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THE LIKELIHOOD RATIO TEST STATISTIC OF MEAN-VARIANCE EFFICIENCY WITHOUT A RISKLESS ASSET*

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The question whether a given portfolio is mean-variance efficient is a basic problem of investment analysis. Mean-variance efficiency is also the basis of the Capital Asset Pricing Model. This paper presents the explicit form of the likelihood ratio test of the hypothesis that a given portfolio, or a particular market index, is ex-ante mean-variance efficient in the case where there is no riskless asset. Geometric relations are illustrated to provide intuition about the constrained maximum likelihood estimators and the test statistic, and two simple economic interpretations of the test are given.

1. Introduction

Since Markowitz (1952, 1959) and Tobin (1958) a fundamental problem of investment analysis has been to decide whether a particular portfolio is dominated in mean-variance space by some other portfolio. Individual investors and portfolio managers who make their portfolio decisions in accordance with mean and variance may wish to test whether a pre-selected portfolio is ex-ante efficient. The concept of mean-variance efficiency is also the basis of a theory of price formation and equilibrium in the capital market. It is well known that the Capital Asset Pricing Model (CAPM) is equivalent to the mean-variance efficiency of the market portfolio.[1] Roll (1977), in a critique of the asset pricing theory tests, argues that the theory is not testable unless the true market portfolio is known and used in the tests. However, even if the market portfolio is identifiable, there is still the question how to test its efficiency.

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[1]See Fama (1976), Roll (1977), and Ross (1977).

Most of the empirical studies of the CAPM use cross-sectional regression to test the ex-ante linear relation between betas and mean returns implied by the mean-variance efficiency of the market portfolio.[2] This approach is subject to a problem of errors in variables since the regressors are estimates of the true values of betas. Gibbons (1982) applies a nonlinear multivariate regression model to Black's (1972) generalized CAPM; this methodology leads naturally to a Likelihood Ratio Test (LRT) of the parameter restrictions and to a maximum likelihood approach for estimation that eliminates, at least in large samples, the problem of errors in variables.[3] Ross (1980) derives the analytical solution for the LRT in the case where there is a riskless asset [the Sharpe (1964) and Lintner (1965) model]. He demonstrates that the LRT has a simple economic interpretation. Ross (1983) and Gibbons and Shanken (1983) derive not only the asymptotic distribution (χ^2) of this test statistic, but also its exact small sample distribution (F). Jobson and Korkie (1982) generalize Ross's (1980) work to test comparative potential performance in the presence of a riskless asset.

This paper derives the exact form of the LRT when there is no riskless asset and presents the economic interpretation of the test. The null hypothesis is that a pre-selected portfolio (or a particular market index) is ex-ante mean-variance efficient. Formally, the test developed here is identical to the LRT suggested by Gibbons (1982).[4] Following Ross (1980), the focus of this paper is on the analytical structure of the maximum likelihood estimators and the LRT and their economic interpretations. Special attention is given to the estimator of the zero-beta return. The zero-beta portfolio with respect to the market index is one of the two determinants of individual assets' expected returns in Black's (1972) model. The constrained maximum likelihood estimator of the zero-beta return is shown to be a zero of a parabola in mean-variance space. This parabola and its construction provide geometric relations between constrained and unconstrained estimators. The relation between the sample frontier and the maximum likelihood estimator of the frontier is also demonstrated. Finally, it is shown that the LRT is a familiar construct in financial economics that measures the distance, in mean-variance space, of the given portfolio from the sample frontier.

The paper is organized as follows: In section 2 the model is described, and some known analyses of the efficient frontier and the zero beta return are

[2] See, for example, Fama and Macbeth (1973), Black, Jensen and Scholes (1972), and Blume and Friend (1973).

[3] See Shanken (1983b) for a further discussion of the errors-in-variables problem in tests of the CAPM.

[4] Gibbons's (1982) actual estimation technique does not revolve around the conditions for the maximization of the likelihood function. His estimators are based upon one-step Gauss–Newton approximation, and they have the same asymptotic properties (as $T \to \infty$) as the maximum likelihood estimators.

presented. In section 3, I introduce and interpret the constrained estimators of the parameters and the zero-beta return, the constrained estimator of the efficient frontier and the LRT. Section 4 includes a summary and conclusions.

2. The model

2.1. The return generating process

The model assumes that the vector $\tilde{X}$ of individual returns on n assets follows a multivariate normal distribution with stationary mean vector E and stationary covariance matrix V. There is no riskless asset, i.e., V is assumed to be positive definite. Given a sample of returns, $(x_1, x_2, \ldots, x_T)$, where x_t is the tth sample period vector of returns, the log-likelihood function is given by

$$\mathscr{L} = -(nT/2)\log(2\pi) - (T/2)\log|V| - \tfrac{1}{2}\sum_{t=1}^{T}(x_t - E)'V^{-1}(x_t - E). \tag{1}$$

The unconstrained maximum likelihood estimators of E and V are the sample mean and covariance matrix,

$$\hat{E} = (1/T)\sum_{t=1}^{T} x_t,$$

$$\hat{V} = (1/T)\sum_{t=1}^{T}(x_t - \hat{E})(x_t - \hat{E})'.$$

2.2. The efficient set

A portfolio $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a vector of investment proportions. It is assumed that there are no restrictions on short sales, hence, the only constraint on this vector is that its elements sum to one: $\sum_{i=1}^{n}\alpha_i = 1$. The return on the portfolio is $\alpha'\tilde{x}$, its expectation is $\alpha'E$ and its variance is $\alpha'V\alpha$. The efficient portfolio frontier, or the efficient set, is the set of portfolios with minimum variance at each level of expected return.[5] To avoid trivial degeneracy assume that at least two assets have different expected returns. With short sales, this enables one to achieve any desirable level of expected return. Every efficient set is completely determined by a mean vector and a covariance matrix; thus,

[5]This definition of the efficient portfolio frontier is given in Roll (1977). Other authors [e.g. Fama (1976)] define this set to be the set of minimum variance portfolios.

given $\hat{E}$ and $\hat{V}$, the frontier of sample efficient portfolios can be easily obtained.

The following properties from mean-variance analysis will be useful. [Derivations can be found in Roll (1977), Merton (1972) or Fama (1976).]

Let E and V be any mean return vector and covariance matrix. In our model these parameters can be the ex-ante parameters, the sample (ex-post) parameters or the constrained maximum likelihood estimators of the ex-ante parameters.

[S1] The efficient set is the set of all portfolios which are the optimal solutions of the following constrained minimization problem for some m:

$$\min p'Vp, \qquad p \in R^n,$$

subject to

$$p'E = m, \quad p'e = 1, \qquad e' = (1,1,1,\ldots,1). \tag{2}$$

[S2] In the mean-variance space the efficient frontier corresponds to a parabola where variance as a function of expected return is

$$\sigma^2(m) = (1/D)(Lm^2 - 2Mm + N),$$

where

$$L \equiv e'V^{-1}e, \quad M \equiv e'V^{-1}E, \quad N \equiv E'V^{-1}E, \quad D \equiv NL - M^2.$$

The function $\sigma^2(m)$ is strictly convex. It has a unique minimum point, with mean return M/L and variance $1/L$, called the 'global minimum variance point'.

[S3] A portfolio α is efficient if it belongs to the efficient set. If α is not the global minimum variance portfolio, then the first-order necessary and sufficient condition for α to be efficient is the existence of scalars k and γ_0 such that

$$E = \gamma_0 e + kV\alpha. \tag{3}$$

Let $\gamma_1 \equiv k(\alpha'V\alpha)$ and $\beta_i = (V\alpha)_i/(\alpha'V\alpha)$, and then (3) is the familiar linear relation between expected returns and betas,

$$E_i = \gamma_0 + \gamma_1\beta_i, \qquad i = 1,2,\ldots,n.$$

[S4] A portfolio p is said to be zero-beta with respect to a portfolio α if there is no correlation between them, i.e., $p'V\alpha = 0$. From (3) it can be shown that, if

α is an efficient portfolio (but not the global minimum variance portfolio), then all zero-beta portfolios with respect to α have the same expected return γ_0 which can be defined as the zero-beta return with respect to α.[6]

3. The constrained maximum likelihood estimators and the likelihood ratio test

In this section I present the constrained maximum likelihood estimators of the parameters and the efficient frontier, and the likelihood ratio test of efficiency. [Detailed derivations of the estimators and the test statistic are given in Kandel (1983).][7]

The null hypothesis is that a pre-selected portfolio α is ex-ante efficient. It is assumed that the sample covariance matrix, $\hat{V}$, is positive definite; a necessary condition is $n < T$. Other technical assumptions are that α is not the (ex-ante) global minimum variance portfolio, α is not sample efficient, and has a sample mean return which differs from that of the sample minimum variance portfolio.[8]

3.1. The constrained maximum likelihood estimators

I begin by defining the constrained maximum likelihood estimators. The essence of this approach is that it directly employs the restriction that the portfolio α is ex-ante efficient. This restriction has important implications for the estimation and testing.

Definition. (E^*, V^*) are the constrained maximum likelihood estimators of the ex-ante parameters (E, V), and γ_0^* is the constrained estimator of the zero-beta return γ_0 if:

(i) the matrix V^* is symmetric and positive definite (and therefore can be a covariance matrix),
(ii) there exists a scalar k^* such that

$$E_i^* = \gamma_0 + k^*(V^*\alpha)_i, \qquad i = 1, 2, \dots, n.$$

(iii) (E^*, V^*) maximize the likelihood function (1) among all the parameters satisfying (i) and (ii).

[6] There is no zero-beta portfolio with respect to the global minimum variance portfolio.

[7] A technical appendix that details the derivations of the estimators and the test statistic will be furnished upon request to the author.

[8] These assumptions rule out only null events. Another implicit assumption is that period-by-period returns on all n assets are observable. Kandel (1984a) presents an analysis of the testability of mean variance efficiency of a market index when the returns on some components of the index itself are not perfectly observable.

Ross (1980) derives the constrained estimators of (E, V) for the case where there is a riskless asset. Suppose r is the rate of return on the riskless asset, then

$$E^*(r) = \left(\alpha'\hat{V}\alpha + (\alpha'\hat{E} - r)^2\right)^{-1} \times \left\{(\alpha'\hat{E} - r)^2 \hat{E} + r(\alpha'\hat{V}\alpha)e + (\alpha'\hat{E} - r)\hat{V}\alpha\right\}, \tag{4}$$

$$V^*(r) = \hat{V} + (\hat{E} - re)(\hat{E} - re)' - uu', \tag{5}$$

where

$$u = \frac{(\alpha'\hat{E} - r)}{\alpha'\hat{V}\alpha + (\alpha'\hat{E} - r)^2}\left\{\hat{V}\alpha + (\alpha'\hat{E} - r)(\hat{E} - re)\right\},$$

$$e' = (1, 1, 1, \ldots, 1).$$

Next, I consider the case without a riskless asset and derive the constrained maximum likelihood estimators. Obviously, the above results hold, but with r replaced by γ_0^* – the constrained maximum likelihood estimator of the zero-beta return.

Theorem 1. Consider the quadratic function $H(m)$,

$$H(m) = Am^2 + Bm + C,$$

where

$$A = -1,$$

$$B = \frac{-\hat{L}(\alpha'\hat{V}\alpha) - \hat{L}(\alpha'\hat{E})^2 + 1 + \hat{N}}{\hat{M} - \hat{L}(\alpha'\hat{E})},$$

$$C = \frac{\hat{M}(\alpha'\hat{V}\alpha) + \hat{M}(\alpha'\hat{E})^2 - (\alpha'\hat{E}) - \hat{N}(\alpha'\hat{E})}{\hat{M} - \hat{L}(\alpha'\hat{E})},$$

and $\hat{N}$, $\hat{M}$ and $\hat{L}$ are the elements of the 'sample information matrix',[9]

$$\begin{pmatrix} \hat{N} & \hat{M} \\ \hat{M} & \hat{L} \end{pmatrix} = \begin{pmatrix} \hat{E}'\hat{V}^{-1}\hat{E} & \hat{E}'\hat{V}^{-1}e \\ e'\hat{V}^{-1}\hat{E} & e'\hat{V}^{-1}e \end{pmatrix}.$$

[9]It is not necessary to invert the matrix $\hat{V}$ in order to calculate the elements of the sample information matrix. Advanced mathematical programming techniques can be used to solve the following two quadratic programming problems:

$$\text{(I)} \quad \begin{array}{l} \min p'\hat{V}p, \quad p \in R^n \\ \text{s.t.} \quad p'e = 1 \end{array} \qquad \text{(II)} \quad \begin{array}{l} \min p'\hat{V}p, \quad p \in R^n \\ \text{s.t.} \quad p'\hat{E} = 0, \quad p'e = 1 \end{array}$$

The minimum of (I) is attained at $\hat{M}/\hat{L}$ and the optimal value is $1/\hat{L}$. The optimal value of (II) is $\hat{N}/(\hat{N}\hat{L} - \hat{M}^2)$. In many practical cases solving these two optimization problems is easier (from the computational aspect) than inverting the whole matrix $\hat{V}$.

(a) *The equation* $H(m)=0$ *has two real solutions*: $\theta_1 < \theta_2$.[10]
(b) *The constrained maximum likelihood estimator of the zero-beta return with respect to* α, γ_0^*, *is a zero of* $H(m)$:

$$\text{if } \alpha'\hat{E} > \hat{M}/\hat{L}, \text{ then } \theta_1 < (\hat{M}/\hat{L}) < (\alpha'\hat{E}) < \theta_2 \text{ and } \gamma_0^* = \theta_1,$$

$$\text{if } \alpha'\hat{E} < \hat{M}/\hat{L}, \text{ then } \theta_1 < (\alpha'\hat{E}) < (\hat{M}/\hat{L}) < \theta_2 \text{ and } \gamma_0^* = \theta_2.$$

(c) *The constrained maximum likelihood estimators of* E *and* V *are*

$$E^* = E^*(\gamma_0^*) \quad \text{and} \quad V^* = V^*(\gamma_0^*),$$

where $E^*(\cdot)$ *and* $V^*(\cdot)$ *are the functions defined in (4) and (5), respectively.*

Proof. See the appendix.

In Theorem 1 the exact form of the constrained maximum likelihood estimator of the zero-beta return, γ_0^*, is derived. Roll (1980) points out that there is some ambiguity in estimating the zero-beta return with respect to an ex-ante efficient portfolio using the ex-post parameters. This difficulty is eliminated when the maximum likelihood approach is used: the given portfolio is, by definition, efficient with respect to the constrained estimators of the ex-ante parameters. In Theorem 2 two other estimators of γ_0 are considered. These alternative estimators are based on two familiar characterizations of γ_0.[11] The first estimator, $\hat{\gamma}_0$, is the sample mean return on the portfolio with the minimum sample variance among the sample zero-beta portfolios with respect to α. Roll (1979) notes that $\hat{\gamma}_0$ is the GLS estimator of γ_0 using $\hat{V}$ as the disturbance covariance matrix in the regression of the vector sample mean returns, $\hat{E}$, on the n-vector of ones, e, and the vector of the sample betas, $\hat{\beta} = (\hat{V}\alpha)/(\alpha'\hat{V}\alpha)$. The second estimator, $\hat{\hat{\gamma}}_0$, is the sample mean return on the unique portfolio which is both sample efficient and sample zero-beta with respect to α. In Theorem 2 it is shown that γ_0^* always lies between the two alternative estimators of γ_0.

Theorem 2. The constrained estimator of γ_0, γ_0^*, *always lies between* $\hat{\gamma}_0$ *and* $\hat{\hat{\gamma}}_0$. *Specifically,*

$$\text{if } \alpha'\hat{E} > \hat{M}/\hat{L}, \text{ then } \hat{\hat{\gamma}}_0 < \gamma_0^* < \hat{\gamma}_0 < (\hat{M}/\hat{L}) < (\alpha'\hat{E}),$$

$$\text{if } \alpha'\hat{E} < \hat{M}/\hat{L}, \text{ then } (\alpha'\hat{E}) < (\hat{M}/\hat{L}) < \hat{\gamma}_0 < \gamma_0^* < \hat{\hat{\gamma}}_0.$$

[10] Under the assumption that $\alpha'\hat{E} \neq \hat{M}/\hat{L}$, the function $H(m)$ is well defined.

[11] The two characterizations are formally presented in Long (1971).

Proof. See Kandel (1983).

Some geometric relations among γ_0^*, $\hat{\gamma}_0$, $\hat{\hat{\gamma}}_0$, and the sample mean and variance are illustrated in fig. 1 using the parabola $H(m)$ and the sample frontier in mean-variance space. Further relations among these estimators and the sample frontier are explored in Roll (1980) and Kandel (1984b). Next I present the relations in the mean-variance space between the unconstrained sample frontier and the constrained maximum likelihood estimator of the frontier determined by E^* and V^*. The discussion begins with an observation about the portfolio α, which is, by definition, on the constrained estimator of the efficient frontier.

Note that (4) and (5) imply that

$$\alpha' E^* = \alpha' \hat{E}, \qquad \alpha' V^* \alpha = \alpha' \hat{V} \alpha.$$

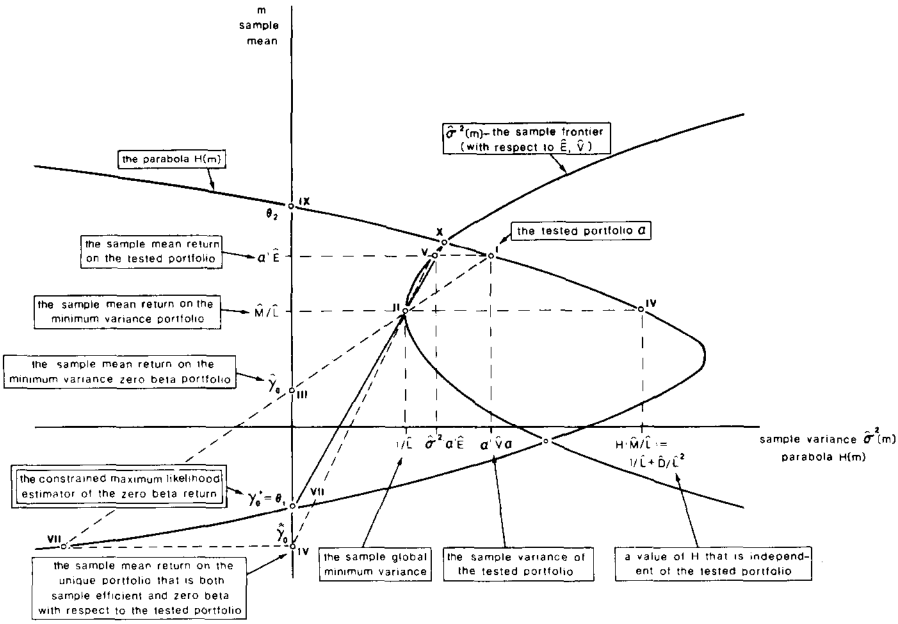

Fig. 1. The parabola $H(m)$, introduced in Theorem 1, is drawn together with the sample frontier, $\hat{\sigma}^2(m)$. The constrained maximum likelihood estimator γ_0^* is a zero of $H(m)$. The parabola $H(m)$ can be determined by the point I which corresponds to α, the point VI which is independent of α, and the point VII whose construction is illustrated above. The point X, a point of intersection of the two parabolas, corresponds to a sample efficient portfolio whose zero-beta return is γ_0^*. The maximum likelihood estimator, γ_0^*, always lies between $\hat{\gamma}_0$ and $\hat{\hat{\gamma}}_0$.

The constrained estimators of the mean and variance of the return on α coincide with the sample moments and the point $(\alpha'\hat{E}, \alpha'\hat{V}\alpha)$ is on the constrained maximum likelihood frontier in mean-variance space. The next theorem shows that the maximum likelihood estimator of the frontier is inside the sample frontier, and it is tangent to it at a single point.

Theorem 3. Consider the two parabolas in mean-variance space corresponding to:

(a) $\hat{\sigma}^2(m)$, *the sample efficient frontier, and*

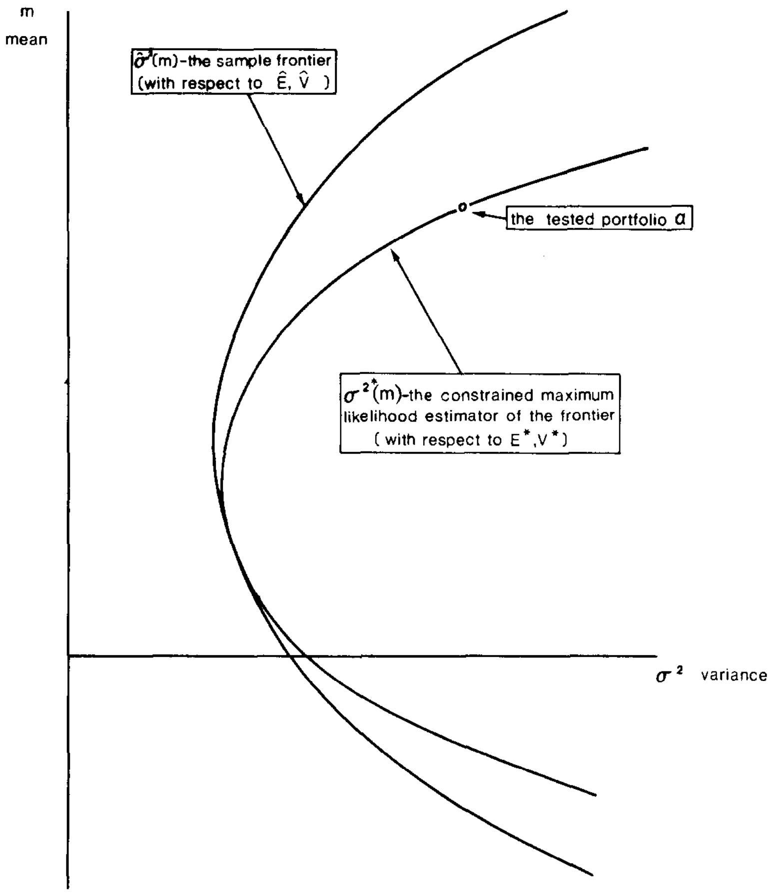

Fig. 2. The constrained maximum likelihood estimator of the efficient frontier, $\sigma^{2*}(m)$, is inside the sample efficient frontier, $\hat{\sigma}^2(m)$, and it is tangent to it at a single point.

(b) $\sigma^{2*}(m)$, *the maximum likelihood estimator of the efficient frontier.*

Then there exists a unique point m_0, *for which* $\hat{\sigma}^2(m_0) = \sigma^{2*}(m_0)$. *If* $m \neq m_0$, *then* $\hat{\sigma}^2(m) < \sigma^{2*}(m)$.

Proof. See Kandel (1983).

Fig. 2 illustrates the relation between the two efficient frontiers in mean-variance space.

3.2. The likelihood ratio test of efficiency

The null hypothesis is that a given portfolio α is ex-ante efficient. Ross (1980) derives a LRT of this hypothesis in the case where there is a riskless asset. The general framework is not changed when such asset is not available. Formally, define H_0 as the set of parameters such that α is efficient,

$$H_0 = \{(E, V) | \alpha \text{ is efficient}\}.$$

If H_1 denotes the alternative hypothesis that α is not efficient, then a LRT statistic is

$$S = 2\left[\max_{H_0 \cup H_1} \log L - \max_{H_0} \log L\right],$$

where the maximizations are over the unconstrained parameter space and the constrained parameter space respectively. Suppose r is the rate of return on the riskless asset, then Ross (1980) shows that the test statistic is

$$S(r) = T \log Q(r),$$

where

$$Q(r) = |V^*|/|\hat{V}| = [1 + x(r)]/[1 + y(r)],$$

$$x(r) = (\hat{E} - re)'\hat{V}^{-1}(\hat{E} - re),$$

$$y(r) = (\alpha'\hat{E} - r)^2/(\alpha'\hat{V}\alpha).$$

Note that $\sqrt{x(r)}$ is the ex-post price of risk, i.e., the excess mean return per unit of standard deviation on an efficient portfolio and $\sqrt{y(r)}$ is the corresponding ratio for α. $Q(r)$ is the ratio of the generalized variances of the restricted and unrestricted models. The test statistic is illustrated in fig. 3.

The next theorem shows that in the absence of a riskless asset the LRT statistic is another familiar construct in financial economics.

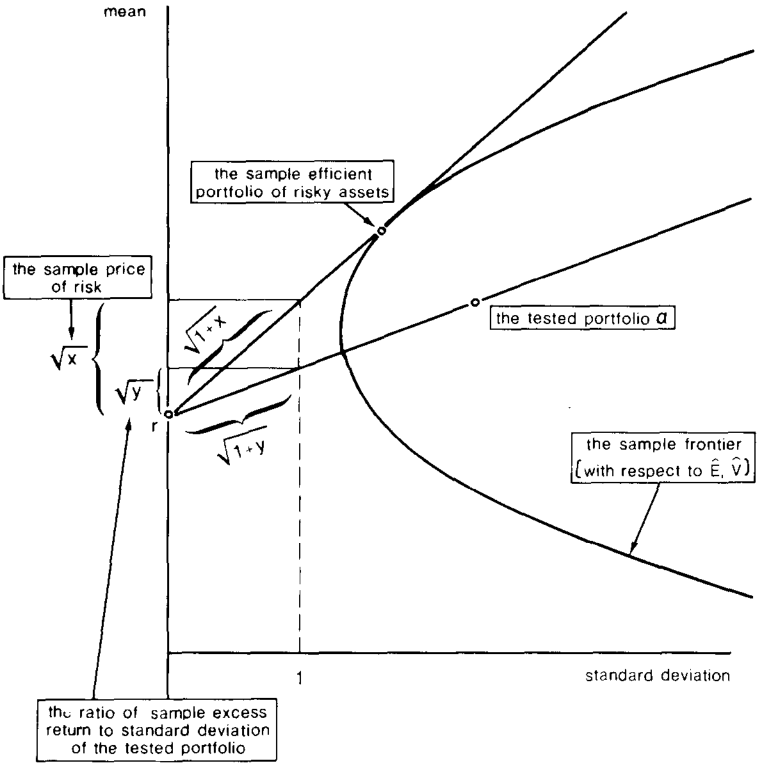

Fig. 3. [Ross (1980)]. When there is a riskless asset whose rate of return is r, the LRT statistic is $S(r)=T\log Q(r)$, where $Q(r)=(1+x(r))/(1+y(r))$. $\sqrt{x(r)}$ is the sample price of risk, and $\sqrt{y(r)}$ is α's ratio of sample excess return to standard deviation.

Theorem 4. (a) *When there is no riskless asset, the statistic of the LRT of the efficiency of a given portfolio* α *is*

$$S=T\log Q(\gamma_0^*),$$

where

$$Q(\gamma_0^*)=\frac{(\hat{M}/\hat{L})-\gamma_0^*}{(1/\hat{L})}\bigg/\frac{(\alpha'\hat{E}-\gamma_0^*)}{\alpha'\hat{V}\alpha}. \tag{6}$$

(b) *Under the null hypothesis the test statistic is asymptotically distributed as* χ^2_{n-2}.

Proof. See Kandel (1983).

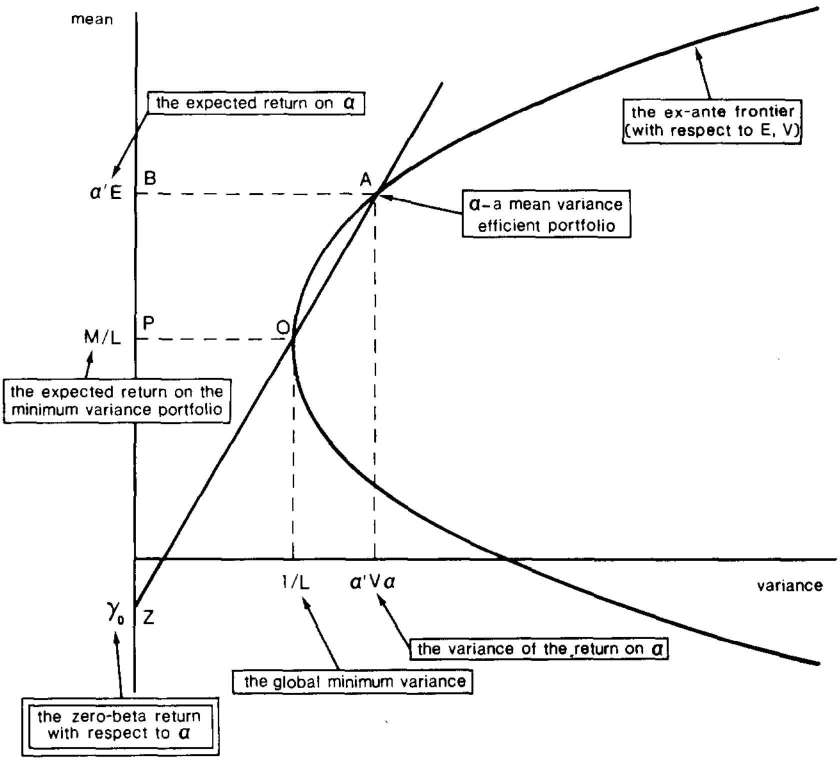

Fig. 4. When α is mean-variance efficient, then the three points A $(\alpha'V\alpha, \alpha'E)$, O $(1/L, M/L)$, and Z $(0, \gamma_0)$ are collinear. In other words the triangle ZOP is similar to the triangle ZBA, or $(\alpha'E - \gamma_0)/\alpha'V\alpha = (M/L - \gamma_0)/(1/L)$.

In fig. 4, it is shown that the null hypothesis implies that the triangle ZAB is similar to the triangle ZOP, or, equivalently,

$$\frac{M/L - \gamma_0}{1/L} = \frac{\alpha'E - \gamma_0}{\alpha'V\alpha}.$$

An intuitive direct test of efficiency of α is to infer whether the 'ex-post' triangles (determined by the sample global minimum point, the portfolio α, and an estimator of γ_0) are 'significantly' dissimilar to each other. Such a test is suggested by Roll (1979), with $\hat{\gamma}_0$ as the estimator of the zero-beta return with respect to α. Eq. (6) shows that the LRT is a test of this sort with γ_0^* (the constrained maximum likelihood estimator) as the estimator of γ_0. The denominator of (6) is the ratio of the sample mean excess (net of γ_0^*) return to variance of α, while the numerator is the ratio of the sample mean excess

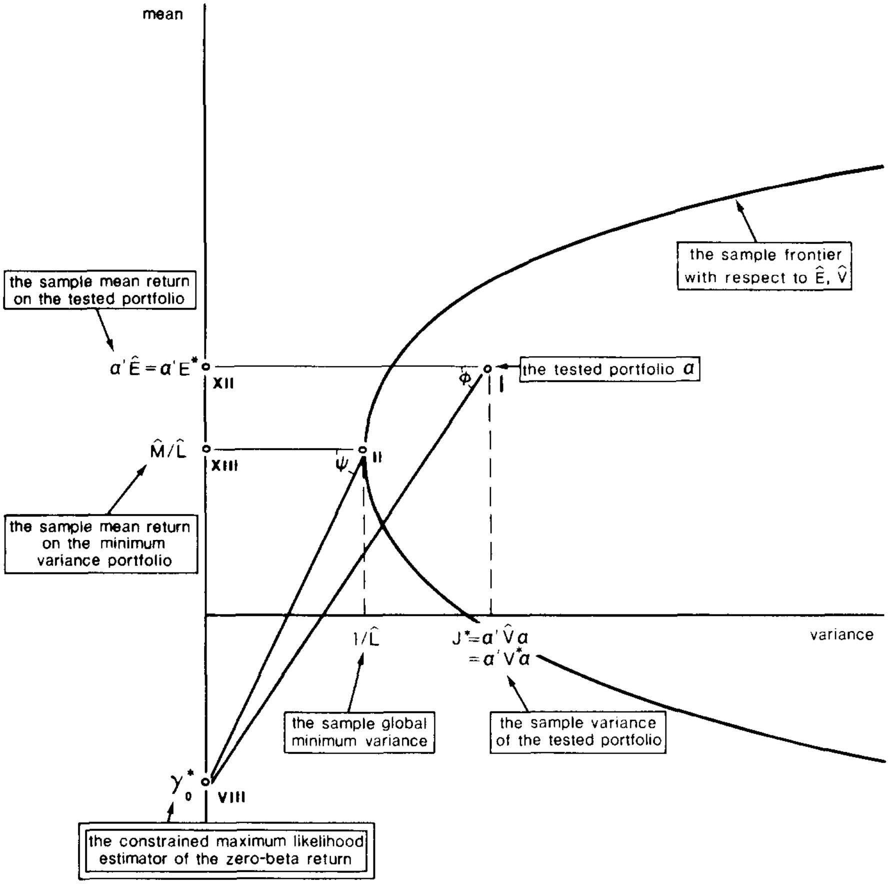

Fig. 5. In the absence of a riskless asset, the statistic of the likelihood ratio test of efficiency of α is $S = T \log Q(\gamma_0^*)$, where $Q(\gamma_0^*) = [(\hat{M}/\hat{L} - \gamma_0^*)/(1/\hat{L})]/[(\alpha'\hat{E} - \gamma_0^*)/\alpha'\hat{V}\alpha] = \tan\psi/\tan\phi$. The LRT is a test that the portfolio α and the minimum variance portfolio have the same ratio of excess return to variance, and thus, it involves the similarity of the triangles (I–VIII–XII) and (II–VIII–XIII).

return to variance of the sample minimum variance portfolio. The LRT is, thus, a test that involves the similarity of the triangles (II–VIII–XIII) and (I–VIII–XII) in fig. 5.

4. Summary and conclusions

This paper extends Ross's (1980) test of efficiency of a given portfolio to the case where there is no riskless asset. The paper derives the exact forms of the maximum likelihood estimators and the LRT statistic. A simple economic interpretation of the test is given. The test statistic is shown to be a familiar construct in financial economics that measures the distance, in mean-variance space, between a given portfolio and the sample frontier. Ambiguity in estimating the zero-beta return, pointed out in Roll (1980), is eliminated with the current procedure as the constrained maximum likelihood estimator of the zero-beta return is unique. Geometric relations between this estimator and two alternative estimators of the zero-beta return and between the unconstrained and constrained efficient frontiers are illustrated to provide intuition about the maximum likelihood estimators and the LRT.

Several directions for future study can be suggested:

(1) A complementary empirical study. The LRT developed in this paper can be used to test the mean variance efficiency of a particular market index whose composition is held constant during the sample period. One may reject the efficiency of such an index by using a small set of portfolios instead of a large set of individual assets. Each individual asset that is employed in the calculation of the return on the market index is assigned to one portfolio, where the relative weights within each portfolio are consistent with those of the given market index. This procedure might reduce the power of the test, but it makes it more manageable and comparable to other empirical studies. It would be interesting to compare the exact values of the estimates and the test statistic, as they are derived here, with Gibbons's (1982) approximated values, and to see whether they lead to any changes in inference.[12]

(2) Maximum likelihood estimation has been used to test linearity by Wald and Lagrangian Multipliers tests, as well as LRT. Stambaugh (1982) reports that Monte Carlo experiments reveal substantial differences in the finite sample distributions of the test statistics. Berndt and Savin (1977) and Shanken (1983a) explore some relations among the different tests. Geometric presentations of these tests, similar to that of the LRT given in this paper, might be helpful in understanding the differences.

(3) In exactly the same way that Ross's test is extended here to the case where there is no riskless asset, it can be extended to testing financial models of the form

$$E_i - r = k_1\beta_i + k_2 d_i,$$

where r is the known rate of return on the riskless asset, beta is calculated

[12] While Gibbons (1982) forms 40 portfolios, there are 41 portfolios in the suggested test. The 41st portfolio includes all assets that are excluded from Gibbons's portfolios but are employed in the actual calculation of the return on the market index. The asymptotic distribution of the statistic is, as in Gibbons (1982), $\chi^2_{(39)}$.

against the market portfolio, and d_i is known with certainty ex-ante. (The vector of expected returns is replaced by the vector of expected excess returns. The n-vector of ones, e, is replaced by another pre-determined vector – d.) An example of such expected return–risk relations is Brennan's (1970) after-tax CAPM which incorporates the effects of dividends and taxes. However, it seems much more difficult to derive the explicit forms of the constrained estimators and the LRT in cases where the vector of expected returns is spanned by the vector of betas and two or more other vectors, [for example, a model with dividends, taxes and margin constraints as in Litzenberger and Ramaswamy (1979)].

(4) The finite sample distributions and moments of the test statistic and the estimators are still unknown. Because of the nonlinearity of the constraints, the exact finite sampling properties are complex, and the prospect for any analytic small sample results seem rather dim.

Appendix: Proof of Theorem 1

Denote by $g(r)$ the maximum value of the likelihood function over the unconstrained parameter space when there exists a riskless asset whose rate of return is r. Let $f(r)$ be the maximum value of the likelihood function over the constrained parameter space, namely, the optimal value of the following optimization problem:

$$f(r)=\max_{E,V,k}\left\{(nT/2)\log(2\pi)-(T/2)\log|V| -\tfrac{1}{2}\sum_{t=1}^{T}(x_t-E)'V^{-1}(x_t-E)\right\},$$

subject to

$$E-re=kV\alpha, \quad V \text{ is symmetric and P.D.}$$

Let $Q(r)=g(r)/f(r)$ be the likelihood ratio.

Ross (1980) shows that the optimal solution of the constrained problem is $(E^*(r),\ V^*(r),\ k^*(r))$, where the functions $E^*(\cdot)$ and $V^*(\cdot)$ are defined in (4) and (5) of section 2, respectively, and

$$k^*(r)=\frac{\alpha'\hat{E}-r}{\alpha'\hat{V}\alpha}.$$

Ross (1980) also derives the likelihood ratio $Q(r)$,

$$Q(r)=\frac{\left[1+(\hat{E}-re)'\hat{V}^{-1}(\hat{E}-re)\right](\alpha'\hat{V}\alpha)}{\alpha'\hat{V}\alpha+(\alpha'\hat{E}-r)^2}.$$

By the definition of the constrained maximum likelihood estimators in the absence of a riskless asset, (E^*, V^*, γ_0^*), it is clear that

$$E^* = E^*(\gamma_0^*), \qquad V^* = V^*(\gamma_0^*),$$

and

$$f(\gamma_0^*) = \max_{\theta \in R} f(\theta).$$

The unconstrained estimators, $\hat{E}$ and $\hat{V}$, are independent of r and therefore, $g(r)$ is also independent of r: $g(r) \equiv g$ for all r. It is obtained that, for every $\theta \in R$,

$$Q(\theta) = g/f(\theta),$$

which implies that the likelihood ratio in the absence of a riskless asset is $Q(\gamma_0^*) = \min_\theta Q(\theta)$. In other words, the constrained estimator, γ_0^* can be found by analyzing the function $Q(\theta)$ instead of the function $f(\theta)$.

To simplify notation I define

$$\hat{L} \equiv e'\hat{V}^{-1}e, \quad \hat{M} \equiv e'\hat{V}^{-1}\hat{E}, \quad \hat{N} \equiv \hat{E}'\hat{V}^{-1}\hat{E},$$

$$\hat{D} \equiv \hat{N}\hat{L} - \hat{M}^2, \quad J \equiv \alpha'\hat{V}\alpha,$$

and then,

$$Q(\theta) = \frac{(1 + \hat{N} - 2\hat{M}\theta + \hat{L}\theta^2)J}{(\alpha'\hat{E} - \theta)^2 + J}.$$

The first-order condition for local optimality, $Q'(\theta) = 0$, implies that

$$(\hat{M} - \hat{L}\gamma_0^*)\left(J + (\alpha'\hat{E} - \gamma_0^*)^2\right) = (\alpha'\hat{E} - \gamma_0^*)(1 + \hat{N} - 2\hat{M}\gamma_0^* + \hat{L}\gamma_0^{*2}),$$

or

$$-\gamma_0^{*2}(\hat{M} - \hat{L}\alpha'\hat{E}) + \gamma_0^*(-J\hat{L} - \hat{L}(\alpha'\hat{E})^2 + 1 + \hat{N})$$

$$+ \left(J\hat{M} + (\alpha'\hat{E})^2\hat{M} - (\alpha'\hat{E}) - \hat{N}(\alpha'\hat{E})\right) = 0.$$

It is assumed that $\alpha'\hat{E} \neq (\hat{M}/\hat{L})$. This, together with the definition of A, B, C and the function H, implies that

$$H(\gamma_0^*) = A\gamma_0^{*2} + B\gamma_0^* + C = 0.$$

$H(m)$ is a strictly concave function. It is easy to show that

$$H(\alpha'\hat{E}) = \alpha'\hat{V}\alpha > 0,$$

and

$$H(\hat{M}/\hat{L}) = \frac{1}{\hat{L}} + \frac{\hat{D}}{\hat{L}^2} > 0,$$

which assure the existence of $\theta_1 < \theta_2$ such that

$$\theta_1 < \min\{\alpha'\hat{E}, \hat{M}/\hat{L}\} < \max\{\alpha'\hat{E}, \hat{M}/\hat{L}\} < \theta_2,$$

and

$$H(\theta_1) = H(\theta_2) = 0.$$

There are two candidates for γ_0^*: θ_1 and θ_2. It is to be checked whether the function $Q(\theta)$ has a local minimum at any of them, and if it has, whether it is a global minimum. By calculating the second derivative,

$$Q''(\theta) = \frac{(\alpha'\hat{V}\alpha)}{\left(\alpha'\hat{V}\alpha + (\alpha'\hat{E} - \theta)^2\right)} \frac{(\hat{L}\alpha'\hat{E} - \hat{M})}{(\alpha'\hat{E} - \theta)},$$

it is easy to show that, if $\alpha'\hat{E} > \hat{M}/\hat{L}$, then

$$\alpha'\hat{E} > \gamma_0^* \quad \text{and} \quad \gamma_0^* = \theta_1 \quad \text{is a local minimum,}$$

and if $\alpha'\hat{E} < \hat{M}/\hat{L}$, then

$$\alpha'\hat{E} < \gamma_0^* \quad \text{and} \quad \gamma_0^* = \theta_2 \quad \text{is a local minimum.}$$

In order to complete the proof, it is shown that γ_0^* is, indeed, a global minimum of $Q(\theta)$. I check that

$$\lim_{\theta \to \pm\infty} Q(\theta) = \lim_{\theta \to \pm\infty} \frac{J(L\theta - M)}{\theta - \alpha'E} = JL > Q(\gamma_0^*),$$

and since the function $Q(\theta)$ has one local minimum and one local maximum, then the local minimum is a global minimum.

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