

## MULTIVARIATE TESTS OF FINANCIAL MODELS A New Approach\*

Michael R. Gibbons

*Stanford University, Stanford, CA 94305, USA*

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A variety of financial models are cast as nonlinear parameter restrictions on multivariate regression models, and the framework seems well suited for empirical purposes. Aside from eliminating the errors-in-the-variables problem which has plagued a number of past studies, the suggested methodology increases the precision of estimated risk premiums by as much as 76%. In addition, the approach leads naturally to a likelihood ratio test of the parameter restrictions as a test for a financial model. This testing framework has considerable power over past test statistics. With no additional variable beyond  $\beta$ , the substantive content of the CAPM is rejected for the period 1926–1975 with a significance level less than 0.001.

### 1. Introduction

A fundamental task for financial economists is characterizing equilibrium relationships between the perceived risks of a financial security and its potential rewards. A feature common to many theories is that the reward or expected return is a linear function of the risks. The risk is usually taken to be a measure of covariability between the asset's return and an appropriately defined hedging portfolio. Thus, a wide class of financial models are of the form:

$$E(\tilde{R}_{it}) = \gamma_0 + \sum_{j=1}^K \beta_{ij} \gamma_j, \quad i = 1, \dots, N,$$

where

$E(\tilde{R}_{it})$  = expected return on the  $i$ th security in period  $t$ ,

$\gamma_0$  = risk-free rate (if it exists) or the expected return on a 'zero-beta' portfolio,

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- $\beta_{ij}$  = a measure (as defined by a particular model) of association between the returns on security  $i$  and the returns on a portfolio designed to hedge risk  $j$ ,
- $\gamma_j$  = premium for risk  $j$  (the nature of the risk is intentionally vague, for this is defined in the context of the particular financial model),
- $N$  = number of securities, and
- $K$  = number of risks.

Examples of such expected return–risk relationships include the Sharpe (1964) — Lintner (1965b) capital asset pricing model (hereafter, CAPM), the Black (1972) CAPM, the arbitrage pricing theory due to Ross (1975, 1976), the Merton (1973) intertemporal asset pricing model, and an extended CAPM which incorporates the effect of dividends [Brennan (1970)].

The empirical investigations into these theoretical models have common features as well. Typically, the above class of financial models is studied through cross-sectional regression methods which relate realized returns on different financial securities to various measures of risks. Since the  $\beta_{ij}$ 's usually are not observable, such methods rely on proxies or estimates which necessarily contain measurement error. While cross-sectional regression methods are modified to account for the measurement errors,<sup>1</sup> there have been few attempts to abandon the basic methodology for another. This paper begins with a development of an alternative conceptual framework.

The potential payoffs from this new methodology seem large. In particular, the methods suggested here not only avoid the errors-in-the-variables problem, but the approach also increases the precision of parameter estimates for the risk premiums (i.e., the  $\gamma$ 's). In addition, the framework lends itself to a likelihood ratio test of the parameter restrictions implied by a particular financial model, and the test statistic seems to have sufficient power to reject some models. The means by which these ends are achieved are through the nonlinear multivariate regression model.

Despite the generality of the title, the bulk of this paper focuses on the CAPM for several reasons. First, new methodology should not only provide theoretical superiority relative to commonly used procedures, but its practical applications should be demonstrated. Second, the new techniques should be applied to an important model, and the CAPM has a long history of theoretical and empirical investigations. Finally, the development of the specific econometric tools for a particular application helps to illuminate the econometric theory being employed.

<sup>1</sup>The initial work on the CAPM is performed by Lintner (1965a) and Douglas (1969). In a detailed study Miller and Scholes (1972) criticize their results due to econometric flaws. Both Black, Jensen, and Scholes (1972) and Fama and MacBeth (1973) contribute to the literature by correcting the difficulties through the grouping procedure. Black and Scholes (1974) extend the method to a financial model which explicitly incorporates dividends, and recently Litzenberger and Ramaswamy (1979) suggest an errors-in-the-variables regression model as an alternative to the grouping procedure.

Sections 2 and 3 focus on the CAPM by first viewing the model from a slightly different viewpoint and then developing the relevant econometric tools. In section 4 the method is applied to a large sample of stock returns, and the appropriateness of asymptotic statistical theory for a finite sample size is confirmed through simulations reported in section 5. In section 6 some additional thoughts on the CAPM are provided. Finally, section 7 demonstrates how the approach can be generalized to a wide variety of financial theories.

## 2. Developing the CAPM hypothesis

For this study the ‘market model’ is assumed to be well specified:

$$\tilde{R}_{it} = \alpha_i + \beta_i \tilde{R}_{mt} + \tilde{\eta}_{it}, \quad i = 1, \dots, N, \quad t = 1, \dots, T, \quad (1)$$

where

$\tilde{R}_{it}$  = return on asset  $i$  in period  $t$ ,

$\tilde{R}_{mt}$  = return on the market portfolio in period  $t$ ,

$\beta_i$  =  $\text{cov}(\tilde{R}_{it}, \tilde{R}_{mt})/\text{var}(\tilde{R}_{mt})$ , and

$\tilde{\eta}_{it}$  = a random disturbance with the following stochastic properties:

$$E(\tilde{\eta}_{it}) = 0,$$

$$E(\tilde{\eta}_{is}\tilde{\eta}_{jt}) = \sigma_{ij} \quad \text{for all } s=t \text{ and for all } i \text{ and } j, \\ = 0 \quad \text{otherwise.}$$

Assuming asset returns are stationary with a multivariate normal distribution and serially uncorrelated is sufficient to justify (1) [Fama (1973)]. Evidence of the approximate normality of *monthly* returns is given by Blattberg and Gonedes (1974) and Fama (1976).

Eq. (1) is a statistical statement rather than one derived from financial theory. Eq. (1) implies

$$E(\tilde{R}_{it}) = \alpha_i + \beta_i E(\tilde{R}_{mt}). \quad (2)$$

The Black CAPM requires the following expected return–risk relationship across assets:

$$E(\tilde{R}_{it}) = \gamma + \beta_i [E(\tilde{R}_{mt}) - \gamma], \quad (3)$$

where

$\gamma \equiv$  expected return on the ‘zero-beta’ portfolio or any portfolio whose return is uncorrelated with the return on the market portfolio,  $m$ .

In terms of (2), the Black model implies the following constraint on the intercept of the market model:

$$\alpha_i = \gamma(1 - \beta_i) \quad \text{for all } i = 1, \dots, N, \quad (4)$$

which is the basis of the subsequent tests of the CAPM. Thus, the formal hypothesis becomes

$$\begin{aligned} H_0: \alpha &= \gamma(\mathbf{1}_N - \beta), \\ H_A: \alpha &\neq \gamma(\mathbf{1}_N - \beta), \end{aligned} \quad (5)$$

where

$$\alpha' \equiv (\alpha_1, \alpha_2, \dots, \alpha_N) \quad (1 \times N \text{ vector}),$$

$$\mathbf{1}'_N \equiv (1, 1, \dots, 1) \quad (1 \times N \text{ vector of ones}), \quad \text{and}$$

$$\beta' \equiv (\beta_1, \beta_2, \dots, \beta_N) \quad (1 \times N \text{ vector}).$$

The CAPM places a nonlinear restriction on a system of  $N$  regression equations.

Relative to previous research, testing (5) represents a different and perhaps a more powerful check on the CAPM. Much of the past work assumes (4) is true and uses the relation to derive a point estimate for  $\gamma$ . This point estimate is then compared with the return on the market portfolio and the risk-free rate in order to generate univariate tests of the CAPM. This study examines the CAPM by testing the restrictions across securities.<sup>2</sup>

In testing the CAPM Fama and Macbeth (1973) rewrite (3) so that it includes  $\beta_i^2$  term as well as the variance of  $\tilde{\eta}_{it}$  (i.e.,  $\sigma_{ii}$ ); that is,

$$E(\tilde{R}_{it}) = \gamma_1 + \gamma_2\beta_i + \gamma_3\beta_i^2 + \gamma_4\sigma_{ii}.$$

They then examine the significance of estimates of  $\gamma_3$  and  $\gamma_4$  as a way of verifying the CAPM. In the multivariate framework, the null hypothesis given in (5) is rejected if  $\gamma_3$  or  $\gamma_4$  differs from zero. For example, if  $\gamma_4$  is not equal to zero and if the two securities have the same  $\beta_i$  but different values for  $\sigma_{ii}$ , then the intercepts differ which is inconsistent with the null hypothesis and which should result in a rejection. Thus, in a multivariate framework  $\beta_i^2$  and  $\sigma_{ii}$  need not be included in (3) to test for the linearity in beta.

<sup>2</sup>The reduction of the parameter space under the null hypothesis may be viewed as the content of the CAPM. That is, given  $\gamma$  and  $\beta$ , one can infer  $\alpha$  without estimating each  $\alpha_i$  individually.

### 3. Developing the econometrics for the CAPM hypothesis

The relevant econometric analysis is presented in this section for testing the CAPM conditional on the statistical model given in (1). In terms of the notation of the previous section,

$$\tilde{\mathbf{R}}_i = \alpha_i \mathbf{1}_T + \beta_i \tilde{\mathbf{R}}_m + \tilde{\boldsymbol{\eta}}_i, \quad i = 1, \dots, N, \quad (6)$$

where

$$\tilde{\mathbf{R}}'_i = (\tilde{R}_{i1}, \tilde{R}_{i2}, \dots, \tilde{R}_{iT}) \quad (1 \times T \text{ vector}),$$

$$\mathbf{1}'_T = (1, 1, \dots, 1) \quad (1 \times T \text{ vector of ones}),$$

$$\tilde{\mathbf{R}}'_m = (\tilde{R}_{m1}, \tilde{R}_{m2}, \dots, \tilde{R}_{mT}) \quad (1 \times T \text{ vector}),$$

$$\tilde{\boldsymbol{\eta}}'_i = (\tilde{\eta}_{i1}, \tilde{\eta}_{i2}, \dots, \tilde{\eta}_{iT}) \quad (1 \times T \text{ vector}), \quad \text{and}$$

$$\tilde{\boldsymbol{\eta}}_i \sim \text{MVN}(\mathbf{0}; \sigma_{ii} \mathbf{I}_T).$$

The hypothesis is

$$H_0: \alpha_i = \gamma(1 - \beta_i), \quad i = 1, \dots, N$$

(i.e., CAPM is consistent with the data),

$$H_A: \alpha_i \neq \gamma(1 - \beta_i), \quad i = 1, \dots, N$$

(i.e., CAPM is not consistent with the data).

Eq. (6) can be viewed as either a system of  $N$  regression equations, or it can be 'stacked' into a single regression of the following form:

$$\begin{bmatrix} \tilde{\mathbf{R}}_1 \\ \tilde{\mathbf{R}}_2 \\ \vdots \\ \tilde{\mathbf{R}}_N \end{bmatrix} = \begin{bmatrix} (\mathbf{1}_T : \tilde{\mathbf{R}}_m) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & (\mathbf{1}_T : \tilde{\mathbf{R}}_m) & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & (\mathbf{1}_T : \tilde{\mathbf{R}}_m) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \\ \vdots \\ \alpha_N \\ \beta_N \end{bmatrix} + \begin{bmatrix} \tilde{\boldsymbol{\eta}}_1 \\ \tilde{\boldsymbol{\eta}}_2 \\ \vdots \\ \tilde{\boldsymbol{\eta}}_N \end{bmatrix}, \quad (7)$$

or

$$\bar{\mathbf{R}}^* = [(\mathbf{1}_T; \bar{\mathbf{R}}_m) \otimes I_N] \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \vdots \\ \alpha_N \\ \beta_N \end{bmatrix} + \tilde{\boldsymbol{\eta}}^*,$$

where

$I_N = N \times N$  identity matrix,

$\mathbf{0} = T \times 2$  matrix of zeroes,

$\bar{\mathbf{R}}^{*'} = (\bar{\mathbf{R}}_1, \bar{\mathbf{R}}_2, \dots, \bar{\mathbf{R}}_N)$  ( $1 \times NT$  vector),

$\tilde{\boldsymbol{\eta}}^{*'} = (\tilde{\boldsymbol{\eta}}'_1, \tilde{\boldsymbol{\eta}}'_2, \dots, \tilde{\boldsymbol{\eta}}'_N)$  ( $1 \times NT$  vector), and

$\otimes$  indicates a Kronecker or direct product operator.

I assume that

$$\begin{aligned} E\{\tilde{\boldsymbol{\eta}}_i \tilde{\boldsymbol{\eta}}'_j\} &= \sigma_{ii} I_T \quad \text{for all } i=j, \\ &= \sigma_{ij} I_T \quad \text{for all } i \neq j, \end{aligned}$$

$$E\{\tilde{\boldsymbol{\eta}}^* \tilde{\boldsymbol{\eta}}^{*'}\} = \Sigma \otimes I_T,$$

where

$\Sigma = N \times N$  contemporaneous covariance matrix with typical element equals  $\sigma_{ij}$ ;  
 $\Sigma$  is assumed to be positive definite symmetric, and

$$\tilde{\boldsymbol{\eta}}^* \sim \text{MVN}(\mathbf{0}; \Sigma \otimes I_T).$$

The arrangement in (7) is used in a seemingly unrelated regression model due to Zellner (1962). Since the set of explanatory variables is identical across the equations, this formulation can be specialized to a multivariate regression model (hereafter, MVRM). With identical regressors across equations and under the alternative hypothesis, ordinary least squares (hereafter, OLS) on each equation like (6) is efficient [Zellner (1962)]. However, under the null hypothesis given in (5), OLS is inefficient relative to a nonlinear MVRM estimation procedure — thanks to the reduction of the parameter space from  $2N$  (under the alternative hypothesis) to  $N+1$  (under the null hypothesis).

Assuming returns have a multivariate normal distribution, the specification of the likelihood function is straightforward under the null hypothesis. To find the

maximum likelihood estimators, first- and second-order conditions for the maximum of the likelihood function need to be determined. Since the actual estimation technique employed later in this paper does not revolve around these conditions, the details of this exercise are not provided here.<sup>3</sup> However, the intuition underlying the procedure is useful. Satisfying the first-order requirements can be likened to iterating back and forth between two sets of regressions with the following algorithm:

- (1) Use OLS equation by equation (with  $T$  observations per equation) where the typical equation is given by (6). This provides the initial estimates for  $\alpha_i$  and  $\beta_i$  ( $i = 1, \dots, N$ ).
- (2) Use the residuals from these  $N$  regression equations to construct an estimate of the contemporaneous covariance matrix,  $\Sigma$ . In order to avoid a singular covariance matrix,  $T$  must be greater than  $N$ .
- (3) Do a generalized least squares version of the Black, Jensen, and Scholes (1972, pp. 100–112) time series estimator using raw returns, not excess returns. That is, compute

$$\hat{\gamma} = \frac{\hat{\alpha}' \hat{\Sigma}^{-1} (\mathbf{t}_N - \hat{\beta})}{(\mathbf{t}_N - \hat{\beta})' \hat{\Sigma}^{-1} (\mathbf{t}_N - \hat{\beta})}, \quad (8)$$

where

$$\begin{aligned} \hat{\alpha}' &= \bar{\mathbf{R}}' - \hat{\beta}' \bar{\mathbf{R}}_m && (1 \times N \text{ vector}), \\ \hat{\beta}' &= (\hat{\beta}_1, \dots, \hat{\beta}_N) && (1 \times N \text{ vector}), \\ \mathbf{t}'_N &= (1, \dots, 1) && (1 \times N \text{ vector of ones}), \\ \bar{\mathbf{R}}_m &= T^{-1} \sum_{t=1}^T R_{mt}, \text{ and} \\ \bar{\mathbf{R}}' &= (T^{-1} \sum_t R_{1t}, T^{-1} \sum_t R_{2t}, \dots, T^{-1} \sum_t R_{Nt}) && (1 \times N \text{ vector}). \end{aligned}$$

- (4) Using  $\hat{\gamma}$  from step 3, perform ‘market model’ regressions ( $N$  of them) with excess returns and without a constant term. That is,

$$(\bar{\mathbf{R}}_i - \hat{\gamma} \mathbf{t}_T) = \hat{\beta}_i (\mathbf{R}_m - \hat{\gamma} \mathbf{t}_T) + \tilde{\varepsilon}_i, \quad i = 1, \dots, N. \quad (9)$$

- (5) With this new set of  $\beta_i$  estimates,  $\hat{\beta}$ , repeat steps 3 and 4. One may iterate back and forth between (8) and (9) until satisfactory convergence is achieved. The estimate of the contemporaneous covariance matrix may also be updated if desired using the residuals from (9).

<sup>3</sup>The interested reader may see Gibbons (1980a).

The above algorithm is only suggestive; the procedure is not desirable due to computational considerations. The five steps demonstrate that the suggested approach is an extension and combination of two procedures outlined by Black, Jensen, and Scholes (1972). Since (8) and the least squares estimator associated with (9) are linear combinations of excess returns, a portfolio interpretation of the estimation scheme would also be possible by examining (8) and (9) at each stage of the iteration.

The actual calculations in this paper use a one-step Gauss–Newton procedure<sup>4</sup> which linearizes the restriction in (4) using a Taylor series expansion about consistent estimators. After the linearization, the problem may be formulated as a general linear hypothesis for a seemingly unrelated regression model. This case is discussed by Theil (1971, pp. 312–317), and the appendix details the approach. A one-step Gauss–Newton method based on consistent estimators has the same asymptotic properties as the maximum likelihood estimator.<sup>5</sup> The one-step approach has the further advantage of easy implementation on many computer software packages without additional programming.

The statistical properties of this estimation scheme are appealing. In contrast to some past approaches, errors-in-the-variables is not a problem since  $\gamma$  and  $\beta$  are estimated simultaneously. The proposed estimator is consistent and asymptotically efficient with an asymptotic normal distribution. The precision of the estimator is improved since a full contemporaneous covariance matrix for the  $\tilde{\eta}_{it}$ 's is incorporated.

At this point two sets of estimators for  $\alpha$  and  $\beta$  exist; one set is restricted by the null hypothesis while the other is estimated under the alternative hypothesis. The object now is to test the null hypothesis. Though several test statistics are available, the statistic employed is a likelihood ratio test (hereafter, LRT) which compares the statistical 'fit' of the unrestricted model with that of the restricted model. If the 'fit' under the null hypothesis is 'close' to that under the alternate, the null hypothesis is not rejected. The measure of the 'fit' is given by the generalized variance.

The appropriate LRT statistic has the form

$$-2 \ln \lambda = T[\ln |\hat{\Sigma}_r| - \ln |\hat{\Sigma}_u|], \quad (10)$$

where

$|\hat{\Sigma}_r|$  = the determinant of the contemporaneous variance–covariance matrix estimated from the residuals of the restricted system, and

$|\hat{\Sigma}_u|$  = the determinant of the contemporaneous variance–covariance matrix estimated from the residuals of the unrestricted system.

<sup>4</sup>I am grateful to Arnold Zellner for suggesting this alternative to me.

<sup>5</sup>The asymptotic properties of a one-step Gauss–Newton method are discussed in detail in Gibbons (1980a, ch. 2). The asymptotic characteristics follow from results reported in Fuller (1976) and Gallant (1975).



It is well known that asymptotically

$$-2 \ln \lambda \sim \chi_{N-1}^2. \quad (11)$$

Eqs. (10) and (11) form the basis of tests of (5).

#### 4. Empirical results

An estimate of the expected return on the 'zero-beta' portfolio and an LRT are computed for ten different five-year subperiods using monthly returns. While the joint distribution of returns is not constant for the whole twentieth century [Fama (1961, p. 132), Gonedes (1973)], Gonedes (1973) and Officer (1972) could not reject the hypothesis that the parameters of (1) are stationary for five to ten years. A first-order Taylor series expansion of  $\alpha_i = \gamma(1 - \beta_i)$  is performed about the consistent estimators; the market model equations are then estimated subject to the linearized restriction. The initial estimator for  $\beta_i$  is unrestricted OLS while  $\gamma$  is estimated by the approach suggested in Black, Jensen, and Scholes (1972). This linearization is developed in detail in the appendix. The estimator derived from this one-step Gauss-Newton procedure is labeled  $\hat{\gamma}^*$ .

The estimation uses monthly stock returns as provided by the Center for Research in Security Prices (hereafter, CRSP). The return on the CRSP equal-weighted index serves as the return on the market portfolio,  $R_{mt}$ .<sup>6</sup> The time period of 1926-75 is divided into ten equal five-year subperiods. Any security for a particular subperiod meets the following criteria:

- (a) it is listed (without any missing returns) continuously for that five-year period,
- (b) its two-digit industry code (SEC) is available from the beginning of the subperiod, and this two-digit code does not change throughout the subperiod, and
- (c) for the last three subperiods (1961-65, 1966-70, 1971-75), the industry codes need only be available at the beginning of 1962.

Criterion (c) is included due to an error in the CRSP data that is being corrected. For the last three subperiods the industry code as of 1962 is assumed constant for these last thirteen years. Any new listing after 1962 is assigned its first

<sup>6</sup>Since the composition of the true market portfolio is unknown, a proxy is used. The following calculations actually investigate the mean-variance efficiency of the market proxy employed. However, using various econometric methods (including the procedure suggested in this paper), Stambaugh (1978) provides evidence that tests of the CAPM are not sensitive to the choice of the proxy.

code for the rest of the period. Criterion (b) is employed to decrease potential parameter shifts due to changes in real production activity.

The above criteria are not stringent enough to reduce the number of assets on the left-hand side (hereafter, LHS) of the market model equations to manageable numbers. With sixty observations per equation and a full contemporaneous covariance matrix for the disturbances across equations, only fifty-eight equations (or 'LHS assets') are feasible; otherwise, there are more parameters than observations, and the covariance matrix becomes singular. Several solutions are available. A subset of securities may be selected, or the covariance matrix may be structured to reduce the number of parameters.<sup>7</sup> The former alternative is selected for this research.

The strategy is to estimate  $\beta_i$  (using sixty months of data) for all securities meeting the above criteria. With these estimates forty groups, each with the same number of securities, are formed ranging from low to high  $\beta_i$  values. An equally weighted portfolio is formed using all securities in a particular class. These forty portfolios are the 'LHS assets' within this *same* five-year period. Since the MVRM approach does not use  $\beta_i$  as an explanatory variable in a regression model, there is no obvious bias<sup>8</sup> from the selection of securities stratified by  $\beta_i$  estimated from the same subperiod in which the test is performed. Such a stratified selection has the advantage of increasing the dispersion of  $\beta_i$  values and making a more powerful test. Of course, the MVRM technique does not require this kind of sample selection; the portfolios formed by Fama and Macbeth (1973) could also be used.

Table 1 summarizes the results for the ten five-year subperiods using the forty portfolios. In the last two columns the LRT of the CAPM hypothesis given in (5) is reported; the parameter restriction implied by the CAPM is rejected at reasonable significance levels in five out of ten subperiods. In the remaining five subperiods, the test statistic is marginally insignificant for three subperiods. Since the sum of independent chi-square statistics is also chi-square, the evidence from the ten subperiods is summarized in an overall statistic provided in the last row of table 1. This value supplies further negative evidence against the restriction.

Thus, even within a limited class of LHS assets and an unspecified alternative hypothesis, table 1 rejects the mean-variance efficiency of the equally weighted NYSE portfolio. To the extent that the CRSP equal-weighted index is an adequate proxy for the market portfolio, this test rejects a special case of the Sharpe-Lintner model (where the risk-free rate is constant), the Black model, and variants of the Black model which conclude that the market portfolio is mean-variance efficient. Furthermore, the evidence is inconsistent with some special cases of the Ross arbitrage pricing theory.

<sup>7</sup>For example, the contemporaneous covariance across market model disturbances may be set to zero unless two securities are in the same industry. Some evidence on the importance of the industry factor is provided by King (1965).

<sup>8</sup>This conclusion is confirmed by both simulation evidence and alternative sorting techniques; details of this investigation are in Gibbons (1980a).

Table 1

Estimates of the expected return on the zero-beta portfolio using the Black, Jensen, and Scholes (1972) estimator ( $\hat{\gamma}_{BJS}$ ) and using a one-step Gauss-Newton estimator ( $\hat{\gamma}^*$ ). Likelihood ratio test (LRT) of the parameter restriction implied by the CAPM. Each subperiod uses 40 equal-weighted portfolios of NYSE securities and the CRSP equal-weighted index as the market portfolio, 1926-1975.

| Time period                                       | $\hat{\gamma}_{BJS}$ | $SE(\hat{\gamma}_{BJS})^a$<br>(uncorrected) | $\hat{\gamma}^*$ | $SE(\hat{\gamma}^*)^b$<br>(unadjusted) | LRT    | $p$ -value <sup>c</sup><br>for LRT |
|---|----------------------|---|------------------|--|--------|------------------------------------|
| 1926/1-1930/12                                    | 0.0004               | 0.00440<br>(0.00440)                        | 0.0124           | 0.00129<br>(0.00127)                   | 75.06  | 0.000                              |
| 1931/1-1935/12                                    | 0.0024               | 0.00827<br>(0.00816)                        | 0.0067           | 0.00280<br>(0.00275)                   | 50.29  | 0.106                              |
| 1936/1-1940/12                                    | -0.0029              | 0.00950<br>(0.00945)                        | -0.0082          | 0.00233<br>(0.00229)                   | 92.50  | 0.000                              |
| 1941/1-1945/12                                    | 0.0085               | 0.00476<br>(0.00452)                        | 0.0115           | 0.00131<br>(0.00129)                   | 43.99  | 0.268                              |
| 1946/1-1950/12                                    | 0.0070               | 0.00351<br>(0.00351)                        | 0.0034           | 0.00134<br>(0.00131)                   | 87.46  | 0.000                              |
| 1951/1-1955/12                                    | 0.0121               | 0.00168<br>(0.00167)                        | 0.0097           | 0.00065<br>(0.00064)                   | 53.31  | 0.063                              |
| 1956/1-1960/12                                    | 0.0129               | 0.00218<br>(0.00216)                        | 0.0081           | 0.00098<br>(0.00096)                   | 69.83  | 0.002                              |
| 1961/1-1965/12                                    | 0.0070               | 0.00254<br>(0.00251)                        | 0.0068           | 0.00127<br>(0.00125)                   | 57.56  | 0.028                              |
| 1966/1-1970/12                                    | 0.0045               | 0.00379<br>(0.00379)                        | 0.0001           | 0.00132<br>(0.00130)                   | 47.01  | 0.177                              |
| 1971/1-1975/12                                    | 0.0096               | 0.00608<br>(0.00606)                        | 0.0061           | 0.00243<br>(0.00239)                   | 52.56  | 0.072                              |
| Overall LRT <sup>d</sup> (390 degrees of freedom) |                      |   |                  |  | 629.57 | 0.000                              |

<sup>a</sup> $SE(\hat{\gamma}_{BJS})$  is based upon the asymptotic distribution [derived in Gibbons (1980b)] while the uncorrected version is given by Black, Jensen, and Scholes (1972).

<sup>b</sup> $SE(\hat{\gamma}^*)$  is adjusted following the suggestion of Gallant (1975) based upon evidence from simulations.

<sup>c</sup>The  $p$ -value represents the probability of a realization greater than the LRT from a chi-square distribution with 39 degrees of freedom.

<sup>d</sup>The overall LRT is just a summation of the LRT for each subperiod. Since the LRT for each subperiod is chi-square with 39 degrees of freedom and independent across subperiods, the overall LRT has a chi-square distribution with 390 degrees of freedom.

Admittedly, the LRT conveys information more from a statistical viewpoint than an economic focus.<sup>9</sup> In an attempt to circumvent this criticism, fig. 1 graphs the departures of the data from the theoretical model while section 5 indicates the power of the LRT and provides some economic intuition about the power.

<sup>9</sup>In some preliminary work Ross (1980) transforms a special case of the above LRT (for the Sharpe-Lintner model) so that it measures the distance of the market proxy from the mean-variance frontier. Unfortunately, the extension to the more general case of the LRT is not trivial.

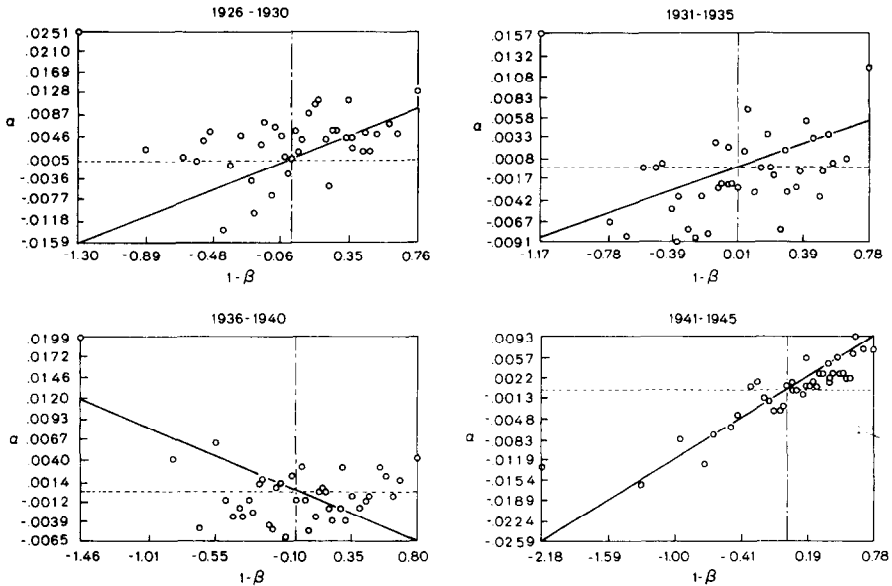


Fig. 1. Scatter plots of the coefficients from the unrestricted versus the restricted market model for ten non-overlapping subperiods from 1926–1975 with 40 portfolios in each subperiod. The slope of the solid straight line in each plot is given by the one-step Gauss-Newton estimate of the expected return from the zero-beta portfolio (*NOT* the ordinary least squares fit of the points), and all points should fall on this line if the CAPM is true. The 40 points on each scatter plot represent the least squares estimates of the market model coefficients when the CAPM restriction is not imposed upon the data. The vertical axis (labeled  $\hat{\alpha}$ ) is the intercept and the horizontal axis (labeled  $1-\hat{\beta}$ ) is one minus the slope coefficient in the market model regression given by  $\hat{R}_{it} = \alpha_i + \beta_i \hat{R}_{mt} + \hat{\eta}_{it}$ . (Note that the scale varies across the scatter plots.)

A scatter plot of unrestricted market model coefficients [i.e.,  $\hat{\alpha}_i$  and  $\hat{\beta}_i$  from (1)] against restricted market model coefficients [i.e.,  $\hat{\alpha}_i$  and  $\hat{\beta}_i$  from (1) subject to (4)] is provided in fig. 1 for each subperiod. These plots require a cautious interpretation, for unrestricted market model coefficients that are apparent ‘outliers’ from the restricted estimates may not represent important deviations from a generalized least squares viewpoint. For example, the appropriate scatter plot for 1931–35 in fig. 1 suggests that the high beta stock (see upper left corner of the plot) is an important deviation from the CAPM hypothesis. However, since the disturbance term of a high beta stock in a market model regression typically has a large variance, this security is not weighted as heavily in the LRT as a low beta stock.<sup>10</sup>

Fig. 1 does point to a potential explanation for the departure from the CAPM. There is some curvature in the scatter plots in that high beta stocks tend to fall

<sup>10</sup>Using a multiple comparisons approach, Gibbons (1980a) formally confirms that this high beta stock does not drive the LRT for this subperiod.

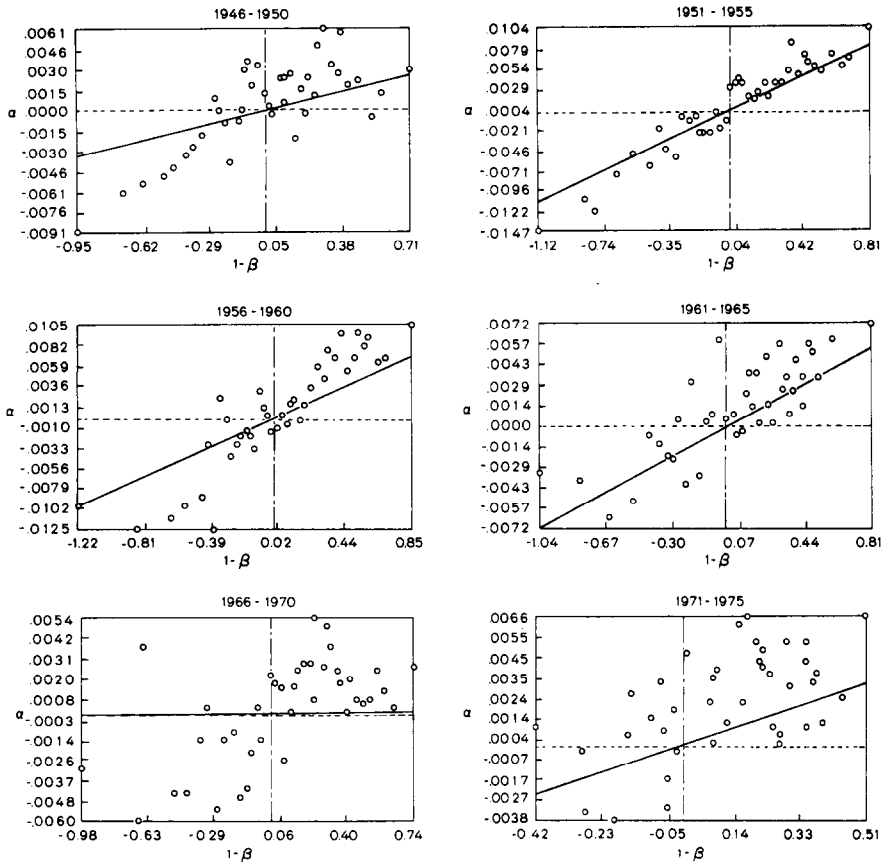


Fig. 1 (continued)

below the straight line while the reverse is true for low beta stocks. This pattern appears for 1946–50, 1951–55, 1956–60, and 1966–70. This curvature may be an artifact of the distinctive pattern in the covariance matrix of the least squares estimators for the market model coefficients (i.e., the correlation between the market model disturbances for high and low beta securities is negative while it is positive for securities with similar beta values). Another interpretation would be that prices of securities reflect risks other than the ‘beta-risk’, and these other risks are related to the level of beta. Future research needs to explore this latter interpretation.

The plots for 1926–30 and 1971–75 indicate that the CAPM tended to misprice all securities. The average returns for these two subperiods are too high according to the financial model.

Table 1 is also interesting since other econometric methods for financial models may be compared with the MVRM approach. One estimator for  $\gamma$  is given by Black, Jensen, and Scholes (1972). This estimator is labeled  $\hat{\gamma}_{\text{BJS}}$ , and an explicit expression is

$$\hat{\gamma}_{\text{BJS}} = \frac{\hat{\alpha}'(\mathbf{t}_N - \hat{\beta})}{(\mathbf{t}_N - \hat{\beta})'(\mathbf{t}_N - \hat{\beta})}$$

The second column of table 1 reports  $\hat{\gamma}_{\text{BJS}}$  for each subperiod while its standard error is in the third column. The uncorrected standard error for  $\hat{\gamma}_{\text{BJS}}$  is calculated following the suggestion of Black, Jensen, and Scholes (1972). Basing the standard error on the asymptotic distribution of  $\hat{\gamma}_{\text{BJS}}$  [derived in Gibbons (1980b)] results in a slightly higher value which is also reported in the third column.

Using the MVRM along with a one-step Gauss–Newton algorithm, the fourth and fifth columns of table 1 indicate point estimates of  $\gamma$  (labeled  $\hat{\gamma}^*$ ) and the associated standard errors. Not only are the asymptotic standard errors reported (see numbers in the parentheses in the fifth column), but some adjustments for finite sample sizes are also provided in the fifth column based on the simulation evidence reported by Gallant (1975)<sup>11</sup> and in section 5.

In all cases the standard errors for  $\hat{\gamma}^*$  are smaller than those for  $\hat{\gamma}_{\text{BJS}}$ . The reduction achieved by the MVRM ranges from 50 to 76 percent. If an investigator accepts the restrictions of the CAPM, then the MVRM technology provides an efficient estimator of the expected return on the zero-beta portfolio.

The MVRM may be used with either portfolios or individual securities. Computational considerations limit the number of assets, and portfolio formation provides one way (but not the only way) to decrease the dimensionality of the problem. However, given the strong rejection of the CAPM in table 1, the fears for the portfolio-based tests [Roll (1976, pp. 62–65), Ross (1978, p. 897)] appear groundless.<sup>12</sup>

Some researchers in financial economics may be surprised that the CAPM fits the data so poorly. Their surprise may stem either from a strong faith in the theory associated with the CAPM or, more likely, from a lack of faith in the power of previous empirical tests of the CAPM. Both Roll (1977, p. 155) and Ross (1978, p. 892), for example, express concern that the existing econometric tests of

<sup>11</sup>Gallant (1975, p. 43) suggests rescaling the variance of  $\hat{\gamma}^*$  by the factor  $T/(T+2)$  where  $T$  is the number of observations per equation and 2 reflects the number of coefficients estimated in each equation. Gallant also suggests the use of Student's  $t$  distribution with  $T-2$  degrees of freedom rather than a normal approximation. While Gallant's estimation technique is not a one-step Gauss–Newton procedure, simulation evidence reported in section 5 confirms the usefulness of Gallant's suggestion for this estimation technique as well.

<sup>12</sup>Gibbons (1980a, ch. 4) repeats the MVRM methodology over the ten subperiods reported in table 1 using 20 individual securities. The overall  $\chi^2$  for this sample also rejects the CAPM restriction. If anything, the sample of individual securities is more supportive of the CAPM than portfolio-based tests — counter to the intuition of the critics of grouping.

the CAPM have little power to reject the model. The proposed methodology, however, seems to have sufficient power.

Furthermore, the statistical assumptions underlying the LRT are reasonable. According to conventional wisdom among empiricists in finance, the market model equation given by (1) is well specified for at least a five-year period, and monthly data does conform reasonably well to a normal distribution. In addition, most empiricists regard a market model using a *portfolio* return as the dependent variable as an even better specification. Nevertheless, various diagnostic tests are performed on (1) for all the regressions used in table 1; these results are reported in Gibbons (1980a). This diagnostic examination searched for non-stationarity of regression coefficients, autocorrelated and/or heteroscedastic disturbances, and non-normal returns. Since the diagnostic fail to uncover any apparent problems, the details of this exercise are not reported here.

## 5. Simulation evidence

Since the statistical properties of the suggested methodology are only determined for large samples, a question naturally arises as to the appropriate interpretation of the above results based on a finite sample size. An extensive simulation study is used to investigate this issue, and a detailed report of the results is provided in Gibbons (1980a, app. A). What follows is a condensed version of that appendix. The focus is on the small sample distributions of the one-step Gauss–Newton estimator (i.e.,  $\hat{\gamma}^*$ ) and the corresponding likelihood ratio statistic.

The Monte Carlo experiments are based on equation system (6) subject to (4) using five equations (i.e.,  $N = 5$ ). Relying on historical evidence from common stock returns to fix the parameters, table 2 reports the values for the six experiments. Each experiment involves 250 replications.<sup>13</sup> The contemporaneous covariance matrix for the market model disturbances is given in table 3. While the numbers in table 3 are arbitrary, other experiments reported in Gibbons (1980a) indicate the results are not sensitive to the actual values selected. The parameter values in tables 2 and 3 along with the normal random number generator from the International Math and Statistical Library provide the data for the simulation.

Table 4 characterizes the small sample distribution of  $\hat{\gamma}^*$ . The second column (labeled  $\bar{\hat{\gamma}}^* + 10^2$ ) of table 4 provides the mean of  $\hat{\gamma}^*$  across all replications for a particular experiment. This estimator of  $\gamma$  is not biased, for the average value of  $\hat{\gamma}^*$  does not systematically deviate from the true value of  $\gamma$  given in parentheses in the second column of table 4. The third column [i.e.,  $\hat{\sigma}^2(\hat{\gamma}^*) \times 10^2$ ] compares the variance of the asymptotic distribution with the sample variance of  $\hat{\gamma}^*$  across all

<sup>13</sup>For each experiment only one time series for  $R_{m_t}$  is generated; the replications create new market model disturbances conditional on the  $R_{m_t}$  values.

Table 2

Parameter input values for each simulation experiment. (The covariance matrix input for the market model disturbances is given in table 3.) In all experiments 250 replications are used. The model has the form  $\tilde{R}_{it} = \alpha_i + \beta_1 \tilde{R}_{mt} + \tilde{\eta}_{it}$ ,  $\forall i = 1, \dots, 5$  and  $\forall t = 1, \dots, T$ , and  $H_0: \alpha_i = \gamma(1 - \beta_i)$ ,  $\forall i = 1, \dots, 5$ .

| Input parameters  |   |   |   |                     |                              |           |           |           |           |           |  |
|-------------------|---|---|---|---------------------|------------------------------|-----------|-----------|-----------|-----------|-----------|--|
| Experiment number | Subperiod from empirical work by Fama-MacBeth (1973) used to suggest parameter values | Number of observations per equation and per replication (T) | Expected return on the zero-beta portfolio ( $\gamma$ ) | $E(\tilde{R}_{mt})$ | $\text{var}(\tilde{R}_{mt})$ | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ |  |
| 1                 | 1935-6/1968   | 60  | 0.0061  | 0.0125              | 0.003688                     | 0.3       | 0.7       | 1.0       | 1.3       | 1.7       |  |
| 2                 | 1935-1945   | 60  | 0.0039  | 0.0159              | 0.007900                     | 0.3       | 0.7       | 1.0       | 1.3       | 1.7       |  |
| 3                 | 1946-1955   | 60  | 0.0087  | 0.0103              | 0.001885                     | 0.3       | 0.7       | 1.0       | 1.3       | 1.7       |  |
| 4                 | 1956-6/1968   | 60  | 0.0060  | 0.0113              | 0.001556                     | 0.3       | 0.7       | 1.0       | 1.3       | 1.7       |  |
| 5 <sup>a</sup>    | 1935-6/1968   | 30  | 0.0061  | 0.0125              | 0.003688                     | 0.3       | 0.7       | 1.0       | 1.3       | 1.7       |  |
| 6 <sup>a</sup>    | 1935-6/1968   | 120   | 0.0061  | 0.0125              | 0.003688                     | 0.3       | 0.7       | 1.0       | 1.3       | 1.7       |  |

<sup>a</sup>Experiments 5 and 6 examine the effect of the number of observations per equation (T).



replications for a particular experiment. The former quantity is provided in parentheses in the third column, and it is an average (through all replications) of the estimates of the asymptotic variance. While reliance on the asymptotic variance overstates the precision of  $\hat{\gamma}^*$ , the problem is not serious. The worst approximation occurs for experiment 5 which uses only thirty observations. Given this apparent overstatement of the precision, a sensible and conservative strategy is to adjust the asymptotic standard errors following the suggestion of Gallant (1975, p. 43); this procedure is discussed in section 4. The last two columns in table 4 help to summarize the results. While there may be some deviations from the asymptotic normal distribution, in no instance are these departures sufficient to make the studentized range test or the chi-square goodness-of-fit test statistically significant.

Table 3  
Contemporaneous covariance input for market model disturbances used in all simulation experiments.

| Equation number |         |         |         |         |         |
|-----------------|---------|---------|---------|---------|---------|
| 1               | 0.00150 |         |         |         |         |
| 2               | 0.00070 | 0.00203 |         |         |         |
| 3               | 0.00042 | 0.00086 | 0.00300 |         |         |
| 4               | 0.00147 | 0.00071 | 0.00139 | 0.00403 |         |
| 5               | 0.00082 | 0.00080 | 0.00039 | 0.00000 | 0.00500 |

Table 4  
Means, variances and normality tests for the sampling distribution of the estimator of the expected return on the zero-beta portfolio ( $\hat{\gamma}^*$ ) using a one-step Gauss–Newton procedure. Each experiment involves 250 replications.

| Experiment number | $\hat{\gamma}^* \times 100$<br>(theoretical value) | $\hat{\sigma}^2(\hat{\gamma}^*) \times 100$<br>(theoretical value) | Studentized range test for normality | $\chi^2$ statistic for normality (p-value) |
|-------------------|--|--|--------------------------------------|--|
| 1                 | 0.700<br>(0.61)                                    | 0.00144<br>(0.0013) <sup>a</sup>                                   | 5.71                                 | 9.01<br>(0.11)                             |
| 2                 | 0.364<br>(0.39)                                    | 0.00131<br>(0.0013) <sup>a</sup>                                   | 6.55                                 | 3.18<br>(0.67)                             |
| 3                 | 0.860<br>(0.87)                                    | 0.00132<br>(0.0010) <sup>a</sup>                                   | 5.71                                 | 2.35<br>(0.80)                             |
| 4                 | 0.642<br>(0.60)                                    | 0.00139<br>(0.0014) <sup>a</sup>                                   | 5.37                                 | 4.66<br>(0.46)                             |
| 5                 | 0.616<br>(0.61)                                    | 0.00364<br>(0.0029) <sup>a</sup>                                   | 5.57                                 | 3.31<br>(0.65)                             |
| 6                 | 0.605<br>(0.61)                                    | 0.00074<br>(0.0007) <sup>a</sup>                                   | 5.20                                 | 3.82<br>(0.58)                             |

<sup>a</sup>These theoretical values are actually for a slightly different estimator whose behavior was almost identical with that of the one-step Gauss–Newton procedure.

In table 5 the small sample distribution of the LRT is characterized under the null hypothesis; theoretically, it has an asymptotic distribution of a chi-square with four degrees of freedom. Except for the last two columns, the format of table 5 is analogous to that of table 4. The test statistic is biased upward relative to its asymptotic mean, and it has greater dispersion than suggested by the theoretical distribution. The chi-square test for goodness-of-fit to a chi-square distribution is significant at the five percent level in two cases, but in one of these two cases, the  $p$ -value is close to 0.05. The main conclusion to draw from table 5 is that the test statistic is inadequate with only thirty observations (experiment 5).

Table 5

Means, variances and the goodness-of-fit tests for the sampling distribution of the likelihood ratio test (LRT) of the CAPM restriction under the null hypothesis. Each experiment involves 250 replications.

| Experiment number | $\overline{LRT}$<br>(theoretical value) | $\hat{\sigma}^2(LRT)$<br>(theoretical value) | $\chi^2$ statistic for a<br>chi-square distribution<br>( $p$ -value) |
|-------------------|---|--|--|
| 1                 | 4.49<br>(4.00)                          | 10.15<br>(8.00)                              | 14.90<br>(0.04)  |
| 2                 | 4.14<br>(4.00)                          | 8.87<br>(8.00)                               | 10.80<br>(0.15)  |
| 3                 | 4.33<br>(4.00)                          | 10.12<br>(8.00)                              | 5.42<br>(0.61)   |
| 4                 | 4.41<br>(4.00)                          | 9.12<br>(8.00)                               | 7.66<br>(0.36)   |
| 5                 | 5.04<br>(4.00)                          | 14.58<br>(8.00)                              | 28.14<br>(0.00)  |
| 6                 | 4.18<br>(4.00)                          | 8.87<br>(8.00)                               | 6.19<br>(0.52)   |

Table 6 provides the small sample power of the LRT. To conveniently parameterize the deviations from the null hypothesis, the model to generate the data under the alternate hypothesis has the form:

$$\alpha_i = \gamma(1 - \beta_i) + \varepsilon_i, \quad i = 1, \dots, 5,$$

where

$$\varepsilon_i \sim N(0, \sigma_\varepsilon^2).$$

The hope is that the power of the test increases as  $\sigma_\varepsilon$  increases.

This formulation of the alternative hypothesis is interesting, for it seems close to one which Ross (1978, p. 892) has in mind when discussing the power of current tests. This alternative also resembles a variety of models that are competitors with the CAPM. For example, the Merton (1973) intertemporal model with one state

Table 6

The simulated probability of failing to reject the CAPM restriction using the likelihood ratio test for different deviations from the null hypothesis. Theoretical significance level at 5%. Each experiment involves 250 replications.  $\tilde{R}_{it} = \alpha_i + \beta_i \tilde{R}_{mt} + \tilde{\eta}_{it}$ ,  $\forall i = 1, \dots, 5$  and  $\forall t = 1, \dots, T$ ,  $H_0: \alpha_i = \gamma(1 - \beta_i)$ ,  $\forall i = 1, \dots, 5$ , and  $H_A: \alpha_i = \gamma(1 - \beta_i) + \varepsilon_i$  where  $\varepsilon_i \sim NID(0, \sigma_\varepsilon^2)$ .

| Deviation of data from null hypothesis ( $\sigma_\varepsilon$ ) | Experiment number |      |      |      |      |      |
|---|-------------------|------|------|------|------|------|
|   | 1                 | 2    | 3    | 4    | 5    | 6    |
| 0.0000  | 0.92              | 0.96 | 0.92 | 0.94 | 0.88 | 0.93 |
| 0.0025  | 0.92              | 0.92 | 0.83 | 0.86 | 0.90 | 0.87 |
| 0.0050  | 0.80              | 0.70 | 0.74 | 0.79 | 0.83 | 0.57 |
| 0.0075  | 0.56              | 0.52 | 0.56 | 0.66 | 0.72 | 0.39 |
| 0.0100  | 0.37              | 0.46 | 0.43 | 0.44 | 0.60 | 0.23 |
| 0.0125  | 0.34              | 0.28 | 0.27 | 0.34 | 0.44 | 0.13 |
| 0.0150  | 0.18              | 0.20 | 0.23 | 0.17 | 0.38 | 0.06 |
| 0.0175  | 0.15              | 0.12 | 0.11 | 0.16 | 0.32 | 0.07 |
| 0.0200  | 0.08              | 0.09 | 0.12 | 0.10 | 0.26 | 0.02 |
| 0.0225  | 0.08              | 0.04 | 0.07 | 0.09 | 0.17 | 0.02 |

variable implies

$$E(\tilde{R}_{it}) = \gamma + \beta_i [E(\tilde{R}_{mt}) - \gamma] + \pi_i [E(\tilde{R}_{Ht}) - \gamma],$$

where

$E(\tilde{R}_{Ht}) \equiv$  expected return on the portfolio used to hedge the risk of the state variable.

For simplicity  $\tilde{R}_{mt}$  and  $\tilde{R}_{Ht}$  are assumed orthogonal so that  $\beta_i$  is the simple regression coefficient. Combining the above equation with (2) implies

$$\alpha_i = \gamma(1 - \beta_i) + \pi_i [E(\tilde{R}_{Ht}) - \gamma], \quad i = 1, \dots, N.$$

Thus,  $\varepsilon_i$  can represent the deviation from the Black (1972) version of the CAPM when the Merton asset pricing relationship is the appropriate model of equilibrium. Further, since there are a variety of alternative models similar in form to that of Merton's,  $\varepsilon_i$  can be viewed as a convenient and rather general way to capture the ability of the LRT to discriminate between the CAPM and various alternative theories. Assuming a particular distribution for  $\varepsilon_i$  is like characterizing the distribution of the excluded risk premiums (e.g.,  $\pi_i [E(\tilde{R}_{Ht}) - \gamma]$ ).

Several conclusions can be drawn using the evidence in table 6. The LRT has the ability to reject the restriction, and this ability increases as  $\sigma_\varepsilon$  increases. Comparing the dispersion of  $\beta_i [E(\tilde{R}_{mt}) - \gamma]$  across  $i$  with  $\sigma_\varepsilon$  may suggest the economic size of  $\sigma_\varepsilon$  needed to reject the null hypothesis. If  $\beta_i$  has a standard

deviation of 0.5 across  $i$ , and if  $E(\tilde{R}_{mt}) - \gamma \simeq 0.006$  [from overall results of Fama and Macbeth (1973)], then a dispersion of 0.003 for  $\sigma_\varepsilon$  is 'large' as long as the dispersion of excluded risk premiums is less than the dispersion of the risk premiums for  $\beta_i$ . By this criterion table 6 suggests the LRT has little chance to detect deviations from the CAPM. Thus, despite the apparently low power of the LRT, the evidence in section 4 still rejects the CAPM — suggesting this financial model may be inadequate in economic as well as statistical terms.<sup>14</sup>

The first row of table 6 (where  $\sigma_\varepsilon = 0$ ) suggests that the theoretical size (five percent) of the test corresponds closely to the size implied by the simulations. The asymptotic approximation seems inadequate for experiment 5 which is based on thirty observations.

The general impression from the simulation study is that the use of asymptotic theory provides an adequate approximation for inferences based on finite sample sizes. Furthermore, the LRT has some power to reject the null hypothesis when departures from the CAPM hypothesis become sufficiently large.

## 6. Some final reflections on the CAPM restriction

This paper has presented a multivariate statistical framework for estimating the expected return on the 'zero-beta portfolio' and for testing a multivariate restriction implied by the CAPM. The multivariate estimation scheme appears promising, for the resulting estimates avoid the errors-in-the-variables problem and have lower standard errors than alternative estimates. More precise information can be gathered from the data when the interdependencies in the market model disturbances are incorporated into the estimator for the expected return on the 'zero-beta portfolio'.

The multivariate methodology also permits tests of the CAPM that are more powerful than past investigations. The CAPM implies a strong cross-sectional restriction on the parameters of the market model, and the model can be judged by the validity of this restriction. Power considerations are of obvious importance in assessing confidence in the empirical results, and many researchers [e.g., Roll (1977, p. 155) and Ross (1978, p. 892)] are concerned about the ability to reject the CAPM even if it is false. The multivariate test does not appear to suffer from a lack of power since the restriction is rejected.

What is the source of this increase in power? Since  $\gamma$  is estimated more precisely, deviations from the theoretical model may loom larger. In addition, the

<sup>14</sup>Table 6 may imply low power for the LRT only because the average deviation from the CAPM is assumed to be zero. In other words,  $\varepsilon_i$  has a mean of zero, and this assumption concerning excluded risk premiums may be unreasonable. To determine the effect of a non-zero mean for  $\varepsilon_i$ , another simulation experiment is performed and reported by Gibbons (1980a). Setting the mean of  $\varepsilon_i$  equal to the average beta premium for the overall period in Fama and MacBeth (1973) (i.e., 0.006) and letting the variance of  $\varepsilon_i$  be zero in order to focus on the effect of a non-zero mean, the LRT still falsely accepts the CAPM in 82 percent of the tests. I wish to thank Charles Plosser for pointing out the potential effect of a non-zero mean as a possible explanation for the apparently low power of the LRT.

multivariate hypothesis provides a strong test of the underlying theory, for a very stringent relation must hold among the parameters of a large number of market model equations.

Of course, there are some philosophical issues as to what it means to reject (or even accept) the CAPM restriction if 'it takes a model to beat a model'. The CAPM restriction given in (5) has been rejected in favor not of another model but of an unspecified alternative hypothesis. The test really asks whether the data are better described with a normal distribution assumption with or without the cross-sectional implication of the CAPM theory. Since the test is not constructive, alternative financial models that are superior to the CAPM are not indicated. The rejection of (5) is at least warning to those using the CAPM for other objectives (e.g., tests of market efficiency). Strict adherence to the CAPM would lead to a structure which is inconsistent with the data, and such misspecification may yield incorrect inferences about a hypothesis not directly related to the CAPM. The use of a statistical model with a more general parameterization would be a sensible precaution. This suggestion complements the advice given by Brown and Warner (1980).

One final point is in order. The multivariate scheme does not require the use of a vague alternative hypothesis. Several financial models may be 'nested' so that rejection of the null hypothesis supports a particular alternative theory. This type of constructive test is provided in Gibbons (1980, app. G) where the CAPM is nested as a special case of the Merton (1973) intertemporal model. The tentative results suggest that the Merton restriction is more consistent with the data than the CAPM. The following section suggests a generalization of the MVRM technology to a wider class of financial models.

## 7. A generalization of the methodology and topics for future research

The general class of financial models that are considered in this section are

$$E(\bar{R}_{it}) = \gamma_0 + \sum_{j=1}^K \beta_{ij} \gamma_j, \quad i = 1, \dots, N. \quad (12)$$

The notation is defined in the first section. To permit inferences concerning a particular financial theory, statistical assumptions must be made. The following relation is assumed to be well specified:

$$\bar{R}_{it} = \beta_{i0} + \sum_{j=1}^K \beta_{ij} \tilde{x}_{jt} + \tilde{\eta}_{it}, \quad i = 1, \dots, N, \quad t = 1, \dots, T. \quad (13)$$

The above equation can be viewed as a single time series regression with  $T$  observations in a pooled system of  $N$  equations. The explanatory variable,  $\tilde{x}_{jt}$ , is intentionally undefined, for it is the random variable which equates  $\beta_{ij}$  in eq. (12)

with  $\beta_{ij}$  in eq. (13). In many financial applications the following structure for the error term seems reasonable:

$$\begin{aligned} E(\tilde{\eta}_{it}) &= 0, & i = 1, \dots, N, & \quad t = 1, \dots, T, \\ E(\tilde{\eta}_{is}\tilde{\eta}_{jt}) &= \sigma_{ij}, & i, j = 1, \dots, N, & \quad s = t, \\ &= 0, & i, j = 1, \dots, N, & \quad s \neq t. \end{aligned} \quad (14)$$

If  $\tilde{R}_{it}$  and  $\tilde{x}_{jt}$  ( $j = 1, \dots, K$ ) have a stationary multivariate normal distribution with serial independence, the above structure follows by implication.<sup>15</sup>

Eqs. (13) and (14) are combined and restated in a form similar to (12). The unconditional expectation of (13) is

$$E(\tilde{R}_{it}) = \beta_{i0} + \sum_{j=1}^K \beta_{ij} E(\tilde{x}_{jt}). \quad (15)$$

Since the left-hand sides of (12) and (15) are equal, the right-hand sides of both may be equated:

$$\beta_{i0} = \gamma_0 + \sum_{j=1}^K \beta_{ij} \gamma_j^* \quad \text{where} \quad \gamma_j^* \equiv \gamma_j - E(\tilde{x}_{jt}). \quad (16)$$

Eq. (16) is the key result, for it is an implication of the financial theory conditional upon the specification given by (13) and (14). The content of the financial model reduces the parameter space to  $K(N+1)+1$  regression coefficients from  $N(K+1)$  parameters of the unrestricted statistical model.

Eq. (16) may be used in two ways. First, the relation permits efficient estimation of the  $\gamma_j^*$ 's (i.e., the risk premiums) by restricting the MVRM given in (13). Second, (16) provides a null hypothesis for testing the substantive content of a particular financial model. The test in its most general form becomes

$$\begin{aligned} H_0: \beta_{i0} &= \gamma_0 + \sum_{j=1}^K \beta_{ij} \gamma_j^*, & i = 1, \dots, N, \\ H_A: \beta_{i0} &\neq \gamma_0 + \sum_{j=1}^K \beta_{ij} \gamma_j^*, & i = 1, \dots, N. \end{aligned} \quad (17)$$

The alternative hypothesis need not take such a general form. The statistical fit of different variants of (12) may be compared by examining the statistical significance of estimates of  $\gamma_j^*$  for a particular  $j$ .

<sup>15</sup>The multivariate normal distribution assumption is only a sufficient condition, for other distributional assumptions (e.g., multivariate Student  $t$ ) imply (13) and (14). Alternatively, (13) and (14) may be postulated, and then diagnostic tests may be performed to confirm the specification.

The primary extensions of this work involve two distinct topics. First, further work on the Merton (1973) asset pricing model is promising. This type of research seems to be important given the rejection of the CAPM and the paucity of any statistical work on intertemporal asset pricing models. Second, through the use of dummy variables, the MVRM approach seems well suited for powerful tests of market efficiency [see Gibbons (1980a, app. H)]. Given the long-standing interest in this subject, such a methodological contribution seems worthy of an actual application.

### Appendix: Development of the MVRM with a linearized restriction for the CAPM

By the linearization of the nonlinear restriction, linear estimation theory applies to the problem. The nonlinearity in each equation arises due to the multiplication of  $\gamma$  and  $\beta_i$ . That is, for any equation  $i$ ,

$$\begin{aligned}\tilde{\mathbf{R}} &= \alpha_i \mathbf{1}_T + \beta_i \tilde{\mathbf{R}}_m + \tilde{\eta}_i \quad \text{subject to} \quad \alpha_i = \gamma(1 - \beta_i), \quad i = 1, \dots, N, \\ \tilde{\mathbf{R}}_i &= \gamma \mathbf{1}_T - \gamma \beta_i \mathbf{1}_T + \beta_i \tilde{\mathbf{R}}_m + \tilde{\eta}_i.\end{aligned}\tag{A.1}$$

Using a Taylor series expansion about some estimates of  $\gamma$  (e.g.,  $\hat{\gamma}_{\text{BJS}}$ ) and  $\beta_i$  (e.g., unrestricted OLS),

$$\gamma \beta_i \simeq \hat{\gamma} \hat{\beta}_i + \hat{\beta}_i (\gamma - \hat{\gamma}) + \hat{\gamma} (\beta_i - \hat{\beta}_i) = \gamma \hat{\beta}_i + \hat{\gamma} \beta_i - \hat{\gamma} \hat{\beta}_i.\tag{A.2}$$

Thus, by substitution of (A.2) into (A.1),

$$(\tilde{\mathbf{R}}_i - \hat{\gamma} \hat{\beta}_i \mathbf{1}_T) \simeq \gamma (1 - \hat{\beta}_i) \mathbf{1}_T + \beta_i (\tilde{\mathbf{R}}_m - \hat{\gamma} \mathbf{1}_T) + \tilde{\eta}_i.$$

With the above relationship  $\gamma$  and  $\beta_i$  ( $i = 1, \dots, N$ ) may be estimated with a seemingly unrelated regression model (hereafter, SURM) using a general linear hypothesis across equations. This SURM has the form

$$\begin{aligned}\begin{bmatrix} \tilde{\mathbf{R}}_1 - \hat{\gamma} \hat{\beta}_1 \mathbf{1}_T \\ \tilde{\mathbf{R}}_2 - \hat{\gamma} \hat{\beta}_2 \mathbf{1}_T \\ \vdots \\ \tilde{\mathbf{R}}_N - \hat{\gamma} \hat{\beta}_N \mathbf{1}_T \end{bmatrix} &= \begin{bmatrix} \tilde{\mathbf{R}}_m - \hat{\gamma} \mathbf{1}_T & \mathbf{0} & \dots & \mathbf{0} & (1 - \hat{\beta}_1) \mathbf{1}_T \\ \mathbf{0} & \tilde{\mathbf{R}}_m - \hat{\gamma} \mathbf{1}_T & \dots & \mathbf{0} & (1 - \hat{\beta}_2) \mathbf{1}_T \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & & \tilde{\mathbf{R}}_m - \hat{\gamma} \mathbf{1}_T & (1 - \hat{\beta}_N) \mathbf{1}_T \end{bmatrix} \\ &\times \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{bmatrix} + \begin{bmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \\ \vdots \\ \tilde{\eta}_N \end{bmatrix},\end{aligned}$$

or

$$y^* = \tilde{Z}\delta + \tilde{\eta}^*,$$

where

$$y^{*'} = (\tilde{R}'_1 - \hat{\gamma}\hat{\beta}'_1\iota'_T, \dots, \tilde{R}'_N - \hat{\gamma}\hat{\beta}'_N\iota'_T),$$

$$\tilde{Z} = [I_N \otimes (\tilde{R}_M - \hat{\gamma}\iota_T) : (\iota_N - \hat{\beta}) \otimes \iota_T], \quad \text{and}$$

$$\delta' = (\beta_1, \beta_2, \dots, \beta_N, \gamma).$$

The restricted SURM estimator for  $\delta$  condition upon  $\Sigma = \hat{\Sigma}$  is

$$\hat{\delta} = [Z'(\hat{\Sigma}^{-1} \otimes I_T)Z]^{-1} Z'(\hat{\Sigma}^{-1} \otimes I_T)y^*.$$

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