THE JOURNAL OF FINANCE • VOL. XLIII, NO. 4 • SEPTEMBER 1988

Estimation Bias Induced by Discrete Security Prices

CLIFFORD A. BALL*

ABSTRACT

Commonly, equilibrium security prices are modeled by continuous-state stochastic processes, while observed prices are rounded into discrete units. This paper models the rounding mechanism and examines the probabilistic structure of the resultant rounded process. We provide accurate and simple estimates of the inflation in estimated variance and kurtosis induced by ignoring rounding. In particular, the maximum-likelihood estimate of security price volatility using rounded prices is developed, and a simulation analysis is performed to examine the small-sample properties of this estimator. For many practical applications, a simple correction for rounding becomes available.

THERE IS AN EXTENSIVE literature in financial economics and statistics devoted to the modeling of equilibrium security price behavior and related parameter estimation. Indeed, economic theory has made much of the optimality of equilibrium prices. However, less insights are available on the effect institutional arrangements exert on the actual functioning of markets. From a statistical perspective, this article addresses one particular aspect of a market's microstructure, viz. the rounding of prices. Reviewing the finance literature, little work has been attempted regarding the effect of discreteness of prices on model parameter estimates. Recent exceptions include Ball, Torous, and Tschoegl [3], Cho and Frees [5], Gottlieb and Kalay [8], and Harris [9].

It is common to model the equilibrium security price by a continuous-time, continuous-state stochastic process. However, observed prices are measured in discrete time and in discrete units, usually to the nearest eighth of a dollar on the major U.S. exchanges. This rounding of prices affects significantly, in some cases, the estimation of a security's expected return, variance, kurtosis, and covariance with other securities. Gottlieb and Kalay [8] detail some of these effects. In this paper, we investigate the probabilistic structure of the rounded process and determine the effect rounding exerts on variance estimation for the underlying equilibrium process. As we shall see for the range of parameter values commonly observed in practice, very simple corrections are available. It must be stressed, however, that, if intraday data are employed or the level of rounding is severe, although estimates are available their precision may be very poor. Variance estimatiion is important intrinsically; moreover, there are ramifications

* Michigan Business School, University of Michigan. The author acknowledges useful conversations with Gary Gottlieb, Lawrence Harris, Clifford Qualls, Adrian Tschoegl, and Michael Woodroffe. Comments from seminar participants at London, Michigan, and Wharton Business Schools were very beneficial. Remaining errors are the author's responsibility. This research was supported by a summer research grant from Michigan Business School.

upon other areas of finance such as option pricing. (For example, see Ball and Torous [2].) The correction for variance estimation subject to rounding may be translated directly to the option-valuation problem.

The paper is organized as follows: Section I introduces our model for the rounding of prices and develops the structure of the rounded process. The rounded process is no longer Markovian, but, due to the Markovian structure of the underlying process, the transition probability of the rounded process is computable. Furthermore, the limiting form of this transition probability is mathematically tractable, and this result provides the framework to investigate the limiting behavior of the rounded process discussed in Section II. In particular, simple formulas are developed for gauging the bias in variance and the resultant kurtosis in returns. This is in direct contrast to the extensive numerical calculations performed by Gottlieb and Kalay. In conjunction with Sheppard's corrections, we eliminate the need for complex numerical calculations and put forward simple accurate alternatives usable in practical cases. Section III outlines some of the ramifications of our model. Specifically, the application to option pricing is discussed, and a range of plausible parameter values is examined. For example, for a security priced at ten dollars per share whose price dynamics are governed by a Geometric Brownian Motion with an annual standard deviation of $\sigma = 0.1587$ and subject to rounding to the nearest eighth of a dollar, the resultant bias in variance estimation is 26.04 percent.

Section IV examines the problem of estimating the variance of a price process on the basis of a set of observed rounded prices. Although the rounded process is not Markovian, we are able to develop a computable formula for the likelihood function for a set of observations. Employing numerical procedures, we implement the maximum-likelihood estimation (MLE) of the variance. Harris [9] also provides a variance estimator subject to rounding. To complete this study, Section V details a careful Monte Carlo simulation experiment designed to examine the small-sample properties of the MLE procedure and to verify the accuracy of various approximate corrections for rounding. Section VI presents our conclusions.

I. The Rounding Model

Let $S(t)$ be the price of a given security at time t. We assume that $\{S(t): t \in [0, \infty)\}$ is a continuous-time, continuous-state Markov process. A subclass of particular interest in financial modeling is the set of stochastic processes with stationary independent increments or perhaps monotonic transforms of such processes. Examples include Arithmetic and Geometric Brownian Motion and the compound Poisson process as introduced in financial economics by Merton [12]. For a discussion of such processes, see Feller [7], especially pp. 179–82. Some of our ensuing results are specific to the Brownian Motion process.

We assume that observations on the security price are taken at equally spaced time intervals although it is straightforward to generalize the model to account for any deterministic method of observation. Consider n equally spaced time periods and define

S_i = equilibrium security price at the end of the ith period,
T_i = observed rounded price at the end of the ith period,
$X_i = S_i - S_{i-1}$, the incremental equilibrium price change through the ith period, and
$K_i = T_i - T_{i-1}$, the incremental observed price change through the ith period,

where $i = 1, 2, \cdots, n$. The level of rounding is \$$d$, and we assume that T_i is the nearest integer multiple of \$$d$ to S_i, $i = 1, 2, \cdots, n$. Let $T_i = S_i + U_i$, so that U_i is the actual rounding at the end of the ith period. Initially, we assume that $S_0 = T_0$; however, we will discuss the relaxation of this assumption in Section III. To illustrate some of the problems induced by rounding, assume that $\{S(t): t \in [0, \infty)\}$ is a Brownian Motion process with zero drift and variance rate σ^2. Accordingly, $S_n = \sum_{i=1}^{n} X_i$, where $\{X_i\}$ are independently and identically distributed normal random variables with mean zero and variance σ^2. The optimal estimate of σ^2 is given by $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2$, and the sampling distribution of $\hat{\sigma}^2$ is well known. For the observed process, we have that $T_n = \sum_{i=1}^{n} K_i$. By contrast, $\{K_i\}$ are not identically distributed, are not independent and, furthermore, are no longer normally distributed. In fact, the $\{K_i\}$ are a set of discrete random variables. Rounding exerts a significant perturbation to the Brownian Motion model.

In general, we may link the $\{X_i\}$ and $\{K_i\}$ processes by means of the rounding mechanism:

$$K_1 = X_1 + U_1,$$

$$K_2 = -U_1 + X_2 + U_2,$$

$$K_3 = -U_2 + X_3 + U_3,$$

and in general for any integer n,

$$K_n = -U_{n-1} + X_n + U_n.$$

We examine the probabilistic structure of the $\{T_n\}$ process via a series of lemmata.

LEMMA 1: *$\{T_n\}$ has a computable transition probability.*

Proof: See Appendix A.

Applying Lemma 1 for Brownian Motion, one can readily show that

$$P[T_n = td \mid T_{n-1} = sd] = \frac{\int_{y=sd-d/2}^{sd+d/2} \left\{ \Phi\left(\frac{td + d/2 - y}{\sigma}\right) - \Phi\left(\frac{td - d/2 - y}{\sigma}\right) \right\} \phi\left(\frac{y}{\sqrt{(n-1)\sigma^2}}\right) dy}{\int_{y=sd-d/2}^{sd+d/2} \phi\left(\frac{y}{\sqrt{(n-1)\sigma^2}}\right) dy}, \quad (1)$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution and $\phi(\cdot)$ is the associated standard normal-density function.

For notational convenience, let

$$P_{n-1,n}(sd,\ td) = P[T_n = td \mid T_{n-1} = sd].$$

The limiting transition matrix of $\{T_n\}$ converges.

LEMMA 2: *Assuming a Browning Motion process for* $\{S(t): t \in [0, \infty)\}$,

$$\lim_{n\to\infty} P_{n-1,n}(sd,\ td) = P(sd,\ td),$$

where

$$P(sd,\ td) = \int_{y=sd-d/2}^{sd+d/2} \frac{1}{d} \times \left\{\Phi\left(\frac{td + d/2 - y}{\sigma}\right) - \Phi\left(\frac{td - d/2 - y}{\sigma}\right)\right\} dy. \quad (2)$$

Proof: See Appendix A.

Next consider the distribution of K_n:

$$P[K_n = kd,\ T_{n-1} = td] = P[T_n = (k + t)d \mid T_{n-1} = td] \times P[T_{n-1} = td].$$

Therefore,

$$P[K_n = kd] = \textstyle\sum_{t=-\infty}^{+\infty} P[T_n = (k + t)d \mid T_{n-1} = td] \times P[T_{n-1} = td]. \quad (3)$$

This can be represented in terms of the transition-probability function of the underlying Markov process using equation (A1). A more elegant representation is available for the limiting distribution of K_n:

$$\begin{aligned}
\lim_{n\to\infty} P[K_n = kd] &= \lim_{n\to\infty} \textstyle\sum_{t=-\infty}^{+\infty} P[T_n = (k+t)d \mid T_{n-1} = td] \\
&\quad \times P[T_{n-1} = td] \\
&= \textstyle\sum_{t=-\infty}^{+\infty} \lim_{n\to\infty} P[T_n = (k+t)d \mid T_{n-1} = td] \\
&\quad \times P[T_{n-1} = td] \\
&= \textstyle\sum_{t=-\infty}^{+\infty} P(td, (k+t)d) \times P[T_{n-1} = td] \\
&= P(td, (k+t)d \times \textstyle\sum_{t=-\infty}^{+\infty} P[T_{n-1} = td] \\
&= P(td, (k+t)d).
\end{aligned}$$

Interchange of limits and infinite sums are legitimate here since all sums are convergent and all terms are positive.

Let K be a discrete random variable with distribution function given by $P[K = kd] = P(td,\ (k + t)d)$. Clearly, $\lim_{n\to\infty} P[K_n = kd] = P[K = kd]$. Under the Brownian Motion assumption, we have the representation given by equation (2) for the distribution of K.

LEMMA 3: *Let U be a uniform random variable defined on the interval* $(-d/2, d/2]$. *Let X be a normal random variable with mean* 0 *and variance* σ^2. *Define* $[U + X]_d$ *to be the nearest integer multiple of d to $U + X$. Assume that U and X are independent. Then, under the assumption of Brownian Motion for the underlying Markov process, we have*

$$K = \lim_{n\to\infty} K_n = [U + X]_d.$$

Proof: See Appendix A.

Consider the following Markov chain: Suppose the underlying Markov process is a special modification of Brownian Motion. Specifically, at the end of each period restart the Markov process not at the point where it was but, rather, according to an independent uniform distribution centered about the rounded observation point with range $(-d/2, d/2]$. Under such a specification, define Q_i to be the observed rounded price at the end of the ith period, and $K_i^Q = Q_i - Q_{i-1}$. Clearly, the $\{Q_i\}$ form an independent increment process and $\{Q_i\}$ is a stationary Markov chain. The sequence $\{K_i^Q\}$ are independently and identically distributed random variables, each with distribution function defined by $[U + X]_d$. Provided that the initial process is perturbed by an independent uniform on $(-d/2, d/2]$, the $\{Q_i\}$ Markov chain will remain in equilibrium. The process $\{Q_i\}$ is a null recurrent, stationary Markov chain. For sufficiently large n, there is no material probabilistic difference between the $\{Q_i\}$ and $\{T_i\}$ processes. We may derive results for the $\{Q_i\}$ process and invoke them in a limiting case for the $\{T_i\}$ process.

It might be useful at this point to contrast these results in terms of the previous literature. Gottlieb and Kalay's [8] Theorem 1 develops $P[K_n = kd \mid T = td]$ under the assumption of Geometric Brownian Motion for the underlying Markov process. The conditional distribution of K_n is tractable. However, the unconditional distribution involves an infinite sum over the values of T_{n-1}; see equation (3). Gottlieb and Kalay use the "uniform approximation" to derive this conditional probability numerically. The elegance of our results does not extend exactly to the Geometric Brownian Motion case. As Gottlieb and Kalay point out, however, for practical purposes we may approximate the Geometric Brownian Motion process by a suitably chosen normal process. We shall see in later sections how accurate these types of approximations become. We conclude this section with a discussion of the actual rounding, U_n.

LEMMA 4: *Under the assumption of an underlying Brownian Motion process, as $n \to \infty$, U_n converges in distribution to a uniform random variable defined on $(-d/2, d/2]$.*

Proof: See Appendix A. This result is also discussed by Harris [9].

II. Properties of the Limiting Increment Process

For this part of the analysis, we maintain the assumption that the underlying Markov process is Brownian Motion. Therefore, invoking equations (1) and (3), we have that

$$P[K_n = kd] = \sum_{t=-\infty}^{+\infty} \left[\int_{y=td-d/2}^{td+d/2} \left\{ \Phi\left(\frac{(t+k)d + d/2 - y}{\sigma}\right) - \Phi\left(\frac{(t+k)d - d/2 - y}{\sigma}\right) \right\} \phi\left(\frac{y}{\sqrt{(n-1)\sigma^2}}\right) dy \right], \tag{4}$$

and, by Lemma 3,

$$\lim_{n\to\infty} P[K_n = kd] = \frac{1}{d}\int_{y=-d/2}^{+d/2} \left\{\Phi\left(\frac{kd + d/2 - y}{\sigma}\right) - \Phi\left(\frac{kd - d/2 - y}{\sigma}\right)\right\} dy$$

$$= P[[U + X]_d = kd]$$

$$= P[\lim_{n\to\infty} K_n = kd] \tag{5}$$

$$= P[K = kd].$$

We now consider the moments of K_n:

$$\mathrm{E}[K_n^m] = \textstyle\sum_{k=-\infty}^{+\infty} (kd)^m \times P[K_n = kd],$$

and

$$\lim_{n\to\infty} \mathrm{E}[K_n^m] = \textstyle\sum_{k=-\infty}^{+\infty} (kd)^m \lim_{n\to\infty} P[K_n = kd]$$

$$= \textstyle\sum_{k=-\infty}^{+\infty} (kd)^m \times P[K = kd].$$

Approximations for the moments of K_n are established in Appendix B.

Gottlieb and Kalay [8] develop similar results in their Theorem 2. They assume a Geometric Brownian Motion process for the underlying Markov process and establish the limiting even moments of K_n conditioned on the known value of T_{n-1}. However, their argument requires the length of each observation period to converge to zero. We now demonstrate the applicability of our simple estimates of the moments of K and contrast them directly with the extensive numerical calculations provided by Gottlieb and Kalay in their Tables I and III. We stress that the Gottlieb and Kalay results are strictly conditional variance estimates. Since Gottlieb and Kalay assume a Geometric Brownian Motion process for the underlying Markov process, we must approximate this process by a Brownian Motion process with variance $\theta = \sigma^2 P^2$, where P is the conditioned price and σ^2 is the instantaneous variance. Table I of Gottlieb and Kalay provides estimates of $\sqrt{\mathrm{E}[K^2]/\theta}$ for various levels of P and σ. Of course, for all practical purposes, the given statistic is a function of the product $\sigma \times P$ only. In Table I below, we list the Gottlieb and Kalay numerically calculated ratio R_1 with the ratio $R_2 = \sqrt{\mathrm{E}[K^2]/\theta}$, where $\mathrm{E}[K^2] = \dfrac{2d\sqrt{\theta}}{\sqrt{2\pi}}$ for various values of $\theta = \sigma^2 P^2$ and $d/\sigma P$. This table gives the ratio of the standard deviation of K to the true standard deviation of the increment of the underlying process. Absent rounding, this ratio is unity. For values of $d/\sigma P$ near 2.5, we do not expect an accurate alignment between R_1 and R_2. As $d/\sigma P$ increases, the ratios appear to close and then to widen, although the mathematics of the problem suggests that for large $d/\sigma P$ the precision of our approximation should improve markedly. Perhaps parts of Table I in Gottlieb and Kalay have been misrecorded. Alternatively, perhaps for the parameter t (not specified in Table I of Gottlieb and Kalay), the limiting results established

Table I

Ratios of the Standard Deviation of K to the True Standard Deviation of the Increment of the Underlying Process for Large Values of $d/\sigma P$[a]

σP	R_1	R_2	$d/\sigma P$
0.005	4.538	4.466	25.0
0.010	3.170	3.158	12.5
0.020	2.235	2.233	6.25
0.030	1.824	1.823	4.17
0.040	1.580	1.579	3.13
0.050	1.420	1.412	2.50

[a] R_1 is the Gottlieb and Kalay numerically calculated ratio, while R_2 is the simple approximation based on Appendix B.

in this paper are not attained for the very small values of σ selected by Gottlieb and Kalay.

It is well known that daily security returns are leptokurtic. To what degree is this phenomenon attributable to the rounding of prices? Employing equation (B7) under Brownian Motion with variance σ^2, we have that

$$E[K^2] = \frac{2d\sigma}{\sqrt{\pi}}$$

and

$$E[K^4] = \frac{2d^3\sigma}{\sqrt{\pi}},$$

subject to an error on the order of $\exp(-d^2/2\sigma^2)$. The kurtosis γ_2 is defined by

$$\gamma_2 = \frac{E[K^4]}{(E[K^2])} - 3.$$

Absent rounding and under our Brownian Motion assumptions, $\frac{E[X_i^4]}{(E[X_i])^2} - 3$ is zero for all i. Under our rounding model,

$$\gamma_2 = \frac{2d^3\sigma}{\sqrt{\pi}} \div \left(\frac{2d\sigma}{\sqrt{\pi}}\right)^2 - 3$$

$$= \frac{d}{\sigma}\sqrt{\frac{\pi}{2}} - 3,$$

subject to a minor error. Gottlieb and Kalay in their Table II compute kurtosis numerically for various values of σ and P under the assumption that price changes are normally distributed with mean zero and variance σ^2P^2. Here, γ_1 is the Gottlieb and Kalay computed kurtosis and γ_2 is the approximation developed herein with variance $\theta = \sigma^2P^2$. Table II displays these statistics for various levels of σP and $d/\sigma P$. We know that our approximations become very accurate as $d/\sigma P$

Table II

Comparison of the Estimated Kurtosis of K[a]

σP	γ_1	γ_2	$d/\sigma P$
0.005	28.339	28.333	25.00
0.010	12.669	12.666	12.50
0.020	4.834	4.833	6.25
0.030	2.223	2.222	4.17
0.040	0.941	0.917	3.13
0.050	0.041	0.133	2.50

[a] γ_1 is the Gottlieb and Kalay numerically computed kurtosis, while γ_2 is the simple approximation based on Appendix B.

increases. The results of Table II are consistent with our intuition. For values of $d/\sigma P$ near 2.5, we do not expect our results to be precise. In fact, the kurtosis is practically zero for such levels of $d/\sigma P$. As $d/\sigma P$ increases, γ_2 converges to γ_1. In summary, for $d/\sigma P \geq 4$, our simple approximation to the kurtosis provides very accurate results.

In this section, we have discussed the moments of K when $d/\sigma P$ is large. Next we extend our analysis to consider the case where $d/\sigma P$ is small. As pointed out in Lemma 3, $K = [U + X]_d$. This is exactly the framework from which to apply Sheppard's correction. For an extensive reference on Sheppard's corrections, see Kendall and Stuart [11], Sections 3.18 through 3.30. Applying the correction under Brownian Motion with variance σ^2,

$$E[K^2] = E([U + X]_d^2) + \frac{d^2}{12} + 0(d^2),$$

where $0(\cdot)$ is the asymptotic-order symbol. However,

$$E[X^2] = \sigma^2 \quad \text{and} \quad E[U^2] = \frac{d^2}{12},$$

so

$$E[K^2] = \sigma^2 + \frac{d^2}{6} + 0(d^2).$$

Alternatively, we may apply a bivariate Sheppard correction to K_n. Recalling our definitions in Section I,

$$K_n = [S_n]_d - [S_{n-1}]_d.$$

Upon approximation of Section 3.30 in Kendall and Stuart [11], we find that

$$\begin{aligned} E[K_n^2] &= E([S_n]_d - [S_{n-1}]_d)^2 \\ &= E([S_n]_d)^2 - 2E([S_n]_d[S_{n-1}]_d) + E([S_{n-1}]_d)^2 \\ &= n\sigma^2 + \frac{d^2}{12} - (2)(n-1)\sigma^2 + (n-1)\sigma^2 + \frac{d^2}{12} + 0(d^2) \\ &= \sigma^2 + \frac{d^2}{6} + 0(d^2). \end{aligned}$$

We will use these approximation results to extend Table I. Clearly,

$$\sqrt{\frac{E[K_n^2]}{\theta^2}} = \sqrt{1 + \frac{d^2}{6\sigma^2 P^2}} + 0(d^2)$$

when we assume that the changes in security prices are normal with mean zero and variance $\theta = \sigma^2 P^2$. Define $R_3 = (1 + d^2/6\sigma^2 P^2)^{1/2}$. Table III, which is an extension of Table I, gives the ratios R_1 and R_3 for various values of $\theta = \sigma^2 P^2$ and $d/\sigma P$. For values of $d/\sigma P$ below 2.5, the Sheppard correction provides a precise estimate of the bias in estimating variance subject to rounding. As $d/\sigma P$ increases, the relative accuracy of the approximation diminishes. However, for values of $d/\sigma P$ above 2.5, we may use the correction R_2. For practical purposes, no complicated numerical tables are required to quantify the bias induced by rounding.

The same Sheppard correction when applied to the kurtosis estimates is of order d^4. Therefore, for small levels of d, the approximate kurtosis is zero. Fortunately, Table II gives accurate predictions of the bias in estimating kurtosis due to rounding for reasonable levels of kurtosis.

In summary, we have documented the limiting distribution of K_n. Expansions involving infinite sums are available for the moments of K. For large values of $d/\sigma P$, the infinite sum may be truncated after one term to yield simple and accurate moment estimates. When truncation after one term becomes inaccurate, we may invoke Sheppard's correction to provide very accurate moment estimates for small values of $d/\sigma P$. In fact, for all values, simple accurate estimates of the moments are available. For all practical purposes, under Geometric Brownian Motion or Brownian Motion, the effects of rounding on the limiting distribution of K_n are correctable simply. It is also worthy to point out that Sheppard's corrections are free of distribution assumptions. Therefore, for small levels of d, whatever the underlying Markov process that drives security prices, a simple correction of $\frac{d^2}{6}$ is appropriate for estimating the variance of K_n.

Table III

Ratios of the Standard Deviation of K to the True Standard Deviation of the Increment of the Underlying Process for Small Values of $d/\sigma P$[a]

σP	R_1	R_3	$d/\sigma P$
0.050	1.420	1.429	2.50
0.070	1.237	1.238	1.79
0.080	1.186	1.186	1.56
0.090	1.149	1.150	1.39
0.110	1.102	1.102	1.14
0.130	1.074	1.074	0.96
0.150	1.056	1.056	0.83
0.220	1.028	1.027	0.57
0.360	1.010	1.010	0.35

[a] R_1 is the Gottlieb and Kalay numerically calculated ratio, while R_3 is the ratio based on Sheppard's correction.

III. Ramifications of the Model

In this section, we restrict our attention to Geometric Brownian Motion or Brownian Motion as underlying models. Since the development of the Center for Research in Security Prices (CRSP) data tapes, most empirical research into stock price behavior has been based on daily data. Certainly for monthly returns the problem of rounding is negligible. To concentrate our thinking in this section, consider the problem of option valuation for common stocks.

Black and Scholes [4] developed a closed-form expression for the value of call and put options written on common stock. Since then, the model has been extended to value both American and European options as well as to account for the payment of dividends by the underlying stock. The only parameters of the option model that are not directly measurable are the risk-free rate and the volatility. The risk-free rate is accurately estimated by the interest rate on the U.S. Treasury Bill maturing at option expiry. Therefore, for practical purposes, the only unknown parameter is the volatility, σ. Cox and Rubinstein [6], in their Appendix 6B, provide a table of all U.S. common stocks with exchange-traded options. The table lists volatility estimates for all these stocks, based on daily data from January 1, 1980 to January 1, 1984. On an annual basis, a low range for σ is 0.15 to 0.20, with a median range of around $\sigma = 0.30$. To enjoy option trading on the Chicago Board Option Exchange (CBOE), the underlying security must be priced at at least ten dollars per share. Special rules go into effect after a stock split. The greatest effects due to rounding occur when σP is small relative to d. On the major U.S. stock exchanges, prices are quoted to the nearest eighth of a dollar. It is not clear, however, that rounding is taking place to the nearest eighth of a dollar. Ball, Torous, and Tschoegl [3] consider the fixing prices of gold in U.S. dollars per ounce at the London market. The maximum precision of pricing in the London gold market is to the nearest twentieth of a dollar. They provide clear evidence that the level of rounding is actually much higher and further may depend upon the price level and the amount of information available to the market. Clearly, on the major U.S. stock exchanges, an eighth of a dollar is merely a lower bound for the value of d.

To assess the effect of rounding in estimating option prices, we shall consider four sets of parameter values, as described in Table IV. We assume 252 trading days per year to convert daily rates to annual rates. Setting $d = \$1/8$ and taking the plausible parameter values given in Table IV, we see that effects due to rounding fall in the range where Sheppard's correction gives very accurate results. For example, when considering parameter set 1, the true value of σP is 0.1. However, the standard deviation of $\lim_{n\to\infty} K_n$ is given by $\sqrt{(0.1)^2 + d^2/6} = 0.1123$, resulting in a bias of 12.3 percent in the standard deviation or 26.04 percent in the variance. These significant biases are translated directly to the option-valuation problem. Fortunately, the correction needed is very simple. Unless intraday data are used, for all plausible parameter values the Sheppard correction will eliminate the bias induced in option pricing due to rounding of prices. An alternative model might integrate the bid/ask spread as a special form of rounding and so further refine the appropriate variance estimate to be inserted into the Black-Scholes formula. For an approach along these lines, see Harris [9].

Table IV

Parameter Values for the Simulation Analysis

Parameter Set	Security Price	Daily Variance	Daily Standard Deviation	Annual Standard Deviation	$d/\sigma P$	σP
1	$10	0.0001	0.0100	0.1587	1.250	0.100
2	$20	0.0001	0.0100	0.1587	0.625	0.200
3	$10	0.0002	0.0141	0.2245	0.884	0.141
4	$20	0.0002	0.0141	0.2245	0.442	0.281

If one is dealing with very low-priced stocks with very low volatility and perhaps is making intraday observations, then the biases may be similar to those in Table I. Gottlieb and Kalay claim that the initial location of the true process within the interval $(-d/2, d/2]$ is not crucial, although they actually express this in terms of different forms of rounding. For relatively large values of σP, the exact location of the initial-equilibrium security price process will not be important. However, for small values of σP, the initial location becomes increasingly relevant. Initially, suppose that $S(0) = nd$. We will examine how long it takes on average before the observed price changes. According to Karlin and Taylor [10], Theorem 5.1, let $\{S(t)\colon t \in [0, \infty)\}$ be a Brownian Motion process with variance rate σ^2, mean zero, and $S(0) = nd$. Let a, b with $a < nd < b$ be given, and let T be the first time the process reaches a or b. Then

$$\mathrm{E}[T \mid S(0) = nd] = \frac{(b - nd)(nd - a)}{\sigma^2}.$$

In order for the price to change, we require $S(t)$ to reach $nd \pm d/2$. Taking $a = nd = d/2$ and $b = nd + d/2$, we have

$$\mathrm{E}[T] = d^2/4\sigma^2.$$

For Geometric Brownian Motion, $\ln S(t) = \sigma B(t)$. Therefore, given $S(0) = nd$, we will see a price change whenever $S(t) = nd \pm d/2$ or, equivalently, whenever $\sigma B(t) = \ln(nd \pm d/2)$. By similar arguments we have

$$\mathrm{E}[T] = \frac{(\ln(nd + d/2) - \ln(nd)) \times (\ln(nd) - \ln(nd - d/2))}{\sigma^2}.$$

However,

$$\ln\left(\frac{(nd + d/2)}{nd}\right) \approx d/2nd,$$

$$\ln\left(\frac{nd}{(nd - d/2)}\right) \approx d/2nd, \quad \text{for} \quad n \text{ large},$$

so

$$\mathrm{E}[T] \approx d^2/4\sigma^2 P^2, \quad \text{where} \quad P = nd.$$

To illustrate, under Geometric Brownian Motion when $\sigma = 0.001$ and $P = \$10$, $\mathrm{E}[T] = 39.06$ days. More dramatically, when $\sigma = 0.001$ and $P = \$1$, $\mathrm{E}[T] = 3{,}906$ days, or 10.7 years. In this case, Gottlieb and Kalay calculate a bias of 1,300 percent. Under such extreme circumstances, the exact location of the initial value of the underlying Markov process is critical.

Under severe rounding, practically all variability is due to the rounding mechanism. The initial location of the underlying Markov process determines the expected length of time before the process changes states. In terms of estimating variance, we are in a particularly difficult situation. If our observation period covers the oscillating price shifts due to rounding, then we grossly overestimate σ^2. Alternatively, if our observation period covers a time of no price change, we estimate σ^2 to be zero. In summary, for very small values of σ^2 the estimation problem is practically hopeless. Fortunately, based on our earlier discussion for daily data, the biases are not extreme.

So far we have concentrated on the distribution of K_n. An important and practical question involves the estimation of σ^2 based on a time series of observed K_is. As alluded to earlier, for very small σ^2 the initial security price position is critical, but, for the levels of σ and P observed in practice, the exact location is not significant.

IV. Variance Estimation

Absent rounding, the variance-estimation problem for Brownian Motion or Geometric Brownian Motion with observations in discrete time is well known and straightforward. We shall assume that $S(0)$ is known and estimate the volatility $\sigma^2 \equiv \theta$ on the basis of the time series of observations $\{K_1, K_2, \cdots, K_n\}$. We begin by assuming that the underlying process is Brownian Motion and later discuss the required modification for the Geometric Brownian Motion process.

Consider n time points $\mathbf{t} = (\tau_1, \tau_2, \cdots, \tau_n)$ and assume that, at time point τ_i, our observation reveals that $S(\tau_i) \in A_i$, where A_i is an arbitrary interval of the real line, for each i. The joint probability of these observations or, equivalently, the likelihood function is given by

$$P[\cup_{i=1}^{n} \{S(\tau_i) \in A_i\}] \equiv L(\mathbf{t}, \theta).$$

A recursive formula is available for the computation of this probability. Define

$$f_m(x_m) = \frac{d}{dx_m} P[\cup_{i=1}^{m-1} \{S(\tau_i) \in A_i\}, S(\tau_m) \leq x_m].$$

By employing the Markov structure of $S(\cdot)$, we have

$$f_m(x_m) = \int_{x_{m-1} \in A_{m-1}} f_{m-1}(x_{m-1}) p(x_{m-1}, x_m)\, dx_{m-1}, \tag{6}$$

where $p(x_{m-1}, x_m)$ is the transition density of the $S(\cdot)$ process at time point τ_{m-1} to τ_m from states x_{m-1} through x_m. We begin the process with $f_1(x_1) = \phi\left(\frac{x_1}{\sigma}\right)$ and iterate to $f_n(x_n)$. Finally,

$$P[\cup_{i=1}^{n} \{S(\tau_i) \in A_i\}] = \int_{x_n \in A_n} f_n(x_n)\, dx_n.$$

The iteration process involves a string of double integrals. Suppose we had modeled the security price by some monotonic transform of Brownian Motion; then we would need only to transform the intervals $A_1, A_2, \cdots, A_n$. For example, for the Geometric Brownian Motion process, $A_i = [\ln(T_i - d/2), \ln(T_i + d/2)]$. On a practical level, missing data can be handled easily by allowing a general sequence of observation points. At the transaction level, there is no reason to expect equally spaced time intervals between observations. As long as the time point and the rounded price are recorded, our methodology remains usable. For illustration purposes, we will consider equally spaced daily observations and constant level of rounding d.

We employ the method of maximum-likelihood estimation using the recursive formula (6) and an m-point Simpson's rule to generate the numerical integrals. The first and second derivatives of $\ln L(\mathbf{t}, \theta)$ were computed numerically, and a series of experiments was performed to select the optimal grid size to form differences as estimates for the derivatives. Finally, we solved the likelihood equation:

$$\frac{\partial \ln L(\mathbf{t}, \theta)}{\partial \theta} = 0 \tag{7}$$

by a simple Newton-Raphson procedure. For all values of θ below the maximum-likelihood estimate (MLE), $\hat{\theta}$, we found that $\dfrac{\partial^2 \ln L(\mathbf{t}, \theta)}{\partial \theta^2}$ was consistently negative. With prudent starting values, the Newton-Raphson procedure converged with six-figure accuracy in less then ten iterations. In computing the first and second differences to estimate the first and second derivatives, increments in θ of 0.01 percent provided stable estimates. Let $\hat{\theta}_m$ be the solution of (7) when the likelihood function is computed using the m-point Simpson's rule. Obviously, $\lim_{m\to\infty} \hat{\theta}_m = \hat{\theta}$ is the maximum-likelihood estimate of $\theta = \sigma^2$. Selecting $m = 1$ is equivalent to ignoring rounding. In other words, $\hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^{n} k_i^2$. Simpson's three-point rule showed a dramatic improvement over the one-point rule. Experiments were performed at various levels of m. For the sets of parameter values in Table IV, a selection of $m = 11$ yielded estimates accurate to six significant figures. To save computer time, $\hat{\theta}_1$ was used as a starting value for estimating $\hat{\theta}_3$. The estimate $\hat{\theta}_3$ then became the starting value to compute $\hat{\theta}_{11}$ by Newton-Raphson. For values of d/σ larger than those given in Table IV, correspondingly larger values of m will be required for accurate estimation.

In addition to estimating $\hat{\theta}$, for large sample sizes we can invoke asymptotic maximum-likelihood theory to approximate the variance of $\hat{\theta}$ given by

$$\left[- \left. \frac{\partial^2 \ln L(t, \theta)}{\partial \theta^2} \right|_{\theta = \hat{\theta}} \right]^{-1}.$$

Next we review to determine whether simple alternative estimates are available.

Recall that in Section I we were concerned with the asymptotic distribution of K_n as $n \rightarrow \infty$. We now consider the joint distribution of $\{K_1, K_2, \cdots, K_n\}$. For large values of d/σ, the convergence of K_n to K is slow. Essentially, the rate of convergence is inversely related to d/σ. There seems to be no simple method-of-moments approach for d/σ significantly greater than 2.5. Of course, the MLE will still provide the optimal estimates, whatever the value of d/σ. However, the precision of the estimates decreases dramatically for large d/σ. Fortunately, for most applications d/σ is less than 2.5. Furthermore, recall that Sheppard's corrections are distribution free. In Section II, we established that

$$E[K_n^2] = \sigma^2 + d^2/6 + 0(d^2).$$

As a candidate for an approximation for $\hat{\theta}$, consider the estimator:

$$\hat{\eta} = \left(\frac{1}{n}\sum_{i=1}^{n} K_i^2\right) - d^2/6.$$

Clearly, $E[\hat{\eta}] = \sigma^2 + 0(d^2)$. It would be useful to estimate the variance of $\hat{\eta}$ also. Unfortunately, the K_is are not independent. However, for large n, they are approximately independent. As an approximation, suppose that $K_i = [U_i + X_i]_d$, where the $\{U_i\}$ are a sequence of independent uniforms and $\{X_i\}$ are independent, normal, random variables each with mean 0 and variance σ^2. For small values of d/σ, this approximation may be reasonable even for K_1. Using Sheppard's corrections (see Kendall and Stuart [11], Section 3.18), we find

$$E[K_i^2] = \sigma^2 + d^2/6 + 0(d^2),$$

$$E[K_i^4] = 3\sigma^4 + \sigma^2 d^2 + 0(d^2).$$

Therefore,

$$\begin{aligned} \operatorname{var}(K_i^2) &= E[K_i^4] - (E[K_i^2])^2, \\ &= 2\sigma^4 + 2/3\sigma^2 d^2 + 0(d^2). \end{aligned}$$

Of course,

$$E[X_i^2] = \sigma^2, \quad E[X_i^4] = 3\sigma^4,$$

and so

$$\operatorname{var}(X_i^2) = 2\sigma^4.$$

The relative inefficiency is given by

$$\frac{\operatorname{var}(K_i^2)}{\operatorname{var}(X_i^2)} = 1 + d^2/3\sigma^2 + 0(d^2).$$

Making the independence approximation,

$$\frac{\operatorname{var}(\hat{\eta})}{\operatorname{var}(\hat{\theta})} = 1 + d^2/3\sigma^2 + 0(d^2), \tag{8}$$

where

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} X_i^2,$$

the maximum-likelihood estimate of σ^2 given observations on the underlying equilibrium process. For illustrative purposes, consider parameter-set 1 values from Table IV. Recall that the percentage bias correction for

$$\hat{\eta} = 1 + d^2/6\sigma^2 + 0(d^2). \tag{9}$$

We have a percentage bias correction of 26.04 percent, and the variance of $\hat{\eta}$ is 52.08 percent bigger than the variance of $\hat{\theta}$. It appears that, for plausible parameter values, the loss in efficiency due to rounding may be severe.

For moderately small values of d/σ, very simple and accurate approximations to both the maximum-likelihood estimator and its asymptotic variance are available. Since under general conditions the method of maximum likelihood is asymptotically efficient, this further suggests that the simple corrections give approximately optimal results in that we provide unbiased minimum-variance estimates of volatility. The next section explores, by means of a Monte Carlo simulation study, the validity of these approximations and corrections across a plausible range of parameter values.

V. Monte Carlo Simulation Analysis

Section IV provided approximations to the maximum-likelihood estimate of volatility under rounding for plausible ranges of σ, P, and d. To measure the success of these approximations and to develop the small-sample properties of the MLE, a Monte Carlo simulation study is performed. We model the underlying Markov process for security prices by a Geometric Brownian Motion process. The parameter-set values are those of Table IV. That is, we consider two initial stock prices of ten dollars and twenty dollars and permit two different instantaneous variances, viz. 1.0×10^{-4} and 2.0×10^{-4} measured on a daily basis. At time n,

$$S(n) = S(0)\exp(\sigma \textstyle\sum_{i=1}^{n} Z_i),$$

where $S(0)$ is the initial price, σ^2 is the instantaneous variance, and $\{Z_1, Z_2, \cdots, Z_n\}$ are a sequence of independent, standard, normal, random variables. The FORTRAN subroutine RANDOM (see Wichman and Hill [13] for details) generated independent, pseudo random numbers, with a uniform distribution on the interval [0, 1]. We employed the Box-Muller transform to generate the independent normals from these numbers. The level of rounding was fixed at an eighth of a dollar, and the $\{T_n\}$ process is generated straightforwardly. For each of the four parameter sets, sample sizes of n = 10, 20, and 30 were used to generate the simulated data. Using the techniques described in Section IV, the maximum-likelihood estimate of σ^2 was calculated for each of the twelve experiments (four parameter sets and three sample sizes). Each experiment was repeated 250 times, and summary information on various statistics is provided in Table V.

Table V

Summary Statistics on the Simulation Analysis[a]

	Minimum	Maximum	Mean	Standard Deviation	Relative Inefficiency	Percentage Bias
	$V_0 = 1.0$, $S(0) = \$10$, $d = \$0.125$.					
Sample Size = 10						
V_1	0.128	2.744	0.987	0.440		
V_2	0.305	3.043	1.240	0.546	53.8%	25.6%
V_3	0.077	2.807	0.985	0.527	43.3%	−2.0%
Sample Size = 20						
V_1	0.334	2.026	1.015	0.326		
V_2	0.392	2.488	1.283	0.409	55.6%	26.4%
V_3	0.154	2.217	1.031	0.398	47.1%	1.6%
Sample Size = 30						
V_1	0.489	1.785	1.009	0.266		
V_2	0.665	2.458	1.273	0.318	43.6%	26.2%
V_3	0.348	1.914	1.004	0.303	29.8%	−0.5%
	$V_0 = 1.0$, $S(0) = \$20$, $d = \$0.125$.					
Sample Size = 10						
V_1	0.103	2.215	0.980	0.400		
V_2	0.193	2.411	1.043	0.417	8.6%	6.4%
V_3	0.157	2.364	0.984	0.417	8.5%	−0.4%
Sample Size = 20						
V_1	0.337	2.177	0.986	0.318		
V_2	0.270	2.234	1.047	0.339	13.5%	6.2%
V_3	0.263	2.168	0.984	0.337	12.0%	−0.2%
Sample Size = 30						
V_1	0.407	2.123	1.011	0.283		
V_2	0.490	2.335	1.068	0.299	11.9%	5.6%
V_3	0.411	2.245	1.004	0.297	10.2%	−0.7%
	$V_0 = 2.0$, $S(0) = \$10$, $d = \$0.125$.					
Sample Size = 10						
V_1	0.334	5.478	2.093	1.012		
V_2	0.451	6.087	2.315	1.111	20.5%	10.61%
V_3	0.260	5.805	2.055	1.084	14.6%	−1.8%
Sample Size = 20						
V_1	0.637	4.732	2.028	0.683		
V_2	0.608	4.578	2.227	0.728	13.4%	9.8%
V_3	0.467	4.507	1.986	0.727	13.2%	−2.1%
Sample Size = 30						
V_1	0.757	3.517	1.999	0.517		
V_2	0.922	4.049	2.255	0.585	28.3%	12.8%
V_3	0.775	3.814	1.999	0.581	26.4%	0.0%
	$V_0 = 2.0$, $S(0) = \$20$, $d = \$0.125$.					
Sample Size = 10						
V_1	0.400	5.937	2.018	0.913		
V_2	0.434	6.331	2.115	0.945	7.2%	4.8%
V_3	0.346	6.295	2.052	0.943	6.9%	1.6%
Sample Size = 20						
V_1	0.714	3.661	1.962	0.573		
V_2	0.718	3.821	2.033	0.598	8.8%	3.6%
V_3	0.659	3.721	1.965	0.598	7.6%	0.2%

Table V—*Continued*

	Minimum	Maximum	Mean	Standard Deviation	Relative Inefficiency	Percentage Bias
		$V_0 = 2.0$,	$S(0) = \$20$,	$d = \$0.125$.		
Sample Size = 30						
V_1	0.939	3.349	2.033	0.472		
V_2	1.078	3.360	2.107	0.489	7.3%	3.6%
V_3	0.988	3.294	2.043	0.490	7.5%	0.5%

[a] V_1 is the (nonobservable in practice) variance estimate when the market is not subject to rounding. The statistic V_2 is the variance estimate ignoring rounding, while V_3 is the maximum-likelihood estimate of variance using rounded prices but accounting for the rounding mechanism. For ease of reading, all V-statistics are multiplied by 10^4.

The Monte Carlo simulation is a very useful tool here. We know the true variance V_0 for each experiment. Three variance estimates were generated:

$$V_1 = \frac{1}{n} \sum_{i=1}^{n} Z_i^2 \times V_0,$$

$$V_2 = \frac{1}{n} \sum_{i=1}^{n} k_i^2,$$

and

$$V_3 = \text{the maximum-likelihood estimate.}$$

Here, V_1 corresponds to the optimal estimate of variance when the market is not subject to rounding. Of course, in a practical situation V_1 is not observable. However, in a simulation analysis it provides a benchmark to assess the impact of rounding. The statistic V_2 is the commonly used estimate of variance ignoring rounding. Finally, V_3 is the maximum-likelihood estimate. The table provides summary information on V_1, V_2, and V_3 for each experiment across the 250 replications. In addition, two other useful statistics are computed for V_2 and V_3. The relative inefficiency of V_i is the ratio of the sample variance of V_i to the sample variance of V_1 for $i = 2, 3$, represented as excess percentage over 100. Similarly, the bias of V_i, $i = 2, 3$, is the ratio of the sample mean of V_i to the sample mean of V_1. Of course, we know the population mean and population variance of V_1, but, for comparison purposes, we feel it is more appropriate to compare the sampling differences.

For all practical purposes and even for the sample sizes as small as $n = 10$, the MLE, V_3, is an unbiased estimator of σ^2. Adjusting equations (8) and (9) for Geometric Brownian Motion, we should expect for V_2

$$\text{relative inefficiency} = 100 \times \quad d^2/3\sigma^2 P^2 \%,$$

$$\text{percentage bias} = 100 \times \quad d^2/6\sigma^2 P^2 \%,$$

where P is the initial security price. Table VI computes the theoretical value of

Table VI

Anticipated Values of Relative Inefficiency and Bias for V_2, the Variance Estimator Ignoring Rounding[a]

V_0	$S(0)$	d	Relative Inefficiency	Percentage Bias
1.0×10^{-4}	\$10	\$0.125	52.08%	26.04%
1.0×10^{-4}	\$20	\$0.125	13.02%	6.51%
2.0×10^{-4}	\$10	\$0.125	26.04%	13.02%
2.0×10^{-4}	\$20	\$0.125	6.51%	3.26%

[a] These figures are computed using equations (8) and (9) for the parameter values given in Table IV.

these statistics for the four selected parameter sets used in the simulation. These theoretical results match very closely with the simulated results and fall well within the limits of sampling error. Across almost all experiments, the relative inefficiency of V_3 is much smaller than that of V_2. We conclude that the maximum-likelihood estimator of variance is in all cases smaller than the variance of V_2 or $\hat{\eta}$, the adjusted version of V_2.

In summary, $\hat{\eta}$, the adjusted-for-bias method of moments estimator, and the MLE are both essentially unbiased estimators of variance. There is an advantage to using the MLE in the sense that it has slightly less variability. However, for the levels of parameters commonly seen in practice, $\hat{\eta}$ is a simple and highly recommended estimate. The theoretical adjustments for bias seem appropriate, and, further, the loss in efficiency due to rounding can be accurately predicted. This has particular importance in applications such as option pricing, where it may be necessary to provide precise confidence intervals on option price estimates. There is evidence that the MLE dominates, and it seems clear that, as the level of rounding becomes more severe, the gains to using the MLE will increase. However, as we have pointed out earlier, the actual loss in efficiency can become so large as to render any type of estimation practically worthless.

VI. Conclusions

Empirical research in finance has, for the most part, ignored the institutional detail of trading at rounded prices. However, several theoretical frameworks, for example the option-pricing model, rely on continuous-time and continuous-state space models for equilibrium financial-security prices. This paper has concentrated on the statistical effects that discrete prices induce when attempting to estimate the parameters of a modeled process for security prices. A key result in Section I establishes the transition probabilities of the rounded price process. These transition probabilities may be computed by numerical methods. The stage is set for a thorough examination of the instantaneous volatility and the development of the limiting rounding process.

A main theme of the paper is the effect that rounding exerts on variance estimation. For severe levels of rounding, the underlying volatility is completely dominated by the rounding mechanism. Fortunately, for most practical cases, the level of rounding is not too severe. However, we do document cases of biases of a twenty-five percent and up to a fifty percent loss in efficiency. Keying on the Markovian structure of the underlying process, we are able to compute the likelihood of a given set of observed rounded prices. The recursive technique employed also allows the numerical implementation of a maximum-likelihood procedure to estimate volatility. Furthermore, very simple and accurate approximations to the MLE are available. A detailed simulation study establishes the validity of these approximations. For a wide range of applications, rounding exerts a simple and removable effect on variance estimation. We advocate strongly the use of these corrections or the implementation of the full maximum-likelihood procedure developed herein.

Appendix A

LEMMA 1: *$\{T_n\}$ has computable transition probability.*

Proof: Without loss in generality, assume that $S(0) = 0$. Let $P(y, t_1; t_2, A) = P[S(t) \in A \mid S(t_1) = y]$ be the transition-probability function of the Markov process $\{S(t): t \in [0, \infty)\}$. Here, $t_2 > t_1 \geq 0$, and A is an interval of the real line. In words, $P(y, t_1; t_2, A)$ is the probability that, at time t_2, the security price falls in the interval A given that, at time t_1, the security price is y. Set $A = (td - d/2, td + d/2]$. Clearly, $P[T_n = td \mid S(n-1) = y] = P(y, n-1; n, A)$. Upon integration,

$$P[T_n = td,\ T_{n-1} = sd] = \int_{y=sd-d/2}^{sd+d/2} P(y,\ n-1;\ n,\ A)p(0,\ 0;\ n-1,\ y)\ dy, \quad \text{(A1)}$$

where $p(\cdot)$ is the density function associated with $P(\cdot)$. However,

$$P[T_{n-1} = sd] = \int_{y=sd-d/2}^{sd+d/2} p(0,\ 0;\ n-1,\ y)\ dy.$$

Therefore, the transition probability for $\{T_n\}$ is given by

$$P[T_n = td \mid T_{n-1} = sd] = \frac{\int_{y=sd-d/2}^{sd+d/2} P(y, n-1; n, A)p(0,0;n-1,y)\,dy}{\int_{y=sd-d/2}^{sd+d/2} p(0,0;n-1,y)\,dy}.$$

Q.E.D.

LEMMA 2: *Assuming a Brownian Motion process for $\{S(t): t \in [0, \infty)\}$,*

$$\lim_{n \to \infty} P_{n-1,n}(sd,\ td) = P(sd,\ td),$$

where

$$P(sd,\ td) = \int_{y=sd-d/2}^{sd+d/2} \frac{1}{d} \times \left\{\Phi\left(\frac{td + d/2 - y}{\sigma}\right) - \Phi\left(\frac{td - d/2 - y}{\sigma}\right)\right\} dy. \quad \text{(A2)}$$

Proof: Equation (1) gives $P_{n-1,n}(sd, td)$. Consider

$$J_{n-1} = \int_{y \in B} \phi\left(\frac{y}{\sqrt{(n-1)\sigma^2}}\right) dy, \quad \text{where} \quad B = [sd - d/2,\ sd + d/2].$$

By the Mean Value Theorem, there exists $\xi_{n-1} \in B$ such that

$$J_{n-1} = d\phi\left(\frac{\xi_{n-1}}{\sqrt{(n-1)\sigma^2}}\right).$$

Let

$$I_{n-1} = \int_{y \in B} \left\{\Phi\left(\frac{td + d/2 - y}{\sigma}\right) - \Phi\left(\frac{td - d/2 - y}{\sigma}\right)\right\} \phi\left(\frac{y}{\sqrt{(n-1)\sigma^2}}\right) dy,$$

$$I_{n-1}(\xi_{n-1}) = \int_{y \in B} \left\{\Phi\left(\frac{td + d/2 - y}{\sigma}\right) - \Phi\left(\frac{td - d/2 - y}{\sigma}\right)\right\} \phi\left(\frac{\xi_{n-1}}{\sqrt{(n-1)\sigma^2}}\right) dy.$$

Let

$$\xi_{n-1}^{\max} = \left\{y \in B\colon \phi\left(\frac{y}{\sqrt{(n-1)\sigma^2}}\right) \text{ is maximum}\right\},$$

$$\xi_{n-1}^{\min} = \left\{y \in B\colon \phi\left(\frac{y}{\sqrt{(n-1)\sigma^2}}\right) \text{ is minimum}\right\}.$$

Define $I_{n-1}(\xi_{n-1}^{\max})$ and $I_{n-1}(\xi_{n-1}^{\min})$ is the same way as $I_{n-1}(\xi_{n-1})$. Certainly,

$$I_n(\xi_n^{\min}) \leq I_n \leq I_n(\xi_n^{\max}),$$

$$I_n(\xi_n^{\min}) \leq I_n(\xi_n) \leq I_n(\xi_n^{\max}), \quad \text{for all} \quad n.$$

Now,

$$\lim_{n \to \infty} \frac{\xi_n^{\max}}{\xi_n^{\min}} = 1,$$

and $\Phi(\cdot)$ is a bounded function; hence,

$$\lim_{n \to \infty} (I_n/J_n) = \lim_{n \to \infty} I_n(\xi_n)/J_n(\xi_n).$$

Therefore,

$$\lim_{n \to \infty} P_{n-1,n}(sd, td) = P(sd, td).$$

Q.E.D.

LEMMA 3: *Let U be a uniform random varible defined on the interval* $(-d/2, d/2]$. *Let X be a normal random variable with mean* 0 *and variance* σ^2. *Define* $[U + X]_d$ *to be the nearest integer multiple of d to* $U + X$. *Assume that U and X are independent. Then, under the assumption of Brownian Motion for the underlying Markov process, we have*

$$K = \lim_{n \to \infty} K_n = [U + X]_d.$$

Proof:

$$P[[U+X]_d = kd] = \int_{y=-d/2}^{+d/2} P[[U+X] = kd \mid U = y] \times f(y)\, dy,$$

where $f(y) = 1/d$, the density of the uniform distribution on $(-d/2, d/2]$. However,

$$P[[U+X]_d = kd \mid U = y] = \Phi\left(\frac{kd + d/2 - y}{\sigma}\right) - \Phi\left(\frac{kd - d/2 - y}{\sigma}\right).$$

Therefore,

$$\begin{aligned} P[[U+X]_d = kd] &= \int_{y=-d/2}^{+d/2} \frac{1}{d} \cdot \left\{\Phi\left(\frac{kd + d/2 - y}{\sigma}\right) - \Phi\left(\frac{kd - d/2 - y}{\sigma}\right)\right\} dy \\ &= P(0, kd) \\ &= P[K = kd]. \end{aligned}$$

Q.E.D.

LEMMA 4: *Under the assumption of an underlying Brownian Motion process, as $n \to \infty$, U_n converges in distribution to a uniform random variable defined on $(-d/2, d/2]$.*

Proof: For $x \in (-d/2, d/2]$, consider $P[U_n \le x]$.

$$P[U_n \le x] = \textstyle\sum_{t=-\infty}^{+\infty} P[U_n \le x \mid T_n = td] \times P[T_n = td].$$

Now

$$P[U_n \le x,\ T_n = td] = \int_{y=td-d/2}^{td+d/2} 1_{[td-d/2,td+x]}(y)\, \phi\left(\frac{y}{\sqrt{n\sigma^2}}\right) dy,$$

where

$$1_{(a,b]}(y) = \begin{matrix} 1 & \text{if} \quad y \in (a, b] \\ 0 & \text{otherwise.} \end{matrix}$$

Therefore,

$$P[U_n \le x \mid T_n = td] = \frac{\int_{y=td-d/2}^{td+d/2} 1_{td-d/2,td+x]}(y)\, \phi\left(\frac{y}{\sqrt{n\sigma^2}}\right) dy}{\int_{y=td-d/2}^{td+d/2} \phi\left(\frac{y}{\sqrt{n\sigma^2}}\right) dy}.$$

By arguments directly similar to those in Lemma 2,

$$\begin{aligned} \lim_{n\to\infty} P[U_n \le x \mid T_n = td] &= \int_{y=td-d/2}^{td+d/2} \frac{1}{d} \times 1_{(td-d/2,td+x]}(y)\, dy. \\ &= \frac{x - d/2}{d}. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} P[U_n \le x] = \textstyle\sum_{t=-\infty}^{+\infty} \lim_{n\to\infty} P[U_n \le t \mid T_n = td] \times P[T_n = td]$$

$$= \frac{x - d/2}{d} \times \textstyle\sum_{t=-\infty}^{+\infty} P[T_n = td]$$

$$= \frac{x - d/2}{d} \quad \text{for} \quad x \in (-d/2, d/2].$$

Clearly, U_n converges in distribution to the uniform on $(-d/2, d/2]$. Q.E.D.

Appendix B

In this appendix, we develop approximations for the moments of K_n. Certainly, for each n, K_n is symmetric about zero, and, therefore, all odd moments of K_n are zero. We first consider the variance of K. By appropriate substitution from Lemma 3 we may write

$$P[K = kd] = \frac{1}{d^*}\left\{\int_{z=kd^*}^{(k+1)d^*} \Phi(z)\,dz - \int_{z=(k-1)d^*}^{kd^*} \Phi(z)\,dz\right\},$$

where $d^* = d/\sigma$. Abramowitz and Stegun [1], in equation 26.2.44, show that

$$\int_{z=0}^{d^*} \Phi(z)\,dz = 2d^* - d^*\Phi(d^*) + \phi(d^*) - \phi(0), \tag{B1}$$

and their equation 26.2.25 demonstrates that,

$$\text{for} \quad x > 2.2, \quad \Phi(x) \ge 1 - \frac{1}{x}\phi(x). \tag{B2}$$

Some algebraic manipulations are required to develop equation (B1). Consider

$$P[K = d] = \frac{1}{d^*}\left\{\int_{z=d^*}^{2d^*} \Phi(z)\,dz - \int_{z=0}^{d^*} \Phi(z)\,dz\right\}.$$

Now

$$\Phi(d^*) \le \frac{1}{d^*}\int_{z=d^*}^{2d^*} \Phi(z)\,dz \le 1$$

and

$$\frac{1}{d^*}\int_{z=0}^{d^*} \Phi(z)\,dz = \frac{\phi(d^*) - \phi(0)}{d^*} + 2 - \Phi(d^*).$$

Upon combination, we see that

$$P[K = d] = \frac{\phi(0) - \phi(d^*)}{d^*} + \varepsilon,$$

where

$$\Phi(d^*) - 1 \leq \varepsilon \leq 2(\Phi(d^*) - 1).$$

However, by equation (B2) for $d^* > 2.2$, $1 - \Phi(d^*) \leq \dfrac{\phi(d^*)}{d^*}$. Therefore,

$$0 < \varepsilon < 2\frac{\phi(d^*)}{d^*},$$

so

$$P[K = d] = \frac{\phi(0)}{d^*} + \eta \tag{B3}$$

where

$$|\eta| < \frac{\phi(d^*)}{d^*}.$$

To continue,

$$\begin{aligned} P[K = 2d] &= \frac{1}{d^*}\left[\int_{z=2d^*}^{3d^*} \Phi(z)\, dz - \int_{z=d^*}^{2d^*} \Phi(z)\, dz\right] \\ &\leq \frac{1}{d^*}[d^* - d^*\Phi(d^*)] \\ &\leq 1 - \Phi(d^*) \\ &\leq \frac{\phi(d^*)}{d^*}, \quad \text{for} \quad d^* > 2.2. \end{aligned} \tag{B4}$$

By similar arguments,

$$\begin{aligned} P[K = kd] &\leq 1 - \Phi((k-1)d^*) \\ &\leq \frac{\phi((k-1)d^*)}{(k-1)d^*}, \quad \text{for} \quad d^* > 2.2. \end{aligned}$$

Consider

$$\textstyle\sum_{k=3}^{\infty} (\mathrm{kd})^2 \times \mathrm{P}[\mathrm{K} = \mathrm{kd}].$$

We have

$$(kd)^2 \times P[K = kd] \leq \frac{2\sigma^2}{d^*}\int_{z=(k-2)d^*}^{(k-1)d^*} z^2\phi(z)\, dz, \tag{B5}$$

whenever

$$((k-2)d^*)^2 \geq \frac{(kd^*)^2}{(k-1)d^*}$$

or, equivalently, whenever $d^* \geq \dfrac{k^2}{2(k-1)(k-2)^2}$. For $k \geq 3$, the above inequality holds whenever $d^* \geq 9/4$. By telescoping (B5) for all $k \geq 3$, we have

$$\textstyle\sum_{k=3}^{\infty} (kd)^2 P[K = kd] \leq \frac{2\sigma^2}{d^*} \int_{z=d^*}^{\infty} z^2 \phi(z)\, dz.$$

Integration by parts reveals that

$$\begin{aligned}\int_{z=d^*}^{\infty} z^2 \phi(z)\, dz &= d^* \phi(d^*) - [1 - \Phi(d^*)] \\ &\leq d^* \phi(d^*) - \frac{\phi(d^*)}{d^*}.\end{aligned}$$

By symmetry,

$$\mathrm{E}[K^2] = 2 \textstyle\sum_{k=1}^{\infty} (kd)^2 P[K = kd].$$

Using (B3), (B4), and (B5), we see that

$$\mathrm{E}[K^2] = 2d\sigma\phi(0) + \text{error},$$

where

$$\begin{aligned}|\,\text{error}\,| &\leq 2d^2\eta + 8d^2 \frac{\phi(d^*)}{d^*} + 4\sigma^2 d^* \left(\frac{\phi(d^*)}{d^*} - \frac{\phi(d^*)}{d^{*3}}\right) \\ &\leq 2d\sigma\{7\phi(d^*)\} \quad \text{for} \quad d^* \geq 9/4.\end{aligned}$$

Put differently,

$$\mathrm{E}[K^2] = \frac{2d\sigma}{\sqrt{2\pi}}, \tag{B6}$$

subject to a maximum percentage error of $700 \times \exp(-d^2/2\sigma^2)$ whenever $d/\sigma \geq 9/4$. For example, when $d/\sigma = 5$, the maximum percentage error is 0.0026 percent. One must also stress that these are not the tightest bounds possible, so in all likelihood the accuracy of the approximation (B6) is much better than these bounds suggest.

The above argument may be generalized straightforwardly to show that

$$\mathrm{E}[K^{2m}] = \frac{2d^{2m-1}\sigma}{\sqrt{2\pi}}, \tag{B7}$$

subject to a maximum percentage error of $C_m \times \exp(-d^2/2\sigma^2)$, where C_m is a constant for $d/\sigma \geq 2.25$.

REFERENCES

1. M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions*. Washington, D.C.: National Bureau of Standards, 1966.
2. C. A. Ball and W. N. Torous. "On Jumps in Common Stock Prices and Their Impact on Call Option Pricing." *Journal of Finance* 40 (March 1985), 155–73.
3. ——— and A. E. Tschoegl. "The Degree of Price Resolution: The Case of the Gold Market." *Journal of Futures Markets* 5 (Spring 1985), 29–43.

4. F. Black and M. Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (May 1973), 637–54.
5. D. Cho and E. Frees. "Estimating the Volatility of Discrete Stock Prices." *Journal of Finance* 43 (June 1988), 451–66.
6. J. Cox and M. Rubinstein. *Options Markets.* Englewood Cliffs, NJ: Prentice-Hall, Inc., 1985.
7. W. Feller. *An Introduction to Probability Theory and Its Applications, Volume II.* New York: Wiley & Sons, 1971.
8. G. Gottlieb and A. Kalay. "Implications of the Discreteness of Observed Stock Prices." *Journal of Finance* 40 (March 1985), 135–54.
9. L. Harris. "Estimation of 'True' Stock Price Variances and Bid-Ask Spreads from Discrete Observations." Unpublished manuscript, University of Southern California, Los Angeles, 1986.
10. S. Karlin and H. Taylor. *A First Course in Stochastic Processes.* New York: Academic Press, 1975.
11. M. Kendall and A. Stuart. *The Advanced Theory of Statistics, Volume 1.* London: Griffin and Company, 1977.
12. R. C. Merton. "Option Pricing When Underlying Stock Returns Are Discontinuous." *Journal of Financial Economics* 3 (January–March 1976), 125–44.
13. B. A. Wichman and I. D. Hill. "An Efficient and Portable Pseudo-Random Number Generator: Algorithm AS183." *Applied Statistics* 31 (1982), 188–90.