THE JOURNAL OF FINANCE • VOL. XLII, NO. 4 • SEPTEMBER 1987

Mean-Variance Spanning

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ABSTRACT

The authors propose a likelihood-ratio test of the hypothesis that the minimum-variance frontier of a set of K assets coincides with the frontier of this set and another set of N assets. They study the relation between this hypothesis, exact arbitrage pricing, and mutual fund separation. The exact distribution of the test statistic is available. The authors test the hypothesis that the frontier spanned by three size-sorted stock portfolios is the same as the frontier spanned by thirty-three size-sorted stock portfolios.

INVESTORS' CHOICES OF PORTFOLIOS of assets and the implications of these choices for assets' prices are major topics in financial economics. Students of the first topic—portfolio theory—often strive to derive separation results, i.e., seek conditions under which each investor allocates all of his or her savings among a small number of separating funds. These separating funds are the same across investors.

The second topic entails a study of the aggregate behavior of security market participants. Students of the pricing issues derive equilibrium restrictions on security prices.

We focus on the relations among mean-variance efficiency, mutual fund separation, and two prominent security-pricing models in finance: the CAPM and the APT.

The investment universe under consideration includes $K + N$ risky assets with returns that have a nonsingular covariance matrix. In addition, it may include a risk-free asset. We are particularly interested in forming K portfolios of the $N + K$ original assets and then studying the relation between the minimum-variance frontier spanned by the K derived assets and the frontier of the original $N + K$ assets. The returns on the derived K assets are denoted by the $K \times 1$ vector $\underline{R}$, and the returns on the other N assets are denoted by the $N \times 1$ vector $\underline{r}$. (The investment opportunity set of the derived K assets and the other N assets is equal to the investment opportunity set of the original $N + K$ assets.) The following linear model is assumed:

$$\underline{r} = \underline{a} + B\underline{R} + \underline{e}, \tag{1}$$

* Huberman is from the University of Chicago and Tel Aviv University; Kandel is from the University of Chicago. An earlier version of this paper was titled "Likelihood Ratio Tests of Asset Pricing and Mutual Fund Separation". We are grateful to Eugene Fama, Wayne Ferson, Jon Ingersoll, Steve Ross, Robert Stambaugh, an anonymous referee, and especially Nai-fu Chen for useful conversations and comments and to the University of Chicago's Center for Research in Security Prices for financial support.

where $\underset{\sim}{r}$, $\underset{\sim}{a}$, and $\underset{\sim}{e}$ are $N \times 1$ vectors, $\underset{\sim}{R}$ is a $K \times 1$ vector, and B is an $N \times K$ matrix. The vectors $\underset{\sim}{r}$, $\underset{\sim}{R}$, and $\underset{\sim}{e}$ are random. The random vector $\underset{\sim}{e}$ is uncorrelated with the random vector $\underset{\sim}{R}$, and the expected value of each element of $\underset{\sim}{e}$ is 0.

Consider the following statements:

1. The minimum-variance frontier of $\underset{\sim}{R}$ intersects the minimum-variance frontier of $\underset{\sim}{R}$ and $\underset{\sim}{r}$.
2. The minimum-variance frontier of $\underset{\sim}{R}$ intersects the minimum-variance frontier of $\underset{\sim}{R}$, $\underset{\sim}{r}$, and the risk-free asset.
3. The minimum-variance frontier of $\underset{\sim}{R}$ is the same as the minimum-variance frontier of $\underset{\sim}{R}$ and $\underset{\sim}{r}$.

The first and second statements are closely related to exact arbitrage pricing, and the third implies (under (1)) K-fund separation. These relations are discussed in Sections I and II, respectively.

In Section I (in Proposition 1), we show that, under (1), the first statement above is equivalent to the existence of a constant w_0 such that

$$\underset{\sim}{a} = w_0[\underset{\sim}{i}_N - B\underset{\sim}{i}_K], \tag{2}$$

where $\underset{\sim}{i}$ is the vector with elements all equal to 1. The dimensionality of these vectors varies with the context. For instance, $\underset{\sim}{i}_K$ is the $K \times 1$ vector with elements all equal to 1.

The second statement above implies that the scalar w_0 is the return on the risk-free asset. In this case, (2) is a set of linear constraints.

In Section II (in Proposition 3), we show that, under (1), the third statement above (spanning) is equivalent to

$$\underset{\sim}{a} = \underset{\sim}{0} \tag{3a}$$

and

$$B\underset{\sim}{i}_K = \underset{\sim}{i}_N, \tag{3b}$$

where $\underset{\sim}{0}$ is the vector with elements all equal to 0.

In Section III, we review multivariate tests of the first two statements and propose a multivariate test of the third statement. The exact small-sample distributions of the test statistics are available for the test of (3) and for tests that assume the presence of a risk-free asset.

In Section IV, the likelihood-ratio test of spanning is illustrated by applying it to the hypothesis that the monthly returns on three size-based indices of NYSE stocks span the minimum-variance frontier of the monthly returns on thirty-three size-sorted portfolios. We test this hypothesis because the size-based indices can potentially proxy for factors in the context of the APT. This factor-proxying property is suggested by the work of Huberman, Kandel, and Karolyi [17], who observe that returns on stocks of firms of a similar size are more correlated with each other than with returns on stocks of firms of different sizes and that the returns on three size-based stock indices capture most of that cross-correlation.

Concluding remarks are offered in Section V.

I. Asset Pricing and Mean-Variance Intersection

The main result of the Capital Asset Pricing Model (CAPM), as developed by Sharpe [32], Lintner [22], and Black [3], is that a capital asset's expected return is linearly related to the covariance of the asset's return with the return on the market portfolio. This linear relation holds for all assets if and only if the market portfolio is on the minimum-variance frontier. (Fama [10], Roll [24], and Ross [27] make this observation.) Therefore, a test of the CAPM may be interpreted as an examination of the distance (in mean-variance space) between a given market index and the minimum-variance frontier of a given set of assets.

The Arbitrage Pricing Theory (APT) of Ross [26, 28] is a one-period model in which the $N \times 1$ vector $\underset{\sim}{r}$ of returns on capital assets satisfies the generating model

$$\underset{\sim}{r} = \underset{\sim}{E} + B\underset{\sim}{f} + \underset{\sim}{e}, \tag{4}$$

where $\underset{\sim}{f}$ is a $K \times 1$ vector of random factors, B is an $N \times K$ matrix of factor loadings, and $\underset{\sim}{e}$ is an $N \times 1$ vector of residuals. With no loss of generality, normalize (4) to obtain $\mathrm{E}\{\underset{\sim}{f}\} = \mathrm{E}\{\underset{\sim}{e}\} = 0$ and $\mathrm{E}\{\underset{\sim}{f}\underset{\sim}{f}'\} = I$, where $\mathrm{E}\{\cdot\}$ denotes expectation and I is the identity matrix, so that $\underset{\sim}{E}$ is the vector of mean returns. Assume further that the matrix B is of rank K. Restrictions on the diagonality of the covariance matrix $\mathrm{E}\{\underset{\sim}{e}\underset{\sim}{e}'\}$ and on the relation between the eigenvalues of that covariance matrix and those of BB' (as N becomes large) are required for proofs of the APT. (Huberman [16] reviews the APT literature.)

Exact arbitrage pricing obtains if an exact linear pricing of $\underset{\sim}{r}$ (and the risk-free asset if such an asset exists) holds with respect to the factors $\underset{\sim}{f}$, i.e.,

$$\underset{\sim}{E} = \underset{\sim}{i} r_0 + B\underset{\sim}{u}, \tag{5}$$

where r_0 is the return on a riskless asset if that asset exists and $\underset{\sim}{u}$ is a $K \times 1$ vector of risk premiums. Chamberlain [6], Chen and Ingersoll [8], and Connor [9] provide conditions under which (5) holds. Exact arbitrage pricing is the tested form of the APT, e.g., by Roll and Ross [25], Chen [7], and Lehmann and Modest [21].

Empirical investigations of the APT often involve the formation of investment positions with payoffs that are intended to mimic the realizations of the K factors, i.e., to be used in place of the factors for pricing the subset's assets (e.g., Lehmann and Modest [21]). Huberman, Kandel, and Stambaugh [18] define and characterize the sets of mimicking positions and discuss their properties.

Grinblatt and Titman [15] and Jobson and Korkie [19] derive results that imply that exact arbitrage pricing is equivalent to the mean-variance efficiency of a portfolio of the mimicking portfolios. Thereby, they show that Chamberlain's [6] result for an infinite economy holds also for a finite set of assets.

The equivalence between the pricing equation (5) and the parameter restriction (2) on the returns-generating model is known for the case $K = 1$ (CAPM, consumption CAPM), as well as for $K > 1$ (intertemporal CAPM, APT). See, for example, Black, Jensen, and Scholes [4], Breeden, Gibbons, and Litzenberger [5], and Shanken [30].

Proposition 1 combines the above results and summarizes the relations among mean-variance intersection, exact linear pricing, and the restrictions on the regression model. To keep the paper self-contained, we provide a simple proof of the proposition.

PROPOSITION 1: *Suppose the linear structure* (1) *holds. The following statements are equivalent.*

1. *There exists a portfolio of the vector of asset returns* $\underline{R}$ *that is on the minimum-variance frontier of* ($\underline{R}$, $\underline{r}$) *but is not the global minimum-variance portfolio.*
2. *Exact linear pricing of* $\underline{r}$ *holds with respect to* $\underline{R}$.
3. *There exists a scalar* w_0 *such that*

$$\underline{a} = w_0[\underline{i}_N - B\underline{i}_K]. \tag{6}$$

The scalar w_0 *is the expected return on any portfolio of* ($\underline{R}$, $\underline{r}$) *with a return that is uncorrelated with the portfolio of* $\underline{R}$ *that is on the minimum-variance frontier of* ($\underline{R}$, $\underline{r}$).

Proof: The following notation is used in the proof. Let $\underline{x}$ and $\underline{y}$ be $n \times 1$ and $m \times 1$ vectors of random variables. The (i, j) element of the $n \times m$ matrix $\text{cov}(\underline{x}, \underline{y})$ is the covariance between x_i and y_j.

It is well known that the following two statements are equivalent.

i) There exists a portfolio $\underline{p}$ of $\underline{R}$ ($\underline{p}'\underline{i} = 1$), with return $y = \underline{p}'\underline{R}$, on the mean-variance frontier of ($\underline{R}$, $\underline{r}$), but y is not the global minimum-variance portfolio.
ii) There exist scalars w_0 and $w \neq 0$ that satisfy

$$\text{E}\{\underline{R}\} = \underline{i}w_0 + \text{cov}(\underline{R}, y)w \tag{7}$$

and

$$\text{E}\{\underline{r}\} = \underline{i}w_0 + \text{cov}(\underline{r}, y)w. \tag{8}$$

This characterization of the mean-variance frontier can be proved by considering the first-order conditions of the optimization problem that seeks a variance-minimizing portfolio with a specified mean.

Since, for every random vector $\underline{x}$, $\text{cov}(\underline{x}, y) = \text{cov}(\underline{x}, \underline{p}'\underline{R}) = \text{cov}(\underline{x}, \underline{R})\underline{p}$, equations (7) and (8) can be rewritten as

$$\text{E}\{\underline{R}\} = \underline{i}w_0 + \text{cov}(\underline{R}, \underline{R})\underline{p}w \tag{9}$$

and

$$\text{E}\{\underline{r}\} = \underline{i}w_0 + \text{cov}(\underline{r}, \underline{R})\underline{p}w. \tag{10}$$

Equation (9) implies that

$$\underline{p}w = \text{cov}(\underline{R}, \underline{R})^{-1}[\text{E}\{\underline{R}\} - \underline{i}w_0]. \tag{11}$$

Substituting (11) into (10) gives

$$\text{E}\{\underline{r}\} = \underline{i}w_0 + \text{cov}(\underline{r}, \underline{R})\text{cov}(\underline{R}, \underline{R})^{-1}[\text{E}\{\underline{R}\} - \underline{i}w_0]. \tag{12}$$

The linear structure (1) implies that

$$B = \mathrm{cov}(\underline{r}, \underline{R})\mathrm{cov}(\underline{R}, \underline{R})^{-1} \tag{13}$$

and

$$\mathrm{E}\{\underline{r}\} = \underline{a} + B\mathrm{E}\{\underline{R}\}. \tag{14}$$

Combine (12) and (13) to infer that the existence of a portfolio of $\underline{R}$ on the frontier of $(\underline{R}, \underline{r})$ is equivalent to exact linear pricing of $\underline{r}$ with respect to $\underline{R}$. Combine (12) to (14) to conclude that the existence of a portfolio of $\underline{R}$ on the frontier is equivalent to the existence of a scalar w_0 that satisfies (6). Equation (8) implies that w_0 is the expected return on any portfolio with a return that is uncorrelated with the intersecting portfolio. Q.E.D.

II. Mutual Fund Separation and Mean-Variance Spanning

The vector of assets $\underline{R}$ is separating relative to the investment universe that consists of the vector $(\underline{R}, \underline{r})$ if, for every portfolio of $(\underline{R}, \underline{r})$ with a return y, there exists a portfolio of $\underline{R}$ with a return z, such that $\mathrm{E}U(z) \geq \mathrm{E}U(y)$ for every concave monotone utility function U. In other words, under K-fund separation, the optimal investment portfolio of each risk-averse individual can be described as a portfolio of the K separating funds. Ross [29] characterizes the class of random variables for which a weaker form of this separation holds.

Mutual fund separation is a stronger property than exact arbitrage pricing relative to the separating funds. Relations between the Arbitrage Pricing Theory and mutual fund separation are pointed out by Chamberlain [6] and Ross [29]. They show that augmentation of the assumptions underlying the APT gives rise to a mutual fund separation.

In this section, we study the relation between mean-variance spanning and K-fund separation, assuming that the multivariate-regression model (1) holds.

PROPOSITION 2: *Suppose that* (1) *and* (3) *hold and the vector of conditional expectations* $\mathrm{E}\{\underline{e} \mid \underline{R}\} = 0$. *Then* $\underline{R}$ *is a separating vector relative to the investment universe that consists of the vector* $(\underline{R}, \underline{r})$.

Proof: Let $y = \underline{p}_R'\underline{R} + \underline{p}_r'\underline{r}$, where $\underline{p}_R'\underline{i}_K + \underline{p}_r'\underline{i}_N = 1$. The return y is a return on a portfolio of $(\underline{R}, \underline{r})$. Let $\underline{p}' = \underline{p}_R' + \underline{p}_r'B$. Equation (3b) implies that $\underline{p}'\underline{i}_K = 1$, so $z = \underline{p}'\underline{R}$ is a return on a portfolio of $\underline{R}$. Moreover, $z = \mathrm{E}\{y \mid z\}$, $y - z = \underline{p}_r'\underline{e}$, and $\mathrm{E}\{\underline{p}_r'\underline{e} \mid z\} = 0$. Apply Jensen's inequality to conclude that $\mathrm{E}U(z) \geq \mathrm{E}U(y)$ for every monotone concave utility function U, which completes the proof. Q.E.D.

An interpretation of (3) in the context of minimum-variance set geometry is provided in Proposition 3.

PROPOSITION 3: *Suppose that* (1) *holds and the minimum-variance frontier of* $\underline{R}$ *contains at least two points with different expected returns. Every minimum-variance portfolio of* $(\underline{R}, \underline{r})$ *is a portfolio of* $\underline{R}$ *if and only if* (3) *holds, i.e.,* $\underline{a} = \underline{0}$ *and* $B\underline{i}_K = \underline{i}_N$.

Proof: Since every minimum-variance frontier of risky assets is spanned by two portfolios, it is sufficient to show that (3) holds if and only if at least two distinct portfolios of $\underline{R}$ are on the minimum-variance frontier of $(\underline{R}, \underline{r})$.

Equations (3) hold if and only if (6) holds with any value of w_0. Apply Proposition 1 to conclude that (3) holds if and only if, for each w_0, there is a minimum-variance portfolio of $\underline{R}$ with a return that is uncorrelated with the assets that have expected returns equal to w_0. Therefore, (3) is equivalent to the existence of more than one minimum-variance portfolio of $\underline{R}$. Q.E.D.

III. Multivariate Tests of Mean-Variance Intersection and Spanning

We apply Propositions 1 and 3 to review multivariate tests of mean-variance intersection and propose a multivariate test of mean-variance spanning.

The following hypotheses are considered for the two vectors of asset returns, $\underline{R} = (R_1, \cdots, R_K)$ and $\underline{r} = (r_1, \cdots, r_N)$.

H_1: $\underline{R}$ spans $(\underline{R}, \underline{r})$.
H_2: $\underline{R}$ intersects $(\underline{R}, \underline{r})$.
H_3: $\underline{R}$ does not intersect $(\underline{R}, \underline{r})$.

If there exists a risk-free asset with a rate of return that is known and equal to r_f and if r_f is not equal to the expected return on the global minimum-variance portfolio of risky assets, then the minimum-variance frontier of all assets is spanned by the risk-free asset and a portfolio of the risky assets. Propositions 1 and 3 can be used to show that the following are equivalent:

1. The mean-variance frontier of $\underline{R}$ and the risk-free asset is equal to the frontier of $\underline{R}$, $\underline{r}$, and the risk-free asset.
2. The mean-variance frontier of $\underline{R}$ intersects the mean-variance frontier of $\underline{R}$, $\underline{r}$, and the risk-free asset.
3. Equation (2) holds with $w_0 = r_f$ in the regression of $\underline{r}$ on the vector $\underline{R}$.
4. Equation (3a) holds in the regression of the vector of excess returns $\underline{r} - r_f \underline{i}$ on the vector $\underline{R} - r_f \underline{i}$.

For the case where a risk-free asset with a known return exists, we consider also H_4 and H_5:

H_4: $\underline{R}$ intersects $(\underline{R}, \underline{r}, r_f)$.
H_5: $\underline{R}$ does not intersect $(\underline{R}, \underline{r}, r_f)$.

It is noteworthy that no assumption about normality or about the correlations of the disturbance terms (the e's) is necessary in (1) in order to derive Propositions 1 and 3. However, in order to conduct tests suggested by H_1 to H_5, one has to specify the distribution of the vector of disturbance terms $\underline{e}$, conditioned on the vector $\underline{R}$. We assume that it is multivariate normal with mean zero and an unknown $N \times N$ covariance matrix V.

Let L_i be the maximized value of the likelihood function under the hypothesis H_i, and let V_i be the associated maximum-likelihood estimator of the covariance matrix V. Let

$$V^{ij} = | V_j | / | V_i |. \tag{15}$$

The maximized log-likelihood values, the L_i's, satisfy $2[L_i - L_j] = T\log(V^{ij})$. These log-likelihood ratios are asymptotically distributed as chi-square under H_i, the null hypothesis against H_j. This chi-square statistic has $2N$ degrees of freedom when H_1 is tested against H_3; it has $N + 1$ degrees of freedom when H_1 is tested against H_2; and it has $N - 1$ degrees of freedom when H_2 is tested against H_3. If the return on a risk-free asset is known, then set $r_0 = w_0$ in (2), and the chi-square statistic has $N - 2$ degrees of freedom when H_4 is tested against H_5.

Proposition 4 provides the small-sample distribution of a monotone transformation of the likelihood-ratio statistic for the case where a risk-free asset with a known return is assumed to exist. It is also in Gibbons, Ross, and Shanken [14] and Jobson and Korkie [19]. The proposition can be proven by applying results from Anderson [1] (Section 8.4).

PROPOSITION 4: *Suppose that there exists a risk-free asset with a known return. Consider testing* H_0: H_4 *against* H_A: H_5. *The statistic* $[1/V^{45} - 1](T - K - N)/N$ *is a monotone transformation of the likelihood-ratio test statistic, which, under* H_4, *has an F distribution with* $N - 1$ *and* $T - K - (N - 1)$ *degrees of freedom.*

When a risk-free asset does not exist, the restriction (2) is nonlinear. For the test of H_0: H_2 against H_A: H_3, the zero-beta rate has to be estimated and the exact small-sample distribution of the test statistic is unknown. Gibbons [12] performs such a test in the case $K = 1$. He resorts to a one-step Gauss-Newton procedure in order to compute the restricted-regression coefficient estimates and thereby calculate the likelihood-ratio test statistic. Kandel [20] also considers this case and obtains an exact maximum-likelihood estimator of the zero-beta rate and closed-form solution for the test statistic. Shanken [30, 31] introduces a multivariate cross-sectional regression test for this case and derives an approximate small-sample distribution for the test statistic. He also obtains bounds on the exact distribution function of the test statistics for this test and for the likelihood-ratio test and an exact maximum-likelihood estimator of the zero-beta rate for the case $K > 1$.

We propose a test of mean-variance spanning. It is an alternative to tests of mean-variance intersection in the case without a risk-free asset. The test involves only linear restrictions on the regression coefficients, and the exact distribution of the test statistic under the null hypothesis is known. The spanning test is a test of a more stringent hypothesis than mean-variance intersection.

PROPOSITION 5: *Suppose that there is no risk-free asset. Consider testing* H_0: H_1 *against* H_A: H_3. *The statistic* $[1/V^{13} - 1](T - K - N)/N$ *is a monotone transformation of the likelihood-ratio test statistic, which, under* H_1, *has an F distribution with* $2N$ *and* $2(T - K - N)$ *degrees of freedom.*

Proof: Apply Theorem 8.4.6 in Anderson [1]. Q.E.D.

IV. A Size-Based Example

In this section, the likelihood-ratio test of spanning is illustrated by applying it to the hypothesis that the monthly returns on three size-based indices of NYSE stocks span the minimum-variance frontier of the monthly returns on thirty-three size-sorted portfolios. Huberman, Kandel, and Karolyi [17] observe that the contemporaneous correlations of the disturbance terms in the market model depend on the difference in the size of the firms with stock returns that are used as independent variables. The more similar the firms are in size, the higher the correlation. A linear structure of stock returns (1) in which similarity of firm size implies similarity of slope coefficients could give rise to this pattern.

We estimate the linear structure (1) with the returns on $K = 3$ size-sorted indices as the explanatory variables and the returns on $N = 30$ size-sorted portfolios as the dependent variables. We then inquire whether the three size-based indices span the minimum-variance frontier of the larger set of size-sorted portfolios. We describe the data in Subsection *A*, discuss temporal variations of the regression coefficients in Subsection *B*, present the results in Subsection *C*, and relate them to the size effect in Subsection *D*.

A. The Data

The raw data consist of monthly returns on all stocks that traded on the New York Stock Exchange (NYSE) from January 1964 until December 1983. At the beginning of each year, we rank the stocks on the NYSE according to the market value of their equity in December of the previous year and construct thirty-three size-sorted sets of stocks. The returns on the equally weighted portfolios of these size-sorted sets are denoted $r_1, \cdots, r_{33}$. ($r_1(t)$ indicates the time-t returns on the portfolio of the smallest stocks.)

The return on the small-stock index is the equally weighted average of $r_1, \cdots, r_{11}$. Similarly, the returns on the medium- and large-stock indices are the equally weighted averages of $r_{12}, \cdots, r_{22}$ and $r_{23}, \cdots, r_{33}$. The returns on the three indices are denoted by the 3×1 vector $\underset{\sim}{R}(t) = (R_1(t), R_2(t), R_3(t))'$.

We consider the multivariate regression (1) with the vector $\underset{\sim}{R}$ as the vector of explanatory variables. As Proposition 5 requires that the contemporaneous covariance matrix of the residuals V be nonsingular, we use only thirty of the thirty-three returns. The vector of independent variables in the example is

$$r(t) = (r_1(t), \cdots, r_5(t), r_7(t), \cdots, r_{16}(t), r_{18}(t), \cdots, r_{27}(t), r_{29}(t), \cdots, r_{33}(t)).$$

B. A Test of Temporal Constancy of Regression Coefficients

The estimation of the regression model (1) entails the assumption that $\underset{\sim}{a}$ and B are constant over the estimation period. We study the temporal constancy of $\underset{\sim}{a}$ and B with a multivariate Chow test. We break the overall period into two subperiods and estimate the following regression:

$$\underset{\sim}{r}_t = \underset{\sim}{a} + \underset{\sim}{a}^1 D_t + (B + B^1 D_t)\underset{\sim}{R}_t + \underset{\sim}{e}_t, \tag{16}$$

where D_t is a dummy variable set equal to one in the first subperiod and to zero in the second subperiod. The vector $\underset{\sim}{a}^1$ and the matrix B^1 are of the same dimensions as the vector $\underset{\sim}{a}$ and the matrix B.

Constancy of the coefficients implies that

$$\underset{\sim}{a}^1 = \underset{\sim}{0} \text{ (the } N \times 1 \text{ zero vector)} \tag{17a}$$

and

$$B^1 = 0 \text{ (the } N \times K \text{ zero matrix).} \tag{17b}$$

The likelihood-ratio test of (17) is similar to the test discussed in Proposition 5. The test statistic is

$$U = |V_0|/|V_A|, \tag{18}$$

where $|V_0|$ and $|V_A|$ denote the determinants of the covariance matrices of the residuals under the restricted system (i.e., (16) and (17)) and under the unrestricted system (i.e., (16) alone).

Unfortunately, the small-sample distribution of the test statistic (under (16) and (17)) is unavailable. Asymptotically, the test statistic

$$U^1 = -(T - 21.5)\ln U$$

has a chi-square distribution with 120 degrees of freedom, where T is the number of observations.

Rao's small-sample approximation (Anderson [1], Section 8.5.4) uses the statistics

$$U^2 = [1/U^{1/s} - 1]((T - 21.5)s - q)/120,$$

where $s = 3.975$ and $q = 59$. Under (16) and (17), the distribution of U^2 is approximately F, with 120 and $(T - 21.5)s - q$ degrees of freedom. Test results for different periods, ranging from the whole twenty-year period to four non-overlapping five-year periods, are reported in Table I. It seems that the coefficients in (1) do change over time.

C. A Test of Mean-Variance Spanning

Temporal instability of regression coefficients is well known in finance. It led researchers to choose fairly short time periods for tests of asset pricing (e.g., Fama and MacBeth [11], Gibbons [12], and Roll and Ross [25]). The justification for this approach is that, although the coefficients vary, their short-run variation is negligible to the extent that the tested pricing relation is unlikely to be rejected

Table I

Multivariate Chow Tests of the Coefficients in Equation (1)[a]

Years	U_1	p-Value	U_2	D.F.	p-Value
64–83	168	.003	1.44	825	.003
64–73	157	.013	1.36	348	.016
74–83	146	.056	1.25	348	.063
64–68	133	.200	1.09	110	.315
69–73	168	.003	1.55	110	.010
74–78	150	.035	1.35	110	.079
79–83	180	.000	1.75	110	.002

[a] Restrictions (17) are imposed on (16) for different test periods. The tests use monthly returns over different test periods. The statistic U_1 is distributed asymptotically as chi-square, with 120 degrees of freedom. The statistic U_2 is distributed approximately as F, with 120 and D.F. degrees of freedom.

due to the temporal variation of the coefficients. We follow this approach in this example.

In Parts A and B of Table II, we report the regression estimates for the ten-year periods 1964 to 1973 and 1974 to 1983. The F-test statistics mentioned in Proposition 5 have 60 and 174 degrees of freedom. They are 1.08 and 0.97 for the first and second ten-year periods. Their p-values are 0.34 and 0.55, so restrictions (3) are not rejected for either period at the conventional significance levels. The point estimates of the slope coefficients display an interesting pattern, discussed by Huberman, Kandel, and Karolyi [17]: the coefficients of R_1 decrease as we go down the table; the coefficients of R_3 increase as we go down the table; and the coefficients of R_2 reach their higher values in the middle of the table.

We also calculate the F-statistics testing restrictions (3) of (1) for the twenty-year period 1964 to 1983 and for the four nonoverlapping five-year subperiods of these twenty years. These F-statistics are 1.6, 0.68, 0.86, 0.59, and 0.91. Their corresponding p-values are 0.0046, 0.93, 0.72, 0.98, and 0.65. The subperiod results can be aggregated. Following Gibbons and Shanken [13], our aggregate test statistic is minus two times the sum of the natural logarithms of the p-values of the nonoverlapping subperiods. This statistic has a chi-square distribution, with twice as many degrees of freedom as the number of time-series observations. Our results for the two ten-year subperiods give rise to the chi-square statistic of 2.88. Under the null hypothesis, the probability of having this value or higher is greater than fifty percent. This chi-square statistic for the four five-year subperiods is 0.95. Under the null hypothesis, the probability of having this value or higher is greater than 99.5 percent.

When ten or five years are used to construct the test statistic (or when the subperiod results are aggregated), the data seem to support the null hypothesis. When twenty years of data are considered, the data do not support the spanning hypothesis. It is possible, however, that the spanning hypothesis is correct but that temporal instability of the coefficients of the underlying returns-generating model, as discussed in Subsection B, accounts for the high values of the test statistic when twenty years of data are used.

D. Mean-Variance Spanning of Size Portfolios and the Size Effect

Banz [2] and Reinganum [23] show that mean returns on small-firm stocks are higher than those on large-firm stocks even after controlling for covariation with the market (beta) as usually measured. Using a multivariate test, Shanken [30] concludes that the CAPM is rejected when size is used to form portfolios. Put differently, the market portfolio is not on the minimum-variance frontier of size-sorted portfolios, contrary to the prediction of the CAPM.

The pattern of covariation of returns on size-sorted portfolios, documented by Huberman, Kandel, and Karolyi [17], leads us to inquire whether the frontier of the size-sorted portfolios is spanned by the three indices. If one accepts that restrictions (3) hold in our size-based example and if one accepts the size-based indices as mimicking portfolios in the context of the APT, then exact arbitrage pricing holds and the size effect is not a mispricing phenomenon. The evidence presented here, however, does not explain the size-related covariation pattern

Table II

Estimates of $r_i = a_i + b_{i1}R_1 + b_{i2}R_2 + b_{i3}R_3 + e_i$, where r_i is the Monthly Return on Size-Sorted Portfolio i and R_j is the Monthly Return on Size-Based Index j[a]

Dependent Variable	a	b_1	b_2	b_3	$b_1 + b_2 + b_3$	F	p	R^2
Part A: Estimation Period 1964 to 1973								
r_1	0.46	1.89	−0.81	−0.16	0.91	4.23	0.02	0.92
	0.19	0.15	0.28	0.20				
r_2	0.28	1.49	−0.45	−0.08	0.96	1.79	0.17	0.93
	0.16	0.12	0.24	0.17				
r_3	0.00	1.26	−0.10	−0.13	1.02	0.15	0.86	0.94
	0.15	0.12	0.23	0.16				
r_4	−0.15	1.24	−0.61	0.40	1.03	1.23	0.30	0.96
	0.12	0.09	0.18	0.12				
r_5	0.00	0.88	0.04	0.10	1.02	0.10	0.91	0.93
	0.16	0.12	0.24	0.17				
r_7	0.12	0.78	0.31	−0.07	1.01	0.49	0.62	0.95
	0.14	0.10	0.20	0.14				
r_8	−0.21	0.77	0.18	0.09	1.05	1.76	0.18	0.95
	0.14	0.10	0.20	0.14				
r_9	−0.14	0.65	0.51	−0.20	0.96	1.57	0.21	0.95
	0.12	0.09	0.18	0.13				
r_{10}	−0.09	0.68	0.36	0.00	1.04	0.68	0.51	0.95
	0.13	0.10	0.20	0.14				
r_{11}	−0.03	0.32	0.67	0.03	1.02	0.17	0.85	0.95
	0.12	0.09	0.18	0.13				
r_{12}	0.21	0.03	1.31	−0.37	0.97	1.55	0.22	0.95
	0.13	0.10	0.19	0.14				
r_{13}	0.06	0.03	1.26	−0.35	0.95	1.00	0.37	0.94
	0.13	0.10	0.19	0.14				
r_{14}	−0.05	0.09	0.99	−0.07	1.01	0.12	0.89	0.95
	0.12	0.09	0.17	0.12				
r_{15}	0.01	−0.01	1.05	−0.07	0.97	0.37	0.69	0.94
	0.12	0.10	0.19	0.13				

Table II (cont.)

Dependent Variable	a	b_1	b_2	b_3	$b_1 + b_2 + b_3$	F	p	R^2
Part A: Estimation Period 1964 to 1973								
r_{16}	−0.26	0.08	0.93	0.00	1.00	3.43	0.04	0.96
	0.10	0.08	0.15	0.11				
r_{18}	−0.12	0.14	0.71	0.15	1.00	0.56	0.57	0.95
	0.12	0.09	0.18	0.13				
r_{19}	0.03	0.07	0.61	0.36	1.04	0.91	0.41	0.95
	0.11	0.09	0.17	0.12				
r_{20}	−0.25	−0.19	1.28	−0.08	1.01	2.26	0.11	0.95
	0.12	0.09	0.18	0.12				
r_{21}	0.17	−0.18	1.15	0.01	0.98	1.47	0.23	0.96
	0.10	0.08	0.16	0.11				
r_{22}	0.05	−0.34	1.07	0.34	1.08	4.26	0.02	0.96
	0.10	0.08	0.15	0.11				
r_{23}	−0.06	0.03	0.35	0.72	1.10	4.35	0.02	0.94
	0.12	0.09	0.18	0.13				
r_{24}	0.08	−0.05	0.44	0.68	1.07	3.89	0.02	0.96
	0.10	0.08	0.15	0.11				
r_{25}	−0.07	0.11	−0.06	1.03	1.07	2.71	0.07	0.93
	0.12	0.09	0.18	0.13				
r_{26}	0.05	−0.01	0.43	0.59	1.00	0.13	0.88	0.95
	0.10	0.08	0.16	0.11				
r_{27}	0.07	−0.15	0.27	0.85	0.97	0.74	0.48	0.94
	0.10	0.08	0.16	0.11				
r_{29}	0.03	0.03	−0.13	1.07	0.97	0.47	0.63	0.94
	0.10	0.08	0.15	0.11				
r_{30}	−0.08	0.00	−0.19	1.17	0.98	0.85	0.43	0.95
	0.09	0.07	0.14	0.10				
r_{31}	0.01	−0.06	−0.07	1.04	0.90	8.80	0.00	0.94
	0.09	0.07	0.14	0.10				
r_{32}	−0.08	0.04	−0.50	1.45	0.99	0.45	0.64	0.94
	0.09	0.07	0.14	0.10				
r_{33}	0.24	0.04	−0.54	1.34	0.83	14.09	0.00	0.87
	0.12	0.09	0.18	0.13				

Part B: Estimation Period 1974 to 1983								
r_1	−0.22	2.45	−1.54	0.12	1.03	0.45	0.64	0.94
	0.26	0.15	0.33	0.21				
r_2	−0.16	1.68	−0.83	0.16	1.01	0.46	0.63	0.96
	0.17	0.10	0.22	0.14				
r_3	−0.05	1.34	−0.41	0.04	0.96	0.67	0.51	0.95
	0.18	0.10	0.23	0.15				
r_4	0.14	1.03	0.01	−0.07	0.97	0.57	0.56	0.95
	0.17	0.09	0.21	0.14				
r_5	0.26	0.93	−0.01	0.06	0.98	1.25	0.29	0.95
	0.17	0.09	0.21	0.14				
r_7	0.02	0.61	0.45	−0.02	1.05	1.32	0.27	0.95
	0.15	0.09	0.20	0.12				
r_8	0.07	0.62	0.39	0.00	1.01	0.25	0.78	0.96
	0.14	0.08	0.18	0.11				
r_9	0.04	0.70	0.31	−0.02	0.99	0.09	0.91	0.95
	0.15	0.08	0.19	0.12				
r_{10}	−0.02	0.25	1.00	−0.27	0.98	0.33	0.72	0.94
	0.16	0.09	0.20	0.13				
r_{11}	0.04	0.37	0.80	−0.19	0.98	0.30	0.74	0.96
	0.13	0.07	0.17	0.11				
r_{12}	0.01	0.11	0.99	−0.10	1.01	0.03	0.97	0.94
	0.14	0.08	0.18	0.12				
r_{13}	−0.19	0.11	1.16	−0.21	1.06	3.28	0.04	0.96
	0.13	0.07	0.16	0.10				
r_{14}	−0.05	−0.12	1.47	−0.32	1.03	0.73	0.49	0.95
	0.13	0.07	0.17	0.11				
r_{15}	0.14	0.17	0.79	0.07	1.04	1.99	0.14	0.96
	0.13	0.07	0.16	0.10				
r_{16}	−0.09	−0.08	1.36	−0.34	0.94	3.56	0.03	0.95
	0.12	0.07	0.16	0.10				
r_{18}	0.12	0.04	1.00	−0.10	0.94	3.21	0.04	0.95
	0.12	0.07	0.16	0.10				
r_{19}	−0.11	0.04	0.81	0.08	0.93	5.49	0.01	0.95
	0.12	0.07	0.15	0.10				
r_{20}	0.25	−0.14	1.01	0.14	1.01	2.70	0.07	0.96

Table II (cont.)

Dependent Variable	a	b_1	b_2	b_3	$b_1 + b_2 + b_3$	F	p	R^2
				Part B: Estimation Period 1974 to 1983				
	0.12	0.06	0.15	0.09				
r_{21}	−0.07	−0.09	0.88	0.20	0.99	0.39	0.67	0.96
	0.11	0.06	0.14	0.09				
r_{22}	−0.01	−0.07	0.59	0.57	1.08	6.15	0.00	0.95
	0.13	0.07	0.16	0.10				
r_{23}	−0.13	0.03	0.60	0.38	1.01	0.64	0.53	0.96
	0.11	0.06	0.14	0.09				
r_{24}	0.05	0.04	0.30	0.71	1.05	2.69	0.07	0.95
	0.12	0.07	0.15	0.10				
r_{25}	0.21	0.05	0.20	0.77	1.02	2.30	0.11	0.95
	0.11	0.06	0.14	0.09				
r_{26}	0.05	0.12	0.04	0.83	0.99	0.12	0.89	0.96
	0.11	0.06	0.14	0.09				
r_{27}	0.10	0.05	−0.06	1.02	1.01	0.90	0.41	0.96
	0.10	0.05	0.12	0.08				
r_{29}	−0.11	0.05	−0.05	0.98	0.98	1.48	0.23	0.97
	0.09	0.05	0.12	0.07				
r_{30}	0.08	−0.17	0.03	1.12	0.99	0.32	0.73	0.95
	0.11	0.06	0.14	0.09				
r_{31}	−0.06	0.01	−0.21	1.16	0.96	2.43	0.09	0.95
	0.10	0.06	0.13	0.08				
r_{32}	−0.15	−0.25	−0.09	1.37	1.04	2.38	0.10	0.96
	0.10	0.06	0.13	0.08				
r_{33}	0.06	0.01	−0.77	1.71	0.94	1.68	0.19	0.89
	0.16	0.09	0.21	0.13				

[a] F is the test statistic that $a = 0$ and $b_1 + b_2 + b_3 = 1$; p is the p-value corresponding to that F-statistic. Standard errors are below the point estimates of the coefficients.

reported by Huberman, Kandel, and Karolyi [17], and it remains for future research to study it closely.

V. Conclusion

This paper puts together results about mean-variance efficiency, exact arbitrage pricing, and K-fund separation. These results imply restrictions on an underlying return-generating process. When the restrictions are linear, the small-sample distributions of some multivariate tests are available.

Most of the empirical research on asset pricing has consisted of tests of mean-variance efficiency. In the absence of a risk-free asset, the restrictions are not linear and the small-sample distribution of the test statistic can only be approximated. The mean-variance spanning test proposed here is a test of a stronger hypothesis than exact pricing, but its statistic is easily computed and the statistic's small-sample distribution is available.

The multivariate test of spanning is applied to the hypothesis that the monthly returns on three size-based indices of NYSE stocks span the minimum-variance frontier of the monthly returns on thirty-three size-sorted portfolios. When twenty years of data are considered, the data do not support the spanning hypothesis. It is possible, however, that the spanning hypothesis is correct but that temporal instability of the coefficients of the underlying return-generating model accounts for the high values of the test statistic when twenty years of data are used. When ten or five years are used to construct the test statistic or when subperiod results are aggregated, the data seem to support the hypothesis.

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