Derivation of Black–Scholes–Merton Option Pricing Formula from Binomial Tree

One way of deriving the famous Black–Scholes–Merton result for valuing a European option on a non-dividend-paying stock is by allowing the number of time steps in the binomial tree to approach infinity.

Suppose that a tree with \( n \) time steps used to value a European call option with strike price \( K \) and life \( T \). Each step is of length \( T/n \). If there have been \( j \) upward movements and \( n - j \) downward movements on the tree, the final stock price is \( S_0u^jd^{n-j} \) where \( u \) is the proportional up movement, \( d \) is the proportional down movement, and \( S_0 \) is the initial stock price. The payoff from a European call option is then

\[
\text{max}(S_0u^jd^{n-j} - K, 0)
\]

From the properties of the binomial distribution, the probability of exactly \( j \) upward movement and \( n - j \) downward movement is

\[
\frac{n!}{(n-j)!j!} p^j (1-p)^{n-j}
\]

It follows that the expected payoff from the call option is

\[
\sum_{j=0}^{n} \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} \text{max}(S_0u^jd^{n-j} - K, 0)
\]

Because the tree represents movements in a risk-neutral world we can discount at this risk-free rate, \( r \) to obtain the option price:

\[
c = e^{-rT} \sum_{j=0}^{n} \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} \text{max}(S_0u^jd^{n-j} - K, 0) \quad (12A.1)
\]

The terms in equation (12A.1) are non-zero when the final stock price is greater than the strike price, that is, when

\[
S_0u^jd^{n-j} > K
\]

or

\[
\ln(S_0/K) > -j \ln(u) - (n - j) \ln(d)
\]

Because \( u = e^{\sigma\sqrt{T/n}} \) and \( d = e^{-\sigma\sqrt{T/n}} \), this condition becomes

\[
\ln(S_0/K) > n\sigma\sqrt{T/n} - 2j\sigma\sqrt{T/n}
\]

or

\[
j > \frac{n}{2} \frac{\ln(S_0/K)}{2\sigma\sqrt{T/n}}
\]

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Equation (12A.1) can therefore be written

\[ c = e^{-rT} \sum_{j>\alpha} \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} (S_0 u^j d^{n-j} - K) \]

where

\[ \alpha = \frac{n}{2} - \frac{\ln(S_0/K)}{2\sigma \sqrt{T/n}} \]

For convenience we define

\[ U_1 = \sum_{j>\alpha} \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} \]  \hspace{1cm} (12A.2)

\[ U_2 = \sum_{j>\alpha} \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} \]  \hspace{1cm} (12A.3)

so that

\[ c = e^{-rT} (S_0 U_1 - KU_2) \]  \hspace{1cm} (12A.4)

Consider first \( U_2 \). As is well known, the binomial distribution approaches a normal distribution as the number of trials approaches infinity. Specifically, when there are \( n \) trials and \( p \) is the probability of success, the probability distribution of the number of successes is approximately normal with mean \( np \) and standard deviation \( \sqrt{np(1-p)} \). The variable \( U_2 \) in equation (12A.3) is the probability of the number of successes being more than \( \alpha \). From the properties of the normal distribution it follows that for large \( n \)

\[ U_2 = N \left( \frac{np - \alpha}{\sqrt{np(1-p)}} \right) \]  \hspace{1cm} (12A.5)

where \( N \) is the cumulative probability distribution function for a standardized normal variable. Substituting for \( \alpha \), we obtain

\[ U_2 = N \left( \frac{\ln(S_0/K)}{2\sigma \sqrt{T/p(1-p)}} + \frac{\sqrt{np(1-p)}}{\sqrt{p(1-p)}} \right) \]  \hspace{1cm} (12A.6)

From the equation for \( p \)

\[ p = \frac{e^{rT/n} - e^{-\sigma \sqrt{T/n}}}{e^{\sigma \sqrt{T/n}} - e^{-\sigma \sqrt{T/n}}} \]

By expanding the exponential functions in a series we see that, as \( n \) tends to infinity, \( p(1-p) \) tends to 1/4 and \( \sqrt{n}(p - 1/2) \) tends to

\[ \frac{(r - \sigma^2/2) \sqrt{T}}{2\sigma} \]
so that in the limit as \( n \) tends to infinity equation (12A.6) becomes

\[
U_2 = N \left( \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \right)
\]

(12A.7)

We now move on to evaluate \( U_1 \). From equation (12A.2)

\[
U_1 = \sum_{j>\alpha} \frac{n!}{(n-j)!j!} (pu)^j [(1-p)d]^{n-j}
\]

(12A.8)

Define

\[
p^* = \frac{pu}{pu + (1-p)d}
\]

(12A.9)

It then follows that

\[
1 - p^* = \frac{(1-p)d}{pu + (1-p)d}
\]

and we can write equation (12A.8) as

\[
U_1 = [pu + (1-p)d]^n \sum_{j>\alpha} \frac{n!}{(n-j)!j!} (p^*)^j (1-p^*)^{n-j}
\]

Since the expected rate of return on the stock in the risk-neutral world is the risk-free rate, \( r \), \( pu + (1-p)d = e^{rT/n} \) and

\[
U_1 = e^{rT} \sum_{j>\alpha} \frac{n!}{(n-j)!j!} (p^*)^j (1-p^*)^{n-j}
\]

This shows that \( U_1 \) involves a binomial distribution where the probability of an up movement is \( p^* \) rather than \( p \). Approximating the binomial distribution with a normal distribution we obtain (similarly to equation 12A.5)

\[
U_1 = e^{rT} \sqrt{n(p^* \alpha)} \left( \frac{n(p^* - \alpha)}{\sqrt{n(p^*(1-p^*))}} \right)
\]

and substituting for \( \alpha \) this gives, similarly to equation (12A.6)

\[
U_1 = e^{rT} N \left( \frac{\ln(S_0/K)}{2\sigma\sqrt{T}} + \frac{\sqrt{n}(p^* - 1/2)}{\sigma\sqrt{T}} \right)
\]

Substituting for \( u, \) and \( d \) in equation (12A.9)

\[
p^* = \left( \frac{e^{rT/n} - e^{-\sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}} \right) \left( \frac{e^{\sigma\sqrt{T/n}}}{e^{rT/n}} \right)
\]
By expanding the exponential functions in a series we see that, as \( n \) tends to infinity, \( p^*(1 - p^*) \) tends to 1/4 and \( \sqrt{n}(p^* - 1/2) \) tends to

\[
\frac{(r + \sigma^2/2)\sqrt{T}}{2\sigma}
\]

with the result that

\[
U_1 = e^{rT}N\left(\frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right)
\]

(12A.10)

From equations (12A.4), (12A.7), (12A.10)

\[
c = S_0N(d_1) - Ke^{-rT}N(d_2)
\]

where

\[
d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}
\]

\[
d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}
\]

This is the Black–Scholes–Merton formula for the valuation of a European call option.