

NUMERICAL PROCEDURES FOR IMPLEMENTING TERM STRUCTURE MODELS I: SINGLE-FACTOR MODELS

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This article presents a new approach for constructing no-arbitrage models of the term structure in terms of the process followed by the short rate, r . The approach, which makes use of trinomial trees, is relatively simple and computationally much more efficient than previously proposed procedures. The advantages of the new approach are particularly noticeable when hedge statistics such as delta, gamma, and vega are computed.

The procedure is appropriate for models where there is some function x of the short rate r that follows a mean-reverting arithmetic process. It can be used for the Ho-Lee model, the Hull-White model, and the Black-Karasinski model. Also, it is a tool that can be used for developing a

wide range of new models.

The key element of the procedure is that it produces a tree that is symmetrical about the expected value of x . A forward induction procedure is used to find the positions of the central nodes at the end of each time step. In the case of the Ho-Lee and Hull-White models, this forward induction procedure is entirely analytic. In the case of other models, it is necessary to use the Newton-Raphson or other iterative search procedure at each time step, but only a small number of iterations are required.

We illustrate the procedure using numerical examples and explain how the models can be calibrated to market data on interest rate option prices.

In recent years there has been a trend toward developing models of the term structure where the initial term structure is an input rather than an output. These models are often referred to as no-arbitrage models.

The first no-arbitrage model was proposed by

Ho and Lee [1986] in the form of a binomial tree of discount bond prices. This model involves one underlying factor and assumes an arithmetic process for the short rate. The Ho and Lee model was extended to include mean reversion by Hull and White [1990]. (Hull and White refer to this as the

extended-Vasicek model.)

One-factor no-arbitrage models where the short rate follows a lognormal process have been proposed by Black, Derman, and Toy [1990] and Black and Karasinski [1991]. Heath, Jarrow, and Morton [1992] develop a model of the term structure in terms of the processes followed by forward rates. Hull and White [1993] show how a range of different one-factor no-arbitrage models can be developed using trinomial trees.

Choosing among the different no-arbitrage models of the term structure involves some difficult trade-offs. A two- or three-factor Heath, Jarrow, and Morton model probably provides the most realistic description of term structure movements, but it has the disadvantage that it is non-Markov (the distribution of interest rates in the next period depends on the current rate and also on rates in earlier periods). This means that the model must be implemented using either Monte Carlo simulation or a non-recombining tree. Computations are very time-consuming, and American-style derivatives are difficult, if not impossible, to value accurately.

Of the one-factor Markov models, those where the interest rate is always non-negative are the most attractive. Yet the only one-factor model that is both capable of fitting an arbitrary initial term structure and analytically tractable is the Hull-White extended-Vasicek model. In this model negative interest rates can occur.

The main purpose of this article is to present numerical procedures that can be used to implement a variety of different term structure models including the Ho-Lee, Hull-White, and Black-Karasinski models. The result is a significant improvement over the trinomial tree procedure suggested in Hull and White [1993]. In a companion sequel article, we show how the procedures here can be extended to model two term structures simultaneously and to represent a family of two-factor models.

I. ONE-FACTOR INTEREST RATE MODELS

Heath, Jarrow, and Morton [1992] provide the most general approach to constructing a one-factor no-arbitrage model of the term structure. Their approach involves specifying the volatilities of all for-

ward rates at all times. The expected drifts of forward rates in a risk-neutral world are calculated from their volatilities, and the initial values of the forward rates are chosen to be consistent with the initial term structure.

Unfortunately, the model that results from the Heath, Jarrow, and Morton approach is usually non-Markov. There are only a small number of known forward rate volatility functions that give rise to Markov models.¹ To develop additional Markov one-factor models, an alternative to the Heath, Jarrow, and Morton [1992] approach has become popular. This involves specifying a Markov process for the short-term interest rate, r , with a drift term that is a function of time, $\theta(t)$. The time-varying drift function is chosen so that the model exactly fits the current term structure.

The Ho and Lee [1986] model can be used to provide an example of the alternative approach. The continuous time limit of the Ho and Lee [1986] model is²

$$dr = \theta(t) dt + \sigma dz$$

In this model all zero-coupon interest rates at all times are normally distributed and have the same variance rate, σ^2 . $\theta(t)$ is chosen to make the model consistent with the initial term structure. As a rough approximation, $\theta(t)$ is the slope of the forward curve at time zero.³

Since Ho and Lee published their work, it has been shown that their model has a great deal of analytic tractability (see, for example, Hull and White [1990]). Define $F(t, T)$ as the instantaneous forward rate at time t for a contract maturing at T . The parameter $\theta(t)$ is given by

$$\theta(t) = F_t(0, t) + \sigma^2 t$$

where the subscript denotes the partial derivative. The price at time t of a discount bond maturing at time T , $P(t, T)$, can be expressed in terms of the value of r at time t :

$$P(t, T) = A(t, T)e^{-r(T-t)}$$

where

$$\log A(t, T) = \log \frac{P(0, T)}{P(0, t)} + (T - t)F(0, t) - \frac{1}{2} \sigma^2 t (T - t)^2$$

Since zero-coupon interest rates are normally distributed, discount bond prices are lognormally distributed. This means that it is possible to use a variant of Black-Scholes to value options on discount bonds. The price, c , at time t of a European call option on a discount bond is given by

$$c = P(t, s)N(h) - XP(t, T)N(h - \sigma_p) \quad (1)$$

where s is the maturity date of the bond underlying the option, X is the strike price, T is the maturity date of the option,

$$h = \frac{1}{\sigma_p} \log \frac{P(t, s)}{P(t, T)X} + \frac{\sigma_p}{2}$$

and

$$\sigma_p^2 = \sigma^2 (s - T)^2 (T - t)$$

The variable σ_p is the product of the forward bond price volatility and the square root of the life of the option.

European options on coupon-bearing bonds can be valued analytically using the approach in Jamshidian [1989]. This approach uses the fact that all bonds are instantaneously perfectly correlated to express an option on a coupon-bearing bond as the sum of options on the discount bonds that make up the coupon-bearing bond.

The Hull-White (extended-Vasicek) model can be regarded as an extension of Ho and Lee that incorporates mean reversion. The short rate, r , follows the process

$$dr = [\theta(t) - ar] dt + \sigma dz$$

in a risk-neutral world. The short rate is pulled toward its expected value at rate a .

There are two volatility parameters, a and σ . The parameter σ determines the overall level of volatility; the reversion rate parameter, a , determines the relative volatilities of long and short rates. A high value of a causes short-term rate movements to damp out quickly, so long-term volatility is reduced. As in the Ho and Lee model, the probability distribution of all rates at all times is normal.

Like Ho and Lee, the Hull-White model has a great deal of analytic tractability. The parameter, $\theta(t)$, is given by

$$\theta(t) = F_t(0, t) + aF(0, t) + \frac{\sigma^2}{2a} (1 - e^{-2at}) \quad (2)$$

The price at time t of a discount bond maturing at time T is given by

$$P(t, T) = A(t, T)e^{-B(t, T)r}$$

where

$$B(t, T) = \frac{1}{a} [1 - e^{-a(T-t)}]$$

and

$$\log A(t, T) = \log \frac{P(0, T)}{P(0, t)} + B(t, T)F(0, t) - \frac{\sigma^2}{4a} (1 - e^{-2at}) B(t, T)^2$$

The price, c , at time t of a European call option on a discount bond is given by Equation (1) with

$$\sigma_p^2 = \frac{\sigma^2}{2a} (1 - e^{-2a(T-t)}) B(T, s)^2$$

As in the case of Ho and Lee, European options on coupon-bearing bonds can be valued analytically using the decomposition approach in Jamshidian [1989].

Another model of the short rate has been suggested by Black, Derman, and Toy [1990]. The continuous time limit of their model is:

$$d\log(r) = \left[\theta(t) + \frac{\sigma'(t)}{\sigma(t)} \log(r) \right] dt + \sigma(t) dz$$

This model has the desirable feature that the short rate cannot become negative, but it has no analytic tractability. The probability distribution of the short rate at all times is lognormal, and the reversion rate, $-\sigma'(t)/\sigma(t)$, is a function of the short rate volatility, $\sigma(t)$, and its derivative with respect to time, $\sigma'(t)$. In practice, the Black, Derman, and Toy model is often implemented with $\sigma(t)$ constant, so $\sigma'(t) = 0$. It then reduces to a lognormal version of Ho and Lee:

$$d\log(r) = \theta(t) dt + \sigma dz$$

Black and Karasinski [1991] decouple the reversion rate and the volatility in the Black, Derman, and Toy model to get:

$$d\log(r) = [\theta(t) - a(t)\log(r)] dt + \sigma(t) dz$$

Black and Karasinski provide a procedure for implementing their model involving a binomial tree and time steps of varying lengths.

One issue that arises in both the Hull-White and Black-Karasinski model is whether a and σ should be functions of time.⁴ The advantage of making these parameters functions of time is that it becomes possible to fit the volatility structure at time zero exactly. The disadvantage is that the volatility term structure at future times is liable to be quite different from the volatility structure today.

We first noticed this when trying to fit the "hump" in cap volatilities by making the reversion rate, a , a function of time. The resulting volatility term structure as seen at a time after the hump proves to be steeply downward-sloping — quite different from the initial volatility structure.

A reasonable approach here may be to introduce a small amount of time dependence into the a and σ parameters without trying to match initial volatilities exactly. For example, we might choose to

set the reversion rate, a , to zero beyond year 7 to reflect the fact that the volatility curve appears to "level out" at about this time. This type of modification to the basic model can easily be accommodated by the tree-building technology that we describe here.

II. BUILDING TREES FOR THE HULL-WHITE MODEL

The Hull-White model is:

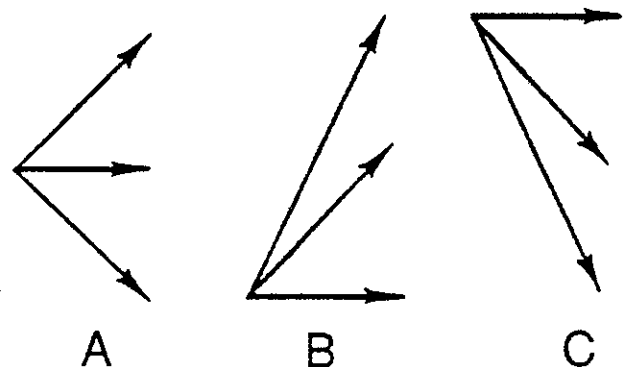
$$dr = [\theta(t) - ar] dt + \sigma dz \quad (3)$$

Although this model has many analytic properties, a tree is necessary to value instruments such as American-style swap options and indexed amortizing rate swaps.

Hull and White [1993] construct a trinomial tree to represent movements in r by using time steps of length Δt and considering at the end of each time step r -values of the form $r_0 + k\Delta r$, where k is a positive or negative integer, and r_0 is the initial value of r . The tree branching can take any of the forms shown in Exhibit 1. Here we improve upon Hull and White [1993] by arranging the geometry of the tree so that the central node always corresponds to the expected value of r . We find that this leads to faster tree construction, more accurate pricing, and much more accurate values for hedge parameters.

The first stage is to build a preliminary tree for r , setting $\theta(t) = 0$ and the initial value of $r = 0$. The process assumed for r during the first stage is therefore

EXHIBIT 1 ALTERNATIVE BRANCHING PROCESSES



$$dr = -ar dt + \sigma dz$$

For this process, $r(t + \Delta t) - r(t)$ is normally distributed. For the purpose of tree construction, we define r as the continuously compounded Δt -period rate. We denote the expected value of $r(t + \Delta t) - r(t)$ as $r(t)M$ and the variance of $r(t + \Delta t) - r(t)$ as V .

We first choose the size of the time step, Δt . We then set the size of the interest rate step in the tree, Δr , as

$$\Delta r = \sqrt{3V}$$

Theoretical work in numerical procedures suggests that this is a good choice of Δr from the standpoint of error minimization.

Our first objective is to build a tree similar to that shown in Exhibit 2, where the nodes are evenly spaced in r and t . To do this, we must resolve which of the three branching methods shown in Exhibit 1 will apply at each node. This will determine the overall shape of the tree. Once this is done, the branching probabilities must also be calculated.

Define (i, j) as the node for which $t = i\Delta t$ and $r = j\Delta r$. Define p_u , p_m , and p_d as the probabilities of the highest, middle, and lowest branches emanating from a node. The probabilities are chosen to match the expected change and variance of the change in r over the next time interval Δt . The probabilities must also sum to unity. This leads to three equations in the three probabilities. When r is at node (i, j) the expected change during the next time step of length Δt is $j\Delta rM$, and the variance of the change is V .

If the branching from node (i, j) is as in Exhibit 1A, the solution to the equations is

$$p_u = \frac{1}{6} + \frac{j^2M^2 + jM}{2}$$

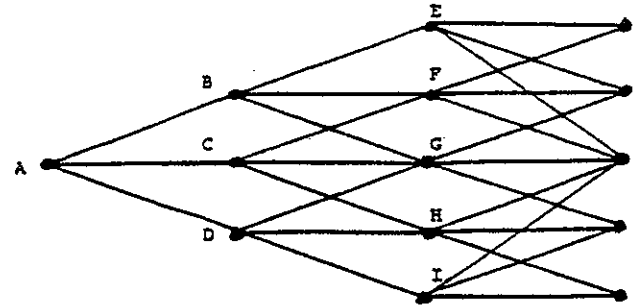
$$p_m = \frac{2}{3} - j^2M^2$$

$$p_d = \frac{1}{6} + \frac{j^2M^2 - jM}{2}$$

respectively.

EXHIBIT 2

TREE WITH $\theta(t) = 0$ WHEN $f(r) = r$, $a = 0.1$, $\sigma = 0.01$, AND $\Delta t = \text{ONE YEAR}$



Node	A	B	C	D	E	F	G	H	I
r	0.00%	1.73%	0.00%	-1.73%	3.46%	1.73%	0.00%	-1.73%	-3.46%
p_u	0.167	0.122	0.167	0.222	0.887	0.122	0.167	0.222	0.087
p_m	0.666	0.656	0.666	0.656	0.026	0.656	0.666	0.656	0.026
p_d	0.167	0.222	0.167	0.122	0.087	0.222	0.167	0.122	0.887

If the branching has the form shown in Exhibit 1B, the solution is

$$p_u = \frac{1}{6} + \frac{j^2M^2 - jM}{2}$$

$$p_m = -\frac{1}{3} - j^2M^2 + 2jM$$

$$p_d = \frac{7}{6} + \frac{j^2M^2 - 3jM}{2}$$

Finally, if it has the form shown in Exhibit 1C, the solution is

$$p_u = \frac{7}{6} + \frac{j^2M^2 + 3jM}{2}$$

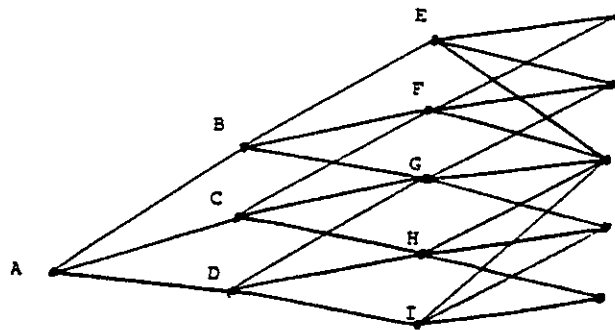
$$p_m = -\frac{1}{3} - j^2M^2 - 2jM$$

$$p_d = \frac{1}{6} + \frac{j^2M^2 + jM}{2}$$

Most of the time the branching in Exhibit 1A

EXHIBIT 3

FINAL TREE WHEN $f(r) = r$, $a = 0.1$, $\sigma = 0.01$, $\Delta t = \text{ONE YEAR}$, AND THE t -YEAR ZERO RATE IS $0.08 - 0.05e^{-0.18t}$



Node	A	B	C	D	E	F	G	H	I
r	3.82%	6.93%	5.20%	3.47%	9.71%	7.98%	6.25%	4.52%	2.79%
P_u	0.167	0.122	0.167	0.222	0.887	0.122	0.167	0.222	0.087
P_m	0.666	0.656	0.666	0.656	0.026	0.656	0.666	0.656	0.026
P_d	0.167	0.222	0.167	0.122	0.087	0.222	0.167	0.122	0.887

is appropriate. When $a > 0$, it is necessary to switch from the branching in Exhibit 1A to the branching in Exhibit 1C when j is large. This is to ensure that the probabilities p_u , p_m , and p_d are all positive. Similarly, it is necessary to switch from the branching in Exhibit 1A to the branching in Exhibit 1B when j is small (i.e., negative and large in absolute value).

Define j_{max} as the value of j where we switch from the Exhibit 1A branching to the Exhibit 1C branching, and j_{min} as the value of j where we switch from the Exhibit 1A branching to the Exhibit 1B branching. It can be shown from the equations that P_u , P_m , and P_d are always positive, providing j_{max} is chosen to be an integer between $-0.184/M$ and $-0.816/M$, and j_{min} is chosen to be an integer between $0.184/M$ and $0.816/M$. (Note that when $a > 0$, $M < 0$.) In practice we find that it is most efficient to set j_{max} equal to the smallest integer greater than $-0.184/M$ and j_{min} equal to $-j_{max}$.

We illustrate the first stage of the tree construction by showing how the tree in Exhibit 2 is constructed for $\sigma = 0.01$, $a = 0.1$, and $\Delta t = \text{one year}$. In this example we set $M = -a\Delta t$ and $V = \sigma^2\Delta t$. This is accurate to order Δt .⁵ The first step in the construc-

tion of the tree is to calculate Δr from Δt . In this case $\Delta r = 0.01\sqrt{3} = 0.0173$.

The next step is to calculate the bounds for j_{max} . These are $0.184/0.1$ and $0.816/0.1$, or 1.84 and 8.16. We set $j_{max} = 2$. Similarly, we set $j_{min} = -2$. The probabilities on the branches emanating from each node are calculated using the equations for p_u , p_m , and p_d .

Note that the probabilities at each node depend only on j . For example, the probabilities at node B are the same as the probabilities at node F. Furthermore, the tree is symmetrical. The probabilities at node D are the mirror image of the probabilities at node B.

This completes the tree for the simplified process. The next stage in the tree construction is to introduce the correct, time-varying drift. To do this, we displace the nodes at time $i\Delta t$ by an amount α_i to produce a new tree, Exhibit 3. The value of r at node (i, j) in the new tree equals the value of r at node (i, j) in the old tree plus α_i . The probabilities on the tree are unchanged. The values of the α_i s are chosen so that the tree prices all discount bonds consistently with the initial term structure observed in the market.

The effect of moving from the tree in Exhibit 2 to the tree in Exhibit 3 is to change the process being modeled from

$$dr = -ar dt + \sigma dz$$

to

$$dr = [\theta(t) - ar] dt + \sigma dz$$

If we define $\hat{\theta}(t)$ as the estimate of θ given by the tree between times t and $t + \Delta t$, the drift in r at time $i\Delta t$ at the midpoint of the tree is $\hat{\theta}(t) - a\alpha_i$, so that⁶

$$[\hat{\theta}(t) - a\alpha_i] \Delta t = \alpha_i - \alpha_{i-1}$$

or

$$\hat{\theta}(t) = \frac{\alpha_i - \alpha_{i-1}}{\Delta t} + a\alpha_i$$

This equation relates the $\hat{\theta}$ s to the α s. In the limit as $\Delta t \rightarrow 0$, $\hat{\theta}(t) \rightarrow \theta(t)$.⁷

To facilitate computations, we define $Q_{i,j}$ as the present value of a security that pays off \$1 if node (i, j) is reached and zero otherwise. The α_i and $Q_{i,j}$ are calculated using forward induction. We illustrate the procedure by showing how the tree in Exhibit 3 is calculated from the tree in Exhibit 2 when the t -year continuously compounded zero-coupon rate is $0.08 - 0.05e^{-0.18t}$. (This corresponds approximately to the U.S. term structure at the beginning of 1994, with one-, two-, and three-year yields of 3.82, 4.51, and 5.09, respectively.)

The value of $Q_{0,0}$ is 1. The value of α_0 is chosen to give the right price for a zero-coupon bond maturing at time Δt . That is, α_0 is set equal to the initial Δt period interest rate. Since $\Delta t = 1$ in this example, $\alpha_0 = 0.0382$. The next step is to calculate the values of $Q_{1,1}$, $Q_{1,0}$, and $Q_{1,-1}$. There is a probability of 0.1667 that the $(1, 1)$ node is reached and the discount rate for the first time step is 3.82%. The value of $Q_{1,1}$ is therefore $0.1667e^{-0.0382} = 0.1604$. Similarly, $Q_{1,0} = 0.6417$, and $Q_{1,-1} = 0.1604$.

Once $Q_{1,1}$, $Q_{1,0}$, and $Q_{1,-1}$ have been calculated, we are in a position to determine α_1 . This is chosen to give the right price for a zero-coupon bond maturing at time $2\Delta t$. Since $\Delta r = 0.0173$ and $\Delta t = 1$, the price of this bond as seen at node B is $e^{-(\alpha_1 + 0.0173)}$. Similarly, the price as seen at node C is $e^{-\alpha_1}$, and the price as seen at node D is $e^{-(\alpha_1 - 0.0173)}$. The price as seen at the initial node A is therefore

$$P(0, 2) = Q_{1,1}e^{-(\alpha_1 + 0.0173)} + Q_{1,0}e^{-\alpha_1} + Q_{1,-1}e^{-(\alpha_1 - 0.0173)} \quad (4)$$

From the initial term structure, this bond price should be $e^{-0.04512 \times 2} = 0.9137$. Substituting for the Q s in Equation (3), we obtain

$$0.1604e^{-(\alpha_1 + 0.0173)} + 0.6417e^{-\alpha_1} + 0.1604e^{-(\alpha_1 - 0.0173)} = 0.9137$$

This can be solved to give $\alpha_1 = 0.0520$.

The next step is to calculate $Q_{2,2}$, $Q_{2,1}$, $Q_{2,0}$,

$Q_{2,-1}$, and $Q_{2,-2}$. These are found by discounting the value of a single \$1 payment at one of nodes E-I back through the tree. This can be simplified by using previously determined Q values.

Consider as an example $Q_{2,1}$. This is the value of a security that pays off \$1 if node F is reached and zero otherwise. Node F can be reached only from nodes B and C. The interest rates at these nodes are 6.93% and 5.20%, respectively. The probabilities associated with the B-F and C-F branches are 0.656 and 0.167. The value at node B of \$1 received at node F is therefore $0.656e^{-0.0693}$. The value at node C is $0.167e^{-0.0520}$, and the present value is the sum of each of these weighted by the present value of \$1 received at the corresponding node. This is

$$0.656e^{-0.0693} \times 0.1604 + 0.167e^{-0.0520} \times 0.6417 = 0.1997$$

Similarly, $Q_{2,2} = 0.0183$, $Q_{2,0} = 0.4737$, $Q_{2,-1} = 0.2032$, and $Q_{2,-2} = 0.0189$.

The next step is to calculate α_2 . After that the $Q_{3,j}$ s can then be computed. We can then calculate α_3 ; and so on.

To express the approach more formally, we suppose the $Q_{i,j}$ s have been determined for $i \leq m$ ($m \geq 0$). The next step is to determine α_m so that at time 0 the tree correctly prices a discount bond maturing at $(m+1)\Delta t$. The interest rate at node (m, j) is $\alpha_m + j\Delta r$ so that the price of a discount bond maturing at time $(m+1)\Delta t$ is given by

$$P(0, m+1) = \sum_{j=-n_m}^{n_m} Q_{m,j} \exp[-(\alpha_m + j\Delta r)\Delta t] \quad (5)$$

where n_m is the number of nodes on each side of the central node at time $m\Delta t$. The solution of this equation is

$$\alpha_m = \frac{\log \sum_{j=-n_m}^{n_m} Q_{m,j} e^{-j\Delta r \Delta t} - \log P(0, m+1)}{\Delta t}$$

Once α_m has been determined, the $Q_{i,j}$ for $i = m+1$ can be calculated using

$$Q_{m+1,j} = \sum_k Q_{m,k} q(k,j) \exp[-(\alpha_m + k\Delta r)\Delta t]$$

where $q(k, j)$ is the probability of moving from node (m, k) to node $(m + 1, j)$, and the summation is taken over all values of k for which this is non-zero.

III. EXTENSION TO OTHER MODELS

We now show how this procedure can be extended to more general models of the form

$$df(r) = [\theta(t) - af(r)] dt + \sigma dz$$

These models have the advantage that they can fit any term structure.⁸ When $f(r) = \log(r)$ the model is a version of the Black and Karasinski [1991] model.

We start by setting $x = f(r)$ so that

$$dx = [\theta(t) - ax] dt + \sigma dz$$

The first stage is to build a tree for x setting $\theta(t) = 0$ and the initial value of $x = 0$. The procedure here is identical to the procedure for building the tree in Exhibit 2.

As in the previous section, we then displace the nodes at time $i\Delta t$ by an amount α_i to provide an exact fit to the initial term structure. The equations for determining α_i and $Q_{i,j}$ inductively are slightly different from those already described. $Q_{0,0} = 1$. Suppose the $Q_{i,j}$ s have been determined for $i \leq m$ ($m \geq 0$). The next step is to determine α_m so that the tree correctly prices an $(m + 1)\Delta t$ discount bond.

Define g as the inverse function of f so that the Δt -period interest rate at the j th node at time $m\Delta t$ is

$$g(\alpha_m + j\Delta x)$$

The period 0 price of a discount bond maturing at time t_{m+1} is given by

$$P(0, m+1) = \sum_{j=-n_m}^{n_m} Q_{m,j} \exp[-g(\alpha_m + j\Delta x)\Delta t] \quad (6)$$

This equation can usually be solved with a small number of iterations using the Newton-Raphson procedure.

When $m = 0$, Equation (6) can be solved to give $\alpha_0 = f(r_0)$, where r_0 is the continuously compounded yield on the Δt maturity discount bond.

Once α_m has been determined, the $Q_{i,j}$ for $i = m + 1$ can be calculated

$$Q_{m+1,j} = \sum_k Q_{m,k} q(k, j) \exp[-g(\alpha_m + k\Delta x)\Delta t]$$

where $q(k, j)$ is the probability of moving from node (m, k) to node $(m + 1, j)$, and the summation is taken over all values of k for which this is non-zero.

Exhibit 4 shows the results of applying the procedure to the model

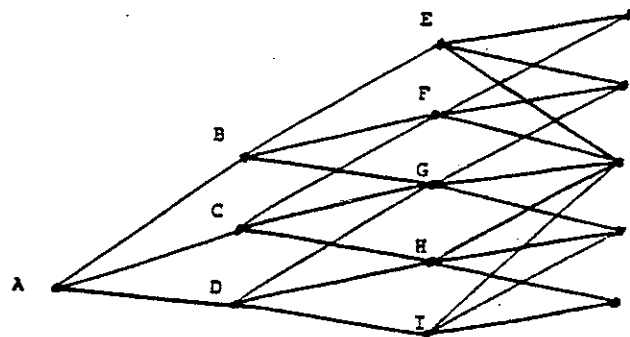
$$d\log(r) = [\theta(t) - a\log(r)] dt + \sigma dz$$

when $a = 0.22$, $\sigma = 0.25$, $\Delta t = 0.5$, and the t -year zero-coupon yield is $0.08 - 0.05e^{-0.18t}$.

The procedures described here can be extended in a number of ways. First, the parameter, a , can be a function of time. This does not affect the positions of the central nodes or Δx . It leads to the probabilities, and possibly the rules for branching, being different at each time step.

EXHIBIT 4

TREE WHEN $f(r) = \text{LOG}(r)$, $a = 0.22$, $\sigma = 0.25$, $\Delta t = 0.5$ YEAR, AND THE t -YEAR ZERO RATE IS $0.08 - 0.05e^{-0.18t}$



Node	A	B	C	D	E	F	G	H	I
x	-3.3725	-2.8751	-3.1813	-3.4875	-2.4300	-2.7362	-3.0424	-3.3486	-3.6548
r	3.43%	5.64%	4.15%	3.06%	8.80%	6.48%	4.77%	3.51%	2.59%
P_0	0.167	0.118	0.167	0.228	0.861	0.118	0.167	0.228	0.081
P_m	0.666	0.654	0.666	0.654	0.058	0.654	0.666	0.654	0.058
P_d	0.167	0.228	0.167	0.118	0.081	0.228	0.167	0.118	0.861

Second, the parameter σ can be made a function of time. The easiest approach here is to make the time step on the tree inversely proportional to that date's σ^2 .

Third, iterative procedures can be devised to choose functions of time for α and σ so that aspects of the initial term structure are matched. (As we explain earlier, we do not recommend this.) Finally, the length of the time step can be changed using a procedure analogous to that outlined in Hull and White [1993]. This might be done to reduce the amount of computation needed for the later periods of a long-maturity instrument.

IV. CALCULATION OF HEDGE STATISTICS

Delta, in this case the partial derivative of the price of a security with respect to the short rate, r , can be calculated directly from the tree in the usual way. Practitioners are usually interested in calculating the partial derivatives of a security price with respect to a number of different shifts in the term structure. A popular approach is to divide the zero curve or the forward curve into a number of sections or "buckets," and to consider changes in the zero curve where there is a small shift in one bucket and the rest of the zero curve is unchanged.

To calculate a generalized delta with respect to a shift in the term structure, we compute the value of the security in the usual way. We then make the shift in the term structure, reconstruct the tree, and observe the change in the security price.

A key feature of our tree-building procedure is that the position of the branches on the tree relative to the central branch and the probabilities associated with the branches do not depend on the term structure. A small change in the term structure affects only the α_i . The result of all this is a control variate effect where the partial derivative is estimated very accurately.^{9,10}

We favor calculating two vega measures: the partial derivatives with respect to the parameters, α and σ . In each case we make a small change to the parameter, reconstruct the tree, and observe the effect on the security price. In the case of σ , a small change affects only the spacing of the nodes; it does not alter the probabilities. In the case of α , a small change affects the probabilities in a symmetrical way; it does

not affect the positions of nodes. In both cases, there is a control variate effect that leads to the partial derivatives being calculated with a high degree of accuracy.

There are many different gamma measures that can be calculated. We favor a single overall measure of curvature: the second partial derivative of the security price with respect to r . This can be calculated directly from the tree.

V. CALIBRATION

The Black-Scholes stock option model has the simplifying feature that it involves only one volatility parameter. The usual procedure for calibrating the model to the market is to infer this parameter from the market prices of actively traded stock options.

The models presented here are more complicated than Black-Scholes in that they involve two volatility parameters, α and σ . The parameter σ determines the overall volatility of the short rate. The parameter α determines the relative volatilities of long and short rates. In practice, both parameters are liable to change over time.

We favor inferring both parameters from broker quotes or other market data on the prices of interest rate options. Our procedure is to choose the values of α and σ that minimize

$$\sum_i (P_i - V_i)^2$$

where P_i is the market price of the i th interest rate option, and V_i is the price given by the model for the i th interest rate option. The minimization is accomplished using an iterative search "hill-climbing" technique. When we calibrate the Hull-White model to the prices of seven at-the-money swap options, we find that the best fit values of α and σ give a root mean square pricing error of about 1% of the option price.

VI. CONCLUSIONS

This numerical procedure for one-factor term structure models can be used for the Hull-White

extended-Vasicek model and for lognormal models similar to those proposed by Black, Derman, and Toy [1990] and Black and Karasinski [1991]. The new approach is simpler and faster than previously suggested approaches. What is more, it gives greater accuracy for both the pricing of interest rate derivatives and hedge parameters.

ENDNOTES

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¹These are the Ho and Lee [1986] and the Hull and White [1990] models.

²Note that we use risk-neutral valuation. All processes are those that would exist in a risk-neutral world.

³A more precise statement is that $\theta(t)$ is the partial derivative with respect to t of the instantaneous futures rate for a contract with maturity t . When interest rates are stochastic, forward and futures rates are not exactly the same.

⁴Like Black and Karasinski, in their original 1990 publication Hull and White provide results for the general case where a and σ are functions of time.

⁵For slightly faster convergence, we can set M and V equal to their exact values:

$$M = e^{-2\Delta t} - 1; \quad V = \sigma^2(1 - e^{-2\Delta t})/2a$$

⁶Equation (2) provides an analytic expression for $\theta(t)$. We prefer not to use this and to construct the tree using the iterative approach described here. This is because it leads to a tree where the initial term structure is matched exactly. If the value of θ at time t is assumed to apply to the time interval between t and $t + \Delta t$, the initial term structure is matched exactly only in the limit as Δt tends to zero.

⁷It is not necessary to calculate θ or $\hat{\theta}$ in order to construct or use the tree.

⁸Not all no-arbitrage models have this property. For example, the extended-CIR model, considered by Cox, Ingersoll, and Ross [1985] and Hull and White [1990], which has the form

$$dr = [\theta(t) - ar]dt + \sigma\sqrt{r} dz$$

cannot fit steeply downward-sloping yield curves. This is because the process is not well-defined when $\theta(t)$ is negative. When r is small, the negative drift makes r become

negative, resulting in imaginary volatilities.

⁹In Hull and White [1993], a small change in the term structure is liable to lead to a change in all the branching probabilities. This introduces "noise," and causes the effect of small changes in the term structure on the price of a derivative to be estimated with much less precision.

¹⁰The control variate approach is a technique for increasing the accuracy of a numerical approximation. If the value of some variable to be approximated, X , is always close to some other variable Y whose value is known, accuracy can often be increased markedly by approximating not X but the *difference* between X and the control variate Y .

REFERENCES

- Black, F., E. Derman, and W. Toy. "A One-Factor Model of Interest Rates and its Application to Treasury Bond Options." *Financial Analysts Journal*, January-February 1990, pp. 33-39.
- Black, F., and P. Karasinski. "Bond and Option Pricing when Short Rates are Lognormal." *Financial Analysts Journal*, July-August 1991, pp. 52-59.
- Cox, J.C., J.E. Ingersoll, Jr., and S.A. Ross. "A Theory of the Term Structure of Interest Rates." *Econometrica*, 53 (1985), pp. 385-407.
- Heath, D., R. Jarrow, and A. Morton. "Bond Pricing and the Term Structure of Interest Rates: A New Methodology." *Econometrica*, 60, 1 (1992), pp. 77-105.
- Ho, T.S.Y., and S.-B. Lee. "Term Structure Movements and the Pricing of Interest Rate Contingent Claims." *Journal of Finance*, 41 (December 1986), pp. 1011-1029.
- Hull, J., and A. White. "One-Factor Interest Rate Models and the Valuation of Interest Rate Derivative Securities." *Journal of Financial and Quantitative Analysis*, 28 (1993), pp. 235-254.
- . "Pricing Interest Rate Derivative Securities." *Review of Financial Studies*, 3, 4 (1990), pp. 573-592.
- Jamshidian, F. "An Exact Bond Option Pricing Formula." *Journal of Finance*, 44 (March 1989), pp. 205-209.