Technical Note No. 10*
Options, Futures, and Other Derivatives
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The Cornish–Fisher expansion to estimate VaR

As indicated in the text, α_i's and β_{ij}'s can be defined so that ΔP for a portfolio containing options is approximated as

$$\Delta P = \sum_{i=1}^{n} \alpha_i \Delta x_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_{ij} \Delta x_i \Delta x_j$$

Define σ_{ij} as the covariance between variable i and j:

$$\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$$

It can be shown that when the Δx_i are multivariate normal

$$E(\Delta P) = \sum_{i,j} \beta_{ij}\sigma_{ij}$$

$$E[(\Delta P)^2] = \sum_{i,j} \alpha_i\alpha_j\sigma_{ij} + \sum_{i,j,k,l} \beta_{ij}\beta_{kl}(\sigma_{ij}\sigma_{kl} + \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk})$$

$$E[(\Delta P)^3] = 3\sum_{i,j,k,l} \alpha_i\alpha_j\beta_{kl}(\sigma_{ij}\sigma_{kl} + \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}) + \sum_{i_1,i_2,i_3,i_4,i_5,i_6} \beta_{i_1i_2}\beta_{i_3i_4}\beta_{i_5i_6}Q$$

The variable, Q, consists of fifteen terms of the form $\sigma_{k_1k_2}\sigma_{k_3k_4}\sigma_{k_5k_6}$ where k_1, k_2, k_3, k_4, k_5, and k_6 are permutations of i_1, i_2, i_3, i_4, i_5, and i_6.

Define μ_P and σ_P as the mean and standard deviation of ΔP so that

$$\mu_P = E(\Delta P)$$

$$\sigma_P^2 = E[(\Delta P)^2] - [E(\Delta P)]^2$$

The skewness of the probability distribution of ΔP, ξ_P, is defined as

$$\xi_P = \frac{1}{\sigma_P^3}E[(\Delta P - \mu_P)^3] = \frac{E[(\Delta P)^3] - 3E[(\Delta P)^2]\mu_P + 2\mu_P^3}{\sigma_P^3}$$

Using the first three moments of ΔP, the Cornish-Fisher expansion estimates the qth percentile of the distribution of ΔP as

$$\mu_P + w_q\sigma_P$$

where

$$w_q = z_q + \frac{1}{6}(z_q^2 - 1)\xi_P$$

and z_q is qth percentile of the standard normal distribution $\phi(0,1)$.

Example

Suppose that for a certain portfolio we calculate $\mu_P = -0.2$, $\sigma_P = 2.2$, and $\xi_P = -0.4$. If we assume that the probability distribution of ΔP is normal, the first percentile of the probability distribution of ΔP is

$$-0.2 - 2.33 \times 2.2 = -5.326$$

In other words, we are 99% certain that

$$\Delta P > -5.326$$

When we use the Cornish-Fisher expansion to adjust for skewness and set $q = 0.01$, we obtain

$$w_q = -2.33 - \frac{1}{6}(2.33^2 - 1) \times 0.4 = -2.625$$

so that the first percentile of the distribution is

$$-0.2 - 2.625 \times 2.2 = -5.976$$

Taking account of skewness, therefore, changes the VaR from 5.326 to 5.976.