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Implications of the Discreteness of Observed Stock Prices

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ABSTRACT

Stock prices on the organized exchanges are restricted to be divisible by ⅛. Therefore, the "true" price usually differs from the observed price. This paper examines the biases resulting from the discreteness of observed stock prices. It is shown that the natural estimators of the variance and all of the higher order moments of the rate of returns are biased. An approximate set of correction factors is derived and a procedure is outlined to show how the correction can be made. The natural estimators of the "beta" and of the variance of the market portfolio, on the other hand, are "nearly" unbiased.

THE BEHAVIOR OF STOCK PRICES has been an issue of interest to the financial economist for many years. This interest resulted in a growing number of empirical studies which attempt to estimate this behavior (e.g., Blattberg and Gonedes [2], Fama [6], Fama and Roll [7, 8], Barnea and Downes [1]). To date, stock price behavior is estimated under the assumption that the observed trading price is the "true" equilibrium price. However, observed stock prices and stock price changes on the organized exchanges are restricted to multiples of ⅛ of a dollar.[1] Therefore, if the "true" distribution of stock prices is continuous, an observed trading price can be different from the "true" price. This paper examines the biases in estimating the moments of stock price changes caused by the discreteness of observed stock prices. The major focus of the paper is in noting this problem, providing a model which explains the source of these biases, and quantifying their size.

Section I demonstrates that due to the discrete nature of observed stock prices the natural estimators for the variance and for the higher order moments of the rate of returns are biased upward. This bias is larger for stocks with lower prices and smaller standard deviation. For instance, assuming that the standard deviation, σ, is 0.001, the stock price is one dollar, the "true" probability distribution of stock prices is lognormal, and the observed prices are as close as possible to the "true" prices, then the natural estimator of σ has expectation 0.01400; hence, it is biased upward by 1300%. Significant biases have important implications in option pricing.

We derive an approximate set of correction factors which can be applied to the

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[1] Provided that the price of the stock is greater than one dollar.

natural estimators to make them nearly unbiased. Using Tables I, II, and III the biases are quantified and a procedure is outlined to show how the corrections can be made. A further approximation good for moments of rates of returns which are calculated on an infinitesimal interval of time is derived in Theorem 2.

Interestingly, the discreteness of observed stock prices has little effect on the estimation of the "market model." The observed price of a well-diversified portfolio is an excellent proxy for its equilibrium price. Therefore, the natural estimator of the portfolio variance is "nearly" unbiased. No significant biases exist for the natural estimator of the portfolio measure of risk (i.e., β) by the above argument, as the covariance is a bilinear form.

Section II contains conclusions and implications for further research. Also, it relates these results to previously documented empirical regularities.

I. The Discontinuity in Price Changes and the Distribution of Stock Returns

In this section, we show that, due to the discrete nature of observed stock prices, the natural estimators for the variance of the rate of return and for its higher order moments are biased. These biases are shown to be very large for stocks with a smaller standard deviation and a low price. We derive the magnitude of this bias and use it to suggest "improved estimators."

A. An Example

Before presenting the model, we give a simple example which highlights the sources of the biases which are caused by ignoring the discreteness of observed stock prices. Let $\{b(t) \mid 0 \leq t \leq 1\}$ be a Brownian motion with unknown variance, σ^2, a known drift $\mu = 0$, and $b(0) = m + ½$, m an integer. Let $\hat{b}(t) = \{k \text{ if } k - ½ < b(t) \leq k + ½\}$.

$$S_n = \sum_{t=0}^{2^n-1} \left(b\left(\frac{t+1}{2^n}\right) - b\left(\frac{t}{2^n}\right)\right)^2$$

is a consistent estimator of σ^2. In fact, $E(S_n) = \sigma^2$ and $\lim_{n\to\infty} S_n = \sigma^2$ with probability 1. Let

$$\hat{S}_n = \sum_{t=0}^{2^n-1} \left(\hat{b}\left(\frac{t+1}{2^n}\right) - \hat{b}\left(\frac{t}{2^n}\right)\right)^2.$$

On the face of it, $\hat{S}_n$ is also a consistent estimator for σ^2, but in fact $\lim_{n\to\infty} \hat{S}_n = +\infty$ with probability 1. Needless to say, $\hat{S}_n$ is not a consistent estimator of σ^2.

B. The Model

We now present our model.

Let $\{\hat{P}(t), \tau_0 \leq t \leq \tau_1\}$ be the stochastic process of a given stock price on $[\tau_0, \tau_1]$ where the prices are restricted to some lattice $\{nd; n \geq 0\}$ and d is the discontinuity in stock price changes (d is ⅛ of a dollar for stocks traded in the

NYSE).[2] These prices are observed at points of time in the set $\{\tau_i' = \tau_0 + i\Delta t;\ 0 \le i \le T, \tau_T' = \tau_1\}$ and Δt is, for example, one day. We assume that the underlying equilibrium price process $\{P(t), \tau_0 \le t \le \tau_1\}$ is a lognormal diffusion, that is to say, it satisfies the following stochastic differential equation,

$$dP = \mu P dt + \sigma P db. \tag{1}$$

Letting $\mu^* = \mu - \frac{1}{2}\sigma^2$, we then have

$$P(t) = P(0)e^{\sigma b(t) + \mu^* t}.$$

To quantify the potential biases which are induced by the discrete nature of stock prices, we have to specify the relationship between $\hat{P}$ and P. We choose the following:

$$\hat{P}(t) = \{nd \quad \text{if} \quad nd - d/2 < P(t) \le nd + d/2\}. \tag{2}$$

That is, the observed price at time t, $\hat{P}(t)$, is the closest one to the "true" price, $P(t)$. This relationship seems logical although we clearly cannot justify it empirically, as $P(t)$ is not observable.

We examine the sensitivity of our results to alternate relationships between $P(t)$ and $\hat{P}(t)$. Specifically, we later consider models (2a) and (2b) below:

$$\hat{P}(t) = \{nd \quad \text{if} \quad nd \le P(t) < (n+1)d\}, \tag{2a}$$

$$\hat{P}(t) = \{nd \quad \text{if} \quad (n-1)d < P(t) \le nd\}. \tag{2b}$$

It is found that the size of the biases is almost identical under each of the three relationships.

Next, we need to assume something about the distribution of $P(t) - d[P(t)/d]$ (where $[x]$ is the integer part of x). One approach would be to assume that we know $P(0)$ (but not $P(t)$) exactly and then use (1). A second approach would be to assume that $P(0)$ has some distribution on the interval $(\hat{P}(0) - d/2, \hat{P}(0) + d/2)$ where $\hat{P}(0)$ is known. Alternatively, we could assume that $P(t) - d[P(t)/d]$ is uniformly distributed. It turns out that for $\sigma P(t)\sqrt{t} > 0.3$ the three assumptions are, for numerical purposes, virtually identical, independent of the distribution chosen in the second approach.[3] This is discussed further in Subsection C. In Appendix A, we show that as $t \to \infty$, $P(t) - d[P(t)/d]$ has a limiting distribution uniform on $(-d/2, d/2]$ (for $\mu^* > 0$), independent of the initial distribution of $P(0)$. We start by formally assuming that $P(0)$ is known. This assumption runs counter to the spirit of the paper, but we justify it by saying that because of the

[2] We consider $\tau_1 - \tau_0$ small enough so as to argue time homogeneity.

[3] We verified numerically the speed of convergence of $P(t) - d\left[\frac{P(t)}{d}\right]$ to the uniform on $[-d/2, d/2]$; we found that for $\sigma P(0)\sqrt{t} > 0.3$,

$$0.05 - 10^{-5} \le \Pr\left(P(t) - d\left[\frac{P(t)}{d}\right] \in (\alpha d, (\alpha + 0.05)d)\right) \le 0.05 + 10^{-5}$$

for $\alpha = -0.5, -0.45, \cdots, 0.45$.

convergence of $P(t) - d[P(t)/d]$ to the uniform any reasonable assumption about the starting price, $P(0)$, would give virtually the same results.

THEOREM 1. *If* $\{\hat{P}(t) = nd\}$ *and* $P(0) = P_0$, *then for* $L \geq 1$,

$$E\{[\hat{P}(t + \Delta t) - \hat{P}(t)]^L\} = \sum_{j=-\infty}^{\infty} (jd)^L I_j, \tag{3a}$$

where

$$I_j = \int_{nd-d/2}^{nd+d/2} h(n, t, P_0, x)\left\{\Phi\left(\frac{1}{\sigma\sqrt{\Delta t}} \ln\left(\frac{(n + j + ½)d}{x}\right) - \frac{\mu^*(\Delta t)}{\sigma}\right)\right.$$
$$\left. - \Phi\left(\frac{1}{\sigma\sqrt{\Delta t}} \ln\left(\frac{(n + j - ½)d}{x}\right) - \frac{\mu^*(\Delta t)}{\sigma}\right)\right\} dx. \tag{3b}$$

Here, Φ is the standard normal distribution and $h(n, t, P_0, x)$ is the density of $P(t)$ given $P(t) \in ((n - ½)d, (n + ½)d]$. We will evaluate $h(n, t, P_0, x)$ in Lemma 1, but point out that for $\sigma P_0\sqrt{t} > 0.3$, $h(n, t, P_0, x) \sim 1/d$.

Proof: Conditional on $P(t) = x$, $x \in ((n - ½)d, (n + ½)d]$. Then

$$\Pr(\hat{P}(t + \Delta t) - \hat{P}(t) = jd \mid P(t) = x)$$
$$= \Pr(P(t + \Delta t) \in ((n + j - ½)d, (n + j + ½)d] \mid P(t) = x)$$
$$= \Pr(xe^{\sigma b(\Delta t)+\mu^*\Delta t} \in ((n + j - ½)d, (n + j + ½)d])$$
$$= \Pr\left[\frac{b(\Delta t)}{\sqrt{\Delta t}} \in \left(\frac{1}{\sigma\sqrt{\Delta t}} \ln\left(\frac{(n + j - ½)d}{x}\right) - \frac{\mu^*(\Delta t)}{\sigma},\right.\right.$$
$$\left.\left.\frac{1}{\sigma\sqrt{\Delta t}} \ln\left(\frac{(n + j + ½)d}{x}\right) - \frac{\mu^*(\Delta t)}{\sigma}\right)\right]$$
$$= \Phi\left(\frac{1}{\sigma\sqrt{\Delta t}} \ln\left(\frac{(n + j + ½)d}{x}\right) - \frac{\mu^*(\Delta t)}{\sigma}\right)$$
$$- \Phi\left(\frac{1}{\sigma\sqrt{\Delta t}} \ln\left(\frac{(n + j - ½)d}{x}\right) - \frac{\mu^*(\Delta t)}{\sigma}\right)$$

as

$$\frac{b(\Delta t)}{\sqrt{\Delta t}} \sim b(1).$$

Then $I_j = \Pr(\hat{P}(t + \Delta t) - \hat{P}(t) = jd \mid P(t) \in ((n - ½)d, (n + ½)d])$ satisfies (3b). By integrating over x the theorem follows.

LEMMA 1.

$$h(n, t, P_0, x) = \frac{\dfrac{d}{dx}\Phi\left(\dfrac{1}{\sigma\sqrt{t}}\ln\left(\dfrac{x}{P_0}\right) - \dfrac{\mu^*\sqrt{t}}{\sigma}\right)}{\Phi\left(\dfrac{1}{\sigma\sqrt{t}}\ln\left(\dfrac{(n+1/2)d}{P_0}\right) - \dfrac{\mu^*\sqrt{t}}{\sigma}\right) - \Phi\left(\dfrac{1}{\sigma\sqrt{t}}\ln\left(\dfrac{(n-1/2)d}{P_0}\right) - \dfrac{\mu^*\sqrt{t}}{\sigma}\right)} \tag{4}$$

Proof: For $x \in ((n - 1/2)d, (n + 1/2)d]$,

$$\Pr(P(t) \in ((n - 1/2)d, x] \mid P(t) \in ((n - 1/2)d, (n + 1/2)d]) = \frac{\Pr(P(t) \in ((n - 1/2)d, x])}{\Pr(P(t) \in ((n - 1/2)d, (n + 1/2)d])}$$

Now

$$\begin{aligned} \Pr(P(t) \in ((n - 1/2)d, x]) &= \Pr(P_0 e^{\sigma b(t) + \mu^* t} \in ((n - 1/2)d, x]) \\ &= \Pr\left(\frac{b(t)}{\sqrt{t}} \in \left(\frac{1}{\sigma\sqrt{t}}\ln\left(\frac{(n-1/2)d}{P_0}\right) - \frac{\mu\sqrt{t}}{\sigma}, \frac{1}{\sigma\sqrt{t}}\ln\left(\frac{x}{P_0}\right) - \frac{\mu^*\sqrt{t}}{\sigma}\right]\right) \\ &= \Phi\left(\frac{1}{\sigma\sqrt{t}}\ln\left(\frac{x}{P_0}\right) - \frac{\mu^*\sqrt{t}}{\sigma}\right) - \Phi\left(\frac{1}{\sigma\sqrt{t}}\ln\left(\frac{(n-1/2)d}{P_0}\right) - \frac{\mu^*\sqrt{t}}{\sigma}\right). \end{aligned} \tag{5}$$

Dividing (5) by $\Pr(P(t) \in ((n - 1/2)d, (n + 1/2)d])$ and differentiating proves the lemma.

Theorem 1 is inconvenient as the moments depend upon P_0 and t. We will initially compute $E((\hat{P}(t + \Delta t) - \hat{P}(t))^L)$ using the uniform approximation that $h(n, t, P_0, x) = d^{-1}$. In the next section, we will discuss the approximation by comparing the computations for selected values of n, P_0, t, and σ using (3) exactly, with the approximations found in Table I.

C. The Biases in the "Estimated" Standard Deviation

To demonstrate the biases in the estimated instantaneous standard deviation of the rate of return of stock prices which are induced by the discreteness of prices, formula (3a,b) (with the approximation that $h(n, t, P_0, x) = d^{-1}$) is evaluated numerically. The biases are shown in Table I and Figure 1.

Table I depicts the values of the measured instantaneous standard deviation of the rate of returns of stock prices for various levels of stock price and "true" instantaneous standard deviation. The underlying equilibrium price process is assumed to be a lognormal diffusion with $\mu = 0$ and the discontinuity in stock price changes to be 1/8 of a dollar.[4] The table is organized as follows. The first row gives the various levels of the true instantaneous standard deviation, which

[4] The calculations were done for various levels of μ and the numbers hardly changed.

Table I

Ratios of the Natural Estimator of the Instantaneous Standard Deviation to the "True" Instantaneous Standard Deviation of Stock Returns, $\hat{\sigma}/\sigma$, Assuming That the Underlying Equilibrium Price Process Is a Lognormal Diffusion with $\mu = 0$

Stock	σ^a							
Price	1×10^{-3}	3×10^{-3}	5×10^{-3}	7×10^{-3}	9×10^{-3}	11×10^{-3}	13×10^{-3}	15×10^{-3}
1.0	14.003	6.026	4.538	3.805	3.344	3.019	2.774	2.581
2.0	7.782	4.123	3.170	2.674	2.356	2.130	1.959	1.823
4.0	5.120	2.891	2.235	1.888	1.665	1.508	1.394	1.311
8.0	3.553	2.040	1.580	1.350	1.225	1.156	1.114	1.087
10.0	3.171	1.824	1.420	1.237	1.149	1.102	1.074	1.056
20.0	2.235	1.311	1.123	1.064	1.039	1.028	1.019	1.014
30.0	1.824	1.149	1.056	1.029	1.018	1.012	1.008	1.006
40.0	1.580	1.087	1.032	1.016	1.010	1.007	1.005	1.003
50.0	1.420	1.056	1.020	1.011	1.006	1.004	1.003	1.002
60.0	1.311	1.039	1.014	1.007	1.004	1.003	1.002	1.002
70.0	1.237	1.029	1.011	1.005	1.003	1.002	1.002	1.001
80.0	1.186	1.022	1.008	1.004	1.002	1.002	1.001	1.001
90.0	1.149	1.017	1.006	1.003	1.002	1.001	1.001	1.001
100.0	1.123	1.014	1.005	1.002	1.002	1.001	1.001	1.001

[a] σ is the assumed standard deviation of the "true" distribution.

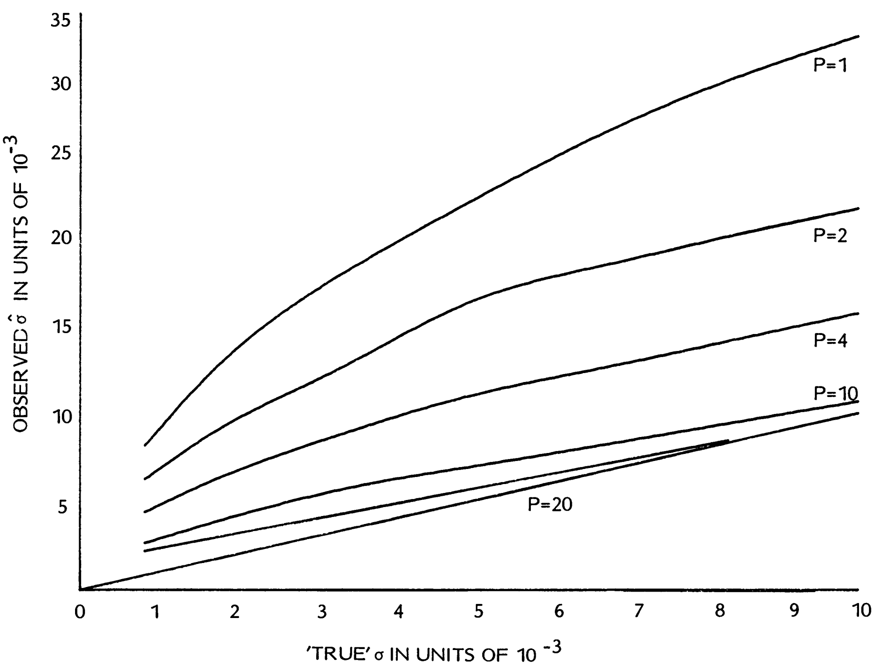

Figure 1. Values of observed instantaneous standard deviation of stock returns $\hat{\sigma}$ (measured on the y axis) as a function of the "true" instantaneous standard deviation, σ (measured on the x axis), for various price levels, P's. The underlying equilibrium price process is assumed to be a lognormal diffusion with $\mu = 0$.

range from 0.001 to 0.015, while the first column lists the various price levels. The corresponding ratios of the measured instantaneous standard deviation to "true" instantaneous standard deviation are contained in the table. For example, if $\sigma = 0.001$ and $P = 1$ the value of the ratio, $\hat{\sigma}/\sigma$, has expectation 14.003. This yields an upward bias of 1300%. For $\sigma = 0.005$ and $P = 20$, $\hat{\sigma}/\sigma = 1.123$; namely, an upward bias of 12.3%.

The relationships between observed instantaneous standard deviation, $\hat{\sigma}$, and the "true" instantaneous standard deviation, σ, are described further in Figure 1. The ratio $\hat{\sigma}/\sigma$ is monotonically decreasing in the product of P and σ. This result is intuitively appealing since the same discontinuity in price changes results in a bigger discontinuity in the rates of return the smaller the stock price is. Similarly, it has a bigger effect the smaller the true variance is.

Table II is an abbreviated version of Table I and is otherwise identical to it except that assumption (2) is replaced with (2a) and then (2b). It is seen that the biases in the observed instantaneous variances are only trivially different from those reported in Table I.

The substitution of d^{-1} for $h(n, t, P_0, x)$ typically has little effect upon the computations. In each case where $\sigma P_0 \sqrt{t} > 0.3$, the ratio of the bias under the

Table II

Ratios of the Natural Estimator of the Instantaneous Standard Deviation to the "True" Instantaneous Standard Deviation of Stock Returns, $\hat{\sigma}/\sigma$, Assuming That the Underlying Equilibrium Price Process Is a Lognormal Diffusion with $\mu = 0$ and Assumption (2) Is Replaced with (2a) and (2b), Respectively

Panel A: The observed stock price is equal to the true stock price rounded down (i.e., (2a)) to the nearest ⅛.

Stock Price	σ[a]		
	1×10^{-3}	7×10^{-3}	15×10^{-3}
1.0	14.025	3.918	2.660
4.0	5.156	1.403	1.324
10.0	3.181	1.243	1.062
20.0	2.239	1.067	1.018
40.0	1.5817	1.018	1.005

Panel B: The observed stock price is equal to the true stock price rounded up (i.e., (2b)) to the nearest ⅛.

Stock Price	σ[a]		
	1×10^{-3}	7×10^{-3}	15×10^{-3}
1.0	13.989	3.688	2.499
4.0	5.084	1.874	1.299
10.0	3.161	1.232	1.050
20.0	2.232	1.061	1.011
40.0	1.579	1.015	1.002

[a] σ is the assumed standard deviation of the "true" distribution.

uniform assumption to the bias without the uniform assumption but with P_0 known differed from 1 by at most 0.005. For $0.1 \le \sigma P_0 \sqrt{t} \le 0.3$, this difference was at most 0.03. For example, for $t = 400$ all the cases presented in Table I (except for $\sigma = 0.001$ and $P_0 = 1.0$, $\sigma = 0.003$ and $P_0 = 1.0$, $\sigma = 0.005$ and $P_0 = 1.0$, and $\sigma = 0.001$ and $P_0 = 2.0$) have $\sigma P_0 \sqrt{t} > 0.1$.[5]

Consider now the second approach where $\hat{P}(0)$ is known. The problem of saying something about the distribution of $P(0)$ given $\hat{P}(0)$ is identical to the problem of saying something about the distribution of $P(t)$ given $\hat{P}(t)$. That is to say it cannot be determined. However, our computational results allow us to get around this problem. Let's focus on the case where $\sigma nd\sqrt{t} > 0.3$.

Define

$$\phi(x) = E\{(\hat{P}(t + \Delta t) - \hat{P}(t))^2 \mid \hat{P}(t) = nd, \quad P(0) = x\}$$

We computed $\phi(x)$ for a range of x, where the size of the range was $4\sigma nd\sqrt{t}$, i.e., at least 1.2. We found that $\phi(x)$ hardly varied as a function of x. In particular $\dfrac{\max \phi(x)}{\min \phi(x)} \le 1.005$.

Now assume that $\hat{P}(0)$ is known. Then $P(0)$ has some distribution on $(\hat{P}(0) - d/2, \hat{P}(0) + d/2]$. Call the distribution F. F has a range of at most $0.125 \ll 1.2$.

So,

$$E(\hat{P}(t + \Delta t) - \hat{P}(t) \mid \hat{P}(0) = nd) = \int_{\hat{P}(0)-1/16}^{\hat{P}(0)+1/16} \phi(x) F(dx).$$

By the above inequality,

$$0.995 \le \frac{\int_{\hat{P}(0)-1/16}^{\hat{P}(0)+1/16} \phi(x) F(dx)}{\phi(\hat{P}(0))} \le 1.005$$

This inequality is independent of F. Further, since the range is 0.125 rather than 1.2, the inequality can probably be tightened by at least a factor of 10.

What this shows is that any of the assumptions below are, for numerical purposes, virtually identical:

1) Given $\hat{P}(0) = x$, then $P(0) = x$.
2) Given $\hat{P}(0) = x$, then $P(0) \sim F$, F a distribution on $(x - d/2, x + d/2]$.
3) Given $\hat{P}(t) = nd$, then $P(t) \sim \text{unif}(nd - d/2, nd + d/2]$.

The observation made in the example (see Section I. *A*) can be refined by Theorem 2.

THEOREM 2. *Assuming* (1) *and the uniform assumption, then for L even and*

[5] It is difficult to include a detailed table as the numbers depend on t and P_0. However, extensive computations were performed (including for various levels of μ) and the results are available from the authors upon request.

positive and $\{\hat{P}(t) = P\}$, $P = nd$, *then as* $\Delta t \to 0$,

$$E((\hat{P}(t + \Delta t) - \hat{P}(t))^L) \sim \frac{2Pd^{L-1}\sigma}{\sqrt{2\pi}} \sqrt{\Delta t}.$$ [6]

Proof: Given in Appendix B.

The practical import of this result is limited unless Δt is very small.

Table I can be used to obtain "improved estimators." Look along the row corresponding most closely to the price range of the security during the period in question; find the column (using linear interpolation if necessary) whose measured standard deviation (i.e., the ratio times the assumed σ) corresponds most closely to the observed standard deviation. The "true" standard deviation corresponding to that column (or its interpolated value) is the "improved estimator."

No significant biases exist for the natural estimator of the variance of a well-diversified portfolio because $P(t + \Delta t) - P(t) - (\hat{P}(t + \Delta t) - \hat{P}(t))$ is independent over the set of securities and is averaged out when considering the price of a well-diversified portfolio.

The above also obtains for the natural estimator of β (portfolio measure of risk) by the above argument and as covariance is a bilinear form. The technical details about these two arguments can be found in Gottlieb and Kalay [10].

D. The Normal Distribution

This section investigates the effects of discreteness of observed stock prices when the underlying distribution of stock returns on any Δt is normal. In fact, since for Δt small the lognormal distribution is very closely approximated by the normal, Theorem 3 is also an analytic approximation of (3a,b).

THEOREM 3. *If* $P(t + \Delta t) - P(t)$ *is distributed normally with mean* $\mu P(t)\Delta t$ *and variance* $\sigma^2 \Delta t P^2(t)$ *and making the uniform assumption, then if* $P(t) = nd$, *we have, for* $L \geq 1$,

$$E\{(\hat{P}(t + \Delta t) - \hat{P}(t))^L\} = \sum_{j=-\infty}^{\infty} (jd)^L I_j' \tag{6}$$

where

$$\begin{aligned} I_j' = \frac{1}{2l} \{ & [(a_j + l)\Phi(a_j + l) - (a_j - l)\Phi(a_j - l) - \eta(a_j - l) + \eta(a_j + l)] \\ & - [(a_{j-1} + l)\Phi(a_{j-1} + l) - (a_{j-1} - l)\Phi(a_{j-1} - l) \\ & - \eta(a_{j-1} - l) + \eta(a_{j-1} + l)]\}. \end{aligned}$$

Here η is the standard normal density,

$$a_j = \frac{(j + 1/2)d - nd\mu\Delta t}{nd\sigma\sqrt{\Delta t}},$$

$$l = \frac{d}{2n\sigma\sqrt{\Delta t}}.$$

[6] $\sim$ means asymptotic to.

Proof: See Gottlieb and Kalay [10].

To demonstrate the biases in the measured standard deviation and kurtosis of stock returns, formula (6) is evaluated numerically. The biases computed are very similar to those documented in Table I.

Table III depicts the relationship between the observed kurtosis of the rate of return of stock prices, $\hat{K}$, and the true standard deviation for various price levels. The first row of this table details the assumed standard deviation, σ, and the first column lists the various price levels. The observed kurtosis, $\hat{K}$, which under the assumed "true" distribution $N(0, \sigma^2)$ is zero, is shown to be biased upward. For example, for $P = 1.0$ and $\sigma = 0.001$, $\hat{K} = 153.695$. For $P = 10$ and $\sigma = 0.003$, $\hat{K} = 2.223$.

The upward bias in the observed kurtosis is documented further in Figure 2. This figure depicts the relationship between observed kurtosis and the "true" standard deviation. Similar to the bias in $\hat{\sigma}$, the upward bias in $\hat{K}$ is shown to be bigger for stocks with lower prices and smaller "true" standard deviation.

II. Conclusion and Implications

This paper demonstrates that the natural estimators of the variance and of the higher order moments of the rate of return of stocks are upward biased due to the discreteness of observed stock prices. These biases are bigger for stocks with low prices and smaller "true" standard deviation. Alternatively, these biases are larger the smaller the time interval in question. In particular, these biases would be huge in studies using transaction to transaction data; they would be significant in studies using daily data; and they would most likely be negligible in studies using monthly data.

These results are important as they shed light on previously documented empirical regularities. Moreover, they have important implications to a large body of empirical research. The related studies can be classified into several groups.

1. The density of the rates of return of stocks is found to depart from normality when the estimation is done using daily data and to be insignificantly different from normal when monthly data are used (e.g., Blattberg and Gonedes [2]). In particular, a positive kurtosis is measured when the estimation utilizes daily data. Moreover, as Barnea and Downes [1] show, when the time interval increases the estimated kurtosis decreases. This regularity can be the result of the discreteness of observed stock prices. As the time interval increases, the "true" standard deviation increases and therefore the measured kurtosis decreases (see Table III).

2. The methodology suggested by Masulis [12] and Brown and Warner [4] for event tests involves standardizing the excess returns by the natural estimator of the standard deviation. Since this natural estimator is biased upward, the test is biased toward the null hypothesis—i.e., finding no effect. This bias will differ across stocks depending on the levels of their prices and the magnitude of their "true" standard deviation.

3. The bias in the natural estimator of σ has to be considered in the implied standard deviation studies (e.g., Latane and Rendleman [11], Chiras and Manaster [5], Schmalensee and Trippi [13], Brenner and Galai [3]). The implied

Table III

Values of Measured Kurtosis of the Rate of Returns of Stock Prices Which Are Induced by the Price Movement Discontinuity. The Underlying Distribution is Assumed to be Normal (0, σ^2). Under These Assumptions the "True" Kurtosis Is Zero.

Stock	σ[a]							
Price	1×10^{-3}	3×10^{-3}	5×10^{-3}	7×10^{-3}	9×10^{-3}	11×10^{-3}	13×10^{-3}	15×10^{-3}
1.0	153.695	49.232	28.339	19.385	14.410	11.245	9.053	7.446
2.0	75.347	23.116	12.669	8.192	5.705	4.122	3.026	2.223
4.0	36.173	10.058	4.834	2.596	1.348	0.619	0.210	0.022
8.0	16.587	3.529	0.941	0.095	−0.071	−0.038	−0.044	−0.016
10.0	12.669	2.223	0.041	−0.055	−0.036	−0.061	−0.022	−0.009
20.0	4.834	0.024	−0.021	0.002	−0.002	0.001	0.001	0.001
30.0	2.22	−0.034	−0.005	−0.0005	0.0009	0.001	0.001	0.002
40.0	0.960	0.027	0.004	0.002	0.007	0.004	0.005	0.002
50.0	0.422	0.018	0.018	0.004	0.006	0.007	0.004	0.004
60.0	0.022	0.013	0.008	0.006	0.005	0.005	0.005	0.004
70.0	−0.055	0.011	0.018	0.008	0.011	0.006	0.005	0.007
80.0	−0.016	0.064	0.030	0.007	0.009	0.009	0.006	0.006
90.0	0.056	0.047	0.017	0.022	0.015	0.012	0.009	0.008
100.0	0.184	0.036	0.027	0.023	0.013	0.013	0.000	0.000

[a] σ is the assumed standard deviation of the "true" distribution.

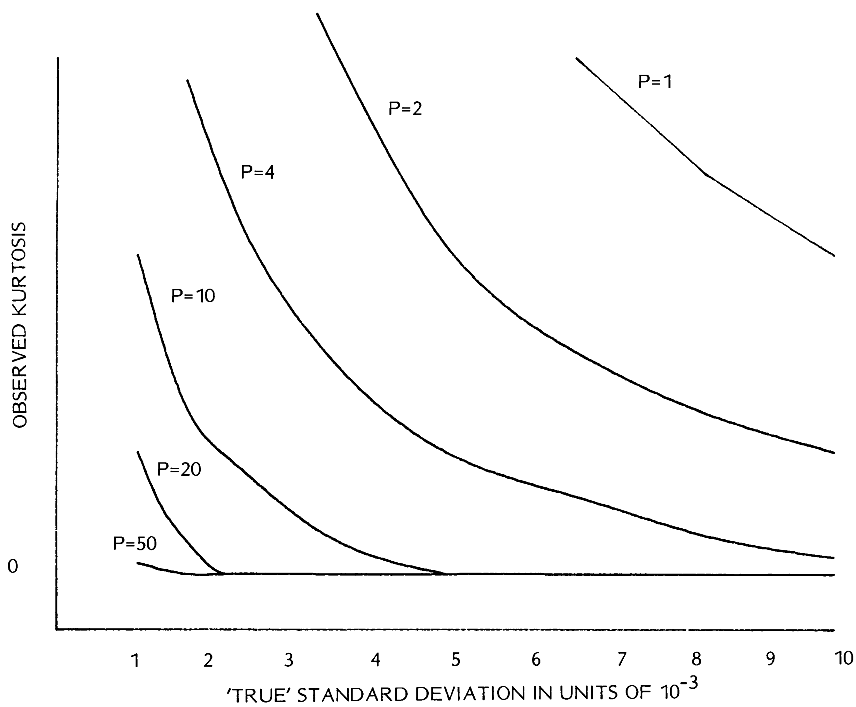

Figure 2. Values of the observed kurtosis of the rates of returns of stock (measured on the y axis) as a function of the true standard deviation (measured on the x axis) for various price levels, P's. The "true distribution" is assumed to be $N(0, \sigma^2)$ and price changes are assumed to be in multiples of ⅛ of a dollar.

standard deviation is an ex ante estimate of the "true" σ obtained from the various option pricing models. The predictive ability of this estimate is frequently compared to that of the natural estimator of σ. The discreteness of stock prices has to be considered for this comparison to be made properly.

The biases in the natural estimator of the variance are especially big when continuous data are utilized. Thus, the obvious next step, which is the subject of a paper in preparation, is to derive unbiased estimators using continuous data.

Appendix A

The uniform assumption is discussed. We begin for reasons of clarity by using the normal model. Then, $P(t) = \sigma b(t) + \mu t + P(0)$. Here $P(0)$ is the initial price, and $b(t)$ is standard Brownian motion. Let

$$Z(t) = P(t) - d[P(t)/d].$$

THEOREM A1. *For* $0 < a < a + \Delta < d$, $\lim_{t\to\infty}\Pr(Z(t) \in (a, a + \Delta)) = \Delta/d$.

Proof: Without loss of generality, set $d = 1$, and condition on $P(0) = P_0$.

$$\Pr(Z(t) \in (a, a + \Delta)) = \textstyle\sum_{n=-\infty}^{+\infty} \Pr(P(t) \in (a + n, a + n + \Delta)). \tag{A1}$$

Choose $0 < b < b + \Delta < 1$. We will show that

$$\lim_{t\to\infty} | \Pr(Z(t) \in (a, a + \Delta)) - \Pr(Z(t) \in (b, b + \Delta)) | = 0 \tag{A2}$$

which will suffice to prove the theorem.

$$\Pr(P(t) \in (n + a, n + a + \Delta)) \tag{A3}$$

$$= \Pr(\sigma b(t) + \mu t + P_0 \in (n + a, n + a + \Delta))$$

$$= \Pr\left(b(t) \in \left(\frac{n}{\sigma} + \frac{a}{\sigma} - \frac{P_0}{\sigma} - \frac{\mu t}{\sigma}, \frac{n}{\sigma} + \frac{a}{\sigma} - \frac{P_0}{\sigma} - \frac{\mu t}{\sigma} + \frac{\Delta}{\sigma}\right)\right) \tag{A3}$$

$$= \Pr\left(\frac{b(t)}{\sqrt{t}} \in \left(\frac{n}{\sigma\sqrt{t}} + \frac{a}{\sigma\sqrt{t}} - \frac{P_0}{\sigma\sqrt{t}} - \frac{\mu\sqrt{t}}{\sigma}, \frac{n}{\sigma\sqrt{t}} + \frac{a}{\sigma\sqrt{t}} - \frac{P_0}{\sigma\sqrt{t}} - \frac{\mu\sqrt{t}}{\sigma} + \frac{\Delta}{\sigma\sqrt{t}}\right)\right).$$

Recall that $\frac{b(t)}{\sqrt{t}}$ has the same distribution as $b(1)$, that is the standard normal distribution. Let Φ be the normal c.d.f. and η the normal density. Let

$$a(n, t) = \frac{n}{\sigma\sqrt{t}} + \frac{a}{\sigma\sqrt{t}} - \frac{P_0}{\sigma\sqrt{t}} - \frac{\mu\sqrt{t}}{\sigma},$$

$$a^+(n, t) = a(n, t) + \frac{\Delta}{\sigma\sqrt{t}},$$

$$b(n, t) = \frac{n}{\sigma\sqrt{t}} + \frac{b}{\sigma\sqrt{t}} - \frac{P_0}{\sigma\sqrt{t}} - \frac{\mu\sqrt{t}}{\sigma},$$

$$b^+(n, t) = b(n, t) + \frac{\Delta}{\sigma\sqrt{t}}.$$

From (A1) and (A3),

$$\Pr(Z(t) \in (a, a + \Delta)) = \textstyle\sum_{n=-\infty}^{+\infty} [\Phi(a^+(n, t)) - \Phi(a(n, t))]$$
$$= \textstyle\sum_{n=-\infty}^{+\infty} \displaystyle\int_{a(n,t)}^{a^+(n,t)} \eta(u)\, du.$$

Similarly,

$$\Pr(Z(t) \in (b, b + \Delta)) = \textstyle\sum_{n=-\infty}^{+\infty} \displaystyle\int_{b(n,t)}^{b^+(n,t)} \eta(u)\, du.$$

Define the functions

$$\alpha(t, u) = \begin{cases} 1 & \text{if} \quad u \in \cup_{n=-\infty}^{+\infty} (a(n, t), a^+(n, t)). \\ 0 & \text{otherwise.} \end{cases}$$

$$\beta(t, u) = \begin{cases} 1 & \text{if} \quad u \in \cup_{n=-\infty}^{+\infty} (b(n, t), b^+(n, t)). \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\Pr(Z(t) \in (a, a + \Delta)) = \int_{-\infty}^{+\infty} \alpha(t, u)\eta(u)\, du,$$

$$\Pr(Z(t) \in (b, b + \Delta)) = \int_{-\infty}^{+\infty} \beta(t, u)\eta(u)\, du.$$

For $\delta = b - a$,

$$\beta(t, u) = \alpha\left(t, u + \frac{\delta}{\sigma\sqrt{t}}\right) \quad \text{as} \quad b(n, t) = a(n, t) + \frac{\delta}{\sigma\sqrt{t}},$$

and

$$b^+(n, t) = a^+(n, t) + \frac{\delta}{\sigma\sqrt{t}}.$$

So,

$$\Pr(Z(t) \in (b, b + \Delta)) = \int_{-\infty}^{+\infty} \beta(t, u)\eta(u)\, du$$
$$= \int_{-\infty}^{+\infty} \alpha\left(t, u + \frac{\delta}{\sigma\sqrt{t}}\right)\eta(u)\, du$$
$$= \int_{-\infty}^{+\infty} \alpha(t, u)\eta\left(u - \frac{\delta}{\sigma\sqrt{t}}\right) du.$$

So,

$$
\begin{aligned}
&|\Pr(Z(t) \in (b, b+\Delta)) - \Pr(Z(t) \in (a, a+\Delta))| \\
&\quad = \left| \int_{-\infty}^{+\infty} \alpha(t, u)\eta\left(u - \frac{\delta}{\sigma\sqrt{t}}\right) du - \int_{-\infty}^{+\infty} \alpha(t, u)\eta(u)\, du \right| \\
&\quad \le \int_{-\infty}^{+\infty} |\alpha(t, u)| \left| \eta\left(u - \frac{\delta}{\sigma\sqrt{t}}\right) - \eta(u) \right| du \\
&\quad \le \int_{-\infty}^{+\infty} \left| \eta\left(u - \frac{\delta}{\sigma\sqrt{t}}\right) - \eta(u) \right| du.
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{t\to\infty} |\Pr(Z(t) \in (b, b+\Delta)) - \Pr(Z(t) \in (a, a+\Delta))| \\
&\qquad \le \lim_{t\to\infty} \int_{-\infty}^{+\infty} \left| \eta\left(u - \frac{\delta}{\sigma\sqrt{t}}\right) - \eta(u) \right| du \\
&\qquad = \int_{-\infty}^{\infty} \lim_{t\to\infty} \left| \eta\left(u - \frac{\delta}{\sigma\sqrt{t}}\right) - \eta(u) \right| du \\
&\qquad = 0. \qquad \text{(A4)}
\end{aligned}
$$

(A4) proves the theorem.

For the lognormal model, we have $P(t) = P(0)e^{\sigma b(t)+\mu^* t}$. Again, let

$$Z(t) = P(t) - d[P(t)/d]. \qquad \text{(A5)}$$

THEOREM A2. *If* $\mu^* > 0$, $0 < a < a + \Delta < d$,

$$\lim_{t\to\infty} \Pr(Z(t) \in (a, a+\Delta)) = \Delta/d.$$

Proof: Again, without loss of generality, set $d = 1$, and condition on $P(0) = P_0$. Choose b as in the previous theorem. We will show that (A2) still holds given definition (A5). Now, reset

$$a(n, t) = \frac{\ln(n+a)}{\sigma\sqrt{t}} - \frac{P_0}{\sigma\sqrt{t}} - \frac{\mu^*\sqrt{t}}{\sigma},$$

$$a^+(n, t) = \frac{\ln(n+a+\Delta)}{\sigma\sqrt{t}} - \frac{P_0}{\sigma\sqrt{t}} - \frac{\mu^*\sqrt{t}}{\sigma},$$

$$b(n, t) = \frac{\ln(n+b)}{\sigma\sqrt{t}} - \frac{P_0}{\sigma\sqrt{t}} - \frac{\mu^*\sqrt{t}}{\sigma},$$

$$b^+(n, t) = \frac{\ln(n+b+\Delta)}{\sigma\sqrt{t}} - \frac{P_0}{\sigma\sqrt{t}} - \frac{\mu^*\sqrt{t}}{\sigma}.$$

Arguing as in the previous theorem, we get

$$\Pr(Z(t) \in (a, a + \Delta)) = \textstyle\sum_{n=-\infty}^{+\infty} \displaystyle\int_{a(n,t)}^{a^{+}(n,t)} \eta(u)\, du,$$

$$\Pr(Z(t) \in (b, b + \Delta)) = \textstyle\sum_{n=-\infty}^{+\infty} \displaystyle\int_{b(n,t)}^{b^{+}(n,t)} \eta(u)\, du.$$

For $\mu^* > 0$, $\lim_{t\to\infty} a(n, t) = -\infty$, $\lim_{t\to\infty} b(n, t) = -\infty$.

So, for arbitrary $M > 0$,

$$\lim_{t\to\infty} \Pr(Z(t) \in (a, a + \Delta)) = \lim_{t\to\infty} \textstyle\sum_{n=M}^{\infty} \displaystyle\int_{a(n,t)}^{a^{+}(n,t)} \eta(u)\, du, \tag{A6}$$

$$\lim_{t\to\infty} \Pr(Z(t) \in (b, b + \Delta)) = \lim_{t\to\infty} \textstyle\sum_{n=M}^{\infty} \displaystyle\int_{b(n,t)}^{b^{+}(n,t)} \eta(u)\, du. \tag{A7}$$

Let

$$e(n, t) = b(n, t) - a(n, t),$$
$$e^{+}(n, t) = b^{+}(n, t) - a^{+}(n, t),$$
$$r(n, t) = e^{+}(n, t) - e(n, t).$$

Now,

$$\int_{b(n,t)}^{b^{+}(n,t)} \eta(u)\, du = \int_{a(n,t)+e(n,t)}^{a^{+}(n,t)+e(n,t)} \eta(u)\, du + \int_{a^{+}(n,t)+e(n,t)}^{a^{+}(n,t)+e(n,t)+r(n,t)} \eta(u)\, du$$

$$= \int_{a(n,t)}^{a^{+}(n,t)} \eta(u + e(n, t))\, du + \int_{a^{+}(n,t)+e(n,t)}^{a^{+}(n,t)+e(n,t)+r(n,t)} \eta(u)\, du \tag{A8}$$

From (A6), (A7), and (A8), we get

$$\lim_{t\to\infty} |\Pr(Z(t) \in (b, b + \Delta)) - \Pr(Z(t) \in (a, a + \Delta))|$$

$$\le \lim_{t\to\infty} \textstyle\sum_{n=M}^{\infty} \displaystyle\int_{a(n,t)}^{a^{+}(n,t)} |\eta(u) - \eta(u + e(n, t))|\, du \tag{A9}$$

$$+ \lim_{t\to\infty} \textstyle\sum_{n=M}^{\infty} |r(n, t)|.$$

The first term on the right-hand side of (A9) is 0 as $\lim_{t\to\infty} e(n, t) = 0$.

For $\delta = b - a$,

$$\sigma\sqrt{t}r(n, t) = \ln(n + a + \delta + \Delta) - \ln(n + a + \Delta)$$
$$- \ln(n + a + \delta) + \ln(n + a)$$
$$= \ln\left(1 + \frac{\Delta}{n + a + \delta}\right) - \ln\left(1 + \frac{\Delta}{n + a}\right)$$

$$= \frac{\Delta}{n + a + \delta} + 0\left(\frac{1}{n^2}\right) - \frac{\Delta}{n + a} + 0\left(\frac{1}{n^2}\right)$$

$$= \frac{-\delta\Delta}{(n + a)(n + a + \delta)} + 0\left(\frac{1}{n^2}\right)$$

$$= 0\left(\frac{1}{n^2}\right).$$

So, $\sigma\sqrt{t}\ \sum_{n=M}^{\infty} | r(n, t) | = 0(1)$. Hence,

$$\lim_{t\to\infty} \sum_{n=M}^{\infty} | r(n, t) | = 0.$$

So, the left-hand side of (A9) is zero, proving the theorem.

Appendix B

This appendix contains the proof of Theorem 2. We need to show that

$$\lim_{\Delta t\to 0} (\Delta t)^{-1/2} \sum_{j=-\infty}^{\infty} (jd)^L \Pr(\hat{P}(t + \Delta t) - \hat{P}(t) = jd \mid \hat{P}(t) = nd) = \frac{2nd^L}{\sqrt{2\pi}}. \tag{B1}$$

We assume throughout that $P(t) \sim$ uniform $(nd - d/2, nd + d/2]$.

We first show that if the above sum is restricted to $j \geq 2$ ($j \leq -2$) the limit is equal to zero.

$$\sum_{j=2}^{\infty} j^L \Pr(\hat{P}(t + \Delta t) - \hat{P}(t) = jd \mid \hat{P}(t) = nd) \tag{B2}$$

$$\leq \sum_{j=2}^{\infty} j^L \Pr(\hat{P}(t + \Delta t) - \hat{P}(t) \geq jd \mid \hat{P}(t) = nd)$$

$$\leq \sum_{j=2}^{\infty} j^L \Pr(\hat{P}(t + \Delta t) - \hat{P}(t) \geq (j - 1)d \mid P(t) = (n + \tfrac{1}{2})d)$$

$$= \sum_{j=2}^{\infty} j^L \Pr\left(e^{\sigma b(\Delta t) + \mu^* \Delta t} \geq \frac{n + j - 1 + \tfrac{1}{2}}{n + \tfrac{1}{2}}\right)$$

$$\leq \sum_{j=2}^{\infty} j^L \Pr\left(e^{\sigma b(\Delta t) + \mu^* \Delta t} - 1 \geq \frac{j - \tfrac{1}{2}}{n + \tfrac{1}{2}}\right)$$

$$\leq \sum_{j=2}^{\infty} j^L \Pr\left(Y(\Delta t) \geq \frac{1}{4(n + 1)}\right) \quad \text{where} \quad Y(\Delta t) = e^{\sigma b(\Delta t) + \mu^* \Delta t} - 1$$

$$\leq \sum_{j=2}^{\infty} j^L \frac{[4(n + 1)]^{L+2}}{j^{L+2}} E(Y(\Delta t))^{L+2}$$

by Chebychev's inequality, $L \geq 2$, even

$$\leq KE(Y(\Delta t))^{L+2}. \tag{B3}$$

Here,

$$K = 4(n + 1)^{L+2} \sum_{j=2}^{\infty} j^{-2} < \infty.$$

Now, using the Taylor expansion of e^x,

$$E(Y(\Delta t))^{L+2} = E\left(\sigma b(\Delta t) + \mu^*(\Delta t) + \frac{[\sigma b(\Delta t) + \mu^*(\Delta t)]^2}{2} + \cdots\right)^{L+2}. \quad \text{(B4)}$$

Now,

$$E(b(\Delta t))^m = \begin{cases} 0 & \text{if } m \text{ odd.} \\ (m-1)(m-3)\cdots 1(\Delta t)^{m/2} & \text{if } m \text{ even.} \end{cases}$$

So, $E\{Y(\Delta t)\}^{L+2} = o(\Delta t)^{1/2}$. Thus,

$$\lim_{\Delta t\to 0}(\Delta t)^{-1/2} \textstyle\sum_{j=2}^{\infty} (jd)^L \Pr(\hat{P}(t+\Delta t) - \hat{P}(t) = jd \mid P(t) = nd) = 0. \quad \text{(B5)}$$

Similarly,

$$\lim_{\Delta t\to 0}(\Delta t)^{-1/2} \textstyle\sum_{j=-\infty}^{-2} (jd)^L \Pr(\hat{P}(t+\Delta t) - \hat{P}(t) = jd \mid P(t) = nd) = 0. \quad \text{(B6)}$$

Now,

$$\lim_{\Delta t\to 0}(\Delta t)^{-1/2}\Pr(\hat{P}(t+\Delta t) - \hat{P}(t) = d \mid \hat{P}(t) = nd)$$

$$= \lim_{\Delta t\to 0}(\Delta t)^{-1/2}\Pr(\hat{P}(t+\Delta t) - \hat{P}(t) \geq d \mid \hat{P}(t) = nd) \quad \text{by (B5)}$$

$$= d^{-1}\lim_{\Delta t\to 0}(\Delta t)^{-1/2} \int_{x=nd-d/2}^{nd+d/2} \Pr(\hat{P}(t+\Delta t) - \hat{P}(t) \geq d \mid P(t) = x)\, dx$$

$$= d^{-1}\lim_{\Delta t\to 0}(\Delta t)^{-1/2}$$

$$\cdot \int_{x=nd-d/2}^{nd+d/2} \Pr(P(t+\Delta t) - P(t) \geq nd + d/2 - x \mid P(t) = x)\, dx$$

(set $y = nd + d/2 - x$)

$$= d^{-1}\lim_{\Delta t\to 0}(\Delta t)^{-1/2} \int_{y=0}^{d} \Pr(P(t+\Delta t) - P(t) \geq y \mid P(t) = nd + d/2 - y)\, dy$$

$$= d^{-1}\lim_{\Delta t\to 0}(\Delta t)^{-1/2} \int_{y=0}^{\varepsilon} \Pr(P(t+\Delta t) - P(t) \geq y \mid P(t) = nd + d/2 - y)\, dy. \quad \text{(B7)}$$

To see the validity of the last equality, note that in (B3) we can replace j by any $\varepsilon > 0$, and the resulting limit is still zero. So, the range of integration in (B7) can be restricted to $[0, \varepsilon]$, $\varepsilon > 0$, arbitrarily small, without changing the value of the limit.

$$= d^{-1}\lim_{\Delta t\to 0}(\Delta t)^{-1/2} \int_{y=0}^{\varepsilon} \Pr\left(e^{\sigma b(\Delta t)+\mu^*\Delta t} \geq 1 \frac{y}{nd + d/2 - y}\right) dy$$

$$= d^{-1}\lim_{\Delta t\to 0}(\Delta t)^{-1/2} \int_{y=0}^{\varepsilon} \Pr\left(b(\Delta t) \geq \frac{1}{\sigma} \ln\left(1 + \frac{y}{nd + d/2 - y}\right) - \frac{\mu^*\Delta t}{\sigma}\right) dy$$

$$= d^{-1}\lim_{\Delta t\to 0}(\Delta t)^{-1/2} \int_{0}^{\varepsilon} \Pr\left(b(1) \geq \frac{1}{\sigma\sqrt{\Delta t}} \ln\left(1 + \frac{y}{nd + d/2 - y}\right) - \frac{\mu^*\sqrt{\Delta t}}{\sigma}\right) dy.$$

Now for arbitrarily small $\beta > 0$, we can choose $\varepsilon > 0$, small enough so that for $y \in [0, \varepsilon]$,

$$\frac{(1-\beta)y}{\sigma(nd + d/2)} - \frac{\mu^*\sqrt{\Delta t}}{\sigma}$$

$$\leq \frac{1}{\sigma}\ln\left(1 + \frac{y}{nd + d/2 - y}\right) - \frac{\mu^*\sqrt{\Delta t}}{\sigma} \leq \frac{(1+\beta)y}{\sigma(nd + d/2)} - \frac{\mu^*\sqrt{\Delta t}}{\sigma}. \quad \text{(B8)}$$

So, letting $\omega = \dfrac{1}{\sigma(nd + d/2)}$,

$$d^{-1}\lim_{\Delta t\to 0}(\Delta t)^{-1/2}\int_{y=0}^{\varepsilon}\Pr\left(b(1) \geq \frac{\omega(1+\beta)y}{(\Delta t)^{1/2}} - \frac{\mu^*\sqrt{\Delta t}}{\sigma}\right)dy$$

$$\leq d^{-1}\lim_{\Delta t\to 0}(\Delta t)^{-1/2}$$

$$\cdot\int_{y=0}^{\varepsilon}\Pr\left(b(1) \geq \frac{1}{\sigma\sqrt{\Delta t}}\ln\left(1 + \frac{y}{nd + d/2 - y}\right) - \frac{\mu^*\sqrt{\Delta t}}{\sigma}\right)dy$$

$$\leq d^{-1}\lim_{\Delta t\to 0}(\Delta t)^{-1/2}\int_{y=0}^{\varepsilon}\Pr\left(b(1) \geq \frac{\omega(1-\beta)y}{(\Delta t)^{1/2}}\right)dy. \quad \text{(B9)}$$

Now,

$$d^{-1}\lim_{\Delta t\to 0}(\Delta t)^{-1/2}\int_{y=0}^{\varepsilon}\Pr\left(b(1) \geq \frac{\omega(1+\beta)y}{(\Delta t)^{1/2}} - \frac{\mu^*(\Delta t)^{1/2}}{\sigma}\right)dy$$

$$= d^{-1}\lim_{\Delta t\to 0}(\Delta t)^{-1/2}\int_{y=0}^{\varepsilon}\frac{1}{\sqrt{2\pi}}\int_{u=\frac{\omega(1+\beta)y}{(\Delta t)^{1/2}} - \frac{\mu^*(\Delta t)^{1/2}}{\sigma}}^{\infty} e^{-u^2/2}\,du\,dy$$

$$= d^{-1}\lim_{\Delta t\to 0}\frac{(\Delta t)^{-1/2}}{\sqrt{2\pi}}\int_{u=\frac{-\mu^*(\Delta t)^{1/2}}{\sigma}}^{\infty}\int_{y=0}^{\frac{u(\Delta t)^{1/2}}{\omega(1+\beta)} + \frac{\mu^*\Delta t}{\omega(1+\beta)\sigma}} e^{-u^2/2}\,dy\,du$$

$$= d^{-1}\lim_{\Delta t\to 0}\frac{(\Delta t)^{-1/2}}{\sqrt{2\pi}}\int_{u=\frac{-\mu^*(\Delta t)^{1/2}}{\sigma}}^{\infty}\left(\frac{u(\Delta t)^{1/2}}{\omega(1+\beta)} + \frac{\mu^*\Delta t}{\omega(1+\beta)\sigma}\right)e^{-u^2/2}\,du$$

$$= d^{-1}\lim_{\Delta t\to 0}\frac{(\Delta t)^{-1/2}}{\sqrt{2\pi}}\,\frac{(\Delta t)^{1/2}}{\omega(1+\beta)}\,e^{-\mu^{*2}\Delta t/2\sigma^2}$$

$$+ d^{-1}\lim_{\Delta t\to 0}\frac{(\Delta t)^{-1/2}}{\sqrt{2\pi}}\,\frac{\Delta t\mu^*}{\omega(1+\beta)\sigma}\int_{u=\frac{-\mu^*(\Delta t)^{1/2}}{\sigma}}^{\infty} e^{-u^2/2}\,du$$

$$= \frac{d^{-1}}{\sqrt{2\pi}\,\omega(1+\beta)}.$$

As $\beta > 0$ was arbitrarily small, arguing from (B9) we get

$$\lim_{\Delta t \to 0}(\Delta t)^{-1/2}\Pr(\hat{P}(t + \Delta t) - \hat{P}(t) = d \mid \hat{P}(t) = nd) = \frac{d^{-1}}{\sqrt{2\pi}}\,\sigma(nd + d/2) = \frac{\sigma(n + 1/2)}{\sqrt{2\pi}}.$$

By similar arguments,

$$\lim_{\Delta t \to 0}(\Delta t)^{-1/2}\Pr(\hat{P}(t + \Delta t) - \hat{P}(t) = -d \mid \hat{P}(t) = nd) = \frac{\sigma(n - 1/2)}{\sqrt{2\pi}}.$$

So,

$$\lim_{\Delta t \to 0} E((\hat{P}(t + \Delta t) - \hat{P}(t))^L \mid \hat{P}(t) = nd)(\Delta t)^{-1/2} = \frac{2\sigma n d^L}{\sqrt{2\pi}} = \frac{2\sigma P d^{L-1}}{\sqrt{2\pi}}.$$

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