Technical Note No. 5*
Options, Futures, and Other Derivatives
John Hull

Calculation of Cumulative Probability in Bivariate Normal Distribution

Define $M(a,\, b;\, \rho)$ as the cumulative probability in a standardized bivariate normal distribution that the first variable is less than a and the second variable is less than b, when the coefficient of correlation between the variables is ρ. Drezner provides a way of calculating $M(a,\, b;\, \rho)$ to four-decimal-place accuracy.[1] If $a \le 0$, $b \le 0$, and $\rho \le 0$,

$$M(a,\, b;\, \rho) = \frac{\sqrt{1-\rho^2}}{\pi} \sum_{i,j=1}^{4} A_i A_j f(B_i,\, B_j)$$

where

$$f(x,\, y) = \exp\left[a'(2x - a') + b'(2y - b') + 2\rho(x - a')(y - b')\right]$$

$$a' = \frac{a}{\sqrt{2(1-\rho^2)}} \qquad b' = \frac{b}{\sqrt{2(1-\rho^2)}}$$

$$A_1 = 0.3253030 \qquad A_2 = 0.4211071 \qquad A_3 = 0.1334425 \qquad A_4 = 0.006374323$$

$$B_1 = 0.1337764 \qquad B_2 = 0.6243247 \qquad B_3 = 1.3425378 \qquad B_4 = 2.2626645$$

In other circumstances where the product of a, b, and ρ is negative or zero, one of the following identities can be used:

$$M(a,\, b;\, \rho) = N(a) - M(a,\, -b;\, -\rho)$$
$$M(a,\, b;\, \rho) = N(b) - M(-a,\, b;\, -\rho)$$
$$M(a,\, b;\, \rho) = N(a) + N(b) - 1 + M(-a,\, -b;\, \rho)$$

In circumstances where the product of a, b, and ρ is positive, the identity

$$M(a,\, b;\, \rho) = M(a,\, 0;\, \rho_1) + M(b,\, 0;\, \rho_2) - \delta$$

can be used in conjunction with the previous results, where

$$\rho_1 = \frac{(\rho a - b)\,\mathrm{sgn}\,(a)}{\sqrt{a^2 - 2\rho ab + b^2}} \qquad \rho_2 = \frac{(\rho b - a)\,\mathrm{sgn}\,(b)}{\sqrt{a^2 - 2\rho ab + b^2}}$$

$$\delta = \frac{1 - \mathrm{sgn}\,(a)\,\mathrm{sgn}\,(b)}{4} \qquad \mathrm{sgn}\,(x) = \begin{cases} +1 & \text{when } x \ge 0 \\ -1 & \text{when } x < 0 \end{cases}$$

[1] Z. Drezner, "Computation of the Bivariate Normal Integral," *Mathematics of Computation*, 32 (January 1978), 277–79. Note that the presentation here corrects a typo in Drezner's paper.