Technical Note No. 8*
Options, Futures, and Other Derivatives
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Analytic Approximation for Valuing American Options

Consider an option on a stock providing a dividend yield equal to q. We will denote the difference between the American and European option price by v. Because both the American and the European option prices satisfy the Black–Scholes differential equation, v also does so. Hence,

$$\frac{\partial v}{\partial t} + (r-q)S\frac{\partial v}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} = rv$$

For convenience, we define

$$\tau = T - t$$
$$h(\tau) = 1 - e^{-r\tau}$$
$$\alpha = \frac{2r}{\sigma^2}$$
$$\beta = \frac{2(r-q)}{\sigma^2}$$

We also write, without loss of generality,

$$v = h(\tau)g(S,\, h)$$

With appropriate substitutions and variable changes, this gives

$$S^2\frac{\partial^2 g}{\partial S^2} + \beta S\frac{\partial g}{\partial S} - \frac{\alpha}{h}g - (1-h)\alpha\frac{\partial g}{\partial h} = 0$$

The approximation involves assuming that the final term on the left-hand side is zero, so that

$$S^2\frac{\partial^2 g}{\partial S^2} + \beta S\frac{\partial g}{\partial S} - \frac{\alpha}{h}g = 0 \tag{1}$$

The ignored term is generally fairly small. When τ is large, $1-h$ is close to zero; when τ is small, $\partial g/\partial h$ is close to zero.

The American call and put prices at time t will be denoted by $C(S,t)$ and $P(S,t)$, where S is the stock price, and the corresponding European call and put prices will be denoted by $c(S,t)$ and $p(S,t)$. Equation (1) can be solved using standard techniques. After boundary conditions have been applied, it is found that

$$C(S,t) = \begin{cases} c(S,t) + A_2\left(\frac{S}{S^*}\right)^{\gamma_2} & \text{when } S < S^* \\ S - K & \text{when } S \geq S^* \end{cases}$$

The variable S^* is the critical price of the stock above which the option should be exercised. It is estimated by solving the equation

$$S^* - K = c(S^*, t) + \left\{1 - e^{-q(T-t)} N[d_1(S^*)]\right\} \frac{S^*}{\gamma_2}$$

iteratively. For a put option, the valuation formula is

$$P(S,t) = \begin{cases} p(S,t) + A_1 \left(\frac{S}{S^{**}}\right)^{\gamma_1} & \text{when } S > S^{**} \\ K - S & \text{when } S \leq S^{**} \end{cases}$$

The variable S^{**} is the critical price of the stock below which the option should be exercised. It is estimated by solving the equation

$$K - S^{**} = p(S^{**}, t) - \left\{1 - e^{-q(T-t)} N[-d_1(S^{**})]\right\} \frac{S^{**}}{\gamma_1}$$

iteratively. The other variables that have been used here are

$$\begin{aligned}
\gamma_1 &= \left[-(\beta - 1) - \sqrt{(\beta - 1)^2 + \frac{4\alpha}{h}}\right] \Big/ 2 \\
\gamma_2 &= \left[-(\beta - 1) + \sqrt{(\beta - 1)^2 + \frac{4\alpha}{h}}\right] \Big/ 2 \\
A_1 &= -\left(\frac{S^{**}}{\gamma_1}\right) \left\{1 - e^{-q(T-t)} N[-d_1(S^{**})]\right\} \\
A_2 &= \left(\frac{S^*}{\gamma_2}\right) \left\{1 - e^{-q(T-t)} N[d_1(S^*)]\right\} \\
d_1(S) &= \frac{\ln(S/K) + (r - q + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}
\end{aligned}$$

Options on stock indices, currencies, and futures contracts are analogous to options on a stock providing a constant dividend yield. Hence the quadratic approximation approach can easily be applied to all of these types of options.