Technical Note No. 17*
Options, Futures, and Other Derivatives
John Hull

The Process for the Short Rate in an HJM Term Structure Model

This note considers the relationship between the HJM model and the models of the short rate. Consider a one-factor continuous time model for forward rates. Because

$$F(t,t) = F(0,t) + \int_0^t dF(\tau,t)$$

and $r(t) = F(t,t)$, it follows from the HJM analysis in the text that

$$r(t) = F(0,t) + \int_0^t v(\tau,t,\Omega_\tau)v_t(\tau,t,\Omega_\tau)\,d\tau + \int_0^t v_t(\tau,t,\Omega_\tau)\,dz(\tau)$$

Differentiating with respect to t and using $v(t,t,\Omega_t) = 0$ we obtain[5]

$$\begin{aligned} dr(t) =& F_t(0,t)\,dt + \left\{\int_0^t [v(\tau,t,\Omega_\tau)v_{tt}(\tau,t,\Omega_\tau) + v_t(\tau,t,\Omega_\tau)^2]\,d\tau\right\} dt \\ &+ \left\{\int_0^t v_{tt}(\tau,t,\Omega_\tau)\,dz(\tau)\right\} dt + [v_t(\tau,t,\Omega_\tau)|_{\tau=t}]\,dz(t) \end{aligned}$$

It is interesting to examine the terms on the right-hand side of this equation. The first and fourth terms are straightforward. The first term shows that one component of the drift in r is the slope of the initial forward rate curve. The fourth term shows that the instantaneous standard deviation of r is $v_t(\tau,t,\Omega_\tau)|_{\tau=t}$. The second and third terms are more complicated, particularly when v is stochastic. The second term depends on the history of v because it involves $v(\tau,t,\Omega_\tau)$ when $\tau < t$. The third term depends on the history of both v and dz.

The second and third terms terms are liable to cause the process for r to be non-Markov. The drift of r between time t and $t+\Delta t$ is liable to depend not only on the value of r at time t, but also on the history of r prior to time t. This means that, when we attempt to construct a tree for r, it is nonrecombining. An up movement followed by a down movement does not lead to the same node as a down movement followed by an up movement.

[5] The stochastic calculus in this equation may be unfamiliar to some readers. To interpret what is going on, we can replace integral signs with summation signs and d's with Δ's. For example, $\int_0^t v(\tau,t,\Omega_\tau)v_t(\tau,t,\Omega_\tau)d\tau$ becomes $\sum_{i=1}^n v(i\Delta t,t,\Omega_i)v_t(i\Delta t,t,\Omega_i)\Delta t$, where $\Delta t = t/n$.