finite sample we analyze the sensitivity of inferences using a likelihood ratio test of Sharpe–Lintner tangency. Ross (1983) demonstrates that the test statistic has a known finite-sample distribution, and he derives a representation of the test statistic in mean–variance space. Combining Ross's geometric representation with the mean–variance analysis described in section 2 [and derived in Kandel and Stambaugh (1986)], we obtain the highest sample correlation between the proxy and a portfolio that reverses the inference about the proxy's tangency.

When monthly data of long periods (26–52 years) are used to test the tangency of the equally weighted or value-weighted NYSE, there are some cases where no rejection region exists at standard test sizes and other cases where the correlation that reverses the inference about the tangency of the NYSE portfolio is quite high. When weekly data of shorter periods (about 6 years) are used, the tangency of both the equally weighted and value-weighted NYSE–AMEX portfolios is rejected. The maximum sample correlation between the NYSE–AMEX proxy and a portfolio inferred to be tangent at the 0.05 level ranges from 0.76 to 0.48 for the value-weighted portfolio and from 0.90 to 0.46 for the equally weighted portfolio.

We extend the preceding sensitivity analysis to test whether a given observable proxy portfolio is correlated at least ρ_0 with the ex ante tangent portfolio of the global universe. We apply the test to weekly returns data for common stocks, with both the equally weighted and the value-weighted NYSE–AMEX indexes as the observable proxies. The null hypothesis is in fact a joint hypothesis that the unobservable benchmark portfolio is (i) the ex ante tangent portfolio and (ii) highly correlated (at least ρ_0) with the NYSE–AMEX index. This hypothesis is almost always rejected for ρ_0 equal to 0.9 and is often rejected for ρ_0 equal to 0.8 and even 0.7.

References

Black, Fischer, 1972, Capital market equilibrium with restricted borrowing, Journal of Business 45, 444–454.

Black, Fischer, Michael C. Jensen and Myron Scholes, 1972, The capital asset pricing model: Some empirical tests, in: Michael C. Jensen, ed., Studies in the theory of capital markets (Praeger, New York).
Breeden, Douglas T., 1979, An intertemporal asset pricing model with stochastic consumption and investment opportunities, Journal of Financial Economics 7, 265–296.
Chamberlain, Gary, 1983, Funds, factors, and diversification in arbitrage pricing models, Econometrica 51, 1305–1323.
Ehrbar, Hans, 1984, Mean–variance efficiency if investors are not required to invest all their money, Working paper (University of Michigan, Ann Arbor, MI).
Fama, Eugene F., 1976, Foundations of finance (Basic Books, New York).
Gibbons, Michael R., 1980, Econometric methods for testing a class of financial models: An application of the nonlinear multivariate regression model, Ph.D. dissertation (University of Chicago, Chicago, IL).
Gibbons, Michael R., 1982, Multivariate tests of financial models: A new approach, Journal of Financial Economics 10, 3–27.
Gibbons, Michael R., Stephen A. Ross and Jay Shanken, 1985, A test of the efficiency of a given portfolio, Working paper (Stanford University, Stanford, CA).
Green, Richard C., 1986, Positively weighted portfolios on the minimum-variance frontier, Journal of Finance 41, 1051–1068.
Grinblatt, Mark and Sheridan Titman, 1986, The relation between mean–variance efficiency and arbitrage pricing, Journal of Business, forthcoming.
Huberman, Gur, Shmuel Kandel and Robert F. Stambaugh, 1986, Mimicking portfolios and exact arbitrage pricing, Journal of Finance, forthcoming.
Ibbotson, Roger G., 1979, The corporate bond market: Structure and returns, Working paper (University of Chicago, Chicago, IL).
Ibbotson, Roger G. and Rex A. Sinquefield, 1982, Stocks, bonds, bills and inflation: The past and the future (Financial Analysts Research Foundation, Charlottesville, VA).
Jensen, Michael C., 1969, Risk, the pricing of capital assets and the evaluation of investment portfolios, Journal of Business 42, 167–247.
Jobson, J.D. and Bob Korkie, 1982, Potential performance and tests of portfolio efficiency, Journal of Financial Economics 10, 433–466.
Kandel, Shmuel, 1984a, The likelihood ratio test statistic of mean–variance efficiency without a riskless asset, Journal of Financial Economics 13, 575–592.
Kandel, Shmuel, 1984b, On the exclusion of assets from tests of the mean–variance efficiency of the market portfolio, Journal of Finance 39, 63–75.
Kandel, Shmuel and Robert F. Stambaugh, 1986, Correlation and mean–variance analysis, Working paper (University of Chicago, Chicago, IL).
Kendall, Sir Maurice and Alan Stuart, 1977, The advanced theory of statistics, Vol. 1 (Macmillan, New York).
Lintner, John, 1965, The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets, Review of Economics and Statistics 47, 13–27.
Long, John B. Jr., 1977, Efficient portfolio choice with differential taxation of dividends and capital gains, Journal of Financial Economics 5, 25–53.
MacKinlay, A. Craig, 1985, An analysis of multivariate financial tests, Working paper (University of Pennsylvania, Philadelphia, PA).
Roll, Richard, 1977, A critique of the asset pricing theory's tests, Part I: On past and potential testability of the theory, Journal of Financial Economics 4, 129–176.
Roll, Richard, 1979, Testing a portfolio for ex ante mean/variance efficiency, TIMS Studies in the Management Sciences 11, 135–149.
Roll, Richard, 1980, Orthogonal portfolios, Journal of Financial and Quantitative Analysis 15, 1005–1023.
Ross, Stephen A., 1976, The arbitrage theory of capital asset pricing, Journal of Economic Theory 13, 341–360.
Ross, Stephen A., 1977, The capital asset pricing model (CAPM), short sale restrictions and related issues, Journal of Finance 32, 177–183.
Ross, Stephen A., 1983, A test of the efficiency of a given portfolio, Working paper (Yale University, New Haven, CT).

Shanken, Jay, 1984, Equilibrium, factors, and arbitrage pricing, Working paper (University of California at Berkeley, Berkeley, CA).

Shanken, Jay, 1985, Multivariate tests of the zero-beta CAPM, Journal of Financial Economics 14, 327–348.

Shanken, Jay, 1986, On the exclusion of assets from tests of the mean variance efficiency of the market portfolio: An extension, Journal of Finance 41, 331–337.

Shanken, Jay, 1987, Multivariate proxies and asset pricing relations: Living with the Roll critique, Journal of Financial Economics 18, this issue.

Sharpe, William F., 1964, Capital asset prices: A theory of market equilibrium under conditions of risk, Journal of Finance 19, 425–442.

Stambaugh, Robert F., 1981, Missing assets, measuring the market, and testing the capital asset pricing model, Ph.D. dissertation (University of Chicago, Chicago, IL).

Stambaugh, Robert F., 1982, On the exclusion of assets from tests of the two-parameter model: A sensitivity analysis, Journal of Financial Economics 10, 237–268.