As explained in the text, Hull and White have proposed a model where the risk-neutral process for the short rate, $r$, is
\[
df(r) = \theta(t) + u - af(r) \, dt + \sigma_1 \, dz_1
\]
where $u$ has an initial value of zero and follows the process
\[
du = -bu \, dt + \sigma_2 \, dz_2
\]
As in the case of one-factor no-arbitrage models, the parameter $\theta(t)$ is chosen to make the model consistent with the initial term structure. The stochastic variable $u$ is a component of the reversion level of $r$ and itself reverts to a level of zero at rate $b$. The parameters $a$, $b$, $\sigma_1$, and $\sigma_2$ are constants and $dz_1$ and $dz_2$ are Wiener processes with instantaneous correlation $\rho$.

This model provides a richer pattern of term structure movements and a richer pattern of volatility structures than one-factor no-arbitrage models. For example, when $f(r) = r, a = 1, b = 0.1, \sigma_1 = 0.01, \sigma_2 = 0.0165, \text{and } \rho = 0.6$ the model exhibits, at all times, a “humped” volatility structure similar to that observed in the market. The correlation structure implied by the model is also plausible with these parameters.

When $f(r) = r$ the model is analytically tractable. The price at time $t$ of a zero-coupon bond that provides a payoff of $\$1$ at time $T$ is
\[
P(t, T) = A(t, T) \exp[-B(t, T)r - C(t, T)u]
\]
where
\[
B(t, T) = \frac{1}{a} \left[1 - e^{-a(T-t)}\right]
\]
\[
C(t, T) = \frac{1}{a(a-b)} e^{-a(T-t)} - \frac{1}{b(a-b)} e^{-b(T-t)} + \frac{1}{ab}
\]
and $A(t, T)$ is as given in the Appendix to this note.

The prices, $c$ and $p$, at time zero of European call and put options on a zero-coupon bond are given by
\[
c = LP(0, s)N(h) - KP(0, T)N(h - \sigma_P)
\]
\[
p = KP(0, T)N(-h + \sigma_P) - LP(0, s)N(-h)
\]
where $T$ is the maturity of the option, $s$ is the maturity of the bond, $K$ is the strike price, $L$ is the bond’s principal
\[
h = \frac{1}{\sigma_P} \ln \frac{LP(0, s)}{P(0, T)K} + \frac{\sigma_P}{2}
\]
and $\sigma_P$ is as given in the Appendix. Because this is a two-factor model, an option on a coupon-bearing bond cannot be decomposed into a portfolio of options on zero-coupon
bonds as described in Technical Note 15. However, we can obtain an approximate analytic valuation by calculating the first two moments of the price of the coupon-bearing bond and assuming the price is lognormal.

**Constructing a Tree**

To construct a tree for the model in equation (1), we simplify the notation by defining $x = f(r)$ so that

$$dx = [\theta(t) + u - ax] dt + \sigma_1 dz_1$$

with

$$du = -bu dt + \sigma_2 dz_2$$

Assuming $a \neq b$ we can eliminate the dependence of the first stochastic variable on the second by defining

$$y = x + \frac{u}{b-a}$$

so that

$$dy = [\theta(t) - ay] dt + \sigma_3 dz_3$$

$$du = -bu dt + \sigma_2 dz_2$$

where

$$\sigma_3^2 = \sigma_1^2 + \frac{\sigma_2^2}{(b-a)^2} + \frac{2\rho\sigma_1\sigma_2}{b-a}$$

and $dz_3$ is a Wiener process. The correlation between $dz_2$ and $dz_3$ is

$$\rho\sigma_1 + \sigma_2/(b-a)$$

$\sigma_3$

A three-dimensional tree for $y$ and $u$ can be constructed on the assumption that $\theta(t) = 0$ and the initial values of $y$ and $u$ are zero. A methodology similar to that for one-factor models can then be used to construct the final tree by increasing the values of $y$ at time $i\Delta t$ by $\alpha_i$. In the $f(r) = r$ case, an alternative approach is to use the analytic expression for $\theta(t)$, given in the Appendix to this note.

Rebonato gives some examples of how the model can be calibrated and used in practice.\(^2\)

APPENDIX
The Functions in the Two-Factor Hull-White Model

The \( A(t, T) \) function is

\[
\ln A(t, T) = \ln \frac{P(0, T)}{P(0, t)} + B(t, T)F(0, t) - \eta
\]

where

\[
\eta = \frac{\sigma_1^2}{4a} (1 - e^{-2at})B(t, T)^2 - \rho \sigma_1 \sigma_2 [B(0, t)C(t, t)B(t, T) + \gamma_4 - \gamma_2]
\]

\[
-\frac{1}{2} \sigma_2^2 [C(0, t)^2 B(t, T) + \gamma_6 - \gamma_5]
\]

\[
\gamma_1 = \frac{e^{-(a+b)T} [e^{(a+b)t} - 1]}{(a+b)(a-b)} - \frac{e^{-2aT} (e^{2at} - 1)}{2a(a-b)}
\]

\[
\gamma_2 = \frac{1}{ab} \left[ \gamma_1 + C(t, T) - C(0, T) + \frac{1}{2} B(t, T)^2 - \frac{1}{2} B(0, t)^2 + \frac{t}{a} - \frac{e^{-a(T-t)} - e^{2at}}{a^2} \right]
\]

\[
\gamma_3 = -\frac{e^{-(a+b)t} - 1}{(a-b)(a+b)} + \frac{e^{-2at} - 1}{2a(a-b)}
\]

\[
\gamma_4 = \frac{1}{ab} \left[ \gamma_3 - C(0, t) - \frac{1}{2} B(0, t)^2 + \frac{t}{a} + \frac{e^{-at} - 1}{a^2} \right]
\]

\[
\gamma_5 = \frac{1}{b} \left[ \frac{1}{2} C(t, T)^2 - \frac{1}{2} C(0, t)^2 + \gamma_2 \right]
\]

\[
\gamma_6 = \frac{1}{b} \left[ \gamma_4 - \frac{1}{2} C(0, t)^2 \right]
\]

where \( F(t, T) \) is the instantaneous forward rate at time \( t \) for maturity \( T \).

The volatility function, \( \sigma_P \), is

\[
\sigma_P^2 = \int_0^t \left\{ \sigma_1^2[B(\tau, T) - B(\tau, t)]^2 + \sigma_2^2[C(\tau, T) - C(\tau, t)]^2 \right. \\
+ 2\rho \sigma_1 \sigma_2 [B(\tau, T) - B(\tau, t)][C(\tau, T) - C(\tau, t)] \right\} d\tau
\]

This shows that \( \sigma_P^2 \) has three components. Define

\[
U = \frac{1}{a(a-b)} [e^{-aT} - e^{-at}]
\]

and

\[
V = \frac{1}{b(a-b)} [e^{-bT} - e^{-bt}]
\]
The first component of $\sigma_p^2$ is

$$\frac{\sigma_1^2}{2a} B(t, T)^2 (1 - e^{-2at})$$

The second is

$$\frac{\sigma_2^2}{2a} \left[ \frac{U^2}{2a} (e^{2at} - 1) + \frac{V^2}{2b} (e^{2bt} - 1) - 2 \frac{UV}{a + b} (e^{(a+b)t} - 1) \right]$$

The third is

$$\frac{2\rho\sigma_1\sigma_2}{a} (e^{-at} - e^{-aT}) \left[ \frac{U}{2a} (e^{2at} - 1) - \frac{V}{a + b} (e^{(a+b)t} - 1) \right]$$

Finally, the $\theta(t)$ function is

$$\theta(t) = F_t(0, t) + aF(0, t) + \phi_t(0, t) + a\phi(0, t)$$

where the subscript denotes a partial derivative and

$$\phi(t, T) = \frac{1}{2} \sigma_1^2 B(t, T)^2 + \frac{1}{2} \sigma_2^2 C(t, T)^2 + \rho\sigma_1\sigma_2 B(t, T) C(t, T)$$