Economic Theory **13**, 577–601 (1999)

Economic Theory

Bargaining with asymmetric information in non-stationary markets★

Daniel Trefler
Department of Economics, University of Toronto, Toronto, M5S 3G7, CANADA
(e-mail: trefler@chass.utoronto.ca)

Received: July 22, 1994; revised version: January 21, 1998

Summary. The Rubinstein and Wolinsky bargaining-in-markets framework is modified by the introduction of asymmetric information and non-stationarity. Non-stationarity is introduced in the form of an arbitrary stochastic Markov process which captures the dynamics of market entry and pairwise matching. A new technique is used for establishing existence and characterizing the unique outcome of a non-stationary market equilibrium. The impact of market supply and demand on bilateral bargaining outcomes and matching probabilities is explored. The results are useful for examining such questions as why coordination failures and macroeconomic output fluctuations are correlated with real and monetary shocks.

Keywords and Phrases: Non-cooperative bargaining, Matching, Asymmetric information, Non-stationarity, Output fluctuations.

JEL Classification Numbers: C72, C78, E20.

In contrast to centralized markets where all participants simultaneously coordinate trade at publicly announced prices, many important markets are characterized by decentralized trading in which participants meet privately to negotiate prices and quantities. Considerable progress has been made in modelling decentralized markets using the Rubinstein and Wolinsky (1985) framework involving a market comprised of many infinitely-lived agents who pair randomly and bargain non-cooperatively over the price of an indivisible good. Despite the fact that the framework has potentially wide applicability

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★ I am indebted to Lee Crabbe, David Levine, Martin Osborne, Michael Peters, John Riley, and a referee for suggesting numerous improvements and clarifying many misunderstandings. An earlier version of this paper appeared as Sections I–III of "Demand Shocks and Equilibrium Output Fluctuations in a Market with Bargaining."

to many economic problems, much of the interest has focused on whether equilibrium allocations of frictionless decentralized markets coincide with competitive allocations.[1] In order to address a larger set of economic issues, this paper extends the bargaining-in-markets framework to include non-stationarity and asymmetric information. In much of the existing literature attention is confined to deterministic entry processes which support a steady state or in which all entry occurs in the initial period.[2] In this paper, non-stationarity is introduced in the form of an arbitrary stochastic Markov process which captures the dynamics of market entry and pairwise matching. The very general specification of this process allows its sample paths to be interpreted as demand and supply shocks to the market. Uncertainty takes two forms: aggregate uncertainty about future demand and supply shocks and private uncertainty about the reservation valuation of the current trading partner. Together, the two modifications enhance the range of questions to which bargaining in markets can be brought to bear.

Consider just three examples. First, in macroeconomics there is an obvious analogy between two-person bargaining inefficiency and macroeconomic coordination failures; yet little attempt has been made to verify whether the two-person bargaining insight holds in a market setting. Shaked and Sutton (1984a, b) integrated macroeconomic insights from a full-information, two-person bargaining game into a stationary market setting. However, in order to fully model the Keynesian world view that coordination failures occur where markets are decentralized, pervaded by uncertainty, and continually buffeted by demand shocks, clearly one must introduce uncertainty and non-stationarity. Second, in several efficiency wage models (see Akerlof and Yellen, 1986), a single uninformed employer is pitted against a group of informed employees, yet it is not clear whether the insights garnered from these models carry over into a decentralized market with many bilateral negotiations. Gale (1986b) highlighted several substantive issues that arise in showing that Weiss' (1980) efficiency-wage explanation of unemployment carries over to a subgame perfect (i.e. certainty) bargaining equilibrium imbedded in a stationary market. To date, no attempt has been made to introduce non-stationary labour demand in order to describe unemployment dynamics. Third, research initiated by Kiyotaki and Wright (1989) has firmly established the usefulness of bargaining in markets for monetary theory. Still to be explored are non-stationary stochastic environments with uncertainty; for example, Trejos and Wright (1992) model dynamics, but not Rubinstein bargaining while Williamson and Wright (1994) model uncertainty, but in a stationary setting. Thus, there has been no

[1] For examples, see Rubinstein and Wolinsky (1985), Gale (1986a, 1987), Binmore and Herrero (1988a, b), Osborne and Rubinstein (1990), Rubinstein and Wolinsky (1990), and McLennan and Sonnenschein (1991). Osborne and Rubinstein (1990) survey the literature on bargaining in markets.

[2] See the literature review below for papers that have dealt with non-stationarity.

analysis of how non-stationary stochastic monetary shocks effect the value of money, output dynamics, and inflation.

In these examples, uncertainty and non-stationarity are inherent to the economic problem, making it difficult to show that two-person bargaining insights carry over into decentralized market settings. While the casting of these bargaining examples within a decentralized market is potentially rewarding, implementation has doubtless been hampered by the information and stationarity assumptions used in much of the literature on bargaining in markets. Attention is usually confined to deterministic entry processes which support a steady state or in which all entry occurs in the initial period.[3] Further, while considerable attention has been given to asymmetric information, it has been cast in terms of uncertainty about trading-partner identities and histories rather than in the more familiar terms of uncertainty about trading-partner reservation valuations e.g. Binmore and Herrero (1988b), Rubinstein and Wolinsky (1990), and Wolinsky (1990). Only Wolinsky (1990) and Samuelson (1992) consider reservation-value uncertainty. The main technical conclusion of this paper establishes existence of a non-stationary market equilibrium and characterizes its unique outcome for a particular extensive-form bargaining game involving one-sided seller uncertainty and offers. The larger contribution is to provide a simple recipe for establishing existence and characterizing the unique market equilibrium for a large class of extensive-form bargaining games.

The outline is as follows. In Section 1, I imbed into a market a very simple extensive form game in which an uninformed seller makes all the offers to an informed buyer. In Section 2, I characterize the generically unique sequential equilibrium outcome for this game. This result is used to establish the existence of a market equilibrium and to characterize its generically unique outcome. The usual proof of the existence of a steady state equilibrium exploits the linearity implied by stationarity to solve explicitly for an equilibrium. The proof is laid out in Section 3 and the impact of market supply and demand on bilateral bargaining outcomes and matching probabilities is explored in Section 4. The steady state proof does not hold in the more general setting where the stochastic process generating market shocks (i.e. entry into the market) is non-stationarity. Section 5 offers a new proof of existence and a new method of characterizing the unique equilibrium. The proof employs well-known, accessible results from the dynamic programming literature. In Section 6, I show how my conclusions can be extended to a large class of extensive-form bargaining games including games with alternating offers, two-sided uncertainty, and outside options.

[3] E.g. Rubinstein and Wolinsky (1985), Gale (1987), Wolinsky (1987), Osborne and Rubinstein (1990), Rubinstein and Wolinsky (1990), Wolinsky (1990), McLennan and Sonnenschein (1991), and Samuelson (1992). More general entry processes appear in Gale (1986a) for the case of no discounting, Gale (1987), and Binmore and Herrero (1988b) for the case of a different equilibrium concept, namely, a "security equilibrium."

1 The model

A buyer and seller match with the intention of jointly producing one unit of an output worth b to the buyer. Each side provides one unit of a costly input. Without loss of generality, take b to be net of the buyer's input cost and let the seller's input cost be zero. b is known only to the buyer and may equal either b_H or b_L. Assume that $b_H > b_L > 0$. The two agents bargain over the price the buyer will pay for the seller's input. In each bargaining period the seller makes an offer p which the buyer either accepts or rejects. If agreement is reached in period t the seller receives $\delta^t p$ and the type i buyer receives $\delta^t(b_i - p)$ where $\delta \in (0, 1)$ is the discount factor. Agents are interpreted as firms with a non-depreciable productive capacity operating in an on-going market. Correspondingly, agents *never* leave the market. After agreement is reached production occurs, the match breaks up, and the agents return to a matching pool to seek other partners with whom to reach agreement.

Matching and market forces

Each period t starts with a bargaining phase and ends with a matching phase. The matching phase begins with the entry of additional agents into the market. Then matching occurs randomly. With probability β_i^t a type i agent $(i = S, L, H)$ is matched to a new partner. Following Rubinstein and Wolinsky (1985), I assume that each agent who is matched anew must abandon his old partner. Thus, for a seller and type i buyer in a match that failed to reach agreement in period t (a) with probability $\beta_S^t\beta_i^t$ the seller and buyer each enter new matches, (b) with probability $\beta_S^t(1 - \beta_i^t)$ only the sellers enters a new match, (c) with probability $(1 - \beta_S^t)\beta_i^t$ only the buyer enters a new match, and (d) with probability $(1 - \beta_S^t)(1 - \beta_i^t)$ the seller and buyer continue bargaining. There are three features of matching that require comment. First, the seller does not know the buyer type and hence does not know β_i^t. Second, the fact that agents exit matches whenever a new partner is found is consistent with the optimal exit decision. After the first round of bargaining in the equilibrium described below agents are indifferent between continuing in the match and exiting.[4] Third, the β_i^t are not exogenous. They may depend on market forces and on the outcome of an unmodelled search intensity decision. To develop this point I will need to describe the evolution of the market.

In each period the market is subject to a shock in the form of new agents entering the market. Let e_i^t be the measure of type i agents $(i = S, L, H)$ entering the market in period t. The e_i^t are just the period t market demand and supply shocks and are modelled as a non-stationary Markov process. In

[4] In an earlier version of this paper (Trefler (1991)) agents chose whether or not to exit a match. This changes the extensive form and adds multiple equilibria. However, it does not alter my method of characterizing a market equilibrium and estabilishing its existence. That is, modelling the exit decision explicitly does not alter the main conclusions of the paper. See Section 6 below for details.

turn, the β_i^t are endogenously determined by the e_i^t. Part of this dependence is via technical features of the matching technology. For example, the measure of matched sellers must always equal the measure of matched buyers for each buyer type:

$$\begin{aligned} \beta_H^t H^t &= \beta_S^t S^t \frac{\beta_H^t H^t}{\beta_L^t L^t + \beta_H^t H^t} \\ \beta_L^t L^t &= \beta_S^t S^t \frac{\beta_L^t L^t}{\beta_L^t L^t + \beta_H^t H^t} \quad . \end{aligned} \tag{1}$$

where $S^t = \sum_{j \le t} e_S^j$, $L^t = \sum_{j \le t} e_L^j$, and $H^t = \sum_{j \le t} e_H^j$ are the measures of sellers, low-valuation buyers, and high-valuation buyers in the market, respectively. Note that $\beta_H^t H^t / (\beta_L^t L^t + \beta_H^t H^t)$ is the probability that a newly matched seller faces a high-valuation buyer. Equation (1) shows one way in which the β_i^t depend on the composition of the matching pool. The β_i^t also depend on the e_i^t via an unmodelled optimal search intensity. The optimal intensity will depend on the marginal-benefit-of-search schedule which is just a derivative of the equilibrium payoffs function. Since the latter depends on the primitives of the market i.e. the e_i^t, the equilibrium search intensity and hence the β_i^t will be related to the e_i^t via a (possibly stochastic) relationship

$$(\beta_S^t, \beta_L^t, \beta_H^t) = F(S^t, L^t, H^t) \quad . \tag{2}$$

Equations (1)–(2) imply that if the composition of the matching pool is an unrestricted non-stationary process then the matching probabilities must be non-stationary. To summarize, the β_i^t depend on market forces and are inherently non-stationary if the market is. I will develop these points below in the context of a specific example.

At this stage I model the dependence of the β_i^t on the e_i^t very generally by treating the β_i^t and e_i^t as correlated random variables. Let $z^t = \{e_i^t, \beta_i^t\}_{i=S,L,H}$ be a period t market shock and let $\lambda(z^{t-1}, z^t)$ be a Markov transition function giving the probability of z^t conditional on last period's shock z^{t-1}. If there is an exact dependence of the β_i^t on the e_i^t such as equations (1)–(2) above, this is modelled by having λ attach probability one to the lower-dimensional space of $\{e_i^t, \beta_i^t\}_{i=S,L,H}$ on which the equations hold.

Histories, beliefs, and strategies

There are two types of histories, the history of the matching pool and the history of the current match. The history of the matching pool in period t, h^t, is the sequence $h^t = (\{e_i^0, \beta_i^0\}_{i=S,L,H}, \ldots, \{e_i^{t-1}, \beta_i^{t-1}\}_{i=S,L,H})$. Let $\tau = 0, 1, \ldots$, index bargaining periods in the current match. A type i agent's history in period τ of the current match, h_i^τ, is just a sequence of seller offers p.

A *belief system* is a function $\pi(h_S^\tau)$ giving the seller's period τ belief that the buyer has a high valuation. I will often write π^τ in place of $\pi(h_S^\tau)$. π satisfies three conditions.

B.1 Beliefs evolve according to Bayes rule as long as the buyer's response is consistent with the strategy of either the low- or high-valuation buyer. If not, the seller is free to form new beliefs.
B.2 The belief system π satisfies Rubinstein's (1985) *never dissuaded once convinced* condition: if the seller concludes with probability one that the buyer is of a certain type, then the seller never changes his belief.
B.3 π is *initially correct* in the sense that the first-period seller belief π^0 equals the objective probability that the newly matched seller faces a high-valuation buyer. Consider a match whose first period of bargaining occurs in period t after history h^t. The probability that a newly matched seller faces a high-valuation buyer is given by $\Pi(h^t) = \Sigma_{j<t}\beta_H^{t-1}e_H^j/\Sigma_{j<t}(\beta_L^{t-1}e_L^j + \beta_H^{t-1}e_H^j)$. The initially correct condition is thus $\pi^0 = \Pi(h^t)$.

By assumption there is always initial uncertainty about the buyer type so that $0 < \Pi < 1$.

A strategy describes an agent's action *during the current match*. Let t be the first period of bargaining in the current match and let $\tau = 0, 1, \ldots$ index bargaining periods in the current match. A *period τ strategy* of a seller, $\sigma_S^\tau(h^{t+\tau}, h_S^\tau)$, is a mapping from available information about the matching pool and current match into a probability measure over seller offers (i.e., the seller can randomize over offers). A period τ strategy of a type i buyer, $\sigma_i^\tau(h^{t+\tau}, h_i^\tau)$, is a mapping from available information into the probability of accepting seller offers. That is, σ_i^τ is the probability of accepting the seller offer and $1 - \sigma_i^\tau$ is the probability of rejecting the seller offer. A *strategy* for a type i agent is $\sigma_i = \{\sigma_i^\tau\}_{\tau=0}^\infty$. Define $\sigma = (\sigma_S, \sigma_L, \sigma_H)$. See Rubinstein and Wolinsky (1985) and Osborne and Rubinstein (1990, pp. 138–146) for further discussion of strategies.

Note that bargaining threat points in bargaining period τ depend on $h^{t+\tau}$ so that the action that agent i takes in period τ depends on $h^{t+\tau}$. Specifically, since $h^{t+\tau}$ evolves with market shocks, an agent will not play the same strategy in every match. It is convenient to use h^t as an index for a match whose first bargaining period occurs in period t after matching pool history h^t. Thus, I will write $\sigma(h^t)$ to indicate the strategy triplet played in such a match. Note that this is a notational convenience rather than a restriction on the domain of σ.

Market equilibrium

A market equilibrium is defined in terms of equilibrium of the game played in each match. These bargaining games have been fully defined except for the payoffs associated with bargaining breakdown. These payoffs are detailed in the next section. Following Osborne and Rubinstein (1990), a sequential equilibrium of the bargaining game indexed by h^t is a pair $(\sigma(h^t), \pi)$ that satisfies restrictions B.1 and B.2 on beliefs and is sequentially rational in the familiar sense. A market equilibrium can now be defined.

Definition. A *market equilibrium* is a strategy triplet $\sigma^* = (\sigma_S^*, \sigma_L^*, \sigma_H^*)$ and a belief system π^* with the following property. For all periods $t > 0$ and all matching pool histories h^t generated by the Markov transition function λ,

(i) $(\sigma^*(h^t), \pi^*)$ is a sequential equilibrium of the bargaining game played by a match whose first period of bargaining occurs in period t after matching pool history h^t, and

(ii) π^* is initially correct ($\pi^0 = \Pi(h^t)$).

Notice that market forces play a central role in determining strategies and outcomes of the bargaining game. Market forces appear as $h^t = (\{e_i^0, \beta_i^0\}_{i=S,L,H}, \ldots, \{e_i^{t-1}, \beta_i^{t-1}\}_{i=S,L,H})$. The e_i^t determine the probability that a newly-matched seller faces a high-valuation buyer ($\Pi(h^t)$), the ratio of sellers to buyers and, via restrictions on λ such as equations (1)–(2), the endogenous matching probabilities β_i^t.

2 The bargaining game

A market equilibrium is in part defined by the bargaining game equilibrium within each match. The bargaining game is defined in the usual way except for the introduction of the following continuation payoffs. For a type i agent let the expected value of continuing in the market be V_i if the agent is currently at the start of the bargaining phase with a new partner and U_i if the agent is currently at the start of the matching phase with no partner. U_i and V_i depend on events in current and future matches and events in future periods of the matching pool. These depend on λ, σ, and h^t. Suppress λ and write $U_i(\sigma, h^t)$ and $V_i(\sigma, h^t)$. Recall that each period starts with a bargaining phase and ends with a matching phase. Since an unmatched agent at the start of the matching phase remains unmatched with probability $(1 - \beta_i^t)$ and is newly matched with probability β_i^t,

$$U_i(\sigma, h^t) = \delta \mathbf{E}_t \Big\{ (1 - \beta_i^t) U_i(\sigma, h^{t+1}) + \beta_i^t V_i(\sigma, h^{t+1}) \Big\} \qquad i = S, L, H \ . \quad (3)$$

Expectations $\mathbf{E}_t$ are over the as yet unannounced e_i^t, β_i^t, and $h^{t+1} = (h^t, \{e_i^t, \beta_i^t\}_{i=S,L,H})$. For a formal discussion of expectations see the Appendix proof of Theorem 1.

The bargaining game has an exogenous counterpart (u_i, v_i) to the endogenous continuation payoffs (U_i, V_i). At the start of each bargaining period τ nature announces $h^{t+\tau}$ which determines the probability $(1 - \beta_i^{t+\tau})(1 - \beta_j^{t+\tau})$ of continuing the game, the probability $\beta_i^{t+\tau}$ of a type i agent receiving termination payoff $v_i(h^{t+\tau})$, and the probability $(1 - \beta_i^{t+\tau})\beta_j^{t+\tau}$ of a type i agent receiving termination payoff $u_i(h^{t+\tau})$. $h^{t+\tau}$ and $\beta_i^{t+\tau}$ evolve according to λ. Define $u = (u_S, u_L, u_H)$ and $v = (v_S, v_L, v_H)$. Let $\Gamma(u, v, \pi^0, h^t)$ be a bargaining game with exogenous termination payoffs (u, v) where play starts in period t so that h^t is an initial condition and π^0 is the initial seller belief that he is facing a high-valuation buyer. I will need a characterization of the bargaining game outcome that is largely independent of any information about u and v. The following will suffice.

Assumption G. For $i = S, L, H$

1) $0 \leq u_i < K$ and $0 \leq v_i < K$ for some constant K.
2) $u_i(h^t) = \delta E_t\{(1-\beta_i^t)u_i(h^{t+1}) + \beta_i^t v_i(h^{t+1})\}$ for all $t > 0$ and h^t.
3) $u_L = v_L = 0$.

Assumption G.2 mimics equation (3) above. Assumption G.3 states that the seller extracts all the surplus from the low-valuation buyer.

The outcome of the game $\Gamma(u, v, \pi^0, h^t)$ is characterized by a sequence $\{p^{(n)}, \sigma_H^{(n)}\}$ and a constant N which have the following interpretation. The seller initially offers $p^{(N)}$ where $p^{(N)}$ is chosen so that the high-valuation buyer is indifferent between accepting $p^{(N)}$ today and $p^{(N-1)}$ tomorrow. The high-valuation buyer accepts $p^{(N)}$ with probability $\sigma_H^{(N)}$ where $\sigma_H^{(N)}$ is chosen so that in the next period seller beliefs are such that the seller offers $p^{(N-1)}$. The low-valuation buyer only accepts offer $p^{(1)} = b_L$. N is the maximum number of bargaining periods and is implicitly part of the seller's strategy.

Lemma 1. *Assume G and $\pi^0 < 1$. For all $t > 0$ and h^t there exists a sequential equilibrium of the game $\Gamma(u, v, \pi^0, h^t)$ and every sequential equilibrium results in the following generically unique outcome.The seller makes a strictly decreasing sequence of offers $(p^{(N)}, p^{(N-1)}, \ldots, p^{(1)})$ where $b_L \leq p^{(n)} < b_H$* for all *n. The high-valuation buyer accepts offer $p^{(n)}$ with probability $\sigma_H^{(n)}$*[25] *where $\sigma_H^{(n)} > 0$ for all n and $\sigma_H^{(1)} = \sigma_H^{(2)} = 1$. The low-valuation buyer accepts only the final offer $p^{(1)} = b_L$.*

N and $\{p^{(n)}, \sigma_H^{(n)}\}_{1 \leq n \leq N}$ are defined recursively in Appendix Definition 1.

3 A steady state market equilibrium

Establishing the existence and characterizing the unique outcome of a non-stationary market equilibrium is simple, but obscured by some technical details. In this section I consider the steady state case where the composition of the matching pool does not change over time. In particular, the matching probabilities are time-invariant ($\beta_i^t = \beta_i$) as is the probability that a newly-matched seller faces a high-valuation buyer ($\Pi(h^t) = \Pi$). The steady state proof is not new (see Osborne and Rubinstein, 1990, Section 7.4), but will help the reader appreciate the non-stationary proof and will help explore the dependence of bilateral bargaining outcomes on market forces. The proof is divided into 3 steps.

Step 1. The first step is to show that Assumption G is satisfied in any market equilibrium (σ^*, π^*) or more precisely, Assumption *G* with (u_i, v_i) replaced by $(U_i(\sigma^*, \cdot),\ V_i(\sigma^*, \cdot))$. Consider Assumption G.1. Since buyers can reject all offers and sellers can always offer $p = 0$, equilibrium strategies must yield $U_i \geq 0$ and $V_i \geq 0$. This fact together with discounting $(\delta < 1)$ and the boundedness of the maximum per-period surplus $(b_H < \infty)$ implies that continuation payoffs are bounded. Assumption G.2 just repeats equation (1) and so must be satisfied. Assumption G.3 is well-known (e.g. Fudenberg, Levine, and Tirole 1985) and follows from the fact that with one-sided seller offers the seller never makes an offer below b_L. The proof appears as Lemma 3 in the Appendix. Thus, any market equilibrium (σ^*, π^*) must be such that $(U_i(\sigma^*, \cdot), V_i(\sigma^*, \cdot))$ satisfies assumption G.

Step 2. Step 1 implies that Lemma 1 must hold for every match in a market equilibrium. One can thus exploit Lemma 1 to find expressions for steady state strategies and continuation payoffs. In steady state the maximum number of bargaining periods per match (N) is constant. I restrict attention to the cases $N = 1, 2$ since these bring out all the essential points. Definition 1 in the Appendix provides the equilibrium strategies. For the steady state case strategies reduce to seller offers

$$p^{(1)} = b_L, \quad p^{(2)} = b_H - \delta(1 - \beta_S)(1 - \beta_H)(b_H - b_L) \tag{4}$$

and high-valuation buyer acceptances

$$\sigma_H^{(1)} = \sigma_H^{(2)} = 1 \ . \tag{5}$$

Let $U_i^{(N)}$ and $V_i^{(N)}$ be continuation payoffs for a given value of N. Since Assumption G.2 is satisfied,

$$U_i^{(N)} = \delta\Big\{(1 - \beta_i)U_i^{(N)} + \beta_i V_i^{(N)}\Big\} \qquad i = S, L, H \ . \tag{6}$$

From Assumptions G.1 and G.3,

$$V_L^{(N)} = 0 \ . \tag{7}$$

Since the high-valuation buyer always accepts the first seller offer (equation (5)),

$$V_H^{(N)} = b_H - p^{(N)} + U_H^{(N)} \qquad N = 1, 2 \ . \tag{8}$$

To understand equation (8) recall that $U_H^{(N)}$ is the return to entering the matching pool without a partner. It is not discounted because it is evaluated in the same period as is $V_H^{(N)}$. For the seller the returns are

$$\begin{aligned} V_S^{(1)} &= b_L + U_S^{(1)} \\ V_S^{(2)} &= \Pi\Big\{p^{(2)} + U_S^{(2)}\Big\} \\ &\quad + (1 - \Pi)\delta\Big\{(1 - \beta_S)\beta_L U_S^{(2)} + \beta_S V_S^{(2)} + (1 - \beta_S)(1 - \beta_L)(b_L + U_S^{(2)})\Big\} \end{aligned} \tag{9}$$

where β_L is used because after rejection the seller is certain the buyer is a low-valuation type.

The last element is the seller choice of N. After each offer and rejection the Bayesian seller downgrades the belief that he faces a high-valuation buyer. The sequence of updated beliefs is $\pi^{(N)} > \cdots > \pi^{(n)} > \cdots > \pi^{(1)} = 0$. The general formula for $\pi^{(n)}$ appears in Appendix Definition 1. Of interest here is $\pi^{(2)}$ which is defined implicitly by

$$V_S^{(2)}(\pi^{(2)}) = b_L + U_S^{(2)}(\pi^{(2)}) \ . \tag{10}$$

That is, when seller beliefs are $\pi^{(2)}$ the seller is indifferent between the sequence of offers $(p^{(2)}, p^{(1)})$ and the offer $p^{(1)}$. Finally, Appendix Definition 1 states that

$$N = 1 \text{ for } \Pi < \pi^{(2)} \text{ and } N = 2 \quad \text{for} \quad \Pi \geq \pi^{(2)} \ . \tag{11}$$

Step 3. A market equilibrium (σ^*, π^*) consists of initially correct beliefs and the sequential equilibrium strategies and beliefs of the game with continuation payoffs $(U_i(\sigma^*, \cdot), V_i(\sigma^*, \cdot))$. Thus, I will have found a market equilibrium if I can find strategies $(p^{(n)}, \sigma_H^{(n)}, N)$ and auxiliary variable $\pi^{(2)}$ satisfying equations (4)–(5) and (10)–(11) together with continuation payoffs satisfying equations (6)–(9). Since these 10 equations are linear in the 10 unknowns $((U_i^{(N)}, V_i^{(N)})_{i=S,L,H}, p^{(N)}, \sigma_H^{(N)}, N, \pi^{(2)})$ it is straightforward to verify that a unique solution exists. Hence a unique market equilibrium exists.

The unique solution is as follows. Define

$$\begin{aligned} \phi_i &= \delta(1-\beta_S)(1-\beta_i) & i &= L, H \\ \Delta_i &= (1-\delta(1-\beta_i))/(1-\delta) & i &= S, H \ . \end{aligned}$$

Δ_i is the effective discount rate after accounting for periods when the agent is not matched. Continuation payoffs are given by

$$\begin{aligned} V_L^{(N)} &= 0 & N &= 1,2 \\ V_H^{(N)} &= \Delta_H(b_H - p^{(N)}) & N &= 1,2 \\ V_S^{(1)} &= \Delta_S p^{(1)}, \\ V_S^{(2)} &= \Delta_S(\Pi p^{(2)} + (1-\Pi)\phi_L p^{(1)}) \end{aligned}$$

where

$$p^{(2)} = b_H - \phi_H(b_H - b_L) \text{ and } p^{(1)} = b_L.$$

N is defined in equation (11) with $\pi^{(2)} = (1-\phi_L)b_L/(p^{(2)} - \phi_L b_L)$.

It is worthwhile to be explicit about matching. Let S, L, and H be the steady state measures of sellers, low-valuation buyers, and high-valuation buyers, respectively. Recall that Π is the probability that a newly matched seller faces a high-valuation buyer. The equation (1) restrictions on matching probabilities together with the definition of Π imply that the following must hold:

$$\begin{aligned} \beta_H H &= \Pi\beta_S S \\ \beta_L L &= (1-\Pi)\beta_S S \\ \Pi &= \beta_H/(\beta_L + \alpha\beta_H) \end{aligned} \tag{12}$$

where $\alpha = H/L$. This completes the proof of Theorem 1 for the steady state case.

4 Market forces and bargaining outcomes

The steady state solution of market equilibrium strategies and payoffs allows for an understanding of how market demand and supply interact with the bilateral bargaining process. The question raised in this section is whether exogenous market forces S, L, and H influence endogenous strategies, bargaining outcomes, and matching probabilities. In addition, I will consider the impact of market forces on the steady state rate of agreements or transactions. Without loss of generality normalize the market so that there is a unit measure of buyers, $L + H = 1$. Consider a rise in demand associated with a rise in the ratio of high- to low-valuation buyers $\alpha = H/L$. Since $V_H^{(N)} > V_L^{(N)}$ it seems likely that a high-valuation buyer will search more intensively than a low-valuation buyer. I will therefore assume $\beta_H > \beta_L$. From equation (12), the matching probabilities will respond endogenously to changes in α. For a concrete example, suppose that only β_S adjusts to changes in α. Using $\alpha = H/L$ and $L + H = 1$ one obtains $\partial H/\partial\alpha = L^2$. Using this fact together with equation (12) to differentiate $\beta_S S = \beta_L L + \beta_H H$ yields $\partial\beta_S/\partial\alpha = (\beta_H - \beta_L)L^2/S$. That is, the rise in demand leads to a higher matching probability for the seller.

Now consider the effect of a rise in demand on the level of transactions. This time assume that the β_i are fixed and let S adjust to satisfy equation (12). When $N = 1$ agreement is reached immediately so that the level of transactions is

$$\begin{aligned} q^{(1)} &= \Pi\beta_H + (1 - \Pi)\beta_L \\ &= (\beta_L^2 + \alpha\beta_H^2)/(\beta_L + \alpha\beta_H) \ . \end{aligned}$$

When $N = 2$ a matched low-valuation buyer only reaches agreement if the match lasts 2 periods i.e. with probability $(1 - \beta_L)(1 - \beta_S)$. Hence

$$\begin{aligned} q^{(2)} &= \Pi\beta_H + (1 - \Pi)\beta_L(1 - \beta_L)(1 - \beta_S) \\ &= q^{(1)} - \beta_L^2(\beta_L + \beta_S(1 - \beta_L))/(\beta_L + \alpha\beta_H) \ . \end{aligned}$$

If the increase in α leaves N unchanged then $\partial q^{(N)}/\partial\alpha > 0$, $N = 1, 2$. That is, an increase in demand raises the steady state level of transactions. For $N = 1$ this occurs because a rise in demand increases the proportion of buyers with high returns to search. For $N = 2$ there is an additional effect related to the failure of agents to coordinate immediate agreement. The measure of matches subject to such coordination failures is proportional to $1 - \Pi$ and is given by $q^{(2)} - q^{(1)}$. Thus, the lower the level of demand the greater the level of coordination failures. This has a Keynesian flavour that is developed in

Trefler (1991). However, in order to fully model the Keynesian world view that coordination failures occur where markets are decentralized, pervaded by uncertainty, and continually buffeted by demand shocks, clearly one must introduce uncertainty and non-stationarity.

5 Market equilibrium: the general case

This section is devoted to proving the existence of a non-stationary market equilibrium and to characterizing the unique equilibrium outcome.

Theorem 1. *There exists a market equilibrium and every market equilibrium results in the following generically unique outcome.The seller makes a strictly decreasing sequence of offers* $(p^{(N)}, p^{(N-1)}, \ldots, p^{(1)})$ where $b_L \leq p^{(n)} < b_H$ *for all n. The high-valuation buyer accepts offer* $p^{(n)}$ *with probability* $\sigma_H^{(n)}$ *where* $\sigma_H^{(n)} > 0$ *for all n and* $\sigma_H^{(1)} = \sigma_H^{(2)} = 1$. *The low-valuation buyer accepts only the final offer* $p^{(1)} = b_L$.

N and $\{p^{(n)}, \sigma_H^{(n)}\}_{1 \leq n \leq N}$ are defined recursively in Appendix Definition 1. Note that N is a function of market histories h^t and that $p^{(N-\tau)}$ and $\sigma_H^{(N-\tau)}$ are functions of $h^{t+\tau}$ so that strategies and outcomes are non-stationary. What follows is a proof of theorem 1. The notationally intensive steps are relegated to the Appendix.

Proof. Consider *existence* and *characterization*. With non-stationarity, the description of strategies and especially continuation payoffs (U_i, V_i) is complicated, making direct verification as in the steady state case impossible. An alternative approach is needed. Let $U = (U_S, U_L, U_H)$ and $V = (V_S, V_L, V_H)$. I begin with an equivalent definition of a market equilibrium couched in terms of continuation payoffs.

> A *market equilibrium* is a strategy triplet σ^* and a belief system π^* such that for all $t > 0$ and h^t, $(\sigma^*(h^t), \pi^*)$ is a sequential equilibrium of the game $\Gamma(u, v, \pi^0, h^t)$ with $u(\cdot) = U(\sigma^*, \cdot)$, $v(\cdot) = V(\sigma^*, \cdot)$, and $\pi^0 = \Pi(h^t)$.

$\pi^0 = \Pi(h^t)$ is the "initially correct" condition.[5]

Let $\hat{\Sigma}$ be the set of strategy triplets such that if all agents play according to $\hat{\sigma} \in \hat{\Sigma}$ the outcome in every match is the one described in the Theorem. It follows that continuation payoffs are the same for all $\hat{\sigma} \in \hat{\Sigma}$ so that $U(\hat{\sigma}, \cdot) = \hat{U}(\cdot)$ for some $\hat{U}$ and $V(\hat{\sigma}, \cdot) = \hat{V}(\cdot)$ for some $\hat{V}$ whenever $\hat{\sigma} \in \hat{\Sigma}$. Let $u = (u_S, u_L, u_H)$ and $v = (v_S, v_L, v_H)$ be the exogenous continuation payoffs in the bargaining game $\Gamma(u, v, \pi^0, h^t)$. Lemma 1 established that if (u, v) satisfies Assumption G then the sequential equilibrium strategy is some $\hat{\sigma} \in \hat{\Sigma}$. Hence $\hat{\sigma}$ yields the payoff $(\hat{U}, \hat{V})$ in a market equilibrium. Let $T(U, V)$

[5] It is now obvious that my definition is strongly influenced by the Osborne and Rubinstein (1990, p. 143) definition, though in my non-stationary setting agents need not play the same (semi-stationary) strategies in every match.

be the mapping from exogenous bargaining game continuation payoffs (u, v) via $\hat{\sigma}$ into endogenous market equilibrium continuation payoffs $(\hat{U}, \hat{V})$. That is, $(\hat{U}, \hat{V}) = T(u, v)$. Step 1 in the Appendix proof of Theorem 1 constructs T. In step 2 of the Appendix proof T is shown to be a contraction mapping. Hence a fixed point exists and is unique. This fixed point is just the continuation payoffs $(\hat{U}, \hat{V}) = T(\hat{U}, \hat{V})$. Using the properties of the fixed point, Step 3 of the Appendix proof shows that $(\hat{U}, \hat{V})$ satisfies Assumption G. By assumption $\pi^0 = \Pi(h^t) < 1$ so that all the conditions of Lemma1 are satisfied for the game $\Gamma(\hat{U}, \hat{V}, \pi^0, h^t)$. Hence, for all $t > 0$ and h^t there exists a sequential equilibrium $(\sigma^*(h^t), \pi^*)$ of $\Gamma(\hat{U}, \hat{V}, \pi^0, h^t)$ with $\sigma^* \in \hat{\Sigma}$. Since $\sigma^* \in \hat{\Sigma}$, $U(\sigma^*, \cdot) = \hat{U}(\cdot)$ and $V(\sigma^*, \cdot) = \hat{V}(\cdot)$. This establishes the existence of a market equilibrium (σ^*, π^*) and its characterization, $\sigma^* \in \hat{\Sigma}$.

Consider *uniqueness*. I will show that every market equilibrium results in the generically unique outcome described in the Theorem. Step 4 of the Appendix proof establishes that if (σ^*, π^*) is a market equilibrium then $(U(\sigma^*, \cdot), V(\sigma^*, \cdot))$ satisfies Assumption G. From the definition of a market equilibrium, for all $t > 0$ and h^t $(\sigma^*(h^t), \pi^*)$ is a sequential equilibrium of $\Gamma(u, v, \Pi(h^t), h^t)$ with $u = U(\sigma^*, \cdot)$ and $v = V(\sigma^*, \cdot)$. Since (u, v) satisfies assumption G, Lemma 1 states that $(\sigma^*(h^t), \pi^*)$ results in the generically unique outcome described in the Theorem. This establishes uniqueness. It also completes the proof of Theorem 1. $\square$

6 The choice of extensive form

In the proof of Theorem 1, three features of the sequential equilibrium of Γ were exploited. First, the equilibrium strategies and payoffs of Γ were represented in terms of a sequence of period τ agreement probabilities $\{\alpha_i^\tau\}_{\tau \geq 0}$, continuation probabilities $\{\rho_i^\tau\}_{\tau \geq 0}$, and returns $\{r_i^\tau\}_{\tau \geq 0}$. For any extensive form game with a known sequential equilibrium, calculation of $\{\alpha_i^\tau, \rho_i^\tau, r_i^\tau\}_{\tau \geq 0}$ is trivial. Second, the equilibrium outcome of the underlying bargaining game was unique: without uniqueness, one cannot expect uniqueness of the market equilibrium outcome. Third, an assumption on (u, v), namely Assumption G, was available which was sufficient to characterize the outcome of the bargaining game $\Gamma(u, v, \cdot, \cdot)$ and which could be verified as holding in any market equilibrium (σ^*, π^*) with $u(\cdot) = U(\sigma^*, \cdot)$ and $v(\cdot) = V(\sigma^*, \cdot)$. To the extent that such an Assumption G can always be found, the method of proof in this paper extends to a large variety of extensive form games e.g. a continuum of buyer types (Fudenberg, Levine, and Tirole, 1985; Gul, Sonnenschein, and Wilson, 1986), one-sided seller uncertainty with alternating offers (Grossman and Perry, 1986), and two-sided uncertainty with alternating offers (Chatterjee and Samuelson, 1987). For the extensive form game used in this paper finding such an Assumption G was easy and while there is no guarantee that it would be easy for all extensive form games, Trefler (1991) provides another example where it poses no challenges. In that paper, I considered a game in which a match ends only if agreement is reached or if at least one partner chooses to exit. Since agents only enter the matching pool if they have no partner in the wings, there is only one payoff U and one contraction mapping T^U i.e. V and T^V do not appear. Thus, identification of an Assumption G and the proof of Theorem 1 are easy. In particular, Assumption G.2 is replaced by the simple condition

$$U_i(\sigma, h^t) \geq \delta \mathbf{E}_t \Big\{ U_i(\sigma, h^{t+1}) \Big\}$$

This example with endogenous bargaining breakdown is indicative of the ease with which the results of this paper can be applied to a variety of extensive form games.

6 Conclusions

The wider applicability of the bargaining-in-markets framework has been hampered by the information and stationarity assumptions used in much of the literature. In order to address a larger set of economic issues in which asymmetric information and non-stationarity are inherent, this paper extended the Rubinstein and Wolinsky (1985) framework to include uncertainty and non-stationarity. Establishing existence and characterizing the unique outcome of a market equilibrium was accomplished by treating agents' continuation payoffs as fixed points of contraction mappings and then by exploiting features of these mappings; thus, the proofs employed accessible results from the dynamic programming literature. While the use of a particular extensive form begs the question of generality, a wide variety of extensive forms seem amenable to the methods of this paper. As such, the larger contribution of this paper is to provide a recipe for establishing existence and characterizing the unique outcome of bargaining with asymmetric information in a non-stationary market. A key feature of the method is that it allows one to explore the impact of market supply and demand on bilateral bargaining outcomes and matching probabilities.

Mathematical Appendix

Throughout I explicitly spell out expectations. For example, in Assumption G.2 $u_i(h^t) = \delta E_t \big\{ (1-\beta_i^t) u_i(h^{t+1}) + \beta_i^t v_i(h^{t+1}) \big\}$ will be written as $u_i(h^t) = \delta \int \big\{ (1-\beta_i^t) u_i(h^{t+1}) + \beta_i^t v_i(h^{t+1}) \big\} \lambda(z^{t-1}, dz^t)$. To understand the dependence of the integrand on z^t note that $h^{t+1} = (h^t, z^t)$ and $z^t = \big\{ e_i^t, \beta_i^t \big\}_{i=S,L,H}$ so that h^{t+1} and β_i^t depend on z^t . Where there is no confusion I will use the short form $u_i = \delta \int \big\{ (1-\beta_i^t) u_i + \beta_i^t v_i \big\} \lambda$.

Proof of Lemma 1

The discussion preceding the statement of Lemma 1 motivates interest in the sequence $\big\{ p^{(n)}, \sigma_H^{(n)} \big\}$ and the constant N. N is defined in terms of a sequence $\big\{ \pi^{(n)} \big\}$ while each $\pi^{(n)}$ is defined in terms of $W_S^{(n)}$, the return to the seller from a sequence of offers $(p^{(n)}, p^{(n-1)}, \dots, p^{(1)})$ and high-valuation buyer acceptances $(\sigma_H^{(n)}, \sigma_H^{(n-1)}, \dots, \sigma_H^{(1)})$. Definition 1 defines $\{p^{(n)}, \sigma^{(n)}, \pi^{(n)}, W_S^{(n)}\}$ recursively.

Definition 1. For $n = 1$, $p^{(1)} = b_L$, $\sigma_H^{(1)} = 1$, $W_S^{(1)} = b_L + u_S$, and $\pi^{(1)} = 0$. For $n > 1$,

$$p^{(n)} = b_H - \delta \int (1 - \beta_H)(1 - \beta_S)(b_H - p^{(n-1)})\lambda$$

$$\sigma_H^{(n)}(\pi) = \begin{cases} \frac{\pi - \pi^{(n-1)}}{(\pi(1-\pi^{(n-1)})} & \text{for } \pi > \pi^{(n-1)} \\ 0 & \text{for } \pi \leq \pi^{(n-1)} \end{cases}$$

$$W_S^{(n)}(\pi) = \pi\sigma_H^{(n)}(\pi)\{p^{(n)} + u_S\} + [1 - \pi\sigma_H^{(n)}(\pi)]\delta \times \int \{(1 - \beta_S)\beta_B^{(n-1)}u_S + \beta_S v_S + (1 - \beta_S)(1 - \beta_B^{(n-1)})W_S^{(n-1)}(\pi^{(n-1)})\}\lambda$$

where $\beta_B^{(n-1)} = \pi^{(n-1)}\beta_H + (1 - \pi^{(n-1)})\beta_L$.

$\pi^{(n)}$ solves $W_S^{(n)}(\pi^{(n)}) = W_S^{(n-1)}(\pi^{(n)})$ and $N(\pi)$ solves $\pi \in [\pi^{(N)}, \pi^{(N+1)})$.

Lemma 2. *Assume G.2. The high-valuation buyer is indifferent between accepting $p^{(n)}$ today and $p^{(n-1)}$ tomorrow.*

Proof. Rejecting the current offer and accepting $p^{(n-1)}$ tomorrow returns

$$\begin{aligned} &\delta \int \{(1 - \beta_H)\beta_S u_H + \beta_H v_H + (1 - \beta_H)(1 - \beta_S)[b_H - p^{(n-1)} + u_H]\}\lambda \\ &\quad = \delta \int \{(1 - \beta_H)(1 - \beta_S)[b_H - p^{(n-1)}]\}\lambda + u_H \\ &\quad = b_H - p^{(n)} + u_H \end{aligned}$$

where the first equality follows from Assumption G.2 and the second equality follows from the definition of $p^{(n)}$. Accepting $p^{(n)}$ today returns $b_H - p^{(n)} + u_H$. Hence, the two returns are equal and the buyer is indifferent. □

Lemma 3. *Assume G.1 and G.2. In any equilibrium of Γ, both types of buyers accept all offers below b_L and the seller never makes such offers.*

Proof. For expositional ease assume $\beta_H > \beta_L$. Consider the sequence $\hat{p}^{(n)} = b_L - \delta \int (1 - \beta_L)(1 - \beta_S)(b_L - \hat{p}^{(n-1)})\lambda$ where $p^{(0)} = 0$.

Step 1: $\lim_n \hat{p}^{(n)} = b_L$.

Proof: Use induction to show that $\{\hat{p}^{(n)}\}$ is an increasing sequence bounded by b_L. $\hat{p}^{(1)} - \hat{p}^{(0)} = b_L - \delta \int (1 - \beta_L)(1 - \beta_S)b_L\lambda > 0$. If $\hat{p}^{(n-1)} > \hat{p}^{(n-2)}$ then $\hat{p}^{(n)} - \hat{p}^{(n-1)} = \delta \int (1 - \beta_L)(1 - \beta_S)(\hat{p}^{(n-1)} - \hat{p}^{(n-2)})\lambda > 0$, as required.

$\hat{p}^{(0)} = 0 < b_L$. If $\hat{p}^{(n-1)} < b_L$ then $\hat{p}^{(n)} < b_L - \delta \int (1-\beta_L)(1-\beta_S)(b_L - b_L)\lambda = b_L$, as required. Hence, $\{\hat{p}^{(n)}\}$ has a limit $\hat{p}^* \le b_L$. Since $\hat{p}^* = b_L - \delta \int (1-\beta_L)(1-\beta_S)(b_L - \hat{p}*)\lambda$ is satisfied for $\hat{p}^* = b_L$, the conclusion follows from the uniqueness of limits.

Proof of Lemma 3. The proof is by induction. Consider $n = 0$. By assumption, the lowest seller offer is zero. (This can be proved rather than assumed.) Knowing that $\hat{p}^{(0)} = 0$ is the lowest offer and being impatient, both buyer types accept $\hat{p}^{(0)}$ as soon as it is offered. Hence, both buyer types accept all offers below $\hat{p}^{(0)}$.

By the inductive hypothesis both buyer types accept all offers below $\hat{p}^{(n-1)}$ and the seller never makes such offers. Since tomorrow a type i buyer accepts all offers below $\hat{p}^{(n-1)}$ and the seller never makes such offers, the most the buyer receives if he rejects the current offer is $\delta \int \{(1-\beta_i)\beta_S u_i + \beta_i v_i + (1-\beta_i)(1-\beta_S)(b_i - \hat{p}^{(n-1)} + u_i)\}\lambda = u_i + \delta \int (1-\beta_i)(1-\beta_S)(b_i - \hat{p}^{(n-1)})\lambda$ where the equality follows from Assumption G.2. Hence, he accepts offer p today if $b_i - p + u_i > u_i + \delta \int (1-\beta_i)(1-\beta_S)(b_i - \hat{p}^{(n-1)})\lambda$ or $p < b_i - \delta \int (1-\beta_i)(1-\beta_S)(b_i - \hat{p}^{(n-1)})\lambda$. Taking the minimum over i, using $\beta_H > \beta_L$, and using the definition of $\hat{p}^{(n)}$, both buyer types accept if $p < \hat{p}^{(n)}$.

Since both buyer types accept all offers below $\hat{p}^{(n)}$, for any such offer $p < \hat{p}^{(n)}$ the seller prefers offer $(p + \hat{p}^{(n)})/2$. Thus, in equilibrium, the seller never makes offers below $\hat{p}^{(n)}$.

The conclusion follows by taking the limit established in Step 1. □

Lemma 4. *Suppose the period τ offer is accepted with probability $\sigma_H^{(n)}(\pi^\tau)$ where π^τ is period τ seller beliefs. If $\sigma^{(n)}(\pi^\tau) > 0$ then $\pi^\tau > \pi^{\tau+1} = \pi^{(n-1)}$. If $\sigma_H^{(n)}(\pi^\tau) = 0$ then $\pi^\tau = \pi^{\tau+1} \le \pi^{(n-1)}$.*

Proof. By Bayesian updating, $\pi^{\tau+1} = (1 - \sigma_H^{(n)}(\pi^\tau))\pi^\tau / [1 - \sigma_H^{(n)}(\pi^\tau)\pi^\tau]$. Consider Definition 2. For $\sigma_H^{(n)}(\pi^\tau) > 0$, $\pi^{\tau+1} = \pi^{(n-1)}$ and $\pi^\tau > \pi^{(n-1)}$. For $\sigma_H^{(n)}(\pi^\tau) = 0$, $\pi^{\tau+1} = \pi^\tau$ and $\pi^\tau \le \pi^{(n-1)}$. □

The next lemma establishes features of $p^{(n)}$, $\sigma_H^{(n)}$, $W_S^{(n)}$, and $\pi^{(n)}$ which will be needed to characterize the bargaining game Γ. Since the lemma has little economic interest *per se*, the proof is presented tersely.

Lemma 5. *Assume G.2. For all* T:

1) $p^{(T)}$, $\sigma_H^{(T)}$, $W_S^{(T)}$, *and* $\pi^{(T)}$ *are unique.*
2) $p^{(1)} = b_L$, $p^{(T-1)} < p^{(T)}$, *and* $\lim_n p^{(n)} = b_H$.
3) $\pi^{(1)} = 0$, $\pi^{(T-1)} < \pi^{T)} < 1$, *and* $\lim_n \pi^{(n)} = 1$.
4) $W_S^{(T)}(\pi) \lesseqgtr W_S^{(T-1)}(\pi)$ *as* $\pi \lesseqgtr \pi^{(T)}$ *for* $\pi \ge \pi^{(T-1)}$.
5) $W_S^{(T)}(\pi) > W_S^{(n)}(\pi)$ *for* $T > n$ *and* $\pi > \pi^{(T)}$.
6) $W_S^{(T)}(\pi^{(T)}) > \delta \int \{(1-\beta_S)\beta_B^{(T)} u_S + \beta_S v_S + (1-\beta_S)(1-\beta_B^{(T)}) W_S^{(T)}(\pi^{(T)})\}\lambda.$

Proof. Consider part (6). For $T = 1$, $W_S^{(1)} = b_L + u_S$ and part (6) states that

$$b_L + u_S > \delta \int (1-\beta_S)(1-\beta_B^{(1)})\lambda b_L + \delta \int \{(1-\beta_S)u_S + \beta_S v_S\}\lambda \ .$$

By Assumption G.2, the last term in the inequality equals u_S so that the inequality reduces to $1 > \delta \int (1-\beta_S)(1-\beta_B^{(1)})\lambda$ which always holds. The proof for $T > 1$ follows from Assumption G.2 and induction.

The proof for part (2) is identical to the proof of Lemma 3, Step 1, with $\{\hat{p}^{(n)}\}$, $\hat{p}^{(0)} = 0$, and b_L replaced by $\{p^{(n)}\}$, $p^{(1)} = b_L$, and b_H, respectively.

I prove the remainder of Lemma 5 by a single inductive argument. Since the proof for $T = 2$ given $p^{(1)} = b_L$, $\sigma_H^{(1)} = 1$, $W_S^{(1)} = b_L + u_S$, and $\pi^{(1)} = 0$ is the same as the proof for T given that the inductive hypothesis holds for $T-1$, I only present the latter proof. By Definition 2, $\pi\sigma_H^{(T)} = (\pi - \pi^{(T-1)}) / (1-\pi^{(T-1)})$ for $\pi \geq \pi^{(T-1)}$. Substituting this into the Definition 2 expression for $W_S^{(T)}(\pi)$ yields

$$\begin{aligned}
&W_S^{(T)}(\pi) - W_S^{(T-1)}(\pi) \\
&\quad = \frac{\pi - \pi^{(T-1)}}{1-\pi^{(T-1)}}(p^{(T)} + u_S) + \frac{1-\pi}{1-\pi^{(T-1)}}\delta \int \{(1-\beta_S)\beta_B^{(T-1)}u_S + \beta_S v_S \\
&\qquad + (1-\beta_S)(1-\beta_B^{(T-1)})W_S^{(T-1)}(\pi^{(T-1)})\}\lambda \\
&\qquad - \frac{\pi - \pi^{(T-2)}}{1-\pi^{(T-2)}}(p^{(T-1)} + u_S) - \frac{1-\pi}{1-\pi^{(T-2)}}\delta \int \{(1-\beta_S)\beta_B^{(T-2)}u_S \\
&\qquad + \beta_S v_S + (1-\beta_S)(1-\beta_B^{(T-2)})W_S^{(T-2)}(\pi^{(T-2)})\}\lambda
\end{aligned}$$

for $\pi \geq \pi^{(T-1)}$. Let $\phi^{(T)}(\pi) = W_S^{(T)}(\pi) - W_S^{(T-1)}(\pi)$. There are four useful facts about $\phi^{(T)}$. First, from Definition 2, $\pi^{(T)}$ solves $\phi^{(T)}(\pi^{(T)}) = 0$. Second, $\phi^{(T)}$ is linear. Third, $\phi^{(T)}(1) = p^{(T)} - p^{(T-1)}$ so that $\phi^{(T)}(1) > 0$ (part (2)). Fourth, $\phi^{(T)}(\pi^{(T-1)}) < 0$ since $W_S^{(T)}(\pi^{(T-1)}) = \delta \int \{(1-\beta_S)\beta_B^{(T-1)}u_S + \beta_S v_S + (1-\beta_S)(1-\beta_B^{(T-1)})W_S^{(T-1)}(\pi^{(T-1)})\}\lambda < W_S^{(T-1)}(\pi^{(T-1)})$ (part (6)).

From these four facts it follows that $\phi^{(T)}$ is increasing, equals zero only once on $(\pi^{(T-1)}, 1)$, and the zero occurs at $\pi^{(T)}$. The following are immediate consequences. $\pi^{(T)}$ is unique. Together with the obvious uniqueness of $p^{(T)}$, this implies that $\sigma_H^{(T)}$ and $W_S^{(T)}$ are unique (Definition 2). This completes part (1). Next, $\pi^{(T-1)} < \pi^{(T)} < 1$ and inspection of the modulus of $\phi^{(T)}$ shows that $\lim_n \pi^{(n)} = 1$. This completes part (3). For $\pi > \pi^{(T-1)}$, $\phi^{(T)}(\pi) \lesseqgtr 0$ as $\pi \lesseqgtr \pi^{(T)}$, thus proving part (4). Part (5) follows from $W_S^{(n)}(\pi) > W_S^{(n-1)}(\pi)$ for $\pi > \pi^{(n)}$ (part (4)) and the fact that $\pi > \pi^{(T)}$ implies $\pi > \pi^{(n)}$ for $T \geq n$ (part (3)) so that $W_S^{(T)}(\pi) > W_S^{(T-1)}(\pi) > \cdots > W_S^{(1)}(\pi)$ for $\pi > \pi^{(T)}$. $\square$

Proof of Lemma 1

The basic insight is adapted from Fudenberg, Levine, and Tirole (1985). Define the game $\Gamma^{(T)}$ played exactly as the game Γ except that:

(i) the game ends in at most T periods, and

(ii) the seller cannot make offers above some $p^{(T)}$.

Define $N(\pi)$ as an N such that $\pi \in [\pi^{(N)}, \pi^{(N+1)})$. For the remainder of the appendix all references to N will refer to $N(\pi)$.

Step 1. Assume $N(\pi^0) \leq T$. *Suppose that in equilibrium both buyer types accept offers* $p \leq p^{(1)}$ *and that the high-valuation (low-valuation) buyer accepts offers* $p \in (p^{(n-1)}, p^{(n)}]$ *with probability* $\sigma_H^{(n)}$ *(zero) for* $n > 1$. *Then the supremum over seller returns is* $W_S^{(N)}(\pi^0)$. $W_S^{(N)}(\pi^0)$ *is attained only by the sequence of offers* $(p^{(N)}, p^{(N-1)}, \ldots, p^{(1)})$ *and, if* $\pi^0 = \pi^{(N)}$, *by* $(p^{(N-1)}, p^{(N-2)}, \ldots, p^{(1)})$ *for* $N > 1$.

Proof of Step 1. Since buyer responses are constant on sets $(p^{(n-1)}, p^{(n)}]$ for $n > 1$ and $[0, p^{(1)}]$, the seller need only consider offers $p = p^{(n)}$ for $1 \leq n \leq T$. Proceed by induction on N and T. For $T = 1$ and 2, use the argument of Fudenberg and Tirole (1983). By the inductive hypothesis, suppose the conclusion follows for $T - 1$ and $N = 1, 2, \ldots, T - 1$. Consider T. For $N = 1, 2, \ldots, T - 1$, $N(\pi^0) \leq T - 1$ and by the inductive hypothesis the conclusion follows. Consider $N = T$. Initial offer $p^{(n)}$ with $n \leq T$ is accepted with probability $\sigma_H^{(n)}(\pi^0) > 0$. For suppose $\sigma_H^{(n)} = 0$. By Lemma 4, $\sigma_H^{(n)} = 0$ implies $\pi^0 \leq \pi^{T-1}$. But $\pi^0 \in [\pi^{(N)}, \pi^{(N+1)})$ so that $\pi^{(N)} \leq \pi^0 \leq \pi^{(T-1)}$ or $N \leq T - 1$ (Lemma 5(3)), a contradiction of $N = T$. After offer $p^{(n)} \leq p^{(T)}$ and rejection $\sigma_H^{(n)} > 0$, next period seller beliefs are $\pi^{(n-1)}$ (Lemma 4). Note that $N(\pi^{(n-1)}) = n - 1$ i.e. $\pi^{(n-1)} \in [\pi^{(n-1)}, \pi^{(n)})$. Since the inductive hypothesis holds for all π^0 provided $N(\pi^0) \leq T - 1$, it holds with π^0 replaced by $\pi^{(n-1)}$ i.e. $N(\pi^{(n-1)}) = n - 1 \leq T - 1$. Hence, in the second and later periods the seller makes the sequence of offers $(p^{(n-1)}, \ldots, p^{(1)})$ or $(p^{(n-2)}, \ldots, p^{(1)})$. Suppose the latter so that offer $p^{(n)}$ is followed by offer $p^{(n-2)}$. Since $p^{(n-2)} < p^{(n-1)}$ (Lemma 5(2)) and since the high-valuation buyer is indifferent between $p^{(n-1)}$ tomorrow and $p^{(n)}$ today (Lemma 2), the high-valuation buyer prefers $p^{(n-2)}$ tomorrow over $p^{(n)}$ today and so rejects $p^{(n)}$, a contradiction of $\sigma_H^{(n)} > 0$. Hence the seller makes a sequence of offers $(p^{(n)}, \ldots, p^{(1)})$. By inspection of the definition of $W_S^{(n)}$ in Definition 2, this returns $W_S^{(n)}(\pi^0)$. By Lemma 5(4) and 5(5) with $N = T$ and $\pi^0 \in (\pi^{(N)}, \ \pi^{(N+1)})$, $\max_{1 \leq n \leq N} W_S^{(n)}(\pi^0) = W_S^{(N)}(\pi^0)$, which is attained by the sequence of offers $(p^{(N)}, \ldots, p^{(1)})$. By Lemma 5(4) and 5(5) with $N = T$ and $\pi^0 = \pi^{(N)}$, $\max_{1 \leq n \leq N} W_S^{(n)}(\pi^0) = W_S^{(N)}(\pi^{(N)}) = W_S^{(N-1)}(\pi^{(N)})$, which is attained by the sequences of offers $(p^{(N)}, \ldots, p^{(1)})$ and $(p^{(N-1)}, \ldots, p^{(1)})$.

Step 2. For $N(\pi^0) \leq T$, *there exists a sequential equilibrium of* $\Gamma^{(T)}$.

Proof of Step 2. Recall that $p^{(1)} = b_L$ and that in bargaining period τ after seller history h_S^τ, seller beliefs are denoted by $\pi(h_S^\tau)$. Show that the following is a sequential equilibrium of $\Gamma^{(T)}$.

Buyers: For $\tau \geq 0$ and current offer p:

$$\sigma_L^\tau = \begin{cases} 0 & \text{for } p > p^{(1)} \\ 1 & \text{for } p \leq p^{(1)} \end{cases} \text{ and } \sigma_H^\tau = \begin{cases} \sigma_H^{(n)}(\pi(h_S^\tau)) & \text{for } p \in (p^{(n-1)}, p^{(n)}], \\ & \quad 12 \leq n \leq T \\ 1 & \text{for } p \leq p^{(1)} \end{cases} .$$

Seller: For $\tau = 0$, offer $p^{(N)}$. For $\tau > 0$ and previous period offer $p \leq p^{(2)}$, offer $p^{(1)}$. For $\tau > 0$, previous period offer $p \in (p^{(n-1)}, p^{(n)}]$, and $3 \leq n \leq T$, offer $p^{(n-1)}$ with probability $\sigma_S^{(n-1)}$ and offer $p^{(n-2)}$ with probability $1 - \sigma_S^{(n-1)}$ where $\sigma_S^{(n-1)} = [p - p^{(n-1)}]/[p^{(n)} - p^{(n-1)}]$.

Along the equilibrium path seller strategies call for the sequence of offers $(p^{(N)}, \ldots, p^{(1)})$. Hence, from Step 1 the seller has no incentive to defect. After any history h_S^τ off the equilibrium path (i.e. after rejection of offer $p \leq b_L$) the sequential equilibrium outcome of $\Gamma^{(T)}$ is independent of how the seller updates his beliefs. For concreteness, the seller strategy above has been chosen so that it is supported by pessimistic conjectures: when offer b_L is rejected the seller offers b_L again.

Consider the low-valuation buyer. From Lemma 3 and Assumption G.3 $(u_L = v_L = 0)$, the most the buyer receives in any sequential equilibrium is zero. σ_L^τ returns zero so that the buyer has no incentive to defect.

Consider the high-valuation buyer. Proceed by induction. For $T = 1, 2$, use the argument of Fudenberg and Tirole (1983). By the inductive hypothesis assume that the above is a sequential equilibrium for $T - 1$ and $N = 1, 2, \ldots, T - 1$ and consider T. If the seller's first offer is $p \leq p^{(T-1)}$ then the seller plays as in the $T - 1$-period game and by the inductive hypothesis the buyer has no incentive to defect. Suppose that the seller's period $\tau = 0$ offer is $p \in (p^{(T-1)}, p^{(T)}]$. Then the sequence of subsequent offers is $(p^{(T-1)}, p^{(T-2)}, \ldots, p^{(1)})$ with probability $\sigma_S^{(T-1)}$ and $(p^{(T-2)}, p^{(T-3)}, \ldots, p^{(1)})$ with probability $1 - \sigma_S^{(T-1)}$. That is, after the first period the seller plays as in the $T - 1$-period game (albeit with a randomized first move) and by the inductive hypothesis the buyer has no incentive to defect after the first period. Since the buyer is indifferent between accepting offer $p^{(n)}$ today and accepting offer $p^{(n-1)}$ tomorrow (Lemma 2), the return to the sequence of offers $(p^{(T-i)}, \ldots, p^{(1)})$ is $b_H - p^{(T-i)} + u_H$ for $i = 1, 2$. Thus, the buyer can do no better than to randomly accept the first period offer p when

$$\begin{aligned} b_H - p + u_H = \delta \int \big\{ &(1 - \beta_H)\beta_S u_H + \beta_H v_H \\ &+ (1 - \beta_H)(1 - \beta_S)[(b_H - p^{(T-1)} + u_H)\sigma_S^{(T-1)} \\ &+ (b_H - p^{(T-2)} + u_H)(1 - \sigma_S^{(T-1)})]\big\}\lambda. \end{aligned}$$

But $\sigma_S^{(T-1)}$ has been chosen to satisfy this equation. (To show this use Assumption G.2 and the definition of $p^{(T)}$.) Hence, the high-valuation buyer has no incentive to defect.

Step 3. For $N(\pi^0) \leq T$, every sequential equilibrium of $\Gamma^{(T)}$ results in the following outcome. The seller makes a strictly decreasing sequence of offers $(p^{(N)}, p^{(N-1)}, \ldots, p^{(1)})$ where $b_L \leq p^{(n)} < b_H$ for all n. The high-valuation buyer accepts offer $p^{(N)}$ with probability $\sigma_H^{(N)}(\pi^0)$ and offer $p^{(n)}$ with probability $\sigma_H^{(n)}(\pi^{(n)})$ for $N > n \geq 1$ if $N > 1$. $\sigma_H^{(1)} = 1$, $\sigma_H^{(2)} = 1$ if $N > 1$, and $\sigma_H^{(n)} \in (0, 1)$ for $N \geq n > 2$ if $N > 2$. The low-valuation buyer only accepts the final offer

$p^{(1)}$. $\{p^{(n)}, \sigma_H^{(n)}\}$ *is unique and N is generically unique so that the outcome is generically unique.*

Proof of Step 3. By genericity I mean $\pi^0 \neq \pi^{(n)}$ for any $n > 1$. Proceed by induction on N and T. For $T = 1, 2$, use the argument of Fudenberg and Tirole (1983). Consider T and the initial ($\tau = 0$) high-valuation buyer response σ_H^0 to offer $p \in (p^{(n-1)}, p^{(n)}]$. The greater the probability of acceptance, the more seller beliefs are revised down after rejection i.e. period $\tau = 1$ seller beliefs π^1 are a decreasing function of σ_H^0. Together with Lemma 4, this implies that $\pi^1 \gtreqless \pi^{(n-1)}$ as $\sigma_H^0 \lesseqgtr \sigma_H^{(n)}$.

Suppose offer $p \in (p^{(n-1)}, p^{(n)}]$ is accepted with probability $\sigma_H^0 > \sigma_H^{(n)}$. Then $\pi^1 < \pi^{(n-1)}$ and by the inductive hypothesis the offer next period is some $p^{(n-k)} \le p^{(n-2)}$. The buyer is indifferent between $p^{(n-1)}$ today and $p^{(n-2)}$ tomorrow. Hence, the buyer prefers $p^{(n-k)} \le p^{(n-2)}$ tomorrow to $p > p^{(n-1)}$ today and defects by choosing $\sigma_H^0 = 0 \le \sigma_H^{(n)} < \sigma_H^0$, a contradiction. Similarly, for $\sigma_H^0 < \sigma_H^{(n)}$ the buyer defects by choosing $\sigma_H^0 = 1 \ge \sigma_H^{(n)} > \sigma_H^0$, a contradiction.

This establishes that the high-valuation buyer response to any offer $p \in (p^{(n-1)}, p^{(n)}]$ is $\sigma_H^{(n)}$ for $n > 1$. The low-valuation buyer rejects offers $p > b_L$ which return $b_L - p + u_L < 0$ ($u_L = 0$ by Assumption G.3). Both buyer types accept offers $p < b_L = p^{(1)}$ (Lemma 3). If either buyer type rejects b_L then the seller can do better offering some $p < b_L$ which both buyer types accept. But then the seller prefers $p' = (p + b_L)/2$ and there is no equilibrium. Hence, in equilibrium both buyer types accept offer b_L. Thus, buyer responses are as required. Further, they are the same as those described in Step 1 and the conclusion for seller offers follows from Step 1.

From Lemma 5(2), $\{p^{(n)}\}$ is a strictly decreasing sequence with $b_L \le p^{(n)} < b_H$ for all n. By Definition 2, $\sigma_H^{(1)} = 1$, $\pi^{(1)} = 0$, and hence $\sigma_H^{(2)} = 1$. Since $0 < \pi^{(n-1)} < \pi^{(n)} < 1$ for $n > 2$ (Lemma 5(3)), $\sigma_H^{(n)}(\pi^{(n)}) \in (0, 1)$ for $n > 2$ (Definition 2). If $N > 2$ then, since $\pi^0 \in [\pi^{(N)}, \pi^{(N+1)})$, $0 < \pi^{(N-1)} < \pi^0 < 1$ so that $\sigma_H^{(N)}(\pi^0) \in (0, 1)$ (Definition 2). Uniqueness of $\{p^{(n)}\}, \sigma_H^{(n)}\}$ follows from Lemma 5(1). Generic uniqueness of N follows from the definition of $N(\pi^0)$, Lemma 5(3), and Step 1.

Proof of Lemma 1. If the two constraints on Γ which define $\Gamma^{(T)}$ (the game ends in at most T periods and the seller cannot make offers above $p^{(T)}$) are not binding then the uniqueness and characterization of the sequential equilibrium outcome of Γ follow from Step 3. By Step 3, for $N(\pi^0) \le T$ the game $\Gamma^{(T)}$ ends in no more than $N(\pi^0)$ periods and the highest offer is $p^{(N)}$. $\{\pi^{(n)}\}$ is independent of T (Definition 2) and, since $N(\pi^0)$ is defined via $\pi^0 \in [\pi^{(N)}, \pi^{(N+1)}), N(\pi^0)$ is also independent of T. In the statement of Lemma 1 it is assumed that $\pi^0 < 1$. Since $\lim_n \pi^{(n)} = 1$ (Lemma 5(3)), $N(\pi^0)$ is finite. It follows that for T large the game ends in fewer than T periods and offers $p > p^{(T)}$ are never made i.e. the two constraints are not binding.

From Step 2, it is straightforward to show that the following is a sequential equilibrium of Γ: both buyers reject offers $p \ge b_H$ and all types follow the strategies outlined in the proof of Step 2 with T set to infinity. This establishes existence. $\square$

Appendix: Proof of Theorem 1

Step 1

This step presents a mapping T from exogeneous bargaining continuation payoffs (u, v) via $\hat{\sigma}$ to endogenous market equilibrium continuation payoffs (U, V). $\hat{\sigma}$ was defined as a strategy triplet yielding the outcome described in the Theorem. I being with some unifying notation. Throughout, $\tau = 0, 1, \ldots, N-1$.

Revenues: $r_S^\tau = p^{(N-\tau)}$; $r_i^\tau = b_i - p^{(N-\tau)}$ for $i = L, H$.

Agreement probabilities: $\alpha_H^\tau = \sigma_H^{(N-\tau)}$; $\alpha_L^\tau = 0$ for $\tau < N-1$ and $\alpha_L^{N-1} = 1$; $\alpha_S^\tau = \pi^\tau \alpha_H^\tau + (1-\pi^\tau)\alpha_L^\tau$.

Continuation probabilities: $\rho_i^1 = 1$; $\rho_i^\tau = \Pi_{k=1}^{\tau-1}(1-\alpha_i^k)(1-\beta_j^{t+k})(1-\beta_i^{t+k})$ where for $i = L$ or H, $\beta_j^{t+k} = \beta_S^{t+k}$ and for $i = S$, $\beta_j^{t+k} = \pi^{(N-k)}\beta_H^{t+k} + (1-\pi^{(N-k)})\beta_L^{t+k}$.

Recall that $z^t = \{e_i^t, \beta_i^t\}_{i=S,L,H}$. Let z^t lie in $Z = (0, M]^3 \times (0,1]^2$ where M is a positive constant. Let B_Z be the Borel algebra for Z and let $\lambda : Z \times B_Z \to [0,1]$ be a publicly-known Markov transition function. For each $z^t \in Z$ and $B \in B_Z$, $\lambda(z^t, B)$ is interpreted as the probability that next period's shock lies in B given that the current shock is z^t . Since β_i^t is a coordinate of z^t, treat it as a function on Z . Use the recursive relation $h^t = (h^{t-1}, z^{t-1})$ to treat N, r_i^τ, α_i^τ, U_i, and V_i as functions on Z . Treat $\rho_i^\tau = \rho_i^\tau(z^{t+1}, \ldots, z^{t+\tau})$ as a function on Z^τ where $Z^\tau = \times_{k=1}^{\tau} Z$ is a product space with typical element $(z^{t+1}, \ldots, z^{t+\tau})$. Derive the probability of $(z^{t+1}, \ldots, z^{t+\tau})$ given z^t from $\lambda(z^t, \cdot)$ in the usual fashion (e.g. Stokey and Lucas (1989, §8.2)) and denote it by $\lambda^\tau(z^t, dz^{t+1} \times \cdots \times dz^{t+\tau})$. Let $B_+(Z)$ be the space of non-negative, bounded, λ-measurable functions on Z, let $B_+^2(Z) = B_+(Z) \times B_+(Z)$, and define the following mappings on $B_+^2(Z)$. For $i = S, L, H$, and $(f^U, f^V) \in B_+^2(Z)$,

$$T_i^U(f^U, f^V)(z^t) = \delta \int \{(1-\beta_i^{t+1})f^U(z^{t+1}) + \beta_i^{t+1} f^V(z^{t+1})\}\lambda(z^t, dz^{t+1}) \ . \quad (13)$$

For $N(z^t) = 1$, $T_i^V(f^U, f^V)(z^t) = r_i^0(z^t) + f^U(z^t)$ and for $N(z^t) > 1$,

$$\begin{aligned} T_i^V(f^U, f^V)(z^t) = {} & \alpha_i^0(z^t)[r_i^0(z^t) + f^U(z^t)] + (1-\alpha_i^0(z^t)) \\ & \times \sum_{\tau=1}^{N(z^t)-1} \delta^\tau \int z^\tau \ \rho_i^\tau(z^{t+1}, \ldots, z^{t+\tau}) \\ & \{(1-\beta_i^{t+\tau})\beta_j^{t+\tau} f^U(z^{t+\tau}) + \beta_i^{t+\tau} f^V(z_i^{t+\tau}) \\ & + (1-\beta_i^{t+\tau})(1-\beta_j^{t+\tau})\alpha_i^\tau(z^{t+\tau})[r_i^\tau(z^{t+\tau}) + f^U(z^{t+\tau})]\} \\ & \times \lambda^\tau(z^t, dz^{t+1} \times \cdots \times dz^{t+\tau}) \ , \end{aligned} \quad (14)$$

where for $i = L$ or H, $\beta_j^{t+\tau} = \beta_S^{t+\tau}$ and for $i = S$, $\beta_j^{t+\tau} = \pi^{(N-\tau+1)}\beta_H^{t+\tau} + (1 - \pi^{(N-\tau+1)})\beta_L^{t+\tau}$.

To interpret T_i^U observe that $T_i^U(U, V)$ is just $U(\cdot, h^{t+1})$ as given in equation (3). To interpret T_i^V recall that $\hat{V}_i$ is the continuation payoff at the start of the period t bargaining phase for a newly-matched type i agent when all agents play according to $\hat{\sigma} \in \hat{\Sigma}$. Use induction on the number of bargaining periods N to verify that $T_i^V(\hat{U}, \hat{V})$ is just $\hat{V}_i$. Since $\tau = 0, 1, ..., N-1$, the Nth bargaining round of any match corresponds to period $\tau = N-1$. As stated in Theorem 1, in the Nth round the seller offers $p^{(1)}$ which both buyer types accept so that $\alpha_i^{N-1} = 1$ for all i. For $N = 1$, agreement is reached immediately returning $r_i^0 + \hat{U}_i$. Turning to equation (14), for $N = 2$, with probability $1 - \alpha_i^0$ agreement is not reached in the first period and discounting occurs. (Also, $\rho_i^1 = 1$.) The three terms in the brackets correspond to the agent losing his partner without finding another which returns $\hat{U}_i$, the agent being newly-matched to another partner which returns $\hat{V}_i$, and the agent retaining the old partner and reaching agreement with probability $\alpha_i^{N-1} = 1$ which returns $r_i^1 + \hat{U}_i$. By the inductive hypothesis, $T_i^V(\hat{U}, \hat{V})$ is just $\hat{V}_i$ for $N = n-1$. Consider $N = n$. By the inductive hypothesis the expected returns in the first $n-1$ periods are given by $T_i^V(\hat{U}, \hat{V})$ with summation from $\tau = 1$ to $\tau = n-2$. Show that the expected returns for the final period ($\tau = n-1$) are expressed by the summation term evaluated at $n-1$. Discounting back $n-1$ periods accounts for δ^{n-1}. Consider $(1-\alpha_i^0)\rho_i^{n-1}$. By the inductive hypothesis the second last period is reached with probability $(1-\alpha_i^0)\rho_i^{n-2}$ so that the final period is reached if the second last period is reached, disagreement occurs, and the match does not breakup i.e. with probability $(1-\alpha_i^0)\rho_i^{n-2}(1-\alpha_i^{n-2})(1-\beta_j^{t+n-2})(1-\beta_i^{t+n-2}) = (1-\alpha_i^0)\rho_i^{n-1}$. Finally, the term in the brackets has the same interpretation as above. Hence, $T_i^V(\hat{U}, \hat{V})$ is just $\hat{V}_i$, the returns to a type i agent newly-matched at the start of the period t bargaining phase when all agents play according to $\hat{\sigma} \in \hat{\Sigma}$.

Define $T_i = (T_i^U, T_i^V)$. The previous discussion demonstrated that if T_i has a fixed point then the fixed point is $(\hat{U}_i, \hat{V}_i)$. T_i links the bargaining payoffs (u_i, v_i) to the market payoffs (U_i, V_i) via $(U_i, V_i) = T_i(u_i, v_i)$. In the definition of a market equilibrium the search for a $(u_i(\cdot), v_i(\cdot))$ which is also a $(U_i(\sigma^*, \cdot), V_i(\sigma^*, \cdot))$ is the search for a fixed point of T_i. The next Step establishes the existence of a fixed point. $\square$

Step 2

The unique fixed point of T_i is $(\hat{U}_i, \hat{V}_i)$ for $i = S, L, H$.

Proof. Show that the T_i are contraction mappings. Recall that $Z = (0, M]^3 \times (0, 1]^2$ and work with $B_+^2(Z)$ and the supnorm. First, show that the T_i map $B_+^2(Z)$ into itself. Measurability of $(\alpha_i^\tau, r_i^\tau, \rho_i^\tau)$ and (λ^τ, λ) follow, respectively, from the measurability of strategies and the definition of a transition function. Hence, by Theorem 8.4 in Stokey and Lucas (1989), if (f^U, f^V) is measurable so is $T_i(f^U, f^V)$. The α_i^τ and ρ_i^τ lie on [0,1] and so are non-negative and bounded. Since the low-valuation buyer only accepts the final offer $p^{(1)} = b_L$, it follows that $\alpha_L^\tau = 0$ for $\tau < N-1$ and $r_L^{N-1} = b_L - b_L = 0$ so that $\alpha_L^\tau r_L^\tau = 0$. Since $p^{(N-\tau)} \in [b_L, b_H]$, r_S^τ and r_H^τ are non-negative and bounded. Hence, if (f^U, f^V) is non-negative and bounded so is $T_i(f^U, f^V)$. Thus, $(f^U, f^V) \in B_+^2(Z)$ implies $T_i(f^U, f^V) \in B_+^2(Z)$.

Second, show that when $\alpha_i^0 < 1$ then the T_i satisfy Blackwell's sufficient conditions for a contraction mapping, namely, monotonicity ((f^U, f^V), $(g^U, g^V) \in B_+^2(Z)$ with $f^U \leq g^U$ and $f^V \leq g^V \Rightarrow T_i(f^U, f^V) \leq T_i(g^U, g^V)$) and discounting (there exists a $(\kappa^U, \kappa^V) \in (0,1)^2$ such that $T_i(f^U + a, f^V + a) \leq T_i(f^U, f^V) + (\kappa^U a, \kappa^V a)$ for all $(f^U, f^V) \in B_+^2(Z)$ and $a > 0$). Both conditions are easy to verify with $\kappa^U = \delta$ and $\kappa^V = \alpha_i^0 + (1 - \alpha_i^0)\delta$. Hence, by the Contraction Mapping Theorem, for $\alpha_i^0 < 1$ the T_i each have a unique fixed point in $B_+^2(Z)$. (See Stokey and Lucas (1989), Theorems 3.2 and 3.3. The Theorems generalize to vector-valued contraction mappings.) If $\alpha_i^0 = 1$, then $N = 1$ and $T_i^V = r_i^0(z^t) + f^U(z^t)$, which is not a contraction mapping; however, the existence of a unique fixed point is easy to establish. Specifically, work with the contraction $S_i^U(f^U) = T_i^U(r_i^0 + f^U, f^U)$ to find a unique fixed point $\hat{f}^U$ and define $\hat{f}^V = r_i^0 + \hat{f}^U$. Then $(\hat{f}^U, \hat{f}^V)$ is the unique fixed point of T_i. From the discussion preceding Step 1, the fixed points of T_i are the $(\hat{U}_i, \hat{V}_i)$. $\square$

Step 3

$(\hat{U}, \hat{V})$ satisfies Assumption G.

Proof. *G.1*: Since T_i maps $B_+^2(Z)$ into itself, the $(\hat{U}_i, \hat{V}_i)$ are non-negative, bounded functions.

G.2: Use $\hat{U}_i = T_i^U(\hat{U}_i, \hat{V}_i)$ to write equation (3) lagged one period as

$$\hat{U}_i(z^{t-1}) = \delta \int \{(1 - \beta_i^t)\hat{U}_i(z^t) + \beta_i^t \hat{V}_i(z^t)\}\lambda(z^{t-1}, dz^t),$$

from which the conclusion follows.

G.3: From the outcome described in the Theorem, the low-valuation buyer reaches agreement only on offer $p^{(1)} = b_L$ so that $\alpha_i^\tau = 0$ for $\tau < N-1$ and $r_i^{N-1} = b_L - b_L = 0$. Hence, $\alpha_L^\tau r_L^\tau = 0$ for all τ. By inspection of the contraction mappings $T_L = (T_i^U, T_i^V)$ in equations (13)–(14) with $\alpha_L^\tau r_L^\tau = 0$, $(f^U, f^V) = (0, 0)$ is the (unique) fixed point. By Step 1, it must be that $\hat{U}_L = \hat{V}_L = 0$. $\square$

Step 4

If (σ^*, π^*) is a market equilibrium then $(U(\sigma^*, \cdot), V(\sigma^*, \cdot))$ satisfies Assumption G.

Proof. *G.1*: A buyer can guarantee himself zero by rejecting every offer. Similarly, a seller can guarantee himself zero by offering $p = 0$ in every period. Thus, $U(\sigma^*, \cdot) \geq 0$ and $V(\sigma^*, \cdot) \geq 0$. Since, $\delta < 1$ and seller offers are bounded above and below, agents' returns must be bounded.

G.2: From the definition of U in equation (1),

$$U_i(\sigma, h^t) = \delta \int \left\{(1 - \beta_i^t) U_i(\sigma, h^{t+1}) + \beta_i^t V_i(\sigma, h^{t+1})\right\} \lambda(z^{t-1}, dz^t)$$

for all σ. Hence it must hold for $\sigma = \sigma^*$.

G.3: Under Assumptions G.1 and G.2, in equilibrium the seller never makes an offer below b_L. (see Lemma 3 in the Appendix). Since every offer p is greater than or equal to b_L, the most a low-valuation buyer receives in any match is zero. Hence, $U_L(\sigma^*, \cdot) \leq 0$ and $V_L(\sigma^*, \cdot) \leq 0$. But by G.1, $U_L(\sigma^*, \cdot) \geq 0$ and $V_L(\sigma^*, \cdot) \geq 0$. $\square$

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