Derivation of Black–Scholes–Merton Option Pricing Formula from Binomial Tree*

One way of deriving the famous Black–Scholes–Merton result for valuing a European option on a non-dividend-paying stock is by allowing the number of time steps in the binomial tree to approach infinity.

Suppose that a tree with n time steps used to value a European call option with strike price K and life T. Each step is of length T/n. If there have been j upward movements and $n-j$ downward movements on the tree, the final stock price is $S_0u^jd^{n-j}$ where u is the proportional up movement, d is the proportional down movement, and S_0 is the initial stock price. The payoff from a European call option is then

$$\max(S_0u^jd^{n-j} - K, 0)$$

From the properties of the binomial distribution, the probability of exactly j upward movement and $n-j$ downward movement is

$$\frac{n!}{(n-j)!j!}p^j(1-p)^{n-j}$$

It follows that the expected payoff from the call option is

$$\sum_{j=0}^{n} \frac{n!}{(n-j)!j!}p^j(1-p)^{n-j}\max(S_0u^jd^{n-j} - K, 0)$$

Because the tree represents movements in a risk-neutral world we can discount at this risk-free rate, r to obtain the option price:

$$c = e^{-rT}\sum_{j=0}^{n} \frac{n!}{(n-j)!j!}p^j(1-p)^{n-j}\max(S_0u^jd^{n-j} - K, 0) \qquad (12A.1)$$

The terms in equation (12A.1) are non-zero when the final stock price is greater than the strike price, that is, when

$$S_0u^jd^{n-j} > K$$

or

$$\ln(S_0/K) > -j\ln(u) - (n-j)\ln(d)$$

Because $u = e^{\sigma\sqrt{T/n}}$ and $d = e^{-\sigma\sqrt{T/n}}$, this condition becomes

$$\ln(S_0/K) > n\sigma\sqrt{T/n} - 2j\sigma\sqrt{T/n}$$

or

$$j > \frac{n}{2} - \frac{\ln(S_0/K)}{2\sigma\sqrt{T/n}}$$

Equation (12A.1) can therefore be written

$$c = e^{-rT} \sum_{j>\alpha} \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} (S_0 u^j d^{n-j} - K)$$

where

$$\alpha = \frac{n}{2} - \frac{\ln(S_0/K)}{2\sigma\sqrt{T/n}}$$

For convenience we define

$$U_1 = \sum_{j>\alpha} \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} u^j d^{n-j} \tag{12A.2}$$

$$U_2 = \sum_{j>\alpha} \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} \tag{12A.3}$$

so that

$$c = e^{-rT}(S_0 U_1 - K U_2) \tag{12A.4}$$

Consider first U_2. As is well known, the binomial distribution approaches a normal distribution as the number of trials approaches infinity. Specifically, when there are n trials and p is the probability of success, the probability distribution of the number of successes is approximately normal with mean np and standard deviation $\sqrt{np(1-p)}$. The variable U_2 in equation (12A.3) is the probability of the number of successes being more than α. From the properties of the normal distribution it follows that for large n

$$U_2 = N\left(\frac{np - \alpha}{\sqrt{np(1-p)}}\right) \tag{12A.5}$$

where N is the cumulative probability distribution function for a standardized normal variable. Substituting for α, we obtain

$$U_2 = N\left(\frac{\ln(S_0/K)}{2\sigma\sqrt{T}\sqrt{p(1-p)}} + \frac{\sqrt{n}\,(p-1/2)}{\sqrt{p(1-p)}}\right) \tag{12A.6}$$

From the equation for p

$$p = \frac{e^{rT/n} - e^{-\sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}}$$

By expanding the exponential functions in a series we see that, as n tends to infinity, $p(1-p)$ tends to $1/4$ and $\sqrt{n}(p-1/2)$ tends to

$$\frac{(r - \sigma^2/2)\sqrt{T}}{2\sigma}$$

so that in the limit as n tends to infinity equation (12A.6) becomes

$$U_2 = N\left(\frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \tag{12A.7}$$

We now move on to evaluate U_1. From equation (12A.2)

$$U_1 = \sum_{j>\alpha} \frac{n!}{(n-j)!j!}(pu)^j[(1-p)d]^{n-j} \tag{12A.8}$$

Define

$$p^* = \frac{pu}{pu + (1-p)d} \tag{12A.9}$$

It then follows that

$$1 - p^* = \frac{(1-p)d}{pu + (1-p)d}$$

and we can write equation (12A.8) as

$$U_1 = [pu + (1-p)d]^n \sum_{j>\alpha} \frac{n!}{(n-j)!j!}(p^*)^j(1-p^*)^{n-j}$$

Since the expected rate of return on the stock in the risk-neutral world is the risk-free rate, r, $pu + (1-p)d = e^{rT/n}$ and

$$U_1 = e^{rT} \sum_{j>\alpha} \frac{n!}{(n-j)!j!}(p^*)^j(1-p^*)^{n-j}$$

This shows that U_1 involves a binomial distribution where the probability of an up movement is p^* rather than p. Approximating the binomial distribution with a normal distribution we obtain (similarly to equation 12A.5)

$$U_1 = e^{rT} N\left(\frac{np^* - \alpha}{\sqrt{np^*(1-p^*)}}\right)$$

and substituting for α this gives, similarly to equation (12A.6)

$$U_1 = e^{rT} N\left(\frac{\ln(S_0/K)}{2\sigma\sqrt{T}\sqrt{p^*(1-p^*)}} + \frac{\sqrt{n}(p^* - 1/2)}{\sqrt{p^*(1-p^*)}}\right)$$

Substituting for u, and d in equation (12A.9)

$$p^* = \left(\frac{e^{rT/n} - e^{-\sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}}\right)\left(\frac{e^{\sigma\sqrt{T/n}}}{e^{rT/n}}\right)$$

By expanding the exponential functions in a series we see that, as n tends to infinity, $p^*(1-p^*)$ tends to 1/4 and $\sqrt{n}(p^* - 1/2)$ tends to

$$\frac{(r+\sigma^2/2)\sqrt{T}}{2\sigma}$$

with the result that

$$U_1 = e^{rT} N\left(\frac{\ln(S_0/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}\right) \tag{12A.10}$$

From equations (12A.4), (12A.7), (12A.10)

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r-\sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

This is the Black–Scholes–Merton formula for the valuation of a European call option.