Technical Note No. 9*
Options, Futures, and Other Derivatives
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Generalized Tree Building Procedure

This note describes a general procedure for constructing a trinomial tree for a variable, x, in the situation where

1. There are nodes at times t_1, t_2,...t_N where $\Delta_i = t_i - t_{i-1}$ and $t_0 = 0$
2. The standard deviation of x between time t_{i-1} and t_i is s_i
3. The drift between time t_{i-1} and t_i is m_i

Suppose that x_0 is the initial value of x. At time t_i the nodes are chosen to be at $x_0 + jq_i$ where j is a positive or negative integer and $q_i = s_i\sqrt{3\Delta_i}$. We will refer to the node where $x = x_0 + jq_i$ at time t_i as the (i, j) node.

If we are at the (i, j) node, the expected value of x at time t_{i+1} is $x_0 + jq_i + m_{i+1}\Delta_{i+1}$. Let the node at time t_{i+1} that is closest to this expected value be the node $(i + 1, k)$ The tree is constructed so that we branch from node (i, j) to one of the three nodes $(i+1, k-1)$, $(i + 1, k)$ and $(i + 1, k + 1)$.

We choose the probabilities on the branches to match the first and second moment. Define p_u, p_m, and p_d as the probabilities on the upper middle and lower branches. It follows that

$$p_u + p_m + p_d = 1$$

$$kq_{i+1} + (p_u - p_d)q_{i+1} = jq_i + M$$

$$k^2q_{i+1}^2 + 2k(p_u - p_d)q_{i+1}^2 + (p_u + p_d)q_{i+1}^2 = V + (jq_i + M)^2$$

where $M = m_{i+1}\Delta_{i+1}$ and $V = s_{i+1}^2\Delta_{i+1}$

The solution to these equations is

$$p_u = \frac{V}{2q_{i+1}^2} + \frac{\alpha^2 + \alpha}{2}$$

$$p_d = \frac{V}{2q_{i+1}^2} + \frac{\alpha^2 - \alpha}{2}$$

$$p_m = 1 - \frac{V}{q_{i+1}^2} - \alpha^2$$

where

$$\alpha = \frac{jq_i + M - kq_{i+1}}{q_{i+1}}$$