

FORWARDS AND EUROPEAN OPTIONS ON CDO TRANCHES

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Abstract

Now that the market for cash and synthetic CDOs is well established, there is increased interest in trading forward contracts and options on CDO tranches. This article develops models for valuing these instruments. The model for valuing European options on CDO tranches has similarities to the standard market model for valuing European swap options and to the model for valuing options on credit default swaps. Once default probabilities, the expected recovery rates and the degree to which defaults tend to cluster have been estimated, it enables traders to calculate option prices from CDO tranche swap spread volatilities and vice versa.

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A credit default swap (CDS) provides protection against default by some particular entity. During the life of the contract the buyer of protection makes periodic payments at some rate, s , times the notional until default or maturity whichever comes first. Typically these payments are made quarterly in arrears. In the event of a default the seller of protection makes a payment equal to the notional times one minus the recovery rate, R , and the buyer of protection makes a final accrual payment to bring the periodic payments up to date.

A synthetic collateralized debt obligation (CDO) is similar to a CDS in that in exchange for a set of periodic payments one can buy protection against losses due to default. The distinction is that in the case of a CDS there is only one entity underlying the contract and the loss covered is the total loss incurred due to default. In a synthetic CDO the claim is on a portfolio of CDSs and the loss covered is only a subset of the total loss on the portfolio. The portion of loss that is covered, known as a tranche, is defined by the attachment point (AP), a_L , and detachment point (DP), a_H . If the total initial notional of the CDS portfolio is P , the seller of protection agrees to cover all losses between $a_L P$ and $a_H P$. In exchange the seller of protection receives payments at rate s on a notional principal that is initially $(a_H - a_L) P$.¹ As with the CDS the periodic payments are usually paid quarterly in arrears. Each loss for which the cumulative loss to date is between $a_L P$ and $a_H P$ reduces the notional on which payments are based and results in a payment from the seller of protection to the buyer of protection equal to the amount of loss covered. Once the cumulative portfolio losses exceed the detachment point no notional remains and all payments stop.

¹ Any CDO tranche for which the attachment point is zero is referred to as an equity tranche. The periodic payment structure for most equity tranches is a spread of 5% per year plus an upfront payment.

Tranches of synthetic CDOs trade actively. Two common CDOs are those based on the iTraxx and the CDX IG indices. The iTraxx index is a portfolio of CDSs on 125 investment grade European firms. The tranches are 0 to 3%, 3 to 6%, 6 to 9%, 9 to 12% and 12 to 22%. The CDX is a similar portfolio of 125 US investment grade firms. In this case the tranches are 0 to 3%, 3 to 7%, 7 to 10%, 10 to 15% and 15 to 30%. The zero to 3% tranche is referred to as the equity tranche, the next tranche is called the mezzanine tranche and all other tranches are senior tranches.

Now that the market for CDOs is well established, a market is developing in forward CDO tranches and options on CDO tranches. A forward CDO contract is the obligation to enter into a CDO on a specified portfolio for a specified spread at a specified future time. The forward contract also specifies which party is the buyer of protection and which is the seller. These contracts are similar to the forward CDS contracts discussed in Schönbucher (2000 and 2003) or Hull and White (2003). If the cumulative losses on the portfolio exceed the forward tranche detachment point before the forward start date, the contract is cancelled. We define the forward CDO spread as the specified spread that causes the forward contract to have a value of zero.

A CDO option is defined analogously to a forward CDO. It is a European option that gives the holder the right to buy or sell protection on a specified tranche of a specified portfolio for a specified future period of time for a certain spread. The option is knocked out if the cumulative losses on the portfolio exceed the tranche detachment point before the option exercise date.

In this article we examine how a forward CDO and a CDO option can be priced.

I. Valuation of Spot and Forward CDO Tranches

Hull and White (2006a) show that the value of a CDO tranche to the seller of protection is the expected present value of the annuity payments (plus any accrual payments in the event of default) less the present value of the expected loss covered by the tranche. The same is true for a forward start CDO tranches.

Define

t : Current time

T_1 : Time at which the forward start CDO tranche starts

T_2 : Time at which the forward start CDO tranche ends

$A(t, T_1, T_2)$: Present value at t of expected periodic payments on the forward start CDO tranche. Payments are made at a rate of 100% per year applied to the outstanding principal.

$L(t, T_1, T_2)$: Present value at t of expected payments due to default on the forward start CDO tranche.

$V(s, t, T_1, T_2)$: Present value at time t of the forward start CDO tranche to the seller of protection when the periodic payments are made at rate s .

The value of the forward start contract to the protection seller is

$$V(s, t, T_1, T_2) = sA(t, T_1, T_2) - L(t, T_1, T_2) \quad (1)$$

We define the forward spread, F , as the value of s that sets the contract value to zero.²

$$F(t, T_1, T_2) = \frac{L(t, T_1, T_2)}{A(t, T_1, T_2)} \quad (2)$$

When $T_1 = t$ the contracts are spot start contracts and the forward spread is the spot spread.

² If we define s as the upfront fee for an equity CDO tranche the value of the contract is

$V(s, t, T_1, T_2) = s + 0.05A(t, T_1, T_2) - L(t, T_1, T_2)$ and the forward fee is $s = L(t, T_1, T_2) - 0.05A(t, T_1, T_2)$.

II. Valuation of Options on CDO Tranches

For any security price g there is a measure under which f/g is a martingale for all security prices f . Suppose that M is the measure when $g = A(t, T_1, T_2)$. Because $L(t, T_1, T_2)$ is also a security price it follows from equation (2) that, F , is a martingale under M and the expected future spread equals the current forward spread.³

Now define C as the price of a European call⁴ option on a forward start CDO tranche with option expiry date T_1 , the forward start date. The periodic payments in the forward start CDO tranche are made at rate K . This option gives the buyer of protection the right but not obligation to enter into the forward start CDO tranche at time T_1 . The optimal exercise strategy at T_1 is to exercise the option and buy protection at cost K if the spot spread for the same protection, s , is greater than K . The gain earned can be monetized by immediately selling protection at the market spread. The expected costs due to default on the two positions net out and an annuity equal to the difference in the spreads is earned. As a result the value of the option at time T_1 is

$$C(T_1, T_1, T_2) = A(T_1, T_1, T_2) \max[s - K, 0] \quad (3)$$

Since C is a price, C/A is a martingale under the measure M and the expected future value of C/A equals its current value.

$$\frac{C(t, T_1, T_2)}{A(t, T_1, T_2)} = E_M \left[\frac{C(T_1, T_1, T_2)}{A(T_1, T_1, T_2)} \right]$$

where E_M denotes expectations under the measure M . Applying equation (3) to this we find the current option price to be

³ This analysis does not apply to the upfront fee charged for equity tranches.

⁴ For reasons that will become obvious, we define a call option as an option that gives the holder the right to buy protection. A put option is an option that gives the holder the right to sell protection.

$$C(t, T_1, T_2) = A(t, T_1, T_2) E_M \left[\max(s - K, 0) \right] \quad (4)$$

Assume that $\ln(s)$ is normally distributed with standard deviation $\sigma\sqrt{T_1 - t}$. Because the expected value of s under our measure M is the current forward spread the expectation can be evaluated resulting in a variant on Black's (1976) model:⁵

$$C(t, T_1, T_2) = A(t, T_1, T_2) \left[F(t, T_1, T_2) N(d_1) - KN(d_2) \right] \quad (5)$$

$$d_1 = \frac{\ln(F(t, T_1, T_2)/K)}{\sigma\sqrt{T_1 - t}} + 0.5\sigma\sqrt{T_1 - t} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T_1 - t}$$

This is analogous to Jamshidian's (1997) result for valuing European options on interest rate swaps.

III. Application to Forward CDOs

To apply the results in section I we must compute the annuity value, A , and the expected loss due to defaults, L . In this section we will illustrate how this can be done. In order to simplify the notation we will assume from now on that $t = 0$.

We must consider whether the underlying portfolio is a portfolio which currently exists and which may have suffered defaults before the option expiry date or if it is a *de novo* portfolio which will come into existence at the option expiry date. An example of the first case would be an option to exit from a tranche of a bespoke portfolio. An example of the second type would be an option to enter into an iTraxx tranche. This latter case is similar to options and futures on the S&P 500 index. The actual composition of the index at the time the forward or option expires is currently unknown but the broad characteristics of the index are known.

⁵ If the option expiry date, τ , is less than T_1 the result is the same except that $\sigma\sqrt{T_1 - t}$ is replaced by $\sigma\sqrt{\tau - t}$.

For forward start CDOs on existing portfolios we can use any model for valuing CDO tranches. For example, consider the use of the one-factor Gaussian copula model for a portfolio of N entities with identical credit risks.⁶ The factor is used to assign a ‘credit quality’ variable to each firm. The credit quality variable for one of the entities is

$$x = aW + \sqrt{1-a^2}Z \quad 0 \leq a \leq 1$$

where W and Z are independent normally distributed random variables with mean zero and standard deviation one. W is a factor common to all entities and Z is an idiosyncratic element. The pairwise correlation between the credit quality of firms in the portfolio is the square of the factor loading, a . The cumulative density of the credit quality variable is set equal to the cumulative default probability so that $N(x) = Q(T)$ or

$$x = N^{-1}[Q(T)]$$

Conditioning on the common factor, W , the probability that x is less than some value X is

$$\text{Prob}(x < X | W) = N \left[\frac{X - aW}{\sqrt{1-a^2}} \right]$$

It follows from this that

$$Q(T|W) = \text{Prob}(t < T|W) = N \left\{ \frac{N^{-1}[Q(T)] - aW}{\sqrt{1-a^2}} \right\} \quad (7)$$

Conditional on W , defaults are independent.

In our example all the underlying CDSs have the same conditional probability of default by time T , $Q(T/W)$. Since defaults are independent the probability of observing j defaults from N names when the probability of a single default is $Q(T/W)$ is

⁶ The assumption that all the firms in the portfolio are identical permits simple calculations of expected loss. Under any other assumptions numerical procedures similar to those discussed in Hull and White (2004) or Andersen *et al* (2003) would have to be used to calculate the expected loss.

$$\pi(j, N | Q(T | W)) = \frac{N!}{j!(N-j)!} Q(T | W)^j [1 - Q(T | W)]^{N-j} \quad (8)$$

The total initial notional of the underlying portfolio is P ; all the CDSs have the same notional principal, P/N , and the same recovery rate, R . The remaining portfolio notional after the j^{th} default, P_j , is

$$P_j(a_L, a_H) = \begin{cases} (a_H - a_L)P & j < m(n_L) \\ a_H P - j(1-R)P/N & m(n_L) \leq j < m(n_H) \\ 0 & j \geq m(n_H) \end{cases} \quad (9)$$

where $m(x)$ as the smallest integer greater than x , a_L is the tranche attachment point, a_H is the tranche detachment point, $n_L = a_L N/(1-R)$ and $n_H = a_H N/(1-R)$.

The conditional probabilities (8) can be used in conjunction with the remaining notional given j defaults in equation (9) to calculate the conditional expected notional outstanding, p , at each period end for the tranche defined by a_L and a_H

$$E[p(t_i | W)] = \sum_{j=0}^N \pi(j, N | Q(t_i | W)) P_j(a_L, a_H) \quad (10)$$

where the forward start CDO tranche starts at time t_0 , has periodic payments at t_1, t_2, \dots and ends at t_n . This expected notional is then used to calculate conditional values of A and L for the tranche

$$A(0, t_0, t_n | W) = \sum_{m=1}^n (t_m - t_{m-1}) \left[E\{p(t_m | W)\} e^{-rt_m} + 0.5 \{E[p(t_{m-1} | W)] - E[p(t_m | W)]\} e^{-r(t_m + t_{m-1})/2} \right] \quad (11)$$

$$L(0, t_0, t_n | W) = \sum_{m=1}^n \{E[p(t_{m-1} | W)] - E[p(t_m | W)]\} e^{-r(t_m + t_{m-1})/2}$$

The unconditional values of A and L are then found by integrating equations (11) over the distribution of W .

$$\begin{aligned}
A(0, t_0, t_n) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(0, t_0, t_n | W) \exp(-W^2 / 2) dW \\
L(0, t_0, t_n) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} L(0, t_0, t_n | W) \exp(-W^2 / 2) dW
\end{aligned}
\tag{12}$$

As an example we apply equations (7) to (12) to generate some sample forward spreads for CDO tranches that mature in 5 years. Contracts starting immediately, and in one- two- three- and four-years are considered. The unconditional default probabilities are generated by assuming the defaults are driven by a Poisson process with intensity 0.005. A constant recovery rate of 0.40 is assumed. This corresponds to spot and forward start CDS spreads of about 30 basis points per year for all maturities. The results for copula correlations of 0.0, 0.1, 0.2, and 0.3 are shown in Table 1.

In most cases the tranche spread is increasing in the tranche start date. This reflects the fact that losses on the portfolio accumulate over time and that the hazard rate is constant. Consider the equity tranche. In determining the spread for forward start contracts we are interested only in cases where the total losses have not already exceeded the tranche detachment point. In our example this corresponds to 7 or fewer defaults leaving 118 or more firms remaining in the portfolio. The constant hazard rate means that the rate at which defaults are observed in the portfolio is almost the same as it was at inception when there were 125 firms. However, the accumulated losses to date make it more likely that the tranche width is smaller. As a result, the forward start 0 to 3% tranche looks similar to a spot start 0 to $x\%$ tranche where $x < 3$. For more senior tranches accumulated losses to date make it more likely that subsequent losses will affect the tranche in question. For example, forward start mezzanine tranches start to look like spot start equity tranches. Further if accumulated losses are sufficiently large the tranche in question starts to suffer the thinning that is observed for the equity tranche. Both these effects tend to raise spreads.

The only exception to the increasing spread phenomenon is the case of the equity tranche when the correlation is relatively high. The effect of high correlation is that defaults tend to be clustered. If one default occurs, it is likely that several defaults occur. Conditional on the fact that there have been less than 8 defaults (the equity tranche has not been

eliminated), the higher the correlation the lower the expected number of defaults. This means that in all cases in which the equity tranche survives the tranche thinning phenomenon discussed above is less significant. The forward start equity tranche looks more like a spot start equity tranche with a shorter life and a corresponding lower spread. The approach for valuing forward CDOs, as we have described it so far, cannot be used for *de novo* portfolios, i.e., portfolios that will come into existence at the option expiry date. The probability of default by time t_0 for a *de novo* portfolio is zero. One way of modifying the approach to handle *de novo* portfolios is to condition the probability of default for a single firm on no defaults prior to time t_0 . Let us denote this conditional probability as Q^*

$$Q^*(T) = \begin{cases} 0 & T < t_0 \\ \frac{Q(T) - Q(t_0)}{1 - Q(t_0)} & T \geq t_0 \end{cases}$$

The Gaussian copula can be implemented based on Q^* rather than Q .

IV. Comparing the Gaussian Copula with a Dynamic Model

An important question is whether the Gaussian copula model is a satisfactory tool for calculating forward CDO spreads. This is considered by Andersen (2006). As a test of the performance of the Gaussian copula model we now compare the forward spreads given by the dynamic model in Hull and White (2006b) with the forward spreads given by the Gaussian copula model.

We fitted the dynamic model to the iTraxx Index and CDO spreads for maturities of 3-, 5-, 7- and 10-years on March 6, 2006 and then used it to calculate index spreads for maturities from one to five years. The index spreads for the five maturities are 12.1, 17.7, 20.0, 28.5, and 35.0 basis points respectively.⁷ The model is also used to calculate

⁷ The model is calibrated so that it provides an exact fit to the index spreads. The 3- and 5-year index spreads are the market spreads.

forward start index and CDO spreads for tranches that start in 0-, 1-, 2-, 3-, and 4-years and mature at the end of the fifth year. These spreads are reported in Table 2.⁸ (The spreads for the tranches that start at time zero are the market spreads.)

We calibrate the copula model to the market data as follows. We select a stepwise hazard rate term structure that replicates the 1- to 5-year index spreads generated by the dynamic model. The hazard rate is λ_1 in the first year. The value of λ_1 is chosen to fit the one-year index spread. The hazard rate is λ_2 in the second year. Holding λ_1 fixed the value of λ_2 is chosen to fit the two year index spread and so on.

Using the term structure of hazard rates we then find the copula correlation required to reproduce each of the 5 CDO tranche spreads. Finally, using the correlation that correctly prices a spot start tranche we calculate the breakeven spreads for forward start tranches as described in Section III. The results are shown in Table 3. A comparison of the results in Tables 2 and 3 reveal that when the Gaussian copula is calibrated to the existing tranche spreads and to the term structure of index spreads the copula model does a remarkably good job of replicating the forward start spreads implied by the more sophisticated dynamic model on the particular day we consider. For one- and two-year forward start contracts the largest difference is 6 basis points and the largest proportional difference is 1.4% of the dynamic model spread. Even for 3- and 4-year forward start contracts the differences are moderate.

V Application to CDO Options

It is not appropriate to use the one-factor Gaussian copula model or any other copula model to value options on CDO tranches. This is because copula models are not dynamic models and have the property that all uncertainty is resolved at time zero. The valuation of an option depends on the possible credit environments at options maturity and how they might evolve from then to the end of the CDO. However, a copula model can be

⁸ Table 2 is based on Table 5 in Hull and White (2006).

used to calculate the values of A and F in equation (5) and the volatility, σ , can be either estimated from historic data or implied from other option prices.

In Hull and White (2006b) we develop a dynamic model of portfolio credit losses and calibrate it to the market. We use the model to value options on CDOs and calculate volatilities implied by the model's prices using equation (5). It is encouraging that the volatilities are relatively constant for options with different strike prices. This indicates that the lognormal assumption for spreads may not be too unreasonable.

Historic volatilities for the iTraxx and CDX spreads for the period July 1, 2005 to December 4, 2006 are shown in Table 4. These volatilities are in theory appropriate for a *de novo* transaction, but experimentation shows that they are reasonable proxies for the appropriate volatilities when an option on an existing portfolio is valued. When the iTraxx volatilities are compared to the implied volatilities in Hull and White (2006b) we find that they are about the same for the mezzanine tranche, while the historical volatilities are slightly lower for the 6% to 9%, and significantly higher for the 9-12% and 12 to 22% tranches.

IV. Conclusions

Little did Fischer Black realize when he published the original version of Black's model in 1976 that he was authoring what has become the most important European option pricing model ever developed. It appears that its significance escaped the attention of the journal editor as well. Black (1976) was the last (and shortest) article in a special double issue of the journal devoted to option pricing.

In the years since 1976 ingenious change of measure arguments have enabled variants of the Black formula to be used in a wide range of situations. The model is now more widely used than the Black-Scholes model. In this paper we have found a new change of measure argument that shows how Black's model can be used for forward and option contracts on CDOs in the fast-growing market for portfolio credit derivatives.

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Table 1							
Spreads for forward start CDO tranches that mature in 5 years based on a Gaussian copula with pairwise correlation ρ . When the start time is zero the spread is a spot spread. All spreads including those for the equity tranche are running fees quoted in basis points.							
ρ	a_L	a_H	CDO Tranche Start Time (Years)				
			0	1	2	3	4
0.0	0.00	0.03	1287	1372	1465	1564	1665
	0.03	0.06	13.7	17.5	23.6	33.8	50.0
	0.06	0.09	0.0	0.0	0.0	0.0	0.0
	0.09	0.12	0.0	0.0	0.0	0.0	0.0
	0.12	0.22	0.0	0.0	0.0	0.0	0.0
0.1	0.00	0.03	1128	1156	1175	1188	1196
	0.03	0.06	104.7	132.0	164.3	197.4	229.4
	0.06	0.09	12.6	16.0	21.0	27.6	35.5
	0.09	0.12	1.7	2.1	2.9	3.9	5.4
	0.12	0.22	0.1	0.1	0.1	0.2	0.3
0.2	0.00	0.03	961.0	945.9	930.7	917.0	904.9
	0.03	0.06	162.6	197.5	226.2	249.5	268.7
	0.06	0.09	45.5	56.6	68.4	80.0	91.0
	0.09	0.12	14.8	18.5	23.1	28.1	33.2
	0.12	0.22	2.3	2.9	3.8	4.8	5.9
0.3	0.00	0.03	812.8	774.8	748.9	728.9	712.4
	0.03	0.06	190.5	220.6	239.1	252.4	261.8
	0.06	0.09	75.1	90.0	102.1	111.7	120.8
	0.09	0.12	34.5	42.1	49.1	56.0	62.4
	0.12	0.22	9.0	11.1	13.4	15.6	17.7

Table 2						
Breakeven tranche spread for forward start indices and CDO tranches that mature in 5 years using a dynamic model calibrated to the iTraxx quotes March 6, 2006. Forward tranches start in t years where $t = 0, 1, 2, 3,$ and 4 . All spreads including those for the equity tranche are running spreads.						
Tranche Start		0	1	2	3	4
a_L	a_H	Breakeven Tranche Spreads				
0	0.03	1241	1532	1854	2482	2706
0.03	0.06	69.1	87.8	118.0	171.8	253.7
0.06	0.09	20.0	25.5	34.5	50.6	78.0
0.09	0.12	11.4	14.6	19.8	29.2	45.7
0.12	0.22	4.5	5.8	7.9	11.7	19.1
Index		35.0	41.3	47.8	59.9	64.1

Table 3							
Breakeven tranche spread for forward start indices and CDO tranches that mature in 5 years using a copula model calibrated to the term structure of index spreads and the iTraxx quotes March 6, 2006. Forward tranches start in t years where $t = 0, 1, 2, 3,$ and 4 . All spreads including those for the equity tranche are running spreads.							
Tranche Start			0	1	2	3	4
a_L	a_H	ρ	Breakeven Tranche Spreads				
0	0.03	0.087	1241	1535	1860	2511	2774
0.03	0.06	0.027	69.1	88.1	119.7	181.8	304.4
0.06	0.09	0.095	20.0	25.5	34.5	51.9	83.8
0.09	0.12	0.150	11.4	14.6	19.7	29.3	45.3
0.12	0.22	0.210	4.5	5.8	7.8	11.5	17.2
Index			35.0	41.3	47.8	60.0	64.1

Table 4					
Historic 5-year CDO tranche spread volatility for the period July 1, 2005 to December 4, 2006.					
Tranche Attachment and Detachment					
	0 – 3%	3 – 6%	6 – 9%	9 – 12%	12 – 22%
iTraxx	62.8%	77.6%	104.3%	206.8%	288.7%
	0 – 3%	3 – 7%	7 – 10%	10 – 15%	15 – 30%
CDX	47.2%	78.4%	111.0%	120.2%	224.0%