THE JOURNAL OF FINANCE • VOL. LVII, NO. 5 • OCTOBER 2002

Survival Bias and the Equity Premium Puzzle

HAITAO LI and YUEWU XU*

ABSTRACT

Previous authors have raised the concern that there could be serious survival bias in the observed U.S. equity premium. Contrary to conventional wisdom, we argue that the survival bias in the U.S. data is unlikely to be significant. To reach this conclusion, we introduce a general framework for modeling survival and derive a mathematical relationship between the ex ante survival probability and the average survival bias. This relationship reveals the fundamental difficulty facing the survival argument: High survival bias requires an ex ante probability of market failure, which seems unrealistically high given the history of world financial markets.

IN AN INFLUENTIAL PAPER, Brown, Goetzmann, and Ross (1995; hereafter BGR) argue that there could be serious survival bias in the observed U.S. equity premium. This claim, if proven true, would have profound implications given the central role the equity premium plays in finance practice and research. For example, the equity premium is of fundamental importance for asset allocation decisions, estimates of the cost of capital, and the current debate about investing Social Security funds in the stock market. Given the equity premium puzzle first raised by Mehra and Prescott (1985), the equity premium is also important for theoretical asset pricing studies.

In this paper, we argue that, contrary to the findings of BGR, the survival bias in the U.S. equity premium is unlikely to be significant and survival cannot explain the equity premium puzzle. To reach this conclusion, we first develop a general framework for modeling survival, then apply it to assess the magnitude of the survival bias in the U.S. data.

Our framework models directly the two key factors that determine the size of the survival bias: (a) the probability of market failure and (b) the losses that investors would suffer in such an event.[1] We assume that market

* Li is from Johnson Graduate School of Management, Cornell University and Xu is from TIAA-CREF. We thank Warren Bailey; Peter Bossaerts; Stephen Brown; Alexei Egorov; Will Goetzmann; Roger Ibbotson; Jonathan Ingersoll; Robert Jarrow; Charles M.C. Lee; Stephen Ross; Mark Schroder; Fan Yu; and seminar participants at Cornell University, the Yale School of Management, and the 11th Annual Financial Economics and Accounting Conference at the University of Michigan Business School for helpful comments. We are especially grateful to the editor Richard Green and the anonymous referee for their comments and suggestions that greatly improved the paper. We are responsible for any errors.

[1] Throughout this paper, we refer to survival as the survival of the whole stock market, given our focus is the equity premium puzzle. However, our modeling framework is general and should be applicable to other survival problems.

closure is triggered by the first jump of a Point process with a stochastic intensity. By allowing the intensity to be a general function of the state variables that describe the state of the economy, our model encompasses various mechanisms of market failure proposed in the existing literature. We also assume that, when the market fails, investors suffer a negative 100 percent return. Therefore, the survival bias in our model serves as an upper bound for the bias in all other models.

Given this general setup, we show that the survival bias in stock returns over a certain time interval is determined by the conditional probability of market failure in that interval. Therefore, to have consistently high survival bias, the ex ante probability of market failure has to be consistently high as well; consequently, the probability of market survival over the long run has to be very small. Specifically, we derive a general mathematical formula which links the ex ante survival probability to the average survival bias. When applied to the U.S. market, this formula shows that (a) the magnitude of the survival bias has been greatly overstated in BGR; and (b) the bias in the equity premium is unlikely to be significant, given existing historical evidence.

The following is the intuition for why the seriousness of the survival problem is exaggerated in BGR and why the BGR model does not predict a large survival bias in the U.S. data. In their model, a market fails when the stock price hits a fixed lower absorbing barrier. The probability of market failure and the survival bias could be high at the initial stage of the market, when the stock price is close to the barrier. If the stock has a positive expected return, however, conditional on survival, the price drifts away from the barrier and eventually is so far above the barrier that the probability of market failure and the survival bias decline to zero. BGR's analysis only estimates the survival bias in the early stage of the market, but ignores the important long-run behavior of the bias.[2]

The real problem with the BGR model, however, is not the fixed lower barrier, but that high survival bias requires an unrealistically high ex ante probability of market failure. We show that this is the fundamental difficulty facing not only the BGR model, but also other survival models with completely different market failure mechanisms.

The rest of this paper is organized as follows. In Section I, we introduce a general survival model. In Section II, we study its implications for the survival bias in the U.S. equity premium. In Section III, we conclude the paper and show in the Appendix that the BGR model can be treated as a special case of our general model.

[2] The survival bias in the BGR model behaves similarly to the credit spread in the defaultable bond model of Merton (1974) in which a firm defaults if its asset value falls below a certain level. It is widely recognized in the literature that Merton's model cannot generate a significant default premium for exactly the same reason: Conditional on a firm's survival, its asset value grows away from the default boundary and the credit spread declines to zero in the long run (see, e.g., Collin-Dufresne and Goldstein (1999)).

I. A General Model of Survival

Survival is a very complicated phenomenon, and there are potentially many different ways to model it. For example, while market failure is modeled as a lower absorbing barrier in BGR, the authors acknowledge that other processes are equally reasonable: "A hyperinflation prelude to market closure may be characterized by an absorbing upper bound. Revolution may in fact be consequent on sustained excessive rates of return realized by domestic and foreign investors" (p. 856). The probability of market failure due to natural disaster or war may not depend on the level of the stock market at all. In this section, we develop a general survival model that is flexible enough to encompass various mechanisms of market failure proposed in the existing literature. Our approach for modeling survival is similar to the "reduced-form" approach for modeling default developed by Jarrow and Turnbull (1995), Jarrow, Lando, and Turnbull (1997), and Duffie and Singleton (1999).

We adopt the standard assumption that the uncertainty in the economy is described by a complete probability space $\{\Omega, \mathcal{F}, P\}$, and that the resolution is given by a filtration $\{\mathcal{F}_t : t \geq 0\}$ which satisfies the usual conditions (see Protter (1990)). We assume that market failure is triggered by the first jump of a Point process with a stochastic intensity, that is, the market failing time τ is the first jump time of a nonexplosive counting process $q_t = \mathbf{1}_{\{t \geq \tau\}}$. The counting process has an intensity λ_t, which is a predictable nonnegative process satisfying $\int_0^t \lambda_s\, ds < \infty$ almost surely for all t. The intensity has the property that the compensated counting process $M_t = q_t - \int_0^t \lambda_s\, ds$ is a local martingale, and M_t is a martingale if $E(\int_0^t \lambda_s\, ds) < \infty$ (see Brémaud (1981) for further details). Intuitively, conditional on market survival until t, the probability of failure in the next small time interval Δt is $\lambda_t \Delta t$.

Let Y_t be a vector of the state variables in R^d that describe the state of the economy, which follows the stochastic differential equation of the form

$$dY_t = \mu(Y_t, t)\, dt + \sigma(Y_t, t)\, dB_t, \tag{1}$$

for some $\mathcal{F}_t$-standard Brownian motion B_t in R^d. We allow the intensity to be a function of the state variables Y_t, that is, $\lambda_t = \Lambda(Y_t, t)$ for some measurable $\Lambda : R^d \times [0,\infty) \to [0,\infty)$.

If the market is still in existence at t, that is, $\tau > t$, then for the closure to occur after some future time T, it must be that $dq_s = 0$ for all $s \in [t, T]$. It thus follows that

$$\Pr[\tau > T \,|\, \mathcal{F}_t] = E_t\left[\exp\left(-\int_t^T \Lambda(Y_s, s)\, ds\right) \middle| \mathcal{F}_t\right]. \tag{2}$$

Following Collin-Dufresne and Goldstein (1999), we introduce the concept of the forward probability of market closure, $f(t, T)$, defined as the probability in per unit of time at t that the market will close between T and $T + dT$, conditional on survival up to T:

$$
\begin{aligned}
f(t,T) &= \lim_{dT \downarrow 0} \left(\frac{1}{dT} \right) \Pr[\tau \in (T, T + dT] \,|\, \tau > T, \mathcal{F}_t] \\
&= \lim_{dT \downarrow 0} \frac{\frac{1}{dT} \Pr[\tau \in (T, T + dT] \,|\, \mathcal{F}_t]}{\Pr[\tau > T \,|\, \mathcal{F}_t]} \\
&= \lim_{dT \downarrow 0} \frac{\frac{1}{dT} \{\Pr[\tau > T \,|\, \mathcal{F}_t] - \Pr[\tau > T + dT \,|\, \mathcal{F}_t]\}}{\Pr[\tau > T \,|\, \mathcal{F}_t]} \\
&= \frac{-\partial \Pr[\tau > T \,|\, \mathcal{F}_t]/\partial T}{\Pr[\tau > T \,|\, \mathcal{F}_t]},
\end{aligned}
\tag{3}
$$

where $f(t,t) = \lambda_t$.[3] The definition of the forward probability implies the following relationship

$$
\Pr[\tau > T \,|\, \mathcal{F}_t] = \exp\left\{ -\int_t^T f(t,s)\, ds \right\},
\tag{4}
$$

which states that the only way that a market can survive from t to T is for it to survive each little time interval $(s, s + ds]$ for all $s \in [t, T]$, conditional on its survival until s. Equation (4) indicates that the ex ante probability of survival from t to T has to be small if there is a high forward probability of market closure in each interval $(s, s + ds]$ for all $s \in [t, T]$. The above equation, derived only from the definition of $f(t,s)$, is very general and holds for any survival model in which the probability density for the stopping time τ exists. A similar relationship exists between the forward probability of market failure and the ex ante probability of survival, when τ follows arbitrary distributions (for details, see p. 30 of Fleming and Harrington (1991)).

Now we link the forward probability of market failure to the survival bias in stock returns. Let $S(t)$ represent the stock price, which could jump to zero at τ. The instantaneous return at t is defined as $dS(t)/S(t-) = R_t\, dt$. For the sake of generality, we leave the process of $S(t)$ unspecified and assume only that it is $\mathcal{F}_t$-adapted.

In the event that $\tau > t$, let $\mu(t)$ represent the time-t unconditional expected return and $\mu^*(t)$ the time-t expected return conditional on survival from t to $t + dt$, both measured in per unit of time. Noting that investors

[3] Grandell (1976, pp. 106–107) shows that $(\partial \Pr[\tau > T|\mathcal{F}_t])/\partial T = E_t[-\exp(-\int_t^T \lambda_s\, ds)\lambda_T|\mathcal{F}_t]$ provided that (a) there is a constant C such that, for all t, $E(\lambda_t^2) < C$; (b) for any $\epsilon > 0$, almost surely every t, $\lim_{\delta \to 0} \Pr(|\lambda(t + \delta) - \lambda(t)| \geq \epsilon) = 0$. These properties are satisfied in most typical models. The authors are very grateful to Darrell Duffie for pointing out this reference.

suffer a negative 100 percent return when the market closes, we have by definition

$$\begin{aligned}\mu(t)\,dt &= E[R_t\,dt\,|\,\mathcal{F}_t, \tau > t] \\ &= E[R_t\,dt\,|\,\mathcal{F}_t, \tau \in (t, t+dt]]\cdot \Pr[\tau \in (t, t+dt]\,|\,\mathcal{F}_t, \tau > t] \\ &\quad + E[R_t\,dt\,|\,\mathcal{F}_t, \tau > t+dt]\cdot \Pr[\tau > t+dt\,|\,\mathcal{F}_t, \tau > t] \\ &= -1\cdot \lambda_t\,dt + \mu^*(t)\,dt\cdot(1-\lambda_t\,dt) = (\mu^*(t) - \lambda_t)\,dt. \end{aligned} \tag{5}$$

Therefore, λ_t can be identified as the ex ante expected survival bias in stock returns at time t. Standing at time t and conditional on market survival from t to s, let $\mu(t,s)$ represent the time-s unconditional expected return and $\mu^*(t,s)$ the time-s expected return conditional on survival from s to $s + ds$, again both measured in per unit of time. Similarly, we have

$$\begin{aligned}\mu(t,s)\,ds &= E[R_s\,ds\,|\,\mathcal{F}_t, \tau > s] \\ &= E[R_s\,ds\,|\,\mathcal{F}_t, \tau \in (s, s+ds]]\cdot \Pr[\tau \in (s, s+ds]\,|\,\mathcal{F}_t, \tau > s] \\ &\quad + E[R_s\,ds\,|\,\mathcal{F}_t, \tau > s+ds]\cdot \Pr[\tau > s+ds\,|\,\mathcal{F}_t, \tau > s] \\ &= -1\cdot f(t,s)\,ds + \mu^*(t,s)\,ds\cdot(1-f(t,s)\,ds) = (\mu^*(t,s) - f(t,s))\,ds. \end{aligned} \tag{6}$$

Therefore, $f(t,s)$ represents the ex ante expected survival bias in stock returns at time s and the survival bias averaged over $[t,T]$ is $[1/(T-t)]\int_t^T f(t,s)\,ds$.

The above analysis shows that over any small time interval, the survival bias in stock returns is determined by the conditional probability of market failure over that interval. This conclusion is very general since we derive it only from the definition of the conditional expectation. By combining it with equation (4), which is another general result, we obtain a mathematical relationship between the ex ante probability of survival and the average survival bias.

PROPOSITION 1: *The ex ante probability of survival and the average survival bias over $[t,T]$ satisfy the following equation:*

$$\frac{1}{T-t}\int_t^T f(t,s)\,ds = -\frac{1}{T-t}\log(\Pr[\tau > T\,|\,\mathcal{F}_t]). \tag{7}$$

Equation (7) makes it very convenient to judge the significance of the survival bias given an estimate of the ex ante survival probability. It shows that the necessary and sufficient condition for a significant survival bias is a small ex ante survival probability. Next, we apply this result to measure the survival bias in the U.S. equity premium.

II. Survival Bias in the U.S. Equity Premium

The importance of the survival bias has been widely recognized in the literature because the BGR model appears to generate significant bias in the equity premium under reasonable model parameters.[4] Contrary to the findings of BGR, however, our model suggests that the survival bias in the U.S. equity premium is unlikely to be significant.

A. The Magnitude of the Survival Bias

In this section, we first explain intuitively why the BGR model does not predict high survival bias in the equity premium, and then apply our general survival model to measure the survival bias in the U.S. equity premium.

According to the BGR model, the log stock price follows a Brownian motion with drift; market survival means that the price stays above a fixed lower absorbing barrier. The probability of market failure and the survival bias could be high in the early stage of the market when the price is close to the barrier. However, under the mild assumption that the stock has a positive expected return, the price will drift away from the barrier once the market has survived long enough. Eventually, the price will be so far above the barrier that the probability of hitting it will approach zero. At this point, the conditional distribution of the stock price will be very close to its unconditional distribution, and there will be little survival bias in the data. The significance of the survival bias has been overstated in BGR: Their conclusion relies only on the bias in the early stage of the market and ignores its important long-run behavior.

In addition, the BGR model also makes other predictions that are not consistent with empirical observations. For example, as just shown, the survival bias concentrates primarily at the early stage of the market and declines rapidly afterwards. Based on this prediction alone, which directly contradicts the existing empirical evidence,[5] we can reject the model even if

[4] Paul Samuelson states in Ross (1997), "There is a survival bias. My hunch is that if you were able to eradicate all of that bias you would remove part of the demonstrated superiority of equities in the last 150 years over alternative investments, such as bonds, money market funds, and bank accounts" (p. 214). Campbell, Lo, and MacKinlay (1997) write, "The authors [BGR] argue that financial economists concentrate on the U.S. stock market precisely because it has survived and grown to become the world's largest market. . . . If this survivorship effect is important, estimates of average U.S. stock returns are biased upwards" (p. 311). Goetzmann and Jorion (1999) write, "A related argument is advanced in Brown, Goetzmann, and Ross (1995), who claim that survival of the series imparts a bias to *ex post* returns. They show that an *ex ante* equity premium of zero can generate a high *ex post* positive premium by simply conditioning on the market surviving an absorbing lower bound over the course of a century" (p. 954).

[5] Siegel (1998) shows that the equity premium, calculated as the difference in compound annual real returns on stocks and bills, averaged 1.9 percent, 3.4 percent, and 6.6 percent over the three major subperiods of the U.S. market since 1802 (1802 to 1870, 1871 to 1925, and 1926 to 1997, respectively).

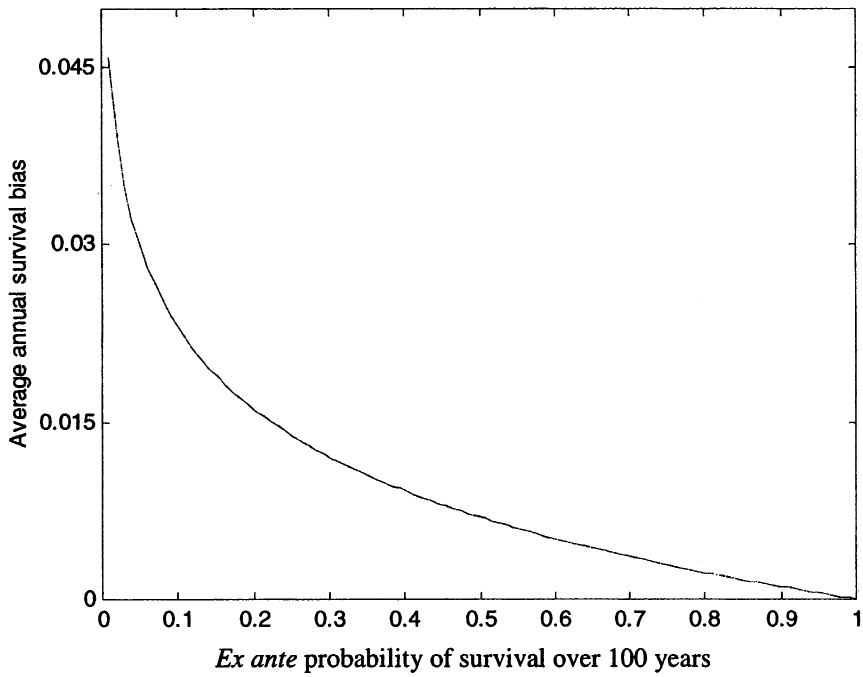

Figure 1. Average annual survival bias and ex ante probability of survival over 100 years. The horizontal axis represents the ex ante probability of survival over 100 years $\Pr[\tau > 100 | \mathcal{F}_0]$ and the vertical axis represents the ex ante expected survival bias averaged over the same period of time $\frac{1}{100}\int_0^{100} f(t,s)\,ds$.

it can generate a high enough survival bias in theory. In the BGR model, the high survival bias in the equity premium is also associated with a strong negative autocorrelation in stock returns. Therefore, survival can explain the equity premium puzzle only if there is significant mean-reversion in stock returns. The fact that the U.S. stock returns exhibited very weak mean-reversion after World War II (see Fama and French (1988)) provides additional evidence that the BGR model cannot generate high survival bias in the equity premium.

Realizing the limitations of the BGR model, we apply our general survival model to the U.S. market. We plot equation (7) in Figure 1, where the horizontal axis represents the ex ante probability of survival for 100 years, $\Pr[\tau > 100 | \mathcal{F}_0]$, and the vertical axis represents the average annual survival bias $\frac{1}{100}\int_0^{100} f(0,s)\,ds$. This mathematical relationship allows us to measure the survival bias based on an estimate of the ex ante survival probability. Following BGR, the ex ante survival probability is taken to be 14/36, which is the ratio of the number of markets that survived the past century to the number of markets that existed at the beginning of the century.[6] Figure 1 shows that for a 14/36 ex ante survival probability, the average annual survival bias is only about 1 percent. Even for a survival probability as low as 1 percent, the survival bias is only 4.6 percent, which is significantly smaller than BGR's estimation. Therefore, if we are to believe that the survival bias is significant, we would have to believe also that it was extremely lucky for the U.S. market to have survived the last century. However, the survival of the U.S. market for almost 200 years suggests that the probability of survival is likely to be large and that the survival bias is unlikely to be significant. This conclusion also applies to the BGR model, since it can be treated as a special case of our general model (see the Appendix for a proof).

B. Other Possible Mechanisms of Market Failure

It may appear that it is the fixed lower barrier that renders the BGR model unable to generate high survival bias. However, that is not the fundamental reason. Our analysis indicates that, to have high survival bias, the ex ante probability of market failure has to be high as well, which implies that over the last century, we should have observed many more market failures than are actually observed in the history of world financial markets. This is the basic difficulty with which both the BGR model and other survival models have to deal. To illustrate this fact, we provide two survival models that look totally different from the BGR model, but make similar predictions about the magnitude of the survival bias.

We assume that market failure is due to exogenous forces such as war or natural disaster in the first model, and due to hyperinflation or a price bubble in the second model. The advantages of our modeling framework become apparent in these two examples. To consider different mechanisms of market failure, we need only change the functional forms of the intensity process.

B.1. Market Failure Due to Exogenous Forces

To model market failure due to exogenous forces such as a natural disaster or a war, we choose the intensity to be independent of the state variables, Y_t. For example, we can assume that λ_t follows the square-root process of Cox, Ingersoll, and Ross (1985; hereafter CIR):

$$d\lambda_t = \kappa(\theta - \lambda_t)\, dt + \sigma_\lambda \sqrt{\lambda_t} dW_t, \tag{8}$$

where W_t is a standard Brownian motion and is independent of B_t in equation (1).

[6] The 14/36 ratio is definitely not a perfect estimate of the ex ante survival probability. It depends on the implicit assumption that different markets disappear independently, and might ignore other markets which have existed. Nevertheless, we follow BGR (1995) in our analysis so that we may compare their results with ours. Without knowing the correlation between the disappearances of different markets, we feel that this is a reasonable starting point.

The probability of market survival until T is given by

$$\begin{aligned}\Pr[\tau > T \mid \mathcal{F}_t] &= E_t\left[\exp\left(-\int_t^T \lambda_s\, ds\right)\Big|\, \mathcal{F}_t\right] \\ &= A(t,T)e^{-B(t,T)\lambda_t} \\ &= \exp\left\{-\int_t^T f(t,s)\, ds\right\}, \end{aligned} \tag{9}$$

where $A(t,T)$ and $B(t,T)$ are the well-known functions in the CIR model and

$$f(t,s) = \lambda_t + \kappa(\theta - \lambda_t)B(t,s) - \frac{\sigma_\lambda^2}{2}\lambda_t B^2(t,s). \tag{10}$$

It can be easily shown that

$$\Pr[\tau > \infty \mid \mathcal{F}_t] = 0 \qquad \text{and} \qquad f(t,\infty) = \frac{2\kappa\theta}{\sqrt{\kappa^2 + 2\sigma_\lambda^2} + \kappa}\ \text{(a constant)}. \tag{11}$$

In this model, the market is in constant danger of dying and as a result, the probability of survival over the long run approaches zero.

The probability of market failure, which decreases with the stock price in the BGR model, is independent of the stock price in this model. Despite such a big difference, Proposition 1 applies to both models, since they are special cases of our general model. Therefore, this particular model does not yield high survival bias for the U.S. equity premium: The ex ante survival probability would have to be unrealistically low given historical record.

B.2. Market Failure Due to Hyperinflation or a Price Bubble

To capture the idea that market failure is caused by hyperinflation or a price bubble, we model the intensity as an increasing function of the stock price. For the sake of simplicity, we assume that the log stock price $X_t = \log(S_t)$ follows the standard Brownian motion

$$dX_t = dW(t), \tag{12}$$

and the intensity of the Point process equals

$$\Lambda(X_t) = \begin{cases} 0 & \text{if } X_t < b \\ \beta & \text{if } X_t \geq b. \end{cases} \tag{13}$$

This means that the market is in danger of disappearing when the stock price is too high. We can also model market failure due to a stock market crash by choosing the intensity as a decreasing function of the stock price, for example,

$$\Lambda(X_t) = \begin{cases} 0 & \text{if } X_t > b \\ \beta & \text{if } X_t \le b. \end{cases} \tag{14}$$

For the intensity defined in (13), if $X_0 = 0$, then the ex ante survival probability equals

$$\begin{aligned} \Pr[\tau > t \,|\, \mathcal{F}_0] &= E_0\left[\exp\left(-\beta \int_0^t \mathbf{1}_{\{X_s \ge b\}}\right) ds \,|\, \mathcal{F}_0\right] \\ &= \mathcal{L}^{-1}\left[e^{-b\sqrt{2s}}\left(\frac{1}{\sqrt{s(s+\beta)}} - \frac{1}{s}\right) + \frac{1}{s}\right] \\ &= F_1 * F_2(t) - \operatorname{erfc}\left(\frac{b}{\sqrt{2t}}\right) + 1, \end{aligned} \tag{15}$$

where

$$F_1(t) = \mathcal{L}^{-1}\left[e^{-b\sqrt{2s}}\frac{1}{\sqrt{s}}\right] = \frac{1}{\sqrt{\pi t}}\exp\left(-\frac{b^2}{2t}\right), \tag{16}$$

$$F_2(t) = \mathcal{L}^{-1}\left[\frac{1}{\sqrt{(s+\beta)}}\right] = \frac{1}{\sqrt{\pi t}}\exp(-\beta t), \tag{17}$$

where "$*$" represents the convolution of the two functions, and $\operatorname{erfc}(x) = (2/\sqrt{\pi})\int_x^\infty e^{-u^2}\,du$.[7]

[7] Karatzas and Shreve (1991, pp. 273–274) show that the Laplace transform of $\Pr[\tau > t \,|\, \mathcal{F}_0]$ equals

$$\int_0^\infty e^{-st} E_0\left[\exp\left(-\beta\int_0^t \mathbf{1}_{\{X_s \ge b\}}\right) ds \,|\, \mathcal{F}_0\right] dt = e^{-b\sqrt{2s}}\left(\frac{1}{\sqrt{s(s+\beta)}} - \frac{1}{s}\right) + \frac{1}{s}.$$

The inverse of the Laplace transform can be found in Abramowitz and Stegun (1970) and Erdelyi (1954).

It can be shown that as $t \to \infty$, both $\Pr[\tau > t | \mathcal{F}_0]$ and $f(0,t)$ converge to zero in this model.[8] Intuitively, the stock price on average spends almost half the time in the region where the market is in constant danger of failing. Therefore, the probability of survival over the long run is very small. On the other hand, if the market is lucky enough to have survived for a long time, then it is likely that the stock price has drifted far away from the dangerous region, and the conditional probability of failure becomes very small.

Again, this model is a special case of our general model, although it differs from BGR in that the probability of market failure increases with the stock price. Therefore, Proposition 1 holds in this model as well, and, as a result, it does not predict high survival bias in the U.S. equity premium either.

III. Conclusion

The U.S. equity premium plays an important role in many areas of finance research and practice. Therefore, the concerns raised by BGR that the equity premium might contain serious survival bias should be studied with great care: If proven true, this hypothesis would have widespread impact.

Based on a general survival model developed in this paper, we show that the fundamental difficulty facing the survival argument is that to have high survival bias, the probability of market survival over the long run has to be extremely small, which seems to be inconsistent with existing historical evidence. Therefore, we argue that contrary to what BGR suggest, the survival bias in the U.S. equity premium is unlikely to be significant and the resultant concerns about the survival problem in the current literature are probably overstated.

Survival has potential effects on many areas of finance research that use historical data. While the focus of this paper is the equity premium, our modeling framework is general and the insights it provides should be useful for understanding other survival-related issues.

[8] Let $z = t\sin^2\varphi$, we have as $t \to \infty$,

$$F_1 * F_2(t) = \frac{2}{\pi}\int_0^{\pi/2} \exp\left(-\frac{b^2}{2t\sin^2\varphi} - \beta t\cos^2\varphi\right) d\varphi \to 0$$

and

$$\frac{\partial}{\partial t}\,(F_1 * F_2(t)) = \frac{2}{\pi}\int_0^{\pi/2}\left(\frac{b^2}{2t^2\sin^2\varphi} - \beta\cos^2\varphi\right)\exp\left(-\frac{b^2}{2t\sin^2\varphi} - \beta t\cos^2\varphi\right) d\varphi \to 0.$$

It is also obvious that as $t \to \infty$, $\mathrm{erfc}(b/\sqrt{2t}) \to 1$ and $(\partial/\partial t)(\mathrm{erfc}(b/\sqrt{2t})) \to 0$.

Appendix: The BGR Model as a Special Case of the General Survival Model

In this section, we show that the BGR model can be treated as a special case of our general model and that Proposition 1 also holds. In the BGR model, the log stock price X_t follows a Brownian motion with drift

$$dX_t = mdt + \sigma dW(t), \tag{A1}$$

and market survival means that X_t stays above some lower barrier b. The market failure time is defined as $\tau = \inf_{s\in[0,t]}\{X(s) = b\}$.[9]

Standing at time 0, and conditional on X_t staying above b from 0 to s, the probability that the market will fail in the next small interval equals

$$\begin{aligned}
f(0,s) &= \lim_{\Delta s\downarrow 0}\frac{1}{\Delta s}\Pr[\tau\in(s,s+\Delta s]|\mathcal{F}_0]\\
&= \lim_{\Delta s\downarrow 0}\frac{1}{\Delta s}\int_b^\infty \pi(\Delta s, x-b,m,\sigma)p^*(x,s|x_0,0)\,dx\\
&= \frac{1}{2}\sigma^2\frac{\partial p^*(x,s|x_0,0)}{\partial x}\bigg|_{x=b}\\
&= -\frac{\partial \Pr[\tau>s|\mathcal{F}_0]/\partial s}{\Pr[\tau>s|\mathcal{F}_0]},
\end{aligned}\tag{A2}$$

where $\pi(\Delta s, x-b,m,\sigma)$ is the probability of the first passage of a Brownian motion with drift m and volatility σ to zero before time $s+\Delta s$, with an initial condition $X(s) = x - b$, and $p^*(x,s|x_0,0)$ is the density of X_s conditional on survival from 0 to s, and $X_0 = x_0$.[10] The proof of the third equality

[9] The BGR model cannot in strict sense be nested by our model, since its stopping time τ does not have an intensity. To have an intensity, τ has to be totally inaccessible: For any sequence of stopping times, the probability that the sequence approaches τ from below is zero. However, if we let $\tau_n = \inf_{s\in[0,t]}\{X(s) = b + n^{-1}\}$, then there is a strictly positive probability that τ_n converges to τ (see Duffie and Lando (2001)).

[10] From Ingersoll (1987), we have the conditional density $p^*(x,t|x_0,0)$ equals

$$p^*(x,t|x_0,0) = \frac{p(x,t|x_0,0)}{\Pr[\tau>t|x_0,0]},$$

where

$$p(x,t|x_0,0) = \frac{1}{\sigma\sqrt{t}}\left[\phi\left(\frac{x-x_0-mt}{\sigma\sqrt{t}}\right) - \exp\left(-\frac{2m(x_0-b)}{\sigma^2}\right)\phi\left(\frac{x+x_0-2b-mt}{\sigma\sqrt{t}}\right)\right],$$

$$\Pr[\tau>t|x_0,0] = \Phi\left(\frac{mt-b+x_0}{\sigma\sqrt{t}}\right) - \exp\left(-\frac{2m(x_0-b)}{\sigma^2}\right)\cdot\Phi\left(\frac{mt-x_0+b}{\sigma\sqrt{t}}\right),$$

and ϕ and Φ are, respectively, the pdf and cdf of the standard normal distribution.

in the above equation can be found in Appendix A of Duffie and Lando (2001) who obtain the important result that although $\lim_{\Delta s\downarrow 0}(1/\Delta s)\pi(\Delta s, x-b, m, \sigma) = 0$ almost surely, $\lim_{\Delta s\downarrow 0}(1/\Delta s)\int_b^\infty \pi(\Delta s, x-b, m, \sigma)p^*(x, s|x_0, 0)\,dx$ is nonzero and equals $\frac{1}{2}\sigma^2[\partial p^*(x, s|x_0, 0)/\partial x]|_{x=b}$. The last equality can be easily verified from the expressions of $p^*(x, s|x_0, 0)$ and $\Pr[\tau > s|\mathcal{F}_0]$. Therefore, in the BGR model, we still have

$$\Pr[\tau > t\,|\,\mathcal{F}_0] = \exp\left\{-\int_0^t f(0, s)\,ds\right\}, \tag{A3}$$

and in this sense, we can treat it as a special case of our general model, except where $f(0,0) = 0$. Given that in our model, the investors suffer a negative 100 percent return when the market fails—which is always greater than or equal to that in the BGR model—the forward probability $f(0, s)$ also serves as the upper bound for the bias in $(s, s + \Delta s]$. Therefore, Proposition 1 holds in the BGR model as well.[11]

The behavior of the survival bias in the BGR model can be understood by examining the forward probability of market failure, which equals

$$f(0, s) = -\frac{\phi\left(\dfrac{ms - b + x_0}{\sigma\sqrt{s}}\right)\left(\dfrac{ms + b - x_0}{2\sigma s^{3/2}}\right) - e^{-[2m(x_0-b)]/\sigma^2}\phi\left(\dfrac{ms + b - x_0}{\sigma\sqrt{s}}\right)\left(\dfrac{ms - b + x_0}{2\sigma s^{3/2}}\right)}{\Phi\left(\dfrac{ms - b + x_0}{\sigma\sqrt{s}}\right) - \exp\left(-\dfrac{2m(x_0 - b)}{\sigma^2}\right)\cdot\Phi\left(\dfrac{ms - x_0 + b}{\sigma\sqrt{s}}\right)}. \tag{A4}$$

As $s \to \infty$, the denominator $(\partial \Pr[\tau > s|\mathcal{F}_0])/\partial s \to 0$, the numerator $\Pr[\tau > s|\mathcal{F}_0] \to 1 - \exp(-[2m(x_0 - b)]/\sigma^2)$, and the forward probability of market failure $f(0, s) \to 0$. Figure 2 plots $f(0, s)$ as a function of s for some concrete model parameters: $m = 0.05$ or 0.10, $\sigma = 0.2$, $b = 0$, and $X(0) = 2\sigma$ or 4σ. The vertical axis represents the forward probability of market failure $f(0, s)$ at year s, and the horizontal axis represents the number of years of survival s. The probability of market failure starts from zero, quickly reaches the maximum, and declines rapidly afterwards. Therefore, after the initial stage, the longer the market survives, the smaller the probability of failure, and, in turn, the smaller the survival bias becomes.

[11] Duffie and Lando (2001) show that in a Merton-type model, if true firm value is observed with noises due to incomplete accounting information and investors can only infer its probability distribution, then the default stopping time has an intensity. Similarly, we can change the BGR model by assuming that a market fails when the true value of the stock hits a lower barrier and investors can only observe the stock price which is the true value plus some noises. Then we can invoke Duffie and Lando's results directly to show that this slightly modified BGR model has an intensity for the stopping time and, therefore, can be completely nested by our model.

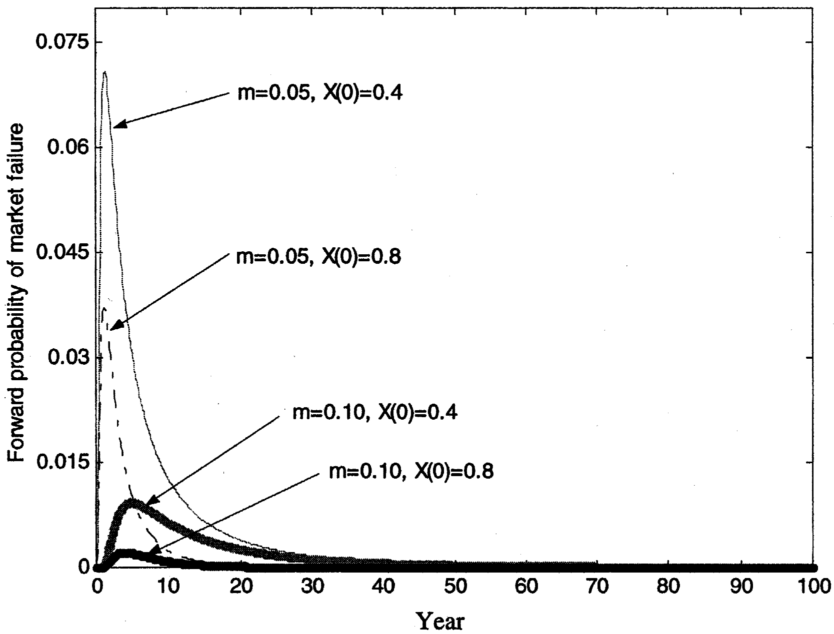

Figure 2. Forward probability of market failure in the BGR model. The horizontal axis represents the number of years of survival s, and the vertical axis represents the forward probability of market failure $f(0,s)$ at year s.

REFERENCES

Abramowitz, Milton, and Irene Stegun, 1970, *Handbook of Mathematical Functions* (U.S. Government Printing Office, Washington, DC).

Brémaud, Pierre, 1981, *Point Processes and Queues: Martingale Dynamics* (Springer-Verlag, New York).

Brown, Stephen, William Goetzmann, and Stephen Ross, 1995, Survival, *Journal of Finance* 50, 853–873.

Campbell, John, Andrew Lo, and Craig MacKinlay, 1997, *The Econometrics of Financial Markets* (Princeton Press, Princeton, NJ).

Collin-Dufresne, Pierre, and Robert Goldstein, 1999, Do credit spreads reflect stationary leverage ratios? Reconciling structural and reduced-form frameworks, Working paper, Carnegie Mellon University.

Cox, John, Jonathan Ingersoll, and Stephen Ross, 1985, A theory of the term structure of interest rates, *Econometrica* 53, 385–407.

Duffie, Darrell, and David Lando, 2001, Term structure of credit spreads with incomplete accounting information, *Econometrica* 69, 633–664.

Duffie, Darrell, and Kenneth Singleton, 1999, Modeling term structures of defaultable bonds, *Review of Financial Studies* 12, 687–720.

Erdelyi, Arthur, 1954, *Tables of Integral Transforms* (McGraw-Hill, New York).

Fama, Eugene, and Kenneth French, 1988, Permanent and temporary components of stock prices, *Journal of Political Economy* 96, 246–273.

Fleming, Thomas, and David Harrington, 1991, *Counting Processes and Survival Analysis* (John Wiley & Sons, Inc., New York).

Goetzmann, William, and Phillipe Jorion, 1999, Global stock markets in the twentieth century, *Journal of Finance* 54, 953–980.

Grandell, Jan, 1976, *Doubly Stochastic Poisson Processes* (Springer-Verlag, New York).

Ingersoll, Jonathan Jr., 1987, *Theory of Financial Decision Making* (Rowman and Littlefield, Totowa, NJ).

Jarrow, Robert, David Lando, and Stuart Turnbull, 1997, A Markov model for the term structure of credit spreads, *Review of Financial Studies* 10, 481–523.

Jarrow, Robert, and Stuart Turnbull, 1995, Pricing options on financial securities subject to default risk, *Journal of Finance* 50, 53–86.

Karatzas, Ioannis, and Steven Shreve, 1991, *Brownian Motion and Stochastic Calculus* (Springer-Verlag, New York).

Mehra, Rajnish, and Edward Prescott, 1985, The equity premium: A puzzle, *Journal of Monetary Economics* 15, 145–161.

Merton, Robert C., 1974, On the pricing of corporate debt: The risk structure of interest rates, *Journal of Finance* 29, 449–470.

Protter, Philip, 1990, *Stochastic Integration and Differential Equations* (Springer-Verlag, New York).

Ross, Philip, 1997, History, as written by the winners, *Forbes*, June 16, 206–214.

Siegel, Jeremy, 1998, *Stocks For The Long Run* (McGraw-Hill, New York).