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Collateral and Credit Issues in Derivatives Pricing*

John Hull and Alan White
Joseph L. Rotman School of Management
University of Toronto

e-mail: hull@rotman.utoronto.ca
awhite@rotman.utoronto.ca

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ABSTRACT

Regulatory changes are increasing the importance of collateral agreements and credit issues in over-the-counter derivatives transactions. This paper considers the nature of derivatives collateral agreements and examines the impact of collateral agreements, two-sided credit risk, funding costs, liquidity, and bid-offer spreads on the valuation of derivatives portfolios.

Key words: derivatives, over-the-counter, credit risk, collateral, funding value adjustment

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1. Introduction

Over forty years have passed since Black and Scholes (1973) and Merton (1973) published their path-breaking model concerned with the valuation European stock options. Few models in finance and economics have been as influential as Black−Scholes−Merton. The model has been extended to value many different types of derivatives on many different underlying assets and has found a wide array of applications.

One assumption made by Black−Scholes−Merton is that the two sides to a derivatives transaction are certain to honor their commitments. It is now recognized that default risk is an important consideration in the bilaterally-cleared over-the-counter market. Many researchers have written in this area. An excellent summary of their work is provided by Gregory (2012).

Banks now take account of counterparty default risk by calculating a credit value adjustment (CVA) for each counterparty. The CVA is the expected loss arising from a default by the counterparty. A bank's own credit risk is considered by a debt (or debit) value adjustment (DVA). The DVA is the expected gain that will be experienced by the bank (expected loss experienced by the counterparty) in the event that the bank defaults on its portfolio of derivatives with a counterparty. CVA and DVA are calculated and hedged in the same way as derivatives by many banks. The calculations typically involve computationally time-consuming Monte Carlo procedures and are discussed by, for example, Canabarro and Duffie (2003), Picault (2005), Labordère (2012), and Hull and White (2012a).

The size of CVA and DVA adjustments can be substantial. For example, JPMorgan-Chase reported a DVA gain (a change in its DVA) of $1.4 billion resulting from an increase in its credit spread in 2011. This increased the bank's net income by about 15% and contributed 2% to its reported return on equity.

One result of the 2008 credit crisis is a regulatory requirement that most standardized over-the-counter derivatives be cleared through central clearing parties (CCPs). A CCP operates similarly to an exchange clearing house and requires the two sides to a derivatives transaction to post both initial margin and variation margin so that the chance of a loss from a default is small.[1] This requirement will reduce the number of derivatives transactions cleared bilaterally.[2] Another requirement that has been proposed is that derivatives, when they are not cleared centrally, be very well collateralized with both initial margin and variation margin being provided by both sides.[3, 4]

One result of these measures is likely to be a reduction in CVA and DVA. However, there are a number of exceptions to the regulations. For example, they will not apply to transactions with non-financial end users and to some foreign exchange transactions. CVA and DVA will therefore continue to be important adjustments for some counterparties.

As the new regulations reduce the effect of counterparty credit risk, they will increase the use of collateral. The likely effect of this change on financial stability is discussed by Cecchetti *et al* (2009), Singh and Aitken (2009), Duffie and Zhu (2011), and Heller and Vause (2012). In this paper, we examine the effect that collateralization has on the valuation of derivatives. If the interest paid on cash collateral is the risk-free rate, no valuation adjustment for collateral agreements is necessary. However, if the interest rate is different from the risk-free rate, both parties to the transaction should make adjustments to their valuations (in addition to CVA and DVA) to reflect the present value of expected gains or losses from the difference between the collateral rate of interest and the risk-free rate. We will refer to the present value of expected loss arising from the interest paid on cash collateral as the collateral rate adjustment (CRA) so that the net value of a bank's derivatives portfolio with a counterparty after all the adjustments is the no-default value minus CVA plus DVA minus CRA.

[1] See Group of Twenty (2009)
[2] In some of its analyses, the International Monetary Fund (2010) assumes that three-quarters of interest rate swaps, two-thirds of credit default swaps, and one-third of other OTC derivatives will be cleared centrally.
[3] See Basel Committee on Banking Supervision (2012).
[4] Agreements specifying the collateral that must be posted in the non-centrally cleared over-the-counter market are usually included in a credit support annex (CSA) of an International Swaps and Derivatives Association master agreement between two market participants.

The main focus of this paper is to show how the arguments used to calculate the no-default value of a derivative can be extended to incorporate changing contractual arrangements. Previous research in this area includes Piterbarg (2010, 2012), Kjaer (2011), Burgard and Kjaer (2011a, 2011b, 2012), and Crépey (2013a, 2013b). Our approach complements earlier research in that we examine equilibrium arguments as well as the no-arbitrage arguments considered by other researchers. Equilibrium arguments are important when unhedgable risks (which arguably include the bank's own credit risk) are considered. We consider collateralization as well as credit risk and consider the implications of the analysis for bid-offer spreads. The expected recovery rates on the derivative for the bank and its counterparty are allowed to be different from that on their other liabilities.

One controversial issue, related to credit risk, is whether it is necessary for a bank (or other market participant) to adjust valuations to reflect funding costs. In practice, many banks do make what is known as a funding value adjustment (FVA).[5] It is argued that if a bank funds a derivatives portfolio by issuing debt, it should ensure that a hedged derivative position earns the rate of interest on the debt rather than the risk-free rate. We relate FVA to a type of DVA which we refer to as DVA2. This is the expected gain to the investors in a derivatives dealer from a default by the dealer on its debt. If the whole of the dealer's credit spread reflects default risk, FVA and DVA2 cancel each other and neither adjustment should be made. If only part of the spread reflects default risk, it is correct to calculate an FVA for the non-default-risk component of the credit spread.

For ease of exposition, the analysis is developed for a single derivative on a non-dividend-paying stock where interest rates are constant. But it can be extended so that it is applicable to derivatives and derivatives portfolios dependent on many underlying market variables. (This is important because CVA and DVA must usually be calculated for the portfolio of outstanding derivatives between two sides, not on a transaction-by-transaction basis.)

[5] In January 2014 JP Morgan reported a charge to income of $1.5 billion as a result of instituting an FVA adjustment. In an investor presentation, a spokesperson said J.P. Morgan was persuaded to make the change as a result of an industry migration toward such an adjustment.

Section 2 of this paper reviews the Black–Scholes (1973) and Merton (1973) arguments. Section 3 shows how they can be extended so that the credit risk of both sides to the transaction is taken into account in both an equilibrium and a no-arbitrage context. Section 4 uses the result of Section 3 to consider the impact of collateral agreements on valuation. Sections 5 and 6 consider the funding value adjustment. Section 7 extends the analysis further to show how bid-offer spreads can be determined. Conclusions are in Section 8.

2. The Basic BSM Argument

We start by considering the Black and Scholes (1973) and Merton (1973) arguments for valuing a derivative dependent on a non-dividend-paying stock when interest rates are constant. The valuation arguments developed by Black and Scholes involve the capital asset pricing model (CAPM). Suppose that f is the price of the derivative and S is the price of the stock. The process assumed for the stock is

$$dS = \mu S dt + \sigma S dz$$

where μ is the expected return on the stock, σ is its volatility, and dz is a Wiener process. An application of Ito's lemma shows that the price of the derivative satisfies

$$\begin{aligned} df &= \mu_f f dt + \sigma S \frac{\partial f}{\partial S} dz \\ \mu_f &= \frac{1}{f}\left[\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right] \end{aligned} \qquad (1)$$

If the continuous time CAPM applies, the expected stock return is

$$\mu = r + \beta_S\left(\mu_M - r\right)$$

where r is the rate of return on a riskless asset, μ_M is the expected return on the market portfolio, and β_S is the stock's beta. Based on the definition of beta and the processes followed by the stock and the derivative, the (instantaneous) beta of the derivative is

$$\beta_f = \frac{S}{f}\frac{\partial f}{\partial S}\beta_S$$

Applying the CAPM to the derivative

$$\mu_f = r + \beta_f\left(\mu_M - r\right) = r + \frac{S}{f}\frac{\partial f}{\partial S}\left(\mu - r\right) \tag{2}$$

Combining equations (1) and (2) leads to the well-known Black−Scholes−Merton (BSM) differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{3}$$

This analysis, which appeared in the original article by Black and Scholes, shows that the derivative price that satisfies equation (3) is the price at which it earns an expected rate of return that is commensurate with its risk. This derivation of equation (3) does not involve any hedging arguments.

We will refer to the solution of differential equation (3) as the no-default value of the derivative, f_{nd}. In general, the solution is given by discounting expected risk-neutral payoffs on the derivative back to the present using the riskless rate. For a European-style derivative maturing at time *T*, the solution is

$$f_{\text{nd}}\left(S_t, t\right) = e^{-r(T-t)} E_r\left[\, f\left(S_T, T\right)\right] \tag{4}$$

where S_t is the stock price at time *t* and E_r is the expectation taken over all paths that the stock price may follow when the stock's expected return is *r*.

An alternative derivation of the differential equation, suggested by Merton (1973), leads to the same result. Consider the case in which a bank has sold a derivative. The short position in the derivative exposes the bank to market risk due to uncertain changes in the price of the underlying stock. This market risk is hedged by buying enough shares to hedge out the market risk of the derivative. The resulting portfolio consists of a short position in the derivative and a position of $\partial f / \partial S$ in the stock. The portfolio value, Π, is

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

An application of Ito's lemma to this portfolio shows that the change in the portfolio value is

$$d\Pi = -\frac{\partial f}{\partial t} dt - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} dt \tag{5}$$

Since this portfolio is riskless, it should earn the same rate of return as other riskless assets, *r*, so that

$$d\Pi = r\Pi dt = \left[-rf + rS \frac{\partial f}{\partial S} \right] dt \tag{6}$$

Combining equations (5) and (6) also leads to the BSM differential equation (3). The solution to this differential equation is the price of the derivative that satisfies the no-arbitrage condition. Merton's analysis forms the basis on which banks have operated their derivatives businesses for many years, buying and selling the derivatives and using delta hedging to manage the risk.

3. Incorporation of Default Risk in Derivative Pricing

We now consider how credit risk affects derivative pricing. For ease of exposition, we consider three cases: the derivative is an asset of the bank (always has a positive value from the bank's point of view), the derivative is a liability of the bank (always has a negative value from the bank's point of view), and the derivative is neither an asset nor a liability of the bank (the value is positive in some situations and negative in others).

Our analysis makes the simplifying assumption that the impact of a default is to reduce the value of a derivative and a bond by known proportional amounts. This allows us to examine key theoretical issues without a great deal of notational complexity. In practice the claim for a derivative in the event of a default is usually the bid or offer no-default value, whichever is more favorable to the non-defaulting party. The claim for a bond is its face value plus accrued interest. A detailed discussion of the impact of close-out value is provided by Burgard and Kjaer (2011b).

We assume for the moment that no collateral is posted.

3.1 The Derivative is an Asset of the Bank

Suppose first that the derivative is an asset of the bank so that it always has a positive value to the bank. If the counterparty may default, the process for the derivative given in equation (1) can be modified to include the possibility of default so that it becomes

$$df = \left(\mu_f f\right)dt + \sigma S \frac{\partial f}{\partial S} dz - \gamma_C f dq \tag{7}$$

where *dq* denotes a jump process that represents the event of default. The probability that a jump occurs in the next interval of length dt is $\lambda_C\, dt$ where λ_C is the counterparty's hazard rate. The size of the jump is one and γ_C is the expected proportional reduction in the value of the derivative in the event of a counterparty default. (For ease of exposition, we assume that the default risk is not systematic. Equivalently we could define λ_C as the risk-neutral hazard rate.) In this case

$$E\left[\frac{df}{f}\right] = \left(\mu_f - \gamma_C \lambda_C\right)dt$$

Combining this with equation (2) results in a modified form of the original BSM differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = \left(r + \gamma_C \lambda_C\right) f$$

Suppose that the counterparty has an outstanding discount bond. The process followed by the price of this bond is

$$dB_C = r_C B_C dt - \eta_C B_C dq$$

where r_C is the instantaneous return earned by the bondholder as long as the bond does not default and η_C is the expected proportional reduction in the value of the bond in the event of a default. Since the default risk is not systematic

$$E\left[\frac{dB_C}{B_C}\right] = \left(r_C - \eta_C \lambda_C\right)dt = rdt$$

This allows us to express the hazard rate in terms of the interest rate on the debt.

$$\lambda_C = \frac{r_C - r}{\eta_C}$$

The differential equation then becomes

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = r_C^* f \tag{8}$$

where

$$r_C^* = r + \left(r_C - r\right)\frac{\gamma_C}{\eta_C}$$

In the particular case where the bond and the derivative are treated in the same way in the event of default $\gamma_C = \eta_C$ so that $r_C^* = r_C$. We can regard r_C^* as an adjusted rate of interest on the bond to allow for differences between the recovery rates on the bond and the derivative.

Equation (8) can also be derived using Merton's hedging arguments. If the counterparty defaults, the bank will suffer a loss. In this case, the bank hedges the counterparty credit risk by shorting an appropriate amount of the discount bond issued by the counterparty. The resulting hedge portfolio consists of a long position in the derivative, a position of $-\partial f/\partial S$ in the stock and a short position in *n* units of the discount bond where *n* is chosen so that $nB_C\eta = \gamma f$. The portfolio value, Π, is

$$\Pi = f - \frac{\partial f}{\partial S} S - nB_C = f\left(1 - \frac{\gamma_C}{\eta_C}\right) - \frac{\partial f}{\partial S}$$

An application of Ito's lemma to this portfolio shows that, as long as there is no default, the change in the portfolio value is

$$d\Pi = \frac{\partial f}{\partial t} dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} dt - r_C \frac{\gamma_C}{\eta_C} f dt \tag{9}$$

The last term in equation (9) is the interest that must be paid on the short position in the counterparty's debt. Since this portfolio is riskless, it should earn the same rate of return, *r*, as other riskless assets so that

$$d\Pi = r\Pi dt = rf\left(1 - \frac{\gamma_C}{\eta_C}\right) dt - rS\frac{\partial f}{\partial S} dt \tag{10}$$

Combining equations (9) and (10) leads to the differential equation (8). The solution is obtained by discounting risk-neutral payoffs at r_C^*. This modified discount rule has been mentioned by other authors such as Hull and White (1995) and Kjaer (2011). For a European-style derivative providing a payoff at time *T*

$$f(S_t, t) = e^{-r_C^*(T-t)} E_r\left[f(S_T, T)\right]$$

which, from equation (4), becomes

$$f(S_t, t) = f_{\text{nd}} e^{-\left(r_C^* - r\right)(T-t)}$$

The variable $r_C^* - r$ can be regarded as the "loss rate" on the derivative. Because CVA is the reduction in the value of the derivative arising from the possibility of a counterparty default it follows that

$$\text{CVA} = f_{\text{nd}} - f_{\text{nd}} e^{-(r_C^* - r)(T-t)}$$

3.2 The Derivative is a Liability of the Bank

Now consider the case in which the derivative is a liability of the bank so that it always has a positive value to the counterparty and a negative value to the bank. The Black−Scholes CAPM derivation of the value of the derivative is essentially unchanged except: the value of the derivative is negative, the hazard rate is λ_B, the bank's hazard rate, and the relevant borrowing rate is that of the bank, r_B. The resulting differential equation is

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = r_B^* f \qquad (11)$$

where

$$r_B^* = r + (r_B - r)\frac{\gamma_B}{\eta_B}$$

Here γ_B and η_B are the expected proportional impact on the value of its derivative and its bonds of a default by the bank. (Because the derivative is a liability, the bank gains γ_B times the value of the derivative from its own default.)

The Merton hedge portfolio argument is also essentially the same except it is now necessary to purchase some of the bank's own debt to hedge the change in derivative value in the event of default. Some researchers are uncomfortable with this. They argue that it may not be possible to use the proceeds from the sale of the derivative to buy the bank's own debt. Their arguments can be countered in several ways. First we can treat the analysis as a thought experiment of what is required to eliminate all risks from the derivative portfolio so that the hedged portfolio can be compared with a riskless asset. Since the value of the derivative is the same whether we hedge it or not, the results of the thought experiment can be used to determine the value even if it is not possible to buy the banks own debt to hedge the default risk.[6]

[6] In this context, it is worth noting that delta hedging and other derivatives hedging strategies rarely work perfectly in practice. Nevertheless, it is assumed that they do work perfectly when hedging arguments are used to value derivatives.

An alternative explanation is that it is not necessary to buy the bank's own debt. If the proceeds from the sale of the derivative can be used as a source of bank financing, the amount of funding the bank requires from external sources is reduced. This reduction in external borrowing reduces the interest payments to external sources. As a result the proceeds from the sale of the derivative earn an effective rate of r_B and equation (11) follows. The final counter is that the Black−Scholes CAPM derivation which produces the same differential equation does not rely on buying or selling the bank's debt. It merely uses the yield on debt to infer the bank's hazard rate.

The general solution to equation (11) for any derivative involves determining the expected risk-neutral payoffs at all times and discounting these at r_B^*. For a European-style derivative providing a payoff at time T, the solution is

$$f\left(S_t, t\right) = e^{-r_B^*(T-t)} E_r\left[f\left(S_T, T\right)\right]$$

If the derivative is always a liability, the credit adjusted value can be determined by merely adjusting the rate used to discount the expected payoffs. Analogously to the asset case

$$f = f_{\text{nd}} + \text{DVA}$$

where

$$\text{DVA} = f_{\text{nd}} e^{-(r_B^* - r)(T-t)} - f_{\text{nd}}$$

Note that DVA is positive because f_{nd} is negative**.**

3.3 The Derivative can be an Asset or a Liability

Finally, consider the case in which the value of the derivative may be positive or negative. Both the Black−Scholes CAPM and the Merton riskless hedging arguments are exactly the same except the argument that applies at any particular time depends on whether the value of the derivative is positive or negative at that time. Whenever the derivative value is positive, equation (8) applies; whenever the value is negative, equation (11) applies. This means that

$$\frac{\partial f}{\partial t}+rS\frac{\partial f}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}=r_C^*\max\left(f,0\right)+r_B^*\min\left(f,0\right) \tag{12}$$

In general:[7]

$$f\left(S_t,t\right)=f_{\text{nd}}-\text{CVA}+\text{DVA}$$

There are three special cases in which a simple discount rate adjustment can be used to value the derivative. The first two are the cases already discussed in which f_{nd} is always positive or always negative. The third special case is the situation in which $r_C^*=r_B^*=r^*$. In this case, equation (12) simplifies to

$$\frac{\partial f}{\partial t}+rS\frac{\partial f}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}=r^* f \tag{13}$$

The value of the derivative can be obtained by discounting risk-neutral payoffs at r^*. When a payoff is expected at a single time *T*:

$$f(S_t,t)=f_{\text{nd}}e^{-\left(r^*-r\right)\left(T-t\right)}$$

and

$$\text{CVA}-\text{DVA}=f_{\text{nd}}-f_{\text{nd}}e^{-\left(r^*-r\right)\left(T-t\right)}$$

3.4 *Extensions*

The analysis we have presented is for a single derivative, dependent on a single underlying asset that provides a payoff at a particular future time. Our aim has been to illustrate the nature of CVA and DVA and the relation between equilibrium and no-arbitrage arguments. The practical problems in evaluating CVA and DVA are evident from information published about Lehman's derivatives portfolio prior to its bankruptcy on September 15, 2008. Lehman entities had over

[7] Note that some adjustment for the possibility that both parties will default during the life of the derivatives portfolio may be necessary in the calculation of CVA and DVA. See, for example, Brigo and Morini (2011).

one million over-the-counter derivatives transactions with about 8,000 different counterparties[8]. Thousands of different market variables were probably involved in the valuation of these derivatives. The differential equations we have presented and their risk-neutral valuation solutions can, with a considerable increase in notational complexity, be extended to accommodate portfolios involving multiple derivatives. In practice, as noted earlier time consuming Monte Carlo simulations are necessary for to evaluate CVA and DVA. One advantage of Monte Carlo simulation is that the details of collateral agreements and accurate rules for determining claims in the event of default can be incorporated.

A further complication is wrong-way/right-way risk. This is where λ_C and possibly λ_B are assumed to be dependent on the value of the derivatives portfolio so that the probability of default by a counterparty or the bank depends on the exposure of the other side. Incorporating wrong-way/right-way risk into even the simple situation considered in Section 3.1 to 3.3 is difficult. Approaches have been suggested by, for example, Cepedes *et al* (2010) and Hull and White (2012a). When a bank does not have a wrong-way-risk model regulators allow for wrong-way/right-way risk by increasing its no-wrong-way-risk CVA estimates by 40%. When the bank does have a wrong-way risk model, it can be used but the impact of wrong-way/right way risk over the no-wrong-way-risk estimate is floored at 20%.

4. Impact of Collateral Agreements

Up to now we have assumed that no collateral is posted. In practice, it is now common to include a credit support annex (CSA) in the agreement covering transactions that are cleared bilaterally. The CSA typically provides formulas governing the amount of collateral that is required by each side at any given time. It is important to recognize that the laws governing derivatives collateral are different from the laws governing physical assets when they are used as security for a loan. When there is a default on a loan, the seizure of collateral by creditors is subject to delays and must be authorized by a bankruptcy court. The collateral posted for derivatives positions is usually in the form of cash or liquid securities. When there is an early termination following an

[8] See Goldstein and Henry (2008).

event of default, collateral posted by the defaulting party is under the control of the non-defaulting party and can immediately be used to compensate the non-defaulting party for its losses.

An International Swaps and Derivatives Association master agreement typically allows the non-defaulting party to claim the cost of replacing the outstanding transactions. This cost is the mid-market value of the transactions at the time of the default, adjusted for the bid-offer spreads that the non-defaulting party would incur when trading with third parties to carry out the replacement. Suppose first that the cost of replacing the transactions is positive (i.e., the portfolio of outstanding transactions, after bid-offer spread adjustments, has a positive value to the non-defaulting party). If the replacement cost is less than the collateral that has been posted, the non-defaulting party incurs no loss and is required to return any excess collateral. If it is greater than the posted collateral, the non-defaulting party is an unsecured creditor for the collateral shortfall. If the cost is negative (i.e., the portfolio of outstanding transactions has a positive value to the defaulting party after bid-offer spread adjustments), there are also two situations to consider. If the collateral posted by the non-defaulting party is greater than the value of the transactions to the non-defaulting party, the non-defaulting party is an unsecured creditor for the excess collateral. If the collateral posted by the non-defaulting party is less than the value of the transactions to the defaulting party, the non-defaulting party is required to pay the amount of the collateral shortfall to the defaulting party.

We start with a simple somewhat idealized situation. We suppose that each side has to post cash collateral equal to $\max(X, 0)$ where X is the mid-market no-default value of the outstanding derivatives to the other side. Collateral is posted continuously right up to the time of default. Furthermore, we assume that the bid-offer spread adjustments referred to above do not apply. We suppose that the interest paid on the collateral exceeds the risk-free rate by a spread s, which may be positive, negative, or zero.

Clearly, in the case where s is zero, collateral arrangements do not affect the value of the derivative. Because we assume that collateral is posted right up to the time of default and there are no bid-offer spread adjustments, no losses are incurred by either side in the event of a default and equation (3) applies. In practice, s is sometimes non-zero (i.e., the interest paid on cash collateral is different from the risk-free rate). The collateralized case is then same as the no-default case considered in Section 1 except that that r+s is paid on the value of the derivative at all times.

First, consider the equilibrium approach to valuation. The bank has an investment f in the derivative and borrowings of f on which an interest of $r+s$ is paid. (If f is negative, the derivative is a liability and the "borrowings" are an investment earning r+s.) As a result, the effective rate of return on the derivative in equation (1) becomes

$$\mu_f = \frac{1}{f}\left[\frac{\partial f}{\partial t} + \mu S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}\right] - (r+s)$$

The equilibrium expected return equals the equilibrium return on the derivative less the equilibrium return on the collateral. The latter is the risk-free rate so that

$$\mu_f = \left[r + \beta_f\left(\mu_M - r\right)\right] - r = \frac{S}{f}\frac{\partial f}{\partial S}(\mu - r)$$

Combining these last two equations leads to the differential equation (3) being modified to

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = (r+s)\, f \qquad (14)$$

Now consider the hedging analysis. The change in the value of the hedge portfolio given by equation (5) becomes

$$d\Pi = -\frac{\partial f}{\partial t}dt - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}dt + (r+s)\, fdt$$

Equation (6) becomes

$$d\Pi = r\Pi dt = \left[-rf + rS\frac{\partial f}{\partial S}\right]dt + rfdt$$

In the first of these equations, the last term reflects the income on the collateral part of the hedged portfolio. In the second equation, the last term reflects the equilibrium expected return on the collateral part of the hedged portfolio. Combining these two equations also leads to equation (14).

Equation (14) shows that derivatives can be valued by discounting risk-neutral payoffs at $r + s$, the interest rate on the collateral. For a European-style derivative providing a payoff at time T, the solution to equation (14) is

$$f\left(S_t, t\right) = e^{-(r+s)(T-t)} E_r\left[f\left(S_T, T\right)\right]$$

We refer to the adjustment for the interest rate on the collateral as the collateral rate adjustment (CRA). It follows that

$$\text{CRA} = f_{\text{nd}}\left(1 - e^{-s(T-t)}\right)$$

In practice, collateral agreements are not as simple as the one we have just considered. An agreement can be one-sided (only one party is required to post collateral) or two-sided (both parties are required to post collateral). Not all collateral is in the form of cash. A collateral agreement usually specifies securities that can be posted in lieu of cash. It also specifies haircuts (i.e., percentage reductions that will be applied to the values of the securities for the purpose of determining their cash equivalent). Returns from the securities that are posted as collateral (income and capital gains) belong to the party that is posting the securities as collateral. The expected return on the securities is by definition their economic return and so no collateral rate adjustment is necessary when securities are posted as collateral. (This is true regardless of the haircut.)

Each day, the collateral already posted is compared with that specified in the agreement. This leads to either some collateral being returned or further collateral being posted. A minimum transfer amount is usually specified to avoid the inconvenience of relatively small daily transfers of collateral being required. Sometimes a threshold, H, is specified. This means that the collateral that has to be posted is $\max(X - H, 0)$. In other circumstances, an independent amount is specified. This is in effect a negative threshold.

The idealized case we considered above is an approximation to a two-sided zero-threshold agreement. This is the type of collateral agreement that has traditionally been used between two derivatives dealers. To a good approximation, CRA can be incorporated into a valuation by discounting at $r + s$ rather than r providing all collateral is posted in the form of cash. In practice, however, the collateral posted between dealers does not usually have to be in the form of cash. Dealers will find it optimal to post cash when $s > 0$ and to post securities when $s < 0$ so that a CRA is necessary only when $s > 0$.

In the idealized case, it can be assumed that CVA and DVA are both zero. But this assumption cannot be made for the two-sided zero-threshold agreements encountered in practice. This is because time typically elapses between the following events:

a) the defaulting party stops posting collateral and stops returning excess collateral; and

b) there is an early termination of outstanding derivatives because of an event of default has occurred

If the derivatives portfolio moves in favor of the non-defaulting party during this period of time, a loss will be incurred and the non-defaulting party will be an unsecured creditor for an amount reflecting the movement. (There is no corresponding gain when the derivatives portfolio moves against the non-defaulting party because the non-defaulting party must return any excess collateral.) Furthermore, the non-defaulting party will also be an unsecured creditor for the bid-offer spread adjustments (referred to earlier) that are made when determining the claim amount.

Our results can be used to reach some conclusions about the relative sizes of CRA and CVA/DVA. The differential equations (8), (11), (13), and (14) have a similar form. Comparing equations (13) and (14), for example, shows that a CRA when $s = x$ in a two-sided zero-threshold agreement is the same as the net credit adjustment, CVA−DVA, when no collateral is posted and the credit spread of each side (adjusted if necessary for differences between the recovery rates on bonds and derivatives) is x.

In practice, when collateral is posted, CVA and DVA are much smaller than in the no-collateral case. As a result, CRA is likely to be considerably larger than the net credit adjustments in the situation we have considered. Furthermore, as new regulations are implemented, collateral requirements between financial institutions are likely to become considerably greater than those given by a two-sided zero-threshold agreement. This will also have the effect of increasing CRA and decreasing the net credit adjustment still further.

In general CRA, like CVA and DVA, must be valued in the same way as a derivative. Monte Carlo simulations must be used to calculate the present value of the expected difference between the interest rate cash flows applicable to the collateral and those that would be applicable to the collateral if the risk-free rate was paid and received. When securities can be posted instead of cash, the extent to which this will happen needs to be estimated.[9]

CRA differs from CVA and DVA in a number of ways. CRA depends on a rate that is contractually defined while CVA and DVA depend on the credit spreads of the two parties and other market variables. When the credit spreads or other market variables change, the CVA and DVA changes result in reportable income. If the rate paid on CRA is the risk-free rate plus a contractual spread, there will be no income effects as a result of changes in market variables. Income effects occur only if the rate paid on collateral is specified as a fixed rate.

Another important difference is that, in the case of two-way zero threshold collateral agreements where all collateral must be cash, CRA is additive across transactions and can therefore be calculated on a transaction-by-transaction basis. Defaults have a very small effect on CRA and can safely be ignored when it is calculated. Because of the impact of netting agreements, CVA and DVA are usually not additive across transactions and must be calculated on a portfolio-by-portfolio basis.[10] Furthermore, collateral arrangements cannot be ignored when CVA and DVA are calculated. Indeed, it is crucially necessary to estimate for each simulation trial the collateral that will be available at each of the default times considered.

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[9] As indicated earlier, dealers are likely to post cash or securities depending on whether s is greater than or less than zero. Non-financial end users are more likely to post cash collateral in all situations.

[10] An exception here is that the net credit adjustment (CVA−DVA) is additive when $r_C^* = r_B^*$ and collateral agreements are symmetrical.

The net value of the derivative portfolio is

$$f\left(S_t,t\right)=f_{\mathrm{nd}}-\mathrm{CVA}+\mathrm{DVA}-\mathrm{CRA}$$

The fair market value of a new transaction with a counterparty is the incremental impact it will have on $f\left(S_t,t\right)$.[11]

5. Funding Value Adjustment

In addition to CVA, DVA, and CRA, many banks make a funding value adjustment (FVA) to reflect the difference between their funding costs and the risk-free rate.

Articles by Burgard and Kjaer (2011, 2012) and Hull and White (2012b, 2013) show that it is not appropriate to make FVA adjustments when determining the value of derivatives. The Burgard and Kjaer papers focus on the mechanics of hedging. Hull and White take a more traditional finance valuation perspective arguing that the fair valuation of derivative products should be related to the economic properties of the contracts and should not be affected by how they are funded or whether the underlying risks can be hedged.

The underlying motivation for making a funding value adjustment is the Merton hedging argument in equation (6). The trader faces borrowing and lending rates that differ from r. In order to focus the discussion on funding value adjustments we will initially abstract from credit risk issues by assuming that default is not possible and that funding rates differ from the risk-free rate for some unknown reason that is not default risk.

To make the analysis as straightforward as possible, we assume that interest rates and credit spreads are constant. (The results continue to apply when this assumption is relaxed.) Suppose that the borrowing and lending rate associated with cash flows related to the derivative are equal

[11] This can be calculated efficiently if all Monte Carlo samples for the last calculation of CVA and DVA are stored because it is then only necessary to simulate the new proposed transaction for the scenarios that were considered.

to r_d and the corresponding rate associated with cash flows related to the stock is r_S. In this case equation (6) must be changed to

$$d\Pi = \left[-r_d f + r_S S \frac{\partial f}{\partial S} \right] dt$$

It is usually assumed that the equity hedge is funded in the repo market.[12] The repo rate can reasonably be assumed to be the risk-free rate, r, so that this becomes

$$d\Pi = \left[-r_d f + rS \frac{\partial f}{\partial S} \right] dt \qquad (15)$$

When this is combined with equation (5), the resulting differential equation is

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = r_d f \qquad (16)$$

For a derivative that has no payoffs before T and has a value equal to $f(T, S_T)$ at time T, the solution to the differential equation at any earlier time, t, is

$$f^*\left(t, S_t\right) = e^{-r_d(T-t)} E_r\left[f\left(T, S_T\right)\right] \qquad (17)$$

where, as before, the notation E_r denotes taking expectations over paths where the stock price has an expected return of r. As a result, $f^*\left(t, S_t\right)$ becomes the trader's no default benchmark price. If the trader sells the derivative at this price, in the absence of credit risk the revenues generated are adequate to cover all the hedging costs including the funding costs.

We define the dealer's credit spread, s_d, as $s_d = r_d - r$. From equations (4) and (17) the FVA-adjusted price is

$$f^*\left(t, S_t\right) = e^{-s_d(T-t)} f_{\mathrm{nd}}\left(T, S_t\right) \qquad (18)$$

[12] If derivatives could be repoed (and this may become possible at some point) we could replace r_d, as well as r_s, by r and FVA would disappear.

The funding value adjustment (FVA) is the amount by which the value determined using equation (4) must be adjusted in order to get the trader's value. Therefore

$$\text{FVA} = f_{\text{nd}}(t, S_t) - f^*(t, S_t)$$

or from equation (18)

$$\text{FVA} = f_{\text{nd}}(t, S_t)\left[1 - \exp\left(-s_d(T-t)\right)\right] \tag{19}$$

We now combine the effects of the dealer's credit risk and funding costs to determine their total economic impact on derivative pricing. Up to now we have defined DVA as the expected gain to the dealer from defaulting on the derivatives. There is another component of DVA: the benefit of defaulting on the funding for the derivative. From now onward we will refer the expected gain from defaulting on derivatives as DVA1 and the expected gain from defaulting on the debt as DVA2. DVA1 is calculated as indicated in earlier sections. We now consider DVA2.

Suppose that the credit spread is entirely compensation for default risk. (We critically examine this assumption in the next section. The expected cost to a lender who is financing the derivative of a default by the dealer between time t and $t+\Delta t$ is $s_d\Delta t$ times the value of the derivative. The cost to the counterparty is a benefit to the dealer. When this benefit is included equation (15) it becomes

$$d\Pi = \left[-r_d f + s_d f + rS\frac{\partial f}{\partial S}\right]dt = \left[-rf + rS\frac{\partial f}{\partial S}\right]dt$$

and equation (16) becomes the usual Black-Scholes-Merton differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf$$

This analysis shows that DVA2 cancels FVA.

When all adjustments are considered, the equation

$$f = f_{\text{nd}} - \text{CVA} + \text{DVA1} + \text{DVA2} - \text{FVA} - \text{CRA}$$

becomes

$$f = f_{\text{nd}} - \text{CVA} + \text{DVA1} - \text{CRA}$$

because DVA2 = FVA. When an instrument provides a single payoff at time T both DVA2 and FVA are

$$f_{\text{nd}}\left[1 - \exp\left(-s_d\left(T-t\right)\right)\right]$$

Once we include the credit adjustment for the debt that must be issued to finance the derivative transaction, we find that the FVA-adjusted price and the non-FVA-adjusted price are the same. Once all the economic factors are considered, the economic value of the derivative transaction is the same for every dealer regardless of their funding cost. Dealers should not make a funding value adjustment.

Although not making a FVA is the right thing to do and makes economic sense, it leaves the trader in an uncomfortable position. The trader buys the option for f and borrows f at an interest rate r_d. The risk is managed using a hedging scheme which assumes that the trader pays an interest rate r. As a result of this, in every period in which the dealer does not default, the trader reports a loss based on the difference between r and r_d. This is offset by the fact that if the dealer does default the trader captures a gain equal to the difference between the value of the option that was purchased and the fraction of the debt that is paid in default. (Actually, it is the investors in the dealer that capture this gain.) Overall, the transaction makes economic sense but, while the dealer is not defaulting, it appears that the trader is losing money.

The reason for this uncomfortable situation is that there has been a risk transfer within the dealer. Within the confines of our modeling, the trader will never default. The options purchased and sold are perfectly matched with a self-financing trading scheme. The funding rate that the trader faces is a result of credit risk incurred elsewhere in the dealer. As a result, the trader is subsidizing other parts of the dealer by paying an interest rate that is too high.

In spite of the arguments showing that the practice of making funding value adjustments has no theoretical basis, the practice persists. The accounting profession now requires banks to make CVA and DVA adjustments to the reported values of their derivatives portfolios.[13] The accounting standards do not seem to accept FVA for reporting purposes. IFRS 13 states[14]

> Th(e) definition of fair value emphasizes that fair value is a market-based measurement, not an entity-specific measurement. When measuring fair value, an entity uses the assumptions that market participants would use when pricing the asset or liability under current market conditions, including assumptions about risk. As a result, an entity's intention to hold an asset or to settle or otherwise fulfil a liability is not relevant when measuring fair value.

One possible outcome is that FVA adjustments will not be allowed for reporting purposes because they do not reflect the economic "fair value" of a portfolio, but will continue to be used for internal decision making. FVA then becomes part of a private valuation that affects a bank's appetite for the transaction and is similar to many other considerations such as the nature of transactions involving the same underlying asset that are already on the bank's books, liquidity considerations, regulatory capital constraints, the bank's ability to hedge the underlying risks, and so on. In the next section we will discuss why liquidity may lead to a form of funding value adjustment.

6. Liquidity Considerations

In deriving the FVA = DVA2 result, we assumed that all of the dealer's credit spread, s_d, was due to default risk. Suppose instead that only some fraction α ($\alpha \leq 1$) of the spread is attributable to default risk and the balance is due to non-economic factors. The benefit to the dealer from default during time Δt is then $\alpha s_d \Delta t$ times the value of the derivative so that equation (15) becomes

[13] See Financial Accounting Standards Board (2006, 2007).
[14] International Accounting Standards Board (2011) .

$$d\Pi = \left[-r_d f + \alpha s_d f + rS\frac{\partial f}{\partial S}\right]dt = \left[-(r+(1-\alpha)s_d)f + rS\frac{\partial f}{\partial S}\right]dt$$

and differential equation (16) becomes

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = [r+(1-\alpha)s_d]f$$

In this case, FVA does not equal DVA2. It is correct to include an "FVA−DVA2" adjustment based on a spread of $(1-\alpha)s_d$. From equation (19), for a derivative promising a payoff at time T

$$\text{FVA} - \text{DVA2} = f_{\text{nd}}\left[1 - \exp(s_d(1-\alpha)(T-t))\right]$$

This adjustment can be regarded as a cost of doing business for the dealer. No benefit is received for $1-\alpha$ of the spread paid by the dealer.

A major reason why α can be expected to be different from zero is liquidity. Investors require higher returns from less liquid bonds. The liquidity of bonds issued by dealers is liable to vary from dealer to dealer and so there is no reason to suppose that liquidity premiums are the same for different dealers. This may be a reason for price differences among dealers. However, we emphasize that the price differences are much smaller than those given by the usual FVA adjustments.

A number of researchers have analyzed the credit spreads for bonds to provide alternative estimates of α. Typically they have tried to attribute the spread of bond yields over Treasury rate to:

1. A tax effect due to the preferential tax treatment of Treasury bonds at the state level
2. The actuarial expected loss due to default
3. A credit risk premium reflecting the market price of default risk
4. A liquidity premium

Practitioners take the first factor is taken into account by using a different benchmark than Treasuries (e.g., the overnight indexed swap rate) when calculating credit spreads. The second and third factors are both relevant in determining the benefit to the dealer of defaulting and should also be reflected in CDS spreads. The fourth factor, the liquidity premium, is actually a catch-all for everything not included in the first three factors. Unfortunately it is difficult to distinguish between the third factor and the fourth factor. They both vary through time and are likely to be correlated.

One of the first studies was by Elton *et al* (2001). This research decomposed the excess spread observed between 1987 and 1996 into two components: the extra return on corporate bonds due to advantageous tax treatment of Treasury bonds, and an unexplained spread component. The excess spread was found by subtracting the expected loss due to default based on Moody's and S&P's historical experience of rates of default and loss in the event of default from the credit spread. The tax premium was based on known state and federal taxes. The researchers found that for five-year industrial bonds the unexplained spread component accounts for about 45% of the observed credit spread. They then apply standard asset pricing models to the corporate bonds and find that about 75% of the unexplained spread component is explained by the Fama-French (1993) three-factor capital asset pricing model. The balance of the unexplained spread which can must be attributed other factors such as liquidity is quite small.

Hull *et al* (2005) using the Merrill Lynch bond data base and a risk-free benchmark that is 10 basis points below the risk-free rate find results that are consistent with those of Elton et al (2005). They estimate that the average total of the credit risk premium and liquidity premium between 1995 and 2004 increased from a low of 38 basis points for AAA/Aaa rated bonds to a high of 264 basis points for CCC/Caa bonds. They argue that much of this risk premium can be explained by systematic risk rather than liquidity.

Driessen (2005) extended Elton *et al* (2001). He carried out an empirical decomposition of observed excess spreads into default, liquidity, risk premium, and tax components. A formal term-structure and asset pricing model was fitted simultaneously to Treasury bond and corporate bond prices. The calculation of the excess spread and the tax treatment of the corporate and Treasury bonds was similar to Elton *et al* (2001). The asset pricing model included as parameters a liquidity premium and a default risk premium. The model was fitted to the weekly prices of corporate and Treasury bonds from February 1991 to February 2000. The prices of 592 corporate bonds issued by 104 investment grade firms were used along with the on-the-run Treasury bonds. The liquidity and default risk parameters of the model were selected to provide the best possible fit to the full data set. The best-fit parameter values were then used to decompose the excess spread into liquidity, risk premium and tax components. The fraction (expressed as a percent) of the total excess spread attributable to liquidity was found to be between 17% and 25%.

Collin-Dufresne *et al* (2001) used Merton's (1974) structural model of default to help decompose the observed yield spread data. The data set was a portfolio of investment-grade bonds in the period 1988 to 1997. If the Merton model is correct, it should be possible to predict credit spreads perfectly from it. However, there is a long literature that shows that while structural models work reasonably well and are very useful for determining which firms will have higher or lower credit spreads, they do not predict the actual size of the credit spread well. Collin-Dufresne *et al* showed that the factors underlying Merton's model explain only about 25% of observed credit spread changes. They find that about 55% of the unexplained changes are explained by some common factor that affects all bond yield spreads. This common factor may reflect changes in the market price of default risk. As bond investors become more or less risk averse they raise or lower the spread they demand on all corporate bonds. The remaining 45% of the unexplained changes (34% of the observed changes) may well be due to liquidity.

Chen *et al* (2009) return to the questions raised by Collin-Dufresne *et al* (2001). They address the question of whether it is possible to modify the standard structural model in a plausible way in order to make theory agree with the observed credit spread puzzle. By introducing time-varying risk aversion in which investors require higher rates of return on all investments during recessions it is found that the model is consistent with the observed excess spreads. As a result, it is possible to argue that most of the excess spread is a risk premium and the liquidity premium is small. Dick-Nielsen *et al* (2012) also find that the liquidity component of credit spreads is small. They used a number of different liquidity measures and a large database of bond trades.

In summary, research on US corporate bond yield spreads finds that up to 30% of the average spread of a corporate bond over Treasuries might be attributed to illiquidity. However, most of these studies used data from before the Financial Industry Regulatory Authority (FINRA) introduced the Trade Reporting and Compliance Engine (TRACE) in 2002. Since the introduction of TRACE, bond market liquidity has improved substantially. As a result it is likely that the liquidity component of bond yield spreads is much smaller than it was pre-2002.

The research described so far provides estimates of the average liquidity premium over time. Practitioners are interested in estimating α at a point in time. One estimate of α can be obtained from the excess of a bond yield spread over the corresponding CDS spread. This is the negative of the CDS–bond basis. The argument for this is that the CDS spread is a pure credit spread whereas bond yield spreads includes an allowance for the illiquidity of the bond. CDSs are also illiquid. Indeed for some companies they are less liquid than bonds.[15] However, illiquidity does not have the same effect on CDS spreads as it does on bond yield spreads. As a bond becomes less liquid an investor becomes less willing to purchase the bond so that the price goes down and the bond yield spread increases. A CDS is a two-way instrument with no net asset. Both buyers and sellers of protection may become less inclined to trade CDSs as they become less liquid, but this should not affect the mid-market spread.

Unfortunately, the basis reflects a number of factors some of which traders should not include in their estimate of α. One indication of this is the fact that sometimes the CDS-bond basis is positive which would give rise to a negative α estimate.[16] Studies such as Euler and Trapp (2009), Collin-Dufresne and Bai (2013), and Augustin (2012)) show that liquidity does play a role in the basis but that many other factors are also important. They also point out that how the bond yield spread is measured (e.g., what the benchmark risk-free rate is assumed to be and whether asset swap spreads are used) has an effect on the results.

The CDS-bond basis is also affected by the fact that CDSs and bonds are not perfect substitutes. If the bond does not trade at par, the loss in the event of default can be different for the CDS than

[15] An analysis of the CDS trades reported by the Depository Trust & Clearing Corporation shows that in many cases the CDS market is very illiquid.

[16] For example, Collin-Dufresne and Bai (2013) report that prior to 2008 the spread for non-investment grade bonds was positive.

the bond. A CDS contract may contain a delivery option, allowing one of a number of bonds to be delivered in the event of default. The recovery on the CDS may therefore differ from that on the actual bond held. The definition of what constitutes default for the bond and the CDS often differs. In the case of a regular asset-swap package, the embedded interest rate swap does not terminate in the event of default, and will have some value.

Our conclusion from this is that the basis is an indicator of factors that are non-economic cost of borrowing, but the basis is also an indicator of some market micro structure phenomena that have something to do with trading but nothing to do with the cost of debt.

As indicated in the previous section, the fair value reported for accounting purposes should be an exit price. The “FVA−DVA2” adjustment should be an adjustment reflecting the non-default-risk component of the credit spread for other dealers rather than for the dealer making the adjustment. One possibility is for market participants to set the “FVA−DVA2” adjustment equal to minus the average CDS-bond basis of the top 10 to 15 derivatives dealers. As already mentioned, the CDS-bond basis is an imperfect measure of the non-default-risk component of a credit spread, but it may be the best measure available.

7. Bid−Offer Spreads

In the previous sections we have presented the arguments that lead to the determination of the fair market value of a derivative where the fair market value is the price at which an investment in the derivative results in an expected rate of return on investment that is commensurate with the risk of the investment. However, a financial institution that is buying or selling a derivative as a service to a customer usually wants to earn a rate of return that is greater than the fair return in order to compensate for the service.

We first consider the case where there is no credit risk. The derivation based on the CAPM, equation (2) has to be modified to incorporate the excess return that is to be earned on the derivative to give

$$\mu_f = \theta + r + \beta_f \left(\mu_M - r\right) = \theta + r + \frac{S}{f}\frac{\partial f}{\partial S}\left(\mu - r\right) \tag{20}$$

where θ is the desired excess rate of return. When equation (20) is used, in the absence of counterparty credit risk the resulting differential equation that the price of the option must satisfy is

$$\frac{\partial f}{\partial t}+rS\frac{\partial f}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}=\left(r+\theta\right)f$$

For a European style derivative the solution to this differential equation is

$$f\left(S_t,t\right)=e^{-(r+\theta)(T-t)}E_r\left[f\left(S_T,T\right)\right]$$

If the bank buys the derivative from the customer (the derivative is an asset for the bank), the bank would like to earn a rate of return that is greater than the fair rate so that $\theta > 0$. In this case, the value of the derivative is lower than the fair market value, i.e., the bank pays the customer a price that is less than the fair market value. If the bank sells the derivative to the customer (the derivative is a liability for the bank), the derivative is acting as a source of funding for the bank and the bank would like to pay a rate of return on its funding that is less than the fair rate so that $\theta < 0$. In this case, the value of the derivative is greater than the fair market value, i.e., the customer pays the bank a price that is more than the fair market value. These adjustments to the price result in a bid-offer spread.

If the derivative may be either an asset or a liability to the bank, the bank will want $\theta=\theta^+>0$ whenever it is an asset and $\theta=\theta^-<0$ whenever it is a liability. In this case the differential equation for the asset price is

$$\frac{\partial f}{\partial t}+rS\frac{\partial f}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}=\left(r+\theta^+\right)\max\left(f,0\right)+\left(r+\theta^-\right)\min\left(f,0\right)$$

This must be solved numerically.

These results can also be derived using the Merton hedging argument. In this case equation (6) has to be modified to

$$d\Pi = \left[-\left(r + \theta \right) f + rS \frac{\partial f}{\partial S} \right] dt$$

That is, the option portion of the hedged portfolio is required to earn a rate of return that is different from the riskless rate while the stock portion still earns the riskless rate.

This analysis can be extended to incorporate credit risk and collateralization.

8. Conclusions

The credit crisis of 2008 has caused derivatives practitioners and their regulators to reconsider many of the ways derivatives are valued and managed. For example, it is now becoming recognized that LIBOR is not the best proxy to use for the risk-free rate when derivatives are valued. Counterparty credit risk and collateral issues in the bilaterally cleared over-the-counter market are receiving more attention and contractual arrangements are changing. This paper shows that the economic arguments underlying derivatives valuation can be extended to incorporate these changing contractual arrangements.

We have shown how the arguments used to produce the no-default value of a derivative can be extended to incorporate collateral agreements as well as credit risk. We have also made the point that, if the whole of a credit spread is compensation for default risk, it is incorrect to make an FVA. However, if part of a credit spread is compensation for liquidity or other factors, an FVA, which is much smaller than that usually calculated, may be appropriate and this can in theory be different for different derivatives dealers.

The results have been presented in the context of a single derivative dependent on a single underlying asset providing no income. The underlying asset is assumed to follow a lognormal diffusion process. The results can be generalized. First, they can be extended to apply to a single derivative or a portfolio of derivatives dependent on many underlying assets.[17] Second,

[17] In practice, netting agreements mean that all derivatives transactions between two parties are treated as a single transaction for the calculation of CVA and DVA.

alternative processes can be assumed for the underlying asset. Third, multifactor extensions of the continuous time CAPM can be used. A point that it is sometimes overlooked in practice is that is not necessary for a market participant to be able to hedge all the underlying risks. Valuations produced on the assumption that all risks can be hedged are the same as those produced using equilibrium models.

New regulations can be expected to reduce, but not eliminate, CVA and DVA. Basel III regulations require banks to hold market risk capital for the credit spread risks underlying CVA.[18] This is likely to mean that banks continue to devote resources to improving their estimates of CVA (and DVA). However, the management of collateral and the calculation of CRA is likely to assume an increased importance as new regulations increase collateral requirements. The impact on derivatives valuation of situations where the interest paid on cash collateral is different from the risk-free rate can be as large as the impact of CVA and DVA.

[18] See Basel Committee on Banking Supervision (2010).

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