Technical Note No. 7*
Options, Futures, and Other Derivatives
John Hull

Differential Equation for Price of a Derivative on a Futures Price

Suppose that the futures price F follows the process

$$dF = \mu F\,dt + \sigma F\,dz \tag{1}$$

where dz is a Wiener process and σ is constant.[1] Because f is a function of F and t, it follows from Ito's lemma that

$$df = \left(\frac{\partial f}{\partial F}\mu F + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial F^2}\sigma^2 F^2\right) dt + \frac{\partial f}{\partial F}\sigma F\,dz \tag{2}$$

Consider a portfolio consisting of

$$\begin{aligned} -1: &\quad \text{derivative} \\ +\frac{\partial f}{\partial F}: &\quad \text{futures contracts} \end{aligned}$$

Define Π as the value of the portfolio and let $\Delta\Pi$, Δf, and ΔF be the change in Π, f, and F in time Δt, respectively. Because it costs nothing to enter into a futures contract,

$$\Pi = -f \tag{3}$$

In a time period Δt, the holder of the portfolio earns capital gains equal to $-\Delta f$ from the derivative and income of

$$\frac{\partial f}{\partial F}\,\Delta F$$

from the futures contract. Define ΔW as the total change in wealth of the portfolio holder in time Δt. It follows that

$$\Delta W = \frac{\partial f}{\partial F}\Delta F - \Delta f$$

The discrete versions of equations (1) and (2) are

$$\Delta F = \mu F\,\Delta t + \sigma F\,\Delta z$$

and

$$\Delta f = \left(\frac{\partial f}{\partial F}\mu F + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial F^2}\sigma^2 F^2\right) \Delta t + \frac{\partial f}{\partial F}\sigma F \Delta z$$

[1] As discussed in the text, the drift is zero in the traditional risk-neutral world.

where $\Delta z = \epsilon\sqrt{\Delta t}$ and ϵ is a random sample from a standardized normal distribution. It follows that

$$\Delta W = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial F^2}\sigma^2 F^2 \right) \Delta t \tag{4}$$

This is riskless. Hence it must also be true that

$$\Delta W = r\Pi\,\Delta t \tag{5}$$

If we substitute for Π from equation (3), equations (4) and (5) give

$$\left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial F^2}\sigma^2 F^2 \right) \Delta t = -rf\,\Delta t$$

Hence

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial F^2}\sigma^2 F^2 = rf$$

This is the equation in the text.