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# Synthesis and Generalization of Structural Results in Inventory Management: A Generalized Convexity Property

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We address a general periodic review inventory control model with the simultaneous presence of the following complications: (a) *bilateral* inventory adjustment options, via procurement orders and salvage sales or returns to the supplier; (b) fixed costs associated with procurement orders and downward inventory adjustments (via salvage sales or returns); and (c) capacity limits associated with upward or downward inventory adjustments. We characterize the optimal adjustment strategy, both for finite and infinite horizon periodic review models, by showing that in each period the inventory position line is to be partitioned into (maximally) five regions.

Our results are obtained by identifying a novel generalized convexity property for the value functions, which we refer to as strong $(C_1K_1, C_2K_2)$-convexity. To our knowledge, we recover most existing structural results for models with exogenous demands as special cases of a unified analysis.

*Key words*: inventory management, convexity, capacity constraint, bilateral adjustment, fixed cost, infinite horizon

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**1. Introduction.** The seminal papers by Arrow et al. [1] and Dvoretzky et al. [8] initiated the field of stochastic inventory theory, more than 65 years ago. These authors proposed a single-item base model with a finite planning horizon in which an order can be placed at the beginning of each period to increase the inventory level. The base model assumes that orders of an arbitrary, unlimited size may be placed and that the associated order costs are *proportional* to the order sizes. Demands are random but independent across time. Additional costs consist of inventory carrying and stockout or backlogging costs, assumed to be *proportional* with the end-of-the-period inventory levels and backlogging sizes, respectively. In the base model, it was shown that a so-called base-stock policy is optimal, in each period. Under such a policy, the inventory level is increased to a "base-stock" level, whenever it is found to be below that level; otherwise, it is optimal not to place any order. Scarf [28] showed that, under backlogging of stockouts, a base-stock policy continues to be optimal in the presence of an order *lead-time*, except that the policy acts on a different inventory measure, the so-called *inventory position* = inventory level plus all outstanding orders.

It was quickly understood that the base model needed to be generalized to address various complications that arise in practice, for example fixed order costs or capacity limits for individual order sizes. When fixed order costs are included to the base model, Scarf [28] and Iglehart [22] showed that, under broad general conditions, an $(s, S)$-policy is optimal, for finite and infinite horizon models, respectively. Under an $(s, S)$-policy, it is optimal to elevate the inventory position to an order-up-to level, $S$, but only if the period's starting inventory position is at or below a second threshold $s < S$ (as opposed to $S$ itself in the absence of fixed order costs). Federgruen and Zipkin [12, 13] showed that order capacity limits result in the optimality of a so-called *modified* base-stock policy: at the beginning of each period, an order is placed to bring the inventory position as close to the base-stock level as is feasible.

But, what if both complications (fixed order costs and capacity limits for individual orders) prevail simultaneously? As Federgruen and Zipkin [13] wrote:

*"If the production costs have a fixed (as well as a variable) component, it might be reasonable to expect that the modified $(s, S)$ policy would be optimal: when the inventory level falls below a critical number $s$, produce enough to bring total stock up to $S$, or as close as possible, given the production capacity; otherwise do not produce."*

However, Wijngaard [38] and later on Shaoxiang and Lambrecht [32] and Shaoxiang [31] identified counterexamples, both in finite and infinite horizon models. Indeed, a more complex structure emerges.

Similarly, some authors, starting with Whisler [37] and Constantinides and Richard [6], have considered settings where inventories may be adjusted downwards (as well as upwards) via sales in secondary channels (jobbers, discounters, outlet stores, etc) or returns to the supplier. Several authors have addressed inventory models with *bilateral* inventory adjustment options, i.e., procurement orders along with salvage sales and/or returns to the suppliers, for example Dai and Yao [7] and Feinberg and Lewis [15, 16], see also the references therein. However, to our knowledge, no one has considered settings where the size of the inventory adjustments is subject to capacity limits, for example.

This paper synthesizes and generalizes the existing literature with exogenously specified demands by addressing a general model with the simultaneous presence of the above-mentioned complications, specifically,

(a) *bilateral* inventory adjustment options, via procurement orders and salvage sales or returns to the supplier;
(b) fixed costs associated with procurement orders and downward inventory adjustments (via salvage sales or returns);
(c) capacity limits associated with upward or downward inventory adjustments.

We provide a full characterization of the optimal inventory adjustment strategy, both for finite and infinite horizon periodic review models, by showing that in each period the inventory position line is to be partitioned into (maximally) five regions: in the most far left (right) region, it is optimal to place an order (initiate a salvage sale) of a specific easily calculable magnitude. In the middle region, it is optimal to avoid any inventory adjustment. Finally, in the second region from the left (right), the policy alternates between intervals where one stays put and those where an order is to be placed (a salvage sale is to be initiated) of a size specified by a given function.

Our results are obtained by identifying a novel generalized convexity property for the value functions, which we refer to as strong $(C_1K_1, C_2K_2)$-convexity. To our knowledge, we recover most existing structural results for models with exogenous demands as special cases of a unified analysis. (To our knowledge, the exceptions are uncapacitated models with non-linear order costs, of a type, different from the fixed-plus-linear structure.)

The remainder of this paper is organized as follows: In Section 2 we review the related literature. Section 3 introduces our general model and the associated notation. Section 4 derives the structure of an optimal policy in a single period model. Section 5 covers a general finite horizon model; this Section also recovers existing structures in the literature as special cases of our general results. Section 6 shows how our structural results extend to stationary infinite horizon models, either under the discounted total cost or the long-run average cost criterion. Section 8 ends the paper with some concluding remarks.

**2. $(C_1K_1, C_2K_2)$-convexity: A generalized convexity property and review of existing literature.** The structural results obtained in this paper are based on our identifying a new generalized concept of convexity.

Definition 1 (($C_1K_1, C_2K_2$)-convexity). Given constants $C_1 > 0, K_1 \geq 0$ and $C_2 > 0, K_2 \geq 0$, a real-valued continuous function $f$ is called *strongly* $(C_1K_1, C_2K_2)$*-convex* if for any $x \geq y, a \in [0, C_1]$ and $b \in (0, C_2]$,

$$f(x+a) + K_1 \geq f(x) + \frac{a}{b}\Big(f(y) - f(y-b) - K_2\Big). \tag{1}$$

Denote $SC_{C_1K_1,C_2K_2}$ as the set of all strongly $(C_1K_1, C_2K_2)$-convex functions. When (1) is required only for $x = y$, we refer to the property as *weak* $(C_1K_1, C_2K_2)$-convexity.

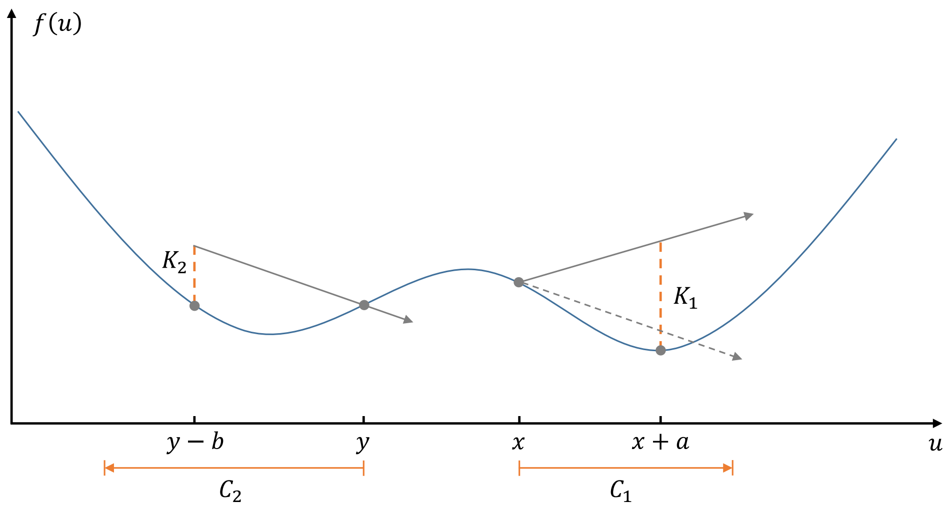


Figure 1. Geometric illustration of strongly $(C_1K_1, C_2K_2)$-convex functions

Figure 1 provides an intuitive way of understanding the strong $(C_1K_1, C_2K_2)$-convexity property. For any two points $y \leq x$, select any point $x + a$ with $a \in (0, C_1]$ and any point $y - b$ with $b \in (0, C_2]$. Raise the function value at point $x + a$ by $K_1$ and draw a ray from $(x, f(x))$ to $(x + a, f(x + a) + K_1)$. Similarly raise the function value at point $y - b$ by $K_2$ and draw a ray from $(y - b, f(y - b) + K_2)$ to $(y, f(y))$. Then $f$ is strongly $(C_1K_1, C_2K_2)$-convex if the slope of the former ray is bigger than or equal to the slope of the latter ray.

The $(C_1K_1, C_2K_2)$-convexity property generalizes many convexity properties, developed since Scarf [28] identified $K$-convexity as the key structural property to establish optimality of the so-called $(s, S)$-policies. Below, we list these earlier convexity properties in Table 1.

It appears that the basic convexity property goes back to Archimedes, in his treatise "On the sphere and cylinder" in the third century B.C.E., see also Heath [20] and Dwilewicz [9]. It arises as a special case of $(C_1K_1, C_2K_2)$-convexity with $C_1 = C_2 = \infty$ and $K_1 = K_2 = 0$. It is, of course, well known that for basic convexity, the weak and strong versions are equivalent: If the inequality in Table 1 holds for all $x = y$—which defines "weak convexity"—it holds for all $x \geq y$, as well. In other words, weak convexity implies strong convexity, and vice versa.

$K$-convexity corresponds with the special case where $C_1 = C_2 = \infty$ and $K_1 \geq 0, K_2 = 0$. The term was coined by Scarf [28] to address models with fixed order costs, but no capacity limits or salvage opportunities. Scarf [28] used the property to show that an $(s, S)$-policy is optimal under convex holding and backlogging costs. Veinott [35] subsequently showed this optimality result for holding and backlogging cost functions that are quasi-convex only, but (nearly) increasing over time. See

Table 1. Summary of Commonly Used Convexity Properties

| Convexity Property | Definition | Related Papers |
|---|---|---|
| convex | $f(x+a) \geq f(x) + \frac{a}{b}[f(y) - f(y-b)]$, $\forall y \leq x,\ a \geq 0,\ b > 0$ | Archimedes (3rd Century B.C.E.) |
| $K$-convex | $f(x+a) + K \geq f(x) + \frac{a}{b}[f(x) - f(x-b)]$, $\forall a \geq 0,\ b > 0$ | Scarf [28], Veinott [35], Kolmogorov and Fomin [23] |
| $CK$-convex | $f(x+a) + K \geq f(x) + \frac{a}{b}[f(x) - f(x-b)]$, $\forall a \in [0, C],\ b > 0$ | Gallego and Scheller-Wolf [18] |
| strongly $CK$-convex | $f(x+a) + K \geq f(x) + \frac{a}{b}[f(y) - f(y-b)]$, $\forall y \leq x,\ a \in [0, C],\ b > 0$ | Gallego and Scheller-Wolf [18], Shaoxiang and Lambrecht [32], Shaoxiang [31] |
| sym-$K$-convex | $f(x+a) + \max\{1, \frac{a}{b}\}K \geq f(x) + \frac{a}{b}[f(x) - f(x-b)]$, $\forall a \geq 0,\ b > 0$ | Chen and Simchi-Levi [3, 4] |
| YD-$(K_1, K_2)$-convex | $f(x+a) + K_1 - \max\{1, \frac{a}{b}\min\{K_1, K_2\}\} \geq f(x) + \frac{a}{b}[f(x) - f(x-b) - K_2]$, $\forall a \geq 0,\ b > 0$ | Ye and Duenyas [39] |
| weak $(K_1, K_2)$-convex or $C(a,b)$-convex | $f(x+a) + K_1 \geq f(x) + \frac{a}{b}[f(x) - f(x-b) - K_2]$, $\forall a \geq 0,\ b > 0$ | Gallego and Özer [17] and Semple [30] |
| strongly $(C_1K_1, C_2K_2)$-convex | $f(x+a) + K_1 \geq f(x) + \frac{a}{b}[f(y) - f(y-b) - K_2]$, $\forall y \leq x,\ a \in [0, C_1],\ b \in (0, C_2]$ | This paper |

also the recent tutorial by Feinberg [14]. Gallego and Sethi [19] extended the $K$-convexity property to functions that are defined on a general Euclidean space $\mathbb{R}^n$, to address multi-product systems with fixed order costs.

Gallego and Scheller-Wolf [18] addressed models with fixed order costs and capacity limits for individual orders (but no salvage opportunities). These authors introduced the $CK$-convexity property, again a special case of our general structure where $C_2 = \infty$ and $K_2 = 0$. Gallego and Scheller-Wolf [18] also pioneered the above distinction between "weak" and "strong" convexity properties.

Chen and Simchi-Levi [3, 4] addressed a periodic review combined inventory control and pricing model in which each period's demand distribution may be controlled by selecting a unit retail price from a closed price interval. The remaining assumptions are identical to those in the Scarf model, i.e., the base inventory model with fixed order costs. Chen and Simchi-Levi [3] covers the finite horizon case, while Chen and Simchi-Levi [4] address the long-run average and discounted profit criterion; the models are confined to the case where the order lead time is zero. The authors consider *affine* price-dependent demand functions, specified as:

$$D_n(p) = \alpha_n d_n(p) + \beta_n, \quad n = 1, 2, \ldots, N, \tag{2}$$

where $d_n(p)$ is a *deterministic* demand function and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of independent random variables whose distributions are independent of the chosen retail price $p_n$. In the finite horizon model of Chen and Simchi-Levi [3], the authors show that the value functions continue to be $K$-convex but only in the special case of an *additive* demand model, i.e., when $\alpha_n = 1$ for all $n$. This implies that an $(s, S)$ policy continues to be optimal in that case. However, $K$-convexity fails to apply in the general affine demand model (2). Indeed, no $(s, S)$ policy is necessarily optimal, contrary to a conjecture by Thomas [34].

For the more general model, the authors identify the sym-$K$-convexity property and show that the value functions satisfy this generalized $K$-convexity property, see Table 1. On that basis, they showed that, in each period $n$, there are two threshold levels $s_n < S_n$ such that no order is placed when the beginning inventory level is above $S_n$ and the inventory level is increased to $S_n$ when it is found to be below $s_n$. However, when the beginning inventory level is between the two thresholds, it is optimal to either refrain from ordering or to elevate the inventory level to $S_n$. In Chen and Simchi-Levi [4], the author showed that in *infinite* horizon settings, with stationary parameters and distributions, an $(s,S)$-policy is optimal in the general affine model (2), following a different approach, aligned with that of Zheng [40].

Returning to inventory models with *exogenously* specified demand variables, Chen and Simchi-Levi [5] addressed a model with bilateral inventory adjustments and fixed costs for each adjustment (but no capacity limits). Their analysis is based on a further generalization of $K$-convexity, introduced by Ye and Duenyas [39] which the authors refer to as $(K_1,K_2)$-convexity. To avoid confusion, we label the property as "YD-$(K_1,K_2)$-convexity" with YD the initials of the authors. In the special case where $K_1=K_2=K$, YD-$(K_1,K_2)$-convexity reduces to sym-$K$-convexity. The authors show that all value functions are YD-$(K_1,K_2)$-convex under minor restrictions for the time-dependence of the fixed adjustment costs $K_1$ and $K_2$. Chen and Simchi-Levi [5] follow Neave [27] who had addressed the same model but failed to provide a complete analysis for the case where the two fixed costs $K_1$ and $K_2$ differ from each other. Similarly, Feinberg and Lewis [16] employed the $YD-(K_1,K_2)$-convexity property to analyze the infinite horizon version of the stochastic cash balance problem.

Ye and Duenyas [39] had introduced their YD-$(K_1,K_2)$-convexity property to analyze a capacity adjustment model, with similar results to those in Chen and Simchi-Levi [5]. Semple [30] introduced the "weak $(K_1,K_2)$-convexity" property as a further generalization of YD-$(K_1,K_2)$-convexity. The author showed, again under the same parameter restrictions as in Ye and Duenyas [39], that all value functions are weakly $(K_1,K_2)$-convex if the terminal value function has this property; moreover, all structural results obtained in Ye and Duenyas [39] can be obtained under this more general convexity property. Clearly, weak $(K_1,K_2)$-convexity is a special case of our "strong $(C_1K_1,C_2K_2)$-convexity" property under the special parameter choices $C_1=C_2=\infty$ and weakening the definitional inequality (1) to hold only for $y=x$. Unbeknownst to Semple, Gallego and Özer [17] had, six years earlier, introduced the same "weak $(K_1,K_2)$-convexity", under the name $C(a,b)$-convexity. The authors used this property to establish optimality of a state-dependent $(s,S)$ policy in an inventory model with advanced demand information.

Caliskan-Demirag et al. [2] introduced a new convexity property that includes the strong $CK$-convexity property of Gallego and Scheller-Wolf [18], Shaoxiang and Lambrecht [32] and Shaoxiang [31], and the sym-$K$ convexity property of Chen and Simchi-Levi [3] as special cases. The authors replace on the right side of inequality (1), the fixed cost $K$, by a general function $\sigma(K,a)$:

$$f(x+a)+\sigma(K,a)\geq f(x)+\frac{a}{b}(f(y)-f(y-b)\quad \text{for any } y\leq x, a\in[0,C], b>0.$$

The authors employ this property, which they refer to as $\sigma(K,z)$-convexity, to characterize the structure of an optimal policy, when there are two possible fixed order costs, $K_1<K_2\leq 2K_1$, with the lower fixed cost $K_1$ applicable iff the order size is below a given threshold. (The model is uncapacitated and inventory adjustments are in the upward direction only.)

Lu and Song [24], subsequently, identified another variant of $\sigma(K,z)$-convexity for a model with a convex piecewise-linear order cost function. These authors refer to their structure as strong $(K,c,q)$-convexity. $K$-approximate convexity, introduced in Lu et al. [25, 26] is a related approach, in *approximate* rather than *exact* dynamic programming. The fundamental idea is to approximate the exact one-period cost structure on the cost-to-go functions, respectively, with a convex function such that the maximal approximation error is at most $K$, and derive bounds for the distance between the exact and approximate value functions. See Caliskan-Demirag et al. [2] and Lu and Song [24] for a review of other models with a non-linear order cost function, different from the standard fixed-plus-linear structure.

Federgruen et al. [11] have employed the strong $(C_1K_1, C_2K_2)$-convexity properly to characterize the structure of the optimal inventory adjustment strategy in a *dual* sourcing setting with salvage opportunities, fixed inventory adjustment costs and capacity limits for orders and salvage batches.

Proposition 1 summarizes the above relationships among the various convexity properties.

PROPOSITION 1. (a) *convexity* $\Rightarrow$ $K$*-convexity* $\Rightarrow$ *sym-*$K$*-convexity* $\Rightarrow$ *YD-*$(K_1, K_2)$*-convexity* $\Rightarrow$ *weak* $(K_1, K_2)$*-convexity* $\Rightarrow$ *strong* $(C_1K_1, C_2K_2)$*-convexity*

(b) *convexity* $\Rightarrow$ *strong* $K$*-convexity* $\Rightarrow$ *strong* $CK$*-convexity* $\Rightarrow$ *strong* $(C_1K_1, C_2K_2)$*-convexity*

Lemma 1 establishes various preservation properties for strongly $(C_1K_1, C_2K_2)$-convex functions.

LEMMA 1 **(Properties of** $SC_{C_1K_1,C_2K_2}$**).** (i) *If* $f(x) \in SC_{C_1K_1,C_2K_2}$*, then* $f(-x) \in SC_{C_2K_2,C_1K_1}$.

(ii) *If* $f(x) \in SC_{C_1K_1,C_2K_2}$*, then* $f(x) \in SC_{C_1'K_1',C_2'K_2'}$ *for any* $C_1' \le C_1, C_2' \le C_2$*,* $K_1' \ge K_1, K_2' \ge K_2$*.*

(iii) *If* $f(x) \in SC_{C_1K_1,C_2K_2}$ *and* $g(x) \in SC_{C_1K_1',C_2K_2'}$*, then for any* $\alpha, \beta \ge 0$*,* $\alpha f(x) + \beta g(x) \in SC_{C_1(\alpha K_1+\beta K_1'),C_2(\alpha K_2+\beta K_2')}$*. As a special case, when* $g(x)$ *is convex, hence* $g(x) \in SC_{C_10,C_20}$*,* $\alpha f(x) + \beta g(x) \in SC_{C_1(\alpha K_1),C_2(\alpha K_2)}$ *for any* $\beta \ge 0$*.*

(iv) *If* $f(x) \in SC_{C_1K_1,C_2K_2}$*, then* $f(x-a) \in SC_{C_1K_1,C_2K_2}$ *for any real number* $a$*. Moreover, for any random variable* $Y$ *with* $\mathbb{E}|f(x-Y)| < \infty$*,* $\mathbb{E}f(x-Y) \in SC_{C_1K_1,C_2K_2}$*.*

*Proof.* Parts (i) and (ii) are immediate.

(iii) Let $h(x) = \alpha f(x) + \beta g(x)$, for any $x \ge y, a \in [0, C_1], b \in (0, C_2]$ we have

$$\begin{aligned}
\Delta &= \alpha K_1 + \beta K_1' + h(x+a) - h(x) - \frac{a}{b}\Big(h(y) - h(y-b) - \alpha K_2 - \beta K_2'\Big) \\
&= \alpha K_1 + \beta K_1' + \alpha f(x+a) + \beta g(x+a) - \alpha f(x) - \beta g(x) \\
&\quad - \frac{a}{b}\Big(\alpha f(y) + \beta g(y) - \alpha f(y-b) - \beta g(y-b) - \alpha K_2 - \beta K_2'\Big) \\
&= \alpha\Big[K_1 + f(x+a) - f(x) - \frac{a}{b}\Big(f(y) - f(y-b) - K_2\Big)\Big] \\
&\quad + \beta\Big[K_1' + g(x+a) - g(x) - \frac{a}{b}\Big(g(y) - g(y-b) - K_2'\Big)\Big] \ge 0
\end{aligned}$$

(iv) Using (iii) this is immediate. $\square$

## 3. Model.

We consider a single-item periodic review model with a single supplier. Extensions with multiple suppliers are addressed in Federgruen et al. [11]. At the beginning of each period, an order may be placed with the supplier, possibly subject to a time-dependent capacity limit. In each period, there may also be a (*limited*) *salvage* option to reduce inventory by sales to a secondary channel (discounters, jobbers, outlet stores, etc.) or returns to the supplier. The lead time is $L$ periods, both for ordering and for salvaging, when available as an option. The cost associated with any given order has a fixed and variable component; similarly, a fixed cost is incurred when a salvage sale is initiated, along with revenues that are proportional with the size of the salvage batch. All stockouts are backlogged. In addition to the ordering and salvaging costs and revenues, there are standard holding and backlogging costs, assumed to be proportional or convexly increasing with the end-of-the-period inventory levels and backlog sizes.

We consider a planning horizon of $N \le \infty$ periods and our objective is to minimize the total expected discounted costs over the full planning horizon. We index the periods *backward* from 1 to $N$. (Section 6 covers the long-run average cost criterion)

The sequence of events in period $n$ is as follows: at the beginning of the period, any order placed [salvage batch initiated] in period $n+L$ is added to [removed from] the inventory. Based on the inventory position (= inventory on hand − backlogs + all outstanding orders), the firm then decides on a new order size, or a salvage quantity to be initiated, if it wants to *reduce* the inventory position. Stochastic demand is then realized and satisfied with on-hand inventory. At the end of the period, any leftover inventory is carried forward to the next period, while any unsatisfied demand is fully backlogged.

We show below that the single inventory position measure suffices to make optimal decisions; moreover, it is never optimal to simultaneously place an order and initiate a salvage batch.

We now state the notation employed in our model:

$$\begin{aligned} K_n, C_n &= \text{fixed cost and capacity limit for an order placed in period } n, \\ K_n^v, C_n^v &= \text{fixed cost and capacity limit for any salvage quantity initiated in period } n, \\ L &= \text{order lead time}, \\ c_n &= \text{unit price charged by the supplier in period } n, \\ c_n^v &= \text{unit revenue received when salvaging inventory in period } n, \\ \alpha &= \text{discount factor, } 0 \le \alpha \le 1. \end{aligned}$$

The sequence of demands $\{D_n\}$ represents *independent* random variables with general distributions. We make the following assumption.

ASSUMPTION 1. $c_n \ge c_n^v, \quad n = N, \ldots, 1.$

This ranking is satisfied in all practical settings and precludes it ever being optimal to place an order and initiate a salvage batch in the same period. (Assume, to the contrary, that in some period $n$, it is optimal to place an order of size $q_n$, along with the initiation of a salvage batch of size $\bar{q}_n$. Under Assumption 1, money is saved by reducing the order to $(q_n - \bar{q}_n)$ and canceling the salvage batch, if $q_n \ge \bar{q}_n$; alternatively, if $\bar{q}_n > q_n$, money may be saved by reducing the salvage batch to $(\bar{q}_n - q_n)$ and canceling the order.)

Settings without actual salvage opportunities may be represented as having such opportunities, however, with $c_n^v = -M$, where $M$ denotes a sufficiently large constant. This representation allows for a unified treatment of models with and without salvage opportunities.

For $n = N, \ldots, 1$, let

$$\begin{aligned} x_n &= \text{the inventory position at the beginning of period } n, \textit{before} \text{ any inventory adjustement;} \\ y_n &= \text{the inventory position at the beginning of period } n, \textit{after} \text{ any inventory adjustmenet.} \end{aligned}$$

Inventory and backlogging related costs are assumed to depend on the end-of-period inventory levels only, it is well known since Scarf [28] that under full backlogging, an equivalent representation of the controllable parts of the total expected discounted cost over the planning horizon is obtained by charging to period $n+L$, the *expected* value of the actual costs incurred at the end of period $n$. This follows from the sample path relationship between $y_n$, the inventory position at the beginning of period $n$, and the inventory level $I_{n-L}$ at the *end* of period $n-L$:

$$I_{n-L} = y_n - D_n^{(L)},$$

where $D_n^{(L)} = D_n + D_{n-1} + \cdots + D_{n-L}$ is the aggregate demand in time interval $[n, n-L]$.

For all $n = N, \ldots, 1$, let

$$\mathcal{L}_n(x_n + q_n) = \text{the expected value of all inventory and backlogging related costs at the end of period } n-L \text{ discounted back to period } n$$

and impose a standard assumption regarding these functions, satisfied for most common cost structures.

ASSUMPTION 2. (i) *The function $\mathcal{L}_n(\cdot)$ is convex and $\mathcal{L}_n(y) = O(|y|^p)$ for some $p \ge 1$, $n = N, \ldots, 1$. Also, $\mathbb{E}(D_n^p) < \infty$ for $n = N, \ldots, 1$.*

(ii) $c_n^v \le -\frac{\partial^- \mathcal{L}_n(x)}{\partial x}$ *for* $n = N, \dots, 1$, *where* $\frac{\partial^- \mathcal{L}_n(x)}{\partial x}$ *denotes the left derivative of the function* $\mathcal{L}_n(\cdot)$.[1]

Assumption 2 (ii) ensures that, in every period $n$, the marginal backlogging cost is in excess of the unit salvage value.

Beyond Assumptions 1 and 2, we need a few additional parameter restrictions.

Assumption 3. *For* $n = N, \dots, 1$,

$$K_n \ge \alpha K_{n-1}, \quad K_n^v \ge \alpha K_{n-1}^v, \tag{3}$$
$$C_n \le C_{n-1}, \quad C_n^v \le C_{n-1}^v. \tag{4}$$

The inequalities (3) were already recognized as essential in the base model with fixed order costs, see Scarf [28] and Zipkin [41]. The inequalities (4) indicate that capacity limits for order and salvage quantities may not decline over time; this is typically the case in practical applications.

To introduce the dynamic programming formulation, define the following value functions:

$f_n(x) =$ the optimal expected discounted total costs in the last $n$ periods, assuming period $n$ is started with an inventory position of $x$ units;
$f_n^1(x) =$ the optimal expected discounted total costs in the last $n$ periods, assuming period $n$ is started with an inventory position of $x$ units and *no* salvage batch is initiated;
$f_n^2(x) =$ the optimal expected discounted total costs in the last $n$ periods, assuming period $n$ is started with an inventory position of $x$ units and a salvage batch *is* initiated.

Clearly, since, as shown, it is never optimal to place an order and to initiate a salvage sale in the same period, we have for $n = N, \dots, 1$:

$$f_n(x) = \min\{f_n^1(x), f_n^2(x)\}, \tag{5}$$
$$f_n^1(x) = \min_{x_n \le y_n \le x_n + C_n} \{K_n \delta(y_n - x_n) + c_n(y_n - x_n) + \mathcal{L}_n(y_n) + \alpha \mathbb{E} f_{n-1}(y_n - D_n)\}, \tag{6}$$
$$f_n^2(x) = \min_{\min\{[x_n - C_n^v]^+, x_n\} \le y_n \le x_n} \{K_n^v \delta(x_n - y_n) + c_n^v(y_n - x_n) + \mathcal{L}_n(y_n) + \alpha \mathbb{E} f_{n-1}(y_n - D_n)\}, \tag{7}$$

for a given terminal value function $f_0(\cdot)$ satisfying:

Assumption 4. *The terminal value function* $f_0(\cdot) \in SC_{C_0 K_0, C_0^v K_0^v}$ *and is non-increasing on the negative half-line.*

The dynamic programming formulation exploits the fact that it is never optimal to simultaneously place a procurement order and to initiate a salvage sale. It also utilizes the simple state dynamics $x_{n-1} = y_n - D_n$. The lower bound for $y_n$ in (7), i.e., $y_n \ge \min\{[x_n - C_n^v]^+, x_n\}$, reflects the fact that, at least in physical inventory models, there are no salvage opportunities when $x_n \le 0$, while salvage opportunities are bounded by $\min\{x_n, C_n^v\}$ when $x_n > 0$. Instead of analyzing the DP (5)–(7) directly, we relax the feasible action set in (7) to $x_n - C_n^v \le y_n \le x_n$, giving rise to the *relaxed* DP:

$$\widetilde{f}_n(x) = \min\{\widetilde{f}_n^1(x), \widetilde{f}_n^2(x)\}, \tag{8}$$
$$\widetilde{f}_n^1(x) = \min_{x_n \le y_n \le x_n + C_n} \{K_n \delta(y_n - x_n) + c_n(y_n - x_n) + \mathcal{L}_n(y_n) + \alpha \mathbb{E} \widetilde{f}_{n-1}(y_n - D_n)\}, \tag{9}$$
$$\widetilde{f}_n^2(x) = \min_{x_n - C_n^v \le y_n \le x_n} \{K_n^v \delta(x_n - y_n) + c_n^v(y_n - x_n) + \mathcal{L}_n(y_n) + \alpha \mathbb{E} \widetilde{f}_{n-1}(y_n - D_n)\}. \tag{10}$$

We first show that this relaxation can be adopted without affecting the optimal policies.

[1] A convex function has left and right derivatives everywhere.

THEOREM 1. *For $i = N, \dots, 1$, let $y_i^*(x_i)$ denote the optimal inventory policy in the relaxed dynamic program (8)–(10) when the inventory position at the beginning of period $i$ is $x_i$, then*
(a) *If $x_i \le 0$, then $y_i^*(x_i) \ge x_i$, i.e., it is optimal not to salvage;*
(b) *If $x_i > 0$, then $y_i^*(x_i) \ge 0$, i.e., it is optimal to maintain a non-negative inventory position.*

*Proof.* (a) Suppose, to the contrary, that $0 < a = x_i - y_i^*(x_i)$. We show that a cost improvement can be achieved on *any* sample path $\omega$, by perturbing the time series $\{y_i^*(x_i), \bar{y}_j = y_j^*(x_j), j = i-1, \dots, 1\}$ to $\{\bar{y}_i = x_i, \bar{y}_j = y_j^*(x_j), j = i-1, \dots, 1\}$. In other words, the perturbation involves the cancellation of the salvage batch in period $i$, and reducing the inventory adjustment in period $i-1$ by $a$ units. Note that after the inventory adjustment in period $i-1$, the remaining sample path until the end of the planning horizon, remains unaltered. Let $\Delta$ denote the incremental costs incurred due to the perturbation,

$$\begin{aligned}\Delta &\le \left[-K_i^v + ac_i^v + a\frac{\partial^- \mathcal{L}_i(0)}{\partial x}\right] + \alpha[K_{i-1}^v + a\max\{-c_{i-1}, -c_{i-1}^v\}] \\ &= -(K_i^v - \alpha K_{i-1}^v) + a\left(c_i^v + \frac{\partial^- \mathcal{L}_i(0)}{\partial x}\right) - a\max\{c_{i-1}, c_{i-1}^v\} < 0 \qquad (11)\end{aligned}$$

To justify the first inequality, note that the first term to its right denotes the cost savings in the first period due to the cancellation of the salvage batch in period $i$. This cancellation results in a saving of $K_i^v$, the fixed cost of this batch and a reduction of the backlog size at the end of period $i$, by $a$ units, at a per-unit saving of at least $\frac{\partial^- \mathcal{L}_i(0)}{\partial x}$; on the other hand, a loss of revenues, hence an *increase* in costs of $ac_i^v$ emerges from the canceled salvage transaction.

The second term to the right of the first inequality in (11) is an *upper bound* for the additional costs incurred in period $i-1$; here, the decrease in the inventory adjustment may save the fixed cost $K_{i-1}$, in case this decrease cancels an order or, at worst, it may initiate a salvage batch in period $i-1$, thus adding $\alpha K_{i-1}^v$ to the total cost. In addition, the modified inventory adjustment results in either a reduction of the variable cost $c_{i-1}$ or an additional revenue $c_{i-1}^v$ per unit. The total additional variable cost in period $i-1$ are therefore bounded from above by $-a\max\{c_{i-1}, c_{i-1}^v\}$.

The second inequality in (11) follows from Assumptions 2 and 3.

(b) Suppose, to the contrary, that $y_i^*(x_i) < 0$. Let $b = -y_i^*(x_i) > 0$. Define

$$z_j = -\sum_{k=j+1}^{i} D_k(\omega) \le 0, \quad j = i, i-1, \dots, 1.$$

Consider the following *modification* to the optimal policy $\delta^*$: in period $i$ reduce the size of the salvage batch by $b$ units; thereafter, stay put until the first period in which $y_j^* \ge z_j$, if any. Let $l = \max\{j \le i-1 : y_j^*(x_j) \ge z_j\}$, where $l = 0$ when this index set is empty. If $l \ge 1$, place an order in period $l$ for $y_j^* - z_j$ units. We distinguish between two cases: (b1) $l \ge 1$ and (b2) $l = 0$.

Proof for case (b1): after period $l$, the modified policy implements the same actions as the original policy $\delta^*$. Let $\Delta$ denote the incremental cost due to the policy perturbation. By part (a) and the definition of the time period $l$, we have for all $j = i-1, \dots, l+1$ that

$$x_j \le y_j^* < z_j. \qquad (12)$$

Note that the sample paths of the modified and the original policies coincide from period $l$ on. Thus, the cost differential $\Delta$ arises due to cost differences in the interval $[i, l]$ only. Thus, let $\Delta = \Delta_1 + \Delta_2 + \Delta_3$, where

$\Delta_1 =$ difference in procurement and salvage costs in periods $i-1, \dots, l$;
$\Delta_2 =$ lost revenues in period $i$ due to the reduction of the salvage batch in that period by $b$ units;
$\Delta_3 =$ difference in backlogging and holding costs in the entire interval $[i, l]$.

Note that, by the definition of the period index $l$:

$$q_l = y_l^* - x_l = y_l^* - (y_{l+1}^* - D_{l+1}) > y_l^* - z_{l+1} + D_{l+1} = y_l^* - z_l > 0.$$

Thus, the original as well as the modified policy initiate a salvage batch in period $i$ and place an order in period $l$, and the salvage batch and order size under the modified policy are *smaller* than their counterparts under the original policy $\delta^*$. Since the modified policy avoids inventory adjustments in the intermediate periods in $(i,l)$, it follows that $\Delta_1 \le 0$. Also $\Delta_2 = bc_i^v$, while $\Delta_3 \le b\frac{\partial^- \mathcal{L}_i(0)}{\partial x}$, since the backlog size at end of period $i$ is $b$ units smaller under the modified policy and at the end of all remaining periods $j = i-1, \dots, l+1$, the modified policy has a smaller backlog size than the original policy $\delta^*$, see (12). Thus, $\Delta \le b\left(c_i^v + \frac{\partial^- \mathcal{L}_i(0)}{\partial x}\right) < 0$ by Assumption 2.

Proof for case (b2): In this case, the modified policy reduces the salvage batch in period $i$ by $b$ units and stays put for the remainder of the planning horizon, ending the planning horizon with an inventory level $z_1 - D_1$, as opposed to an ending inventory level $y_1^* - D_1$ under the original policy. The proof for case (b1) shows that the modified policy incurs a lower total of procurement, salvage, holding and backlogging costs. However, in this case, $\Delta$ contains the additional differential $f_0(z_1 - D_1) - f_0(y_1^* - D_1) \le 0$, by Assumption 4 and $y_1^* < z_1$. □

In view of Theorem 1, we proceed without loss of optimality, with the relaxed dynamic program (8)–(10), omitting the $\sim$ sign on top of the value functions $\widetilde{f}(\cdot), \widetilde{f}_1(\cdot), \widetilde{f}_2(\cdot)$.

**4. The single period problem.** It follows from the dynamic programming recursions (8)–(10) that, in each period $n$, we face an optimization problem of the following structure

$$g_1(x) = \min_{y\in[x,x+C_1]}\{K_1\delta(y-x) + \beta_1(y-x) + g(y)\}, \tag{13}$$
$$g_2(x) = \min_{y\in[x-C_2,x]}\{K_2\delta(x-y) + \beta_2(y-x) + g(y)\}, \tag{14}$$
$$g_0(x) = \min\{g_1(x), g_2(x)\} \tag{15}$$

with $g_1(\cdot) = f_n^1(\cdot), g_2(\cdot) = f_n^2(\cdot), g_0(\cdot) = f_n(\cdot), \beta_1 = c_n, \beta_2 = c_n^v, K_1 = K_n, K_2 = K_n^v, C_1 = C_n, C_2 = C_n^v$ and $g(y) = \mathcal{L}_n(y) + \alpha\mathbb{E}f_{n-1}(y - D_n)$.

We now analyze this single stage optimization problem (13)–(15), under the assumption that the terminal cost formulation $g(\cdot)$ has the strong $(C_1K_1, C_2K_2)$-convexity property for specific parameter values $C_1, K_1, C_2, K_2$.

Define auxiliary functions

$$\widetilde{g}_1(x) = K_1 + \min_{y\in[x,x+C_1]}\{\beta_1(y-x) + g(y)\}, \tag{16}$$
$$\widetilde{g}_2(x) = K_2 + \min_{y\in[x-C_2,x]}\{\beta_2(y-x) + g(y)\}, \tag{17}$$

as counterparts of $g_1(x)$ and $g_2(x)$, under *definitive* inventory adjustment, i.e., *definitively* incurring fixed costs for ordering or salvaging, respectively, and let $A_i(x) = \widetilde{g}_i(x) - g(x)$ be the increase in minimal cost if forced to order (for $i = 1$) or salvage (for $i = 2$).

To characterize the structure of an optimal policy, we need to define some critical points, with the convention that the infimum (supremum) of an empty set equals $+\infty$ $(-\infty)$.

Definition 2 (Critical Points). For a continuous function $g(\cdot) \in SC_{C_1K_1,C_2K_2}$ and any $\beta_1, \beta_2$, define

$$B = \inf\left\{\arg\min_y\{\beta_1 y + g(y)\}\right\}, \qquad b = \inf\{x:\ A_1(x) \ge 0\}, \qquad \bar{b} = \sup\{x:\ A_1(x) < 0\}, \tag{18}$$
$$S = \sup\left\{\arg\min_y\{\beta_2 y + g(y)\}\right\}, \qquad s = \sup\{x:\ A_2(x) \ge 0\}, \qquad \underline{s} = \inf\{x:\ A_2(x) < 0\}. \tag{19}$$

These critical points play important roles in the structure of the optimal strategy. By its definition, $B$ is the (smallest) global minimizer of $\widetilde{g}_1(x)$ if $C_1 = \infty$, i.e., the smallest order-up-to level for sufficiently small $x$ if ordering is better than staying put. Similarly, $S$ is the (largest) global minimizer of $\widetilde{g}_2(x)$ if $C_2 = \infty$, .i.e., the biggest salvage-down-to level for sufficiently large $x$ if salvaging is better than staying put; $b$ is the smallest among all inventory levels where ordering is not better than staying put; $\bar{b}$ is the largest among all inventory levels where ordering is better than staying put; $s$ is the largest among all inventory levels where salvaging is not better than staying put; $\underline{s}$ is the smallest among all inventory levels where salvaging is better than staying put.

Note that $b = \bar{b}$ $[\underline{s} = s]$ if the function $A_1(\cdot)$ $[A_2(\cdot)]$ has a single root. We have observed this single root property to hold in all problem instances we have encountered, see Section 7. It can, however, not be guaranteed, for general $(C_1K_1, C_2K_2)$-convex functions, which may have many local optima, see Figure 1.

The Proposition below characterizes the ranking of the critical points, which is important when developing the optimal policy structure.

PROPOSITION 2 **(Critical Points)**. *Assume $\beta_1 \geq \beta_2$ and $g(\cdot) \in SC_{C_1K_1, C_2K_2}$, then*
(i) $-\infty \leq b \leq \bar{b} \leq \underline{s} \leq s \leq \infty$;
(ii) $-\infty \leq b \leq B \leq S \leq s \leq \infty$;
(iii) *If $C_2 = \infty$ and $K_1 \geq K_2$, then $\bar{b} \leq B$; if $C_1 = \infty$ and $K_1 \leq K_2$, then $S \leq \underline{s}$;*
(iv) *If $C_1 = \infty$ and $K_2 = 0$, then $b = \bar{b}$; if $C_2 = \infty$ and $K_1 = 0$, then $\underline{s} = s$. If $C_1 = C_2 = \infty$ and $K_1 = K_2 = 0$, then $b = \bar{b} = B, S = \underline{s} = s$.*

In this Proposition, (i) ranks four critical points. (ii) ranks and locates the global minimizers $B$ and $S$ between $b$ and $s$. (iii) and (iv) lead to simple policy structures, in certain special cases, which will be discussed later.

To prove this Proposition, we first need some auxiliary lemmas. Note that by definition we have

$$g_1(x) = \min\{g(x),\ \widetilde{g}_1(x)\}, \qquad A_1(x) < 0 \quad \forall x < b, \qquad A_1(x) \geq 0 \quad \forall x > \bar{b}, \tag{20}$$
$$g_2(x) = \min\{g(x),\ \widetilde{g}_2(x)\}, \qquad A_2(x) < 0 \quad \forall x > s, \qquad A_2(x) \geq 0 \quad \forall x < \underline{s}. \tag{21}$$

The following lemma shows that all regions where it is optimal to order (order regions) are to the left of all regions where it is optimal to salvage inventory (salvage regions).

LEMMA 2 **(Separation of Order/Salvage Regions)**. *Assume $\beta_1 \geq \beta_2$ and $g(\cdot) \in SC_{C_1K_1, C_2K_2}$, then*
(i) *if $\widetilde{g}_2(y) < g(y)$ for some $y$, then $g(x) \leq \widetilde{g}_1(x)$ for any $x \geq y$;*
(ii) *if $\widetilde{g}_1(y) < g(y)$ for some $y$, then $g(x) \leq \widetilde{g}_2(x)$ for any $x \leq y$.*

*Proof.* (i) Given $\widetilde{g}_2(y) < g(y)$, by the definition of $\widetilde{g}_2(\cdot)$ we have

$$\widetilde{g}_2(y) = K_2 + \beta_2(-b) + g(y - b) < g(y) \quad \text{for some } b \in (0, C_2],$$

where $b$ cannot take the value of 0 because $K_2 \geq 0$. Equivalently,

$$g(y) - g(y - b) - K_2 > -\beta_2 b.$$

Hence by strong $(C_1K_1, C_2K_2)$-convexity of $g(\cdot)$, for any $x \geq y$ and $a \in [0, C_1]$ we have

$$K_1 + g(x + a) - g(x) \geq \frac{a}{b}\Big(g(y) - g(y - b) - K_2\Big) \geq -\beta_2 a \geq -\beta_1 a,$$

where the last inequality follows from $\beta_1 \geq \beta_2$. Equivalently,

$$K_1 + \beta_1 a + g(x + a) \geq g(x).$$

As this holds for any $a \in [0, C_1]$, we obtain $\widetilde{g}_1(x) \geq g(x)$. It can also be verified that if $K_1 > 0$, we have strict inequality as $\widetilde{g}_1(x) > g(x)$. Case (ii) can be proved in a similar way and the details are omitted here. $\square$

Intuitively, (i) shows that if salvaging is better than staying put at a given level $y$, then staying put is better than ordering at or above $y$. In other words, ordering is never optimal above a "salvaging" point. Similarly, (ii) shows that if ordering is better than staying put at a given level $y$, then staying put is better than salvaging at or below $y$, i.e., salvaging is never optimal below an "ordering" point.

The following corollary shows that if at a given level $y$, salvaging is strictly preferred, it is optimal not to order for any inventory level $x > y$. Similarly, if at a given level $y$, ordering is strictly preferred, it is optimal not to salvage for any inventory level $x < y$.

COROLLARY 1. *Assume $\beta_1 \geq \beta_2$ and $g(\cdot) \in SC_{C_1K_1,C_2K_2}$, then*
(i) *if $g_2(y) < g_1(y)$ for some $y$, then $g_2(x) \leq g_1(x)$ for any $x \geq y$;*
(ii) *if $g_1(y) < g_2(y)$ for some $y$, then $g_1(x) \leq g_2(x)$ for any $x \leq y$.*

*Proof.* To verify (i), notice that $g_2(y) < g_1(y)$ implies $\widetilde{g}_2(y) < g(y)$ since $g_1(y) \leq g(y)$ and $g_2(y) = \min\{g(y), \widetilde{g}_2(y)\}$. By Lemma 2 (i), $g(x) \leq \widetilde{g}_1(x)$, which implies $g_2(x) \leq g_1(x)$ since $g_2(x) \leq g(x)$ and $g_1(x) = \min\{g(x), \widetilde{g}_1(x)\}$. Similarly, we can prove (ii): $g_1(y) < g_2(y)$ implies $\widetilde{g}_1(y) < g(y)$ since $g_2(y) \leq g(y)$ and $g_1(y) = \min\{g(y), \widetilde{g}_1(y)\}$. By Lemma 2 (ii), $g(x) \leq \widetilde{g}_2(x)$, which implies $g_1(x) \leq g_2(x)$ since $g_1(x) \leq g(x)$ and $g_2(x) = \min\{g(x), \widetilde{g}_2(x)\}$. □

Certain monotonicities of the functions concerned play an important role in formulating optimal policy structure, as are shown in the lemma below.

LEMMA 3 **(Monotonicity)**. *Assume $g(\cdot) \in SC_{C_1K_1,C_2K_2}$ and finite $|\bar{b}|, |\underline{s}|$,* [2] *then*
(i) *if $K_2 = 0$, $\beta_1 x + g(x)$ is strictly decreasing on $(-\infty, \bar{b})$;*
(ii) *if $K_1 = 0$, $\beta_2 x + g(x)$ is strictly increasing on $(\underline{s}, \infty)$.*

*Proof.* Here we prove (i) as (ii) can be shown similarly, and we prove the general case where $K_2 \geq 0$ noted by the footnote. Consider $x_1 < x_2 < \bar{b}$ with $x_2 - x_1 \leq C_2$, then there exists $b_0 \in (x_2, \bar{b})$ such that $A_1(b_0) < 0$ by the definition of $\bar{b}$ and the continuity of $A_1(\cdot)$. Hence we have

$$g(b_0) > \widetilde{g}_1(b_0) = K_1 + \beta_1(z - b_0) + g(z),$$

for some $z \in (b_0, b_0 + C_1]$. Note that $z$ cannot take the value of $b_0$ since otherwise $K_1 < 0$. Equivalently,

$$\beta_1 b_0 + g(b_0) > K_1 + \beta_1 z + g(z).$$

Then by the strong $(C_1K_1, C_2K_2)$-convexity of $\beta_1 x + g(x)$ we have

$$\beta_1 b_0 + g(b_0) > K_1 + \beta_1 z + g(z) \geq \beta_1 b_0 + g(b_0) + \frac{z - b_0}{x_2 - x_1}\Big((\beta_1 x_2 + g(x_2)) - (\beta_1 x_1 + g(x_1)) - K_2\Big),$$

which implies

$$\beta_1 x_2 + g(x_2) < \beta_1 x_1 + g(x_1) + K_2,$$

i.e., $\beta_1 x + g(x)$ is strictly non-$K_2$-increasing on $(-\infty, \bar{b})$. Specially, if $K_2 = 0$, $\beta_1 x + g(x)$ is strictly decreasing on $(-\infty, \bar{b})$. □

We are now ready for the proof of Proposition 2.

*Proof of Proposition 2.*

(i) First, we show $\bar{b} \leq \underline{s}$ by contradiction. Suppose $\bar{b} > \underline{s}$, then by the definition of $\bar{b}$ and $\underline{s}$ in (18) and (19), respectively, and the continuity of $A_1(\cdot)$ and $A_2(\cdot)$, there exist $x$ and $y$ such that $\underline{s} < x < y < \bar{b}$ for which $A_2(x) < 0$ and $A_1(y) < 0$, or $\widetilde{g}_2(x) < g(x)$ and $\widetilde{g}_1(y) < g(y)$. This contradicts Lemma 2 and hence $\bar{b} \leq \underline{s}$. Next, we show $b \leq \bar{b}$ also by contradiction. Assume $b > \bar{b}$, then by the definition of $b$ in (18), $A_1(z) < 0$ for any $z \in (\bar{b}, b)$, which contradicts the definition of $\bar{b}$. Hence we have $b \leq \bar{b}$. We can prove $\underline{s} \leq s$ in a similar way.

[2] Finite $|\bar{b}|$ and $|\underline{s}|$ can be implied by $A_1(x) < 0$ for some $x$ and $A_2(y) < 0$ for some $y$, respectively.

(ii) First, we show $B \le S$. Let

$$h_1(y) = \beta_1 y + g(y), \quad h_2(y) = \beta_2 y + g(y) = h_1(y) - (\beta_1 - \beta_2)y,$$

which are both strongly $(C_1K_1, C_2K_2)$-convex according to Lemma 1.(iii). Then by (18) and (19) we have

$$B = \inf\{\arg\min_y h_1(y)\}, \quad S = \sup\{\arg\min_y h_2(y)\},$$

which imply that $h_1(x) > h_1(B)$ for all $x < B$. Then for any $x < B$, we have

$$h_2(x) = h_1(x) - (\beta_1 - \beta_2)x > h_1(B) - (\beta_1 - \beta_2)x \ge h_1(B) - (\beta_1 - \beta_2)B = h_2(B),$$

where the second inequality follows from $\beta_1 \ge \beta_2$. This implies that $B \le S$ by the definition of $S$.

Next, we show $b \le B$ and $S \le s$. For $b \le B$, suppose on the contrary $b > B$, then by (20) we have $A_1(B) < 0$, hence

$$g(B) > \widetilde{g}_1(B) = K_1 + \min_{B \le y \le B + C_1} \{\beta_1 y + g(y)\} - \beta_1 B = K_1 + g(B),$$

where the last equality follows from the fact that $B$ is a global minimizer of $\beta_1 y + g(y)$. This contradicts $K_1 \ge 0$ and hence it should be $b \le B$. In a similar way we can show $S \le s$.

(iii) We prove the case where $C_2 = \infty$ and $K_1 \ge K_2$ by contradiction; the other case where $C_1 = \infty$ and $K_1 \le K_2$ can be proved in the same way. Assuming $\bar{b} > B$, there exists $x \in (B, \bar{b})$ such that $A_1(x) < 0$ by the definition of $\bar{b}$ in (18). Then

$$g(x) > \widetilde{g}_1(x) = K_1 + g(z) + \beta_1(z - x) \tag{22}$$

for some $z \in (x, x + C_1]$. Notice that $z$ cannot take value of $x$ because that results in $K_1 < 0$. By the definition of $B$ in (18) and $z > x > B$, we have

$$g(z) + \beta_1 z \ge g(B) + \beta_1 B,$$

or equivalently,

$$g(B) - g(z) \le \beta_1(z - B). \tag{23}$$

By strong $(C_1K_1, \infty K_2)$-convexity of $g(\cdot)$ we have

$$K_1 + g(z) \ge g(x) + \frac{z - x}{x - B}\Big(g(x) - g(B) - K_2\Big),$$

or equivalently,

$$\begin{aligned} g(x) &\le \frac{x - B}{z - B}\Big(K_1 + g(z)\Big) + \frac{z - x}{z - B}\Big(g(B) + K_2\Big) \\ &= K_1 + g(z) + \frac{z - x}{z - B}\Big(g(B) + K_2 - K_1 - g(z)\Big) \\ &\le K_1 + g(z) + \frac{z - x}{z - B}\Big(g(B) - g(z)\Big) \\ &\le K_1 + g(z) + \beta_1(z - x), \end{aligned}$$

where the second inequality follows from the assumption $K_1 \ge K_2$ and the last inequality follows from (23). This contradicts (22), thus we have shown $\bar{b} \le B$.

(iv) We first prove by contradiction the case where $K_2 = 0$; the other case where $K_1 = 0$ can be shown in the same way. By part (i), it suffices to show that the assumption $b < \bar{b}$ results in a contradiction. By the definition of $b$ and $\bar{b}$ in (18) there exist $x$ and $y$ such that $b \le x < y < \bar{b}$ and $A_1(x) \ge 0$, $A_1(y) < 0$. Since $K_2 = 0$ and $x < y < \bar{b}$, Lemma 3 (i) implies

$$\beta_1 x + g(x) > \beta_1 y + g(y). \tag{24}$$

By (iii) of this Lemma, $\bar{b} \le B$, thus, since $C_1 = \infty$ and since $B$ is a global minimizer of the function $\beta_1 y + g(y)$,

$$\widetilde{g}_1(x) = K_1 + \beta_1 B + g(B) - \beta_1 x, \tag{25}$$
$$\widetilde{g}_1(y) = K_1 + \beta_1 B + g(B) - \beta_1 y. \tag{26}$$

Noticing the definition of $A_1(\cdot)$ in Definition 2, $A_1(x) \ge 0$ and $A_1(y) < 0$ together with (24)–(26) yield

$$K_1 + \beta_1 B + g(B) \ge \beta_1 x + g(x) > \beta_1 y + g(y) > K_1 + \beta_1 B + g(B),$$

a clear contradiction. Hence, $b = \bar{b}$.

Next, we consider the case where $K_1 = K_2 = 0$. We prove $\bar{b} = B$; the equality $S = \underline{s}$ can be shown in the same way. First notice that $\bar{b} \le B$ by (iii) of this Lemma. Suppose, to the contrary, $\bar{b} < B$, then by the definition of $\bar{b}$ in (18) there exists $x \in (\bar{b}, B)$ that $A_1(x) \ge 0$, or $\widetilde{g}_1(x) \ge g(x)$ by the definition of $A_1(\cdot)$. With $K_1 = 0$, this implies that

$$\widetilde{g}_1(x) = \beta_1 B + g(B) - \beta_1 x \ge g(x),$$

which contradicts the definition of $B$, as $x < B$. Hence $\bar{b} = B$. □

We now proceed to the optimal single-period policy structure, in the following Theorem.

Theorem 2 **(Single Period Optimal Policy Structure)**. *Assume $\beta_1 \ge \beta_2$ and $g(\cdot) \in SC_{C_1K_1,C_2K_2}$, then $g_0(x)$ and the corresponding minimizer $y^*(x)$ are characterized by Table 2 and Figure 2, in which $\widetilde{g}_1(\cdot)$ and $\widetilde{g}_2(\cdot)$ are defined by (16) and (17), respectively. If $y^*(x)$ is specified as a two-element set $\{\cdot,\cdot\}$, either one of the two elements may apply. Let*

$$B(x) = \inf \mathcal{B}(x) \text{ where } \mathcal{B}(x) = \underset{x \le y \le x + C_1}{\arg\min} \{\beta_1 y + g(y)\}, \tag{27}$$
$$S(x) = \sup \mathcal{S}(x) \text{ where } \mathcal{S}(x) = \underset{x - C_2 \le y \le x}{\arg\min} \{\beta_2 y + g(y)\} \tag{28}$$

*denote minimizers of $\widetilde{g}_1(x)$ and $\widetilde{g}_2(x)$, respectively. Let $b(x) = B(x) - x$ and $s(x) = x - S(x)$ denote the corresponding order and salvage quantity.*

Table 2. Single period optimal policy structure

| $x$ | $(-\infty, b)$ | $[b, \bar{b})$ | $[\bar{b}, \underline{s}]$ | $(\underline{s}, s]$ | $(s, \infty)$ |
|---|---|---|---|---|---|
| $g_0(x)$ | $\widetilde{g}_1(x)$ | $\min\{\widetilde{g}_1(x), g(x)\}$ | $g(x)$ | $\min\{\widetilde{g}_2(x), g(x)\}$ | $\widetilde{g}_2(x)$ |
| $y^*(x)$ | $B(x)$ | $\{B(x), x\}$ | $x$ | $\{S(x), x\}$ | $S(x)$ |

*Proof.*

- $x \in (-\infty, b)$. $x < b$ implies that $A_1(x) < 0$ by (20), so $\widetilde{g}_1(x) < g(x)$ and by Lemma 2 $g(x) \le \widetilde{g}_2(x)$. It follows that $g_0(x) = g_1(x) = \widetilde{g}_1(x)$ and $y^*(x) = B(x)$, the minimizer of $\widetilde{g}_1(x)$.
- $x \in [b, \bar{b})$. By the definition of $\bar{b}$ in (18), there exists $y \in (x, \bar{b})$ such that $A_1(y) < 0$, i.e., $\widetilde{g}_1(y) < g(y)$. Then $g(x) \le \widetilde{g}_2(x)$ by Lemma 2. It is therefore optimal to either place an order or to keep the inventory position unaltered. The minimizer $y^*(x)$ therefore equals $B(x)$ or $x$.

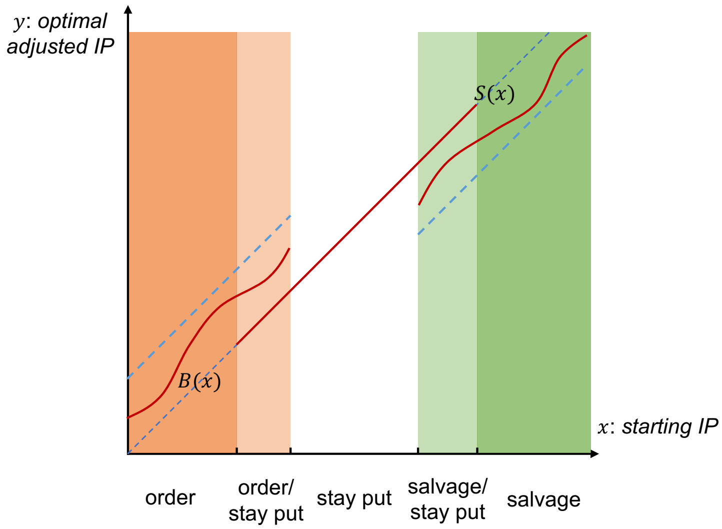


FIGURE 2. Illustration of single period optimal policy structure

- $x \in [\bar{b}, \underline{s}]$. $x \geq \bar{b}$ implies that $A_1(x) \geq 0$ by (20), so $\widetilde{g}_1(x) \geq g(x)$. Similarly $x \leq \underline{s}$ implies that $A_2(x) \geq 0$ by (21), so $\widetilde{g}_2(x) \geq g(x)$. Therefore $g_0(x) = g_1(x) = g_2(x) = g(x)$ and $y^*(x) = x$.
- $x \in (\underline{s}, s]$. By the definition of $\underline{s}$ in (19), there exists $y \in (\underline{s}, x)$ such that $A_2(y) < 0$, i.e., $\widetilde{g}_2(y) < g(y)$. Then $g(x) \leq \widetilde{g}_1(x)$ by Lemma 2. Therefore it is optimal to either initiate a salvage batch or stay put, and the minimizer $y^*(x)$ equals $S(x)$ or $x$.
- $x \in (s, \infty)$. $x > s$ implies that $A_2(x) < 0$ by (21), so $\widetilde{g}_2(x) < g(x)$ and by Lemma 2 $g(x) \leq \widetilde{g}_1(x)$. It hence follows that $g_0(x) = g_2(x) = \widetilde{g}_2(x)$ and $y^*(x) = S(x)$, the minimizer of $\widetilde{g}_2(x)$. □

In other words, four critical points partition the inventory position line into five regions. In the two extreme regions, $(-\infty, b)$ and $(s, \infty)$, a positive order or salvage transaction needs to be initiated, respectively; in the middle region, $[\bar{b}, \underline{s}]$, it is optimal to stay put; in the second region, $[b, \bar{b})$, it is optimal to either order or to stay put, and in the fourth region, $(\underline{s}, s]$, it is optimal to either initiate a salvage transaction or to stay put. Within the latter two regions, it is possible that the optimal policy alternates several times between ordering or salvaging versus staying put, a phenomenon already discovered in simpler models without salvage opportunities, see e.g., Shaoxiang and Lambrecht [32] and Shaoxiang [31].

As mentioned, if the functions $A_1(\cdot)$ and $A_2(\cdot)$ have a *single* root, $b = \bar{b}$ and $\underline{s} = s$, so that the second and fourth region vanish. In all of our numerical experience, this single root property prevails. In this case, the five-region policy simplifies to a three-region policy, and Table 2 and Figure 2 simplify to the following Table 3 and Figure 3. However, Ye and Duenyas [39], dealing with the special case of our model with unrestricted order sizes, identified an instance where a five-region policy emerges because the functions $A_1(\cdot)$ and $A_2(\cdot)$ have multiple roots.

TABLE 3. Simplified optimal policy structure

| $x$ | $(-\infty, b)$ | $[b, s]$ | $(s, \infty)$ |
|---|---|---|---|
| $g_0(x)$ | $\widetilde{g}_1(x)$ | $g(x)$ | $\widetilde{g}_2(x)$ |
| $y^*(x)$ | $B(x)$ | $x$ | $S(x)$ |

The following monotonicity properties enable further simplification when computing an optimal policy.

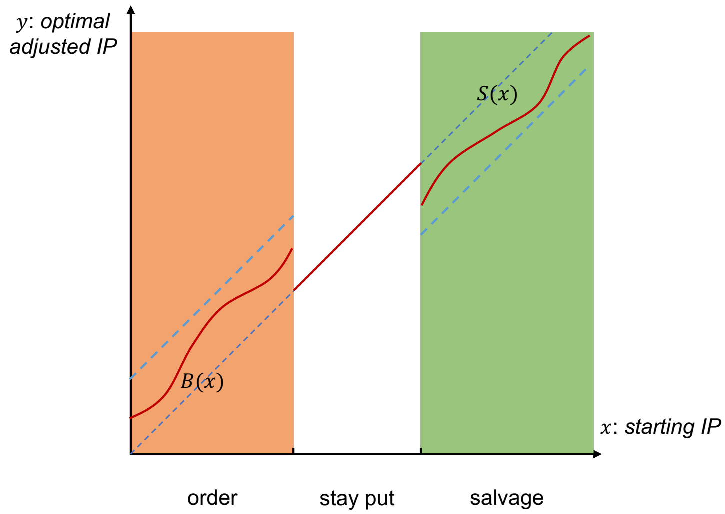


FIGURE 3. Illustration of single period optimal policy structure

PROPOSITION 3 **(Monotonicity).** *(a) The functions $B(\cdot)$ and $S(\cdot)$ are increasing for all $n = N, \ldots, 1$.*

*(b) The optimal order-up-to level $y^*(x)$ is increasing in $x$, almost everywhere, for all $n = N, \ldots, 1$. Lack of full monotonicity may occur in terms of downward jumps and these may arise at the, at most finitely many, breakpoint values where the optimal policy switches between ordering and staying put, or between staying put and salvaging.*

*(c) If the function $g(\cdot)$ is convex, $b(\cdot)$ is decreasing and $s(\cdot)$ is increasing.*

*Proof.* (a) We prove the monotonicity of the function $B(\cdot)$; the proof for the function $S(\cdot)$ is analogous. It suffices to show that the family of sets $\{\mathcal{B}(x) : -\infty < x < \infty\}$ is increasing in the standard partial order $\geq^p$ for subsets of a lattice, see Vives [36] (p 23): for a pair of sets $\mathcal{B}_1, \mathcal{B}_2$, $\mathcal{B}_1 \geq^p \mathcal{B}_2$ if for any $b_1 \in \mathcal{B}_1$ and $b_2 \in \mathcal{B}_2$, $\sup(b_1, b_2) \in \mathcal{B}_1$ and $\inf(b_1, b_2) \in \mathcal{B}_2$. Note that the feasibility intervals $[x, x + C]$, subsets of the real line $\mathbb{R}$, are increasing in $x$. Since the minimand in (27) is independent of $x$, hence has decreasing differences in $(x, y)$, the monotonicity of the sets $\{\mathcal{B}(x) : -\infty < x < \infty\}$ follows from Theorem 2.3 (b), in combination with Remark 10, in Vives [36].
(b) An immediate corollary of part (a) is that $y^*(\cdot)$ is increasing on any interval on which the optimal policy prescribes "ordering" or any interval on which it prescribes "staying put". The remaining characterization of the function $y^*(\cdot)$ is immediate.
(c) We prove that $b(x)$ is decreasing in $x$. The monotonicity proof for $s(\cdot)$ is analogous. Similar to the proof of part (a), define

$$\Omega(x) = \arg\min_{0 \leq q \leq C_1} \{\beta_1(x + q) + g(x + q)\} \tag{29}$$

and note that $b(x) = \inf \Omega(x)$. Since $g(\cdot)$ is convex, it has increasing differences in $(x, q)$. Applying Theorem 2.3 (b) in Vives [36] to a minimization problem, we get that the sets $\{\Omega(x) : -\infty < x < \infty\}$ are decreasing in the partial order $\geq^p$, defined in the proof of part (a). In particular, $b(\cdot)$ is decreasing as well. $\square$

Downward jumps of the function $y^*(\cdot)$ in a few points, may indeed occur, as exhibited by the common Example in Shaoxiang and Lambrecht [32] and Shaoxiang [31]: the order-up-to level exhibits a downward jump at $x = 6$. Note that this example pertains to an infinite horizon model with stationary inputs, and hence, a fortiori, in finite horizon models with non-stationary inputs.

In spite of the fact that the order-up-to policy $y^*(\cdot)$ fails to be *perfectly monotone*, the structure in Proposition 3 may be exploited to simplify the dynamic programming calculations, no less than in models where perfect monotonicity can be shown. Assuming $y^*(\cdot)$ is calculated on a grid $\{x_1, x_2, \ldots\}$, we may exploit the fact that $y^*(x_i) \in [y^*(x_{i-1}), \infty) \cup \{x_i\}$.

The convexity assumption of the function $g(\cdot)$ is usually satisfied in a true single-period setting, where $g(y) = \mathcal{L}(y)$. Unfortunately, it often fails in multi-period settings. The monotonicity of the function $g(\cdot)$ implies that every interval on which it is optimal to order may be partitioned into two (possibly empty) subintervals: in the first subinterval, it is optimal to order up to capacity; in the second subinterval the order quantity decreases. Similarly, any interval in which it is optimal to salvage, may be partitioned into two (possibly empty) subintervals: in the first subinterval, the salvage quantity increases; if this quantity reaches the capacity level, there is a second subinterval on which the salvage quantity equals the capacity level.

Based on Theorem 2 and the previous lemmas, we have the following three corollaries that capture special cases where the optimal policy takes on simpler or more specific forms.

First, as mentioned, a setting without a salvage option corresponds with the parameter choices $\beta_2 = -M, K_2 = 0, C_2 = \infty$. In this case, $\underline{s} = \infty$, and the four-region structure in Table 2 reduces to three regions only. Similar simplifications due to $\underline{s} = \infty$ arise in the special cases discussed below.

Corollary 2 **(No-Salvage Models)**. *When there is no salvage option and $g(\cdot) \in SC_{C_1 K_1, \infty 0}$, the structure of the optimal policy in the one-period problem is displayed by the first three columns in Table 2, since $\underline{s} = \infty$.*

Corollary 3 **(Uncapacitated Models)**. *When $C_1 = C_2 = \infty$, part of the optimal policy structure in Theorem 2 takes on simpler forms summarized by Table 4.*

Table 4. Special optimal policy structures when $C_1 = C_2 = \infty$

(a) When $K_1 \geq K_2$ (If $K_2 = 0$, $b = \bar{b}$ and the shaded column disappears)

| $x$ | $(-\infty, b)$ | $[b, \bar{b})$ | $[\bar{b}, \underline{s}]$ | $(\underline{s}, s]$ | $(s, \infty)$ |
|---|---|---|---|---|---|
| $g_0(x)$ | $\widetilde{g}_1(x)$ | $\min\{\widetilde{g}_1(x), g(x)\}$ | $g(x)$ | $\min\{\widetilde{g}_2(x), g(x)\}$ | $\widetilde{g}_2(x)$ |
| $y^*(x)$ | $B$ | $\{B, x\}$ | $x$ | $\{S(x), x\}$ | $S$ |

(b) When $K_1 \leq K_2$ (If $K_1 = 0$, $\underline{s} = s$ and the shaded column disappears)

| $x$ | $(-\infty, b)$ | $[b, \bar{b})$ | $[\bar{b}, \underline{s}]$ | $(\underline{s}, s]$ | $(s, \infty)$ |
|---|---|---|---|---|---|
| $g_0(x)$ | $\widetilde{g}_1(x)$ | $\min\{\widetilde{g}_1(x), g(x)\}$ | $g(x)$ | $\min\{\widetilde{g}_2(x), g(x)\}$ | $\widetilde{g}_2(x)$ |
| $y^*(x)$ | $B$ | $\{B(x), x\}$ | $x$ | $\{S, x\}$ | $S$ |

(c) When $K_1 = K_2 = 0$

| $x$ | $(-\infty, B)$ | $[B, S]$ | $(S, \infty)$ |
|---|---|---|---|
| $y^*(x)$ | $B$ | $x$ | $S$ |

*Proof.* In this case we clearly have

$$B(x) = \inf\{\arg\min_{y \geq x}\{\beta_1 y + g(y)\}\} = B, \quad \text{for } x \leq B;$$
$$S(x) = \inf\{\arg\min_{y \leq x}\{\beta_2 y + g(y)\}\} = S, \quad \text{for } x \geq S.$$

By Proposition 2 (ii), for $x < b \le B$, $y^*(x) = B$; for $x > s \ge S$, $y^*(x) = S$. This verifies the structure in the two outer regions for both $K_1 \ge K_2$ and $K_1 \le K_2$. For the shaded regions in subtable (a) and (b):

- When $K_1 \ge K_2$, , $\bar{b} \le B$ by Proposition 2 (iii), hence for any $x < \bar{b} \le B$, $g_0(x) = \widetilde{g}_1(x)$ and $y^*(x) = B$. Specially, if $K_2 = 0$, Proposition 2 (iv) indicates $b = \bar{b}$, and the shaded region in Table 4 (a) does not exist.
- When $K_1 \le K_2$, $S \le \underline{s}$ by Proposition 2 (iii), hence for any $x > \underline{s} \ge S$, $g_0(x) = \widetilde{g}_2(x)$ and $y^*(x) = S$. Specially, if $K_1 = 0$, Proposition 2 (iv) indicates $s = \underline{s}$, and the shaded region in Table 4 (b) does not exist.

For the special case where $K_1 = K_2 = 0$, as given by subtable (c) simply follows from Proposition 2 (iv). □

When there are no capacity limits but a fixed cost for ordering or salvaging does exist (as in subtables (a) and (b)), the following simplifications arise: the two outer regions have simple constant order-up-to and salvage-down-to levels $B$ and $S$, respectively. Depending on the relative size of $K_1$ and $K_2$, the second or fourth region also has a specific target adjustment level. Finally, when either $K_1$ or $K_2$ is zero, the second or fourth region does not exist. This makes the corresponding ordering or salvaging decision a simple "$(s, S)$"–type policy. Furthermore, when there are no fixed costs, subtable (c) displays a three-region structure where both ordering and salvaging decisions become "base stock"–type policies.

The characterization in Table 4 is similar to that in Theorem 1 in Ye and Duenyas [39], with additional simplifications indicated when one or both of the fixed costs are zero, see also Semple [30]. Dai and Yao [7] consider a continuous review variant of this model where the demand process is given by a Brownian motion; the authors also confine themselves to stationary models under the long-run average cost criterion, further assuming that $L = 0$. For this case, they establish optimality of the following 4 threshold policy: there exist threshold $d < D < U < u$, such that inventory is increased (decreased) to $D$ ($U$) when it reaches the level $d$ ($u$); no inventory adjustment is made as long as the inventory level resides in $(d, u)$.

Corollary 4 **(No-Fixed Costs Models)**. *When either $K_1 = 0$ or $K_2 = 0$, part of the optimal structure can be characterized with more specificity, as is shown in Table 5, in which*

$$\bar{B}(x) = \inf\{\mathop{\arg\min}_{\bar{b} \le y \le x + C_1} \{\beta_1 y + g(y)\}, \qquad \text{for } x \ge \bar{b} - C_1;$$
$$\underline{S}(x) = \sup\{\mathop{\arg\min}_{x - C_2 \le y \le \underline{s}} \{\beta_2 y + g(y)\}, \qquad \text{for } x \le \underline{s} + C_2;$$

$$\mathbf{1}_b^+ = \mathbf{1}(b > \bar{b} - C_1), \qquad \mathbf{1}_b^- = \mathbf{1}(b < \bar{b} - C_1);$$
$$\mathbf{1}_s^+ = \mathbf{1}(s > \underline{s} + C_2), \qquad \mathbf{1}_s^- = \mathbf{1}(s < \underline{s} + C_2).$$

*Proof.* We first consider the case where $K_2 = 0$; the case where $K_1 = 0$ is symmetric and can be shown similarly. When $K_2 = 0$, by Lemma 3 (i), $\beta_1 x + g(x)$ is strictly decreasing on $(-\infty, \bar{b})$.

- $x < \min\{\bar{b} - C_1, b\}$. $x < b$ implies that $g_0(x) = \widetilde{g}_1(x)$ by the general optimal policy in Table 2. Since $\beta_1 y + g(y)$ is strictly decreasing on $(-\infty, \bar{b})$ and $x + C_1 < \bar{b}$, clearly $y^*(x) = x + C_1$.
- $\min\{\bar{b} - C_1, b\} \le x < \max\{\bar{b} - C_1, b\}$. It is presumed that $\bar{b} - C_1 \ne b$ since otherwise this interval is empty and there is nothing to show. Then there are two cases to consider:

(a) $b < \bar{b} - C_1$. The interval is $b \le x < \bar{b} - C_1$. Clearly $x \in [b, \bar{b})$ so $g_0(x) = \min\{\widetilde{g}_1(x), g(x)\}$ by the general optimal policy in Table 2. By the same argument as in the previous interval, if an order is placed, it is optimal to place a full capacity order. Therefore $y^*(x) \in \{x + C_1, x\}$.

TABLE 5. Special optimal policy structures (partly) when $K_1 = 0$ or/and $K_2 = 0$

(a) When $K_2 = 0$. (Structure on $[\bar{b}, \infty)$ same as in Table 2)

| $x$ | $(-\infty, \min\{\bar{b} - C_1, b\})$ | $[\min\{\bar{b} - C_1, b\}, \max\{\bar{b} - C_1, b\})$ | $[\max\{\bar{b} - C_1, b\}, \bar{b})$ |
|---|---|---|---|
| $g_0(x)$ | $\widetilde{g}_1(x)$ | $\widetilde{g}_1(x)$ | $\min\{\widetilde{g}_1(x), g(x)\}$ |
| $y^*(x)$ | $x + C_1$ | $\{x + C_1, x\}\mathbf{1}_b^- + \bar{B}(x)\mathbf{1}_b^+$ | $\{\bar{B}(x), x\}$ |

(b) When $K_1 = 0$. (Structure on $(-\infty, \underline{s}]$ same as in Table 2)

| $x$ | $(\underline{s}, \min\{\underline{s} + C_2, s\}]$ | $(\min\{\underline{s} + C_2, s\}, \max\{\underline{s} + C_2, s\}]$ | $(\max\{\underline{s} + C_2, s\}, \infty)$ |
|---|---|---|---|
| $g_0(x)$ | $\min\{\widetilde{g}_2(x), g(x)\}$ | $\widetilde{g}_2(x)$ | $\widetilde{g}_2(x)$ |
| $y^*(x)$ | $\{\underline{S}(x), x\}$ | $\{x - C_2, x\}\mathbf{1}_s^+ + \underline{S}(x)\mathbf{1}_s^-$ | $x - C_2$ |

(c) When $K_1 = K_2 = 0$ and $C_1 < \infty, C_2 = \infty$. (Structure on $[\bar{b}, \infty)$ same as in Table 2)

| $x$ | $(-\infty, \bar{b} - C_1)$ | $[\bar{b} - C_1, \bar{b})$ |
|---|---|---|
| $g_0(x)$ | $\widetilde{g}_1(x)$ | $\widetilde{g}_1(x)$ |
| $y^*(x)$ | $x + C_1$ | $\bar{b}$ |

(b) $b > \bar{b} - C_1$. The interval is $\bar{b} - C_1 \le x < b$. Again $x < b$ implies that $g_0(x) = \widetilde{g}_1(x)$ by the general optimal policy in Table 2. Since $\bar{b} \le x + C_1$ and $\beta_1 y + g(y)$ is strictly decreasing on $(-\infty, \bar{b})$, $y^*(x) = \bar{B}(x)$.

- $\max\{\bar{b} - C_1, b\} \le x < \bar{b}$. Clearly $x \in [b, \bar{b})$ so $g_0(x) = \min\{\widetilde{g}_1(x), g(x)\}$ by the general optimal policy in Table 2. Since $\bar{b} \le x + C_1$ and $\beta_1 y + g(y)$ is strictly decreasing on $(-\infty, \bar{b})$, if it is optimal to place an order then $y^*(x) = \bar{B}(x) \in [\bar{b}, x + C_1]$. Thus, $y^*(x) \in \{\bar{B}(x), x\}$.

Next we prove the optimal policy structure given by Table 5 (c) under $K_1 = K_2 = 0$ and $C_1 < \infty, C_2 = \infty$. Notice that this is a special case of subtable (a), where we also have $K_1 = 0$ and $C_2 = \infty$. We only need to show $b = \bar{b} = \bar{B}(x), \forall x \in [\bar{b} - C_1, \bar{b}]$ so that subtable (a) becomes subtable (c). First we show $\beta_1 x + g(x)$ is increasing on $(\bar{b}, \infty)$, which directly implies $\bar{B}(x) = \bar{b}, \forall x \in [\bar{b} - C_1, \bar{b}]$ by the definition of $\bar{B}$. To see this, it follows from (20) and $K_1 = 0$ that for any $x > \bar{b}$ and $y \in [x, x + C_1]$,

$$A_1(x) \ge 0 \Rightarrow \beta_1 y + g(y) - \beta_1 x \ge g(x) \Leftrightarrow \beta_1 x + g(x) \le \beta_1 y + g(y).$$

Then we show $b = \bar{b}$. By Lemma 3 (i), $\beta_1 x + g(x)$ is strictly decreasing on $(-\infty, \bar{b})$. Therefore for any $x \le \bar{b} - C_1$,

$$\beta_1 x + g(x) > \beta_1 y + g(y) \Rightarrow \beta_1 y + g(y) - \beta_1 x < g(x),\ \forall y \in (x, x + C_1] \Rightarrow \widetilde{g}_1(x) < g(x) \Rightarrow A_1(x) < 0.$$

This implies $b \ge \bar{b} - C_1$ noticing the definition of $b$ in (18). Suppose $\bar{b} - C_1 \le b < \bar{b}$. By the definition of $b$ and $\bar{b}$ in (18) there exist $x$ and $y$ such that $\bar{b} - C_1 \le b \le x < y < \bar{b}$ and $A_1(x) \ge 0$, $A_1(y) < 0$. It is shown above that $\bar{B}(s) = \bar{b}, \forall s \in [\bar{b} - C_1, \bar{b}]$, hence

$$\widetilde{g}_1(x) = \beta_1 \bar{b} + g(\bar{b}) - \beta_1 x, \quad \widetilde{g}_1(y) = \beta_1 \bar{b} + g(\bar{b}) - \beta_1 y.$$

Therefore

$$\begin{aligned} A_1(x) \ge 0 &\Rightarrow \widetilde{g}_1(x) \ge g(x) \Rightarrow \beta_1 \bar{b} + g(\bar{b}) - \beta_1 x \ge g(x), \\ A_1(y) < 0 &\Rightarrow \widetilde{g}_1(y) < g(y) \Rightarrow \beta_1 \bar{b} + g(\bar{b}) - \beta_1 y < g(y), \end{aligned}$$

which imply the following obvious contradiction:

$$\beta_1 \bar{b} + g(\bar{b}) \ge \beta_1 x + g(x) > \beta_1 y + g(y) > \beta_1 \bar{b} + g(\bar{b}),$$

where the middle inequality follows from Lemma 3 (i) as $x < y < \bar{b}$. Hence, $b = \bar{b}$. $\square$

Observe that $B(x)$ $[S(x)]$ denotes the optimal inventory position to order up to [salvage down to] when the period is started with an inventory position of $x$ units and assuming one is committed to initiate an order [a salvage batch]. $\bar{B}(x)$ $[\underline{S}(x)]$ restricts the choice for the optimal order-up-to [salvage-down-to] levels to those above [below] $\bar{b}$ $[\underline{s}]$. Corollary 4 shows that, when $K_2 = 0$, the (ordering) half line $(-\infty, \bar{b})$ may be partitioned into three intervals, see Table 5 (a): in the left most interval, it is optimal to place a maximum size order and in the right most interval, it is optimal to place an order or to stay put (but salvaging is suboptimal). In the middle interval, it is optimal to place an order when $b > \bar{b} - C_1$; when $b \le \bar{b} - C_1$, it is optimal to either place a maximum size order $(C_1)$ or to stay put. A similar specification may be provided for the (salvage) half line $(\underline{s}, +\infty)$ when $K_1 = 0$, see Table 5 (b). When $K_1 = K_2 = 0$ and $C_1 < \infty, C_2 = \infty$, Table 5 (c) shows that the (ordering) half line $(-\infty, \bar{b})$ displays a modified base-stock policy for the ordering decision.

**5. The multi period problem.** The $(C_1K_1, C_2K_2)$-convexity is preserved under the minimization operations specified by (13)–(15). This enables us to extend the structural results, above, to general multi-period planning horizons.

Proposition 4 **(Preservation of strong $(C_1K_1, C_2K_2)$-convexity).** *Assuming $\beta_1 \ge \beta_2$, if $g(\cdot)$ is strongly $(C_1K_1, C_2K_2)$-convex, then*

$$\begin{aligned} g_1(x) &= \min_{y \in [x, x+C_1']} \{K_1\delta(y-x) + \beta_1(y-x) + g(y)\}, \\ g_2(x) &= \min_{y \in [x-C_2', x]} \{K_2\delta(x-y) + \beta_2(y-x) + g(y)\}, \\ g_0(x) &= \min\{g_1(x), g_2(x)\} \end{aligned}$$

*are also strongly $(C_1K_1, C_2K_2)$-convex for any $C_1' \ge C_1, C_2' \ge C_2$.*

*Proof of Proposition 4.* See the Appendix.

We are now ready for our main result.

Theorem 3 **(Multi Period Optimal Policy Structure).** (a) *Assume $f_0(\cdot) \in SC_{C_0K_0, C_0^vK_0^v}$ and $f_0(x) = O(|x|^p)$ for some integer $p \ge 1$. Then $f_n(x) \in SC_{C_nK_n, C_n^vK_n^v}$ and $f_n(x) = O(|x|^p)$ for $n = N, N-1, \ldots, 1$.*

(b) *In each period $n = N, N-1, \ldots, 1$, the optimal policy structure is as defined in Theorem 2 and Proposition 3 (a) and (b), with $g_1(\cdot) = f_n^1(\cdot), g_2(\cdot) = f_n^2(\cdot), g_0(\cdot) = f_n(\cdot), \beta_1 = c_n, \beta_2 = c_n^v, K_1 = K_n, K_2 = K_n^v, C_1 = C_n, C_2 = C_n^v$ and $g(y) = \mathcal{L}_n(y) + \alpha\mathbb{E}f_{n-1}(y - D_n)$.*

*Proof.* (a) We prove this theorem by induction. By our assumption, the theorem holds for $n = 0$. Suppose the result holds for period $n-1$, i.e., $f_{n-1}(\cdot) \in SC_{C_{n-1}K_{n-1}, C_{n-1}^vK_{n-1}^v}$ and $f_{n-1}(x) = O(|x|^p)$. We first prove that $f_n(x) = O(|x|^p)$. Since $f_{n-1}(x) = O(|x|^p)$, there exists a constant $A > 0$ such that $|f_{n-1}(x)| \le A|x|^p$; so that $|\mathbb{E}f_{n-1}(y - D_n)| \le A\mathbb{E}|y - D_n|^p \le A\mathbb{E}(|y| + D_n)^p = A\sum_{l=0}^p \binom{p}{l}\mathbb{E}D_n^{p-l}|y|^l \le B\max\{|y|^p, 1\}$ for some constant $B > 0$. Since $\mathcal{L}_n(y) = O(|y|^p)$ by Assumption 2 (i), there exists a constant $C > 0$ such that $|\mathcal{L}_n(y)| \le C|y|^p$. Let $y^*(x)$ achieve the minimum in (9), then $|f_n^1(x)| \le K_n + c_n|y^*| + |\mathcal{L}_n(y^*)| + \alpha B|y^*|^p \le K_n + c_n(|x| + C_n) + C(|x| + C_n)^p + \alpha B\max\{1, (|x| + C_n)^p\} = O(|x|^p)$, thus $f_n^1(x) = O(|x|^p)$. By similar argument, $f_n^2(x)$ and hence $f_n(x)$ are also $O(|x|^p)$.

We then approve that $f_n(x) \in SC_{C_nK_n, C_n^vK_n^v}$. Since $f_{n-1}(\cdot) \in SC_{C_{n-1}K_{n-1}, C_{n-1}^vK_{n-1}^v}$, by Lemma 1 (iii), (iv) and Assumption 3,

$$\alpha\mathbb{E}f_{n-1}(y - D_n) \in SC_{C_{n-1}(\alpha K_{n-1}), C_{n-1}^v(\alpha K_{n-1}^v)} \subset SC_{C_nK_n, C_n^vK_n^v}. \tag{30}$$

Since $\mathcal{L}_n(\cdot)$ is convex, by Lemma 1 (iii) we have

$$g(y) = \mathcal{L}_n(y) + \alpha\mathbb{E}f_{n-1}(y - D_n) \in SC_{C_nK_n, C_n^vK_n^v}. \tag{31}$$

It then follows from Proposition 4 that $f_n^1(\cdot), f_n^2(\cdot), f_n(\cdot) \in SC_{C_nK_n, C_n^vK_n^v}$.
(b) Immediate from Theorem 2 and Proposition 3 (a) and (b). $\square$

Pursuant to Proposition 3 in Section 4, we discussed the implications of the everywhere monotonicity property of Proposition 3 (a), and the almost everywhere monotonicity property of Corollary 3 (b). The same observations pertain to the general multi-period setting. Proposition 3 (c) fails to apply to the general multi-period model, since the convexity assumption, there, typically fails.

**6. The infinite horizon model: minimizing total expected discounted costs as well as long-run average costs.** In this section, we prove that all of our structural results carry over to *stationary* infinite horizon models, assuming either the present value of all costs and revenues is to be minimized, or the long-run average cost value.

In extending our results from finite horizon to infinite horizon models, we follow the approach in Huh et al. [21], closely; we therefore adopt much of the notation there.

A deterministic Markov policy $\delta$ is a sequence of decision rules $\{\delta_1, \delta_2, \ldots,\}$ such that in period $t$, $\delta_t$ prescribes a specific feasible action to any potential state of the system. Under a given Markov policy $\delta$ and starting state $s$, let $\phi(S_t, A_t)$ denote the net costs charged in period $t$ when $S_t$ is the state of the system, and $A_t$ the action (order size, salvage batch) chosen, then. Let $J_\alpha(\delta, s) = \mathbb{E}_\delta[\sum_{t=1}^{\infty} \alpha^t \phi(S_t, A_t)]$ denote the expected infinite-horizon present value of costs under policy $\delta$ when starting in state $s$. A policy $\delta^\alpha$ is called *discounted cost optimal* under a given discount factor $\alpha$, if, simultaneously, for every starting state $s \in S$,

$$J_\alpha(\delta^\alpha, s) = \inf_\delta J_\alpha(\delta, s).$$

The *long-run average cost* under a Markov policy $\delta$ and starting state $s \in S$ is defined as

$$\Phi(\delta, s) = \limsup_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \phi(S_t, A_t).$$

A stationary policy $\delta^*$ is *long-run average cost optimal* if, simultaneously for all $s \in S$

$$\Phi(\delta^*, s) = \inf_{s' \in S} \inf_\delta \Phi(\delta, s').$$

We show the existence of a stationary discounted cost optimal policy, for any discount factor $\alpha < 1$, as well as the existence of a stationary long-run average cost optimal policy and the even stronger *preservation property* establishing a strong relationship between the two optimality criteria. We show that our model has the preservation property in that there exists a stationary policy $\delta^*$ satisfying the following properties.

(i) $\delta^*$ is "long-run average cost optimal" stationary in the sense that

$$\Phi(\delta^*, s) = \inf_{s' \in S} \inf_\delta \Phi(\delta, s') \tag{32}$$

for all $s \in S$, and

(ii) $\delta^*$ is "limit discount optimal" in the following sense: for any starting state $s$ and any $\alpha_m \uparrow 1$, there exit a subsequence $\{\alpha_{m_k}\}$ and a sequence $\{s_k\}$ converging to $s$ such that

$$\delta^*(s) = \lim_{k\to\infty} \delta^{\alpha_{m_k}}(s_k). \tag{33}$$

Theorem 4 **(Infinite Horizon Optimality)**. (a) *(Discounted Cost Optimality) For every* $0 < \alpha < 1$*, there exits a sequence of* finite-horizon *optimal policies* $\{\delta^\alpha(\cdot)\}$ *that converges pointwise to a discounted cost optimal policy* $\delta^\alpha(\cdot)$ *as* $T$ *approaches* $\infty$*. The discounted optimal policy* $\delta^\alpha(\cdot)$ *has the structure described in Theorem 3.*

(b) *(Long-Run Average Cost Optimality) There exits a stationary long-run average cost optimal policy $\delta^*$. Moreover, the preservation property described in (32) and (33) holds.*

Theorem 4 corresponds with Theorem 3.1 in Huh et al. [21] where it is shown to hold for any inventory management Markov Decision Process (MDP) that satisfies Assumptions 1 and 2, as well as Condition (SC) there. The authors show that under these three conditions, the MDP satisfies the conditions in Schäl [29]. The framework addressed in Huh et al. [21] is very broad and, in some ways, more general than the broad model addressed in this paper: it allows for demand distributions and capacity values that are Markov modulated, i.e., determined by an underlying world state variable which evolves according to a given Markov chain; it also allows for combined inventory control and pricing problems, where, as discussed in Section 2, in each period a price level is chosen along with an inventory adjustment and where the price level may impact the demand distribution. However, Huh et al. [21] did not allow for *salvage opportunities*, i.e., *bilateral* inventory adjustments.

To ensure that Assumption 1 in Huh et al. [21] is satisfied, we merely require the additional Assumption:

Assumption 5. *In the stationary infinite-horizon model, per definition, the sequence $\{D_n\}$ is assumed to be i.i.d. as a random variable $D$, and $C_n = C$ for all $n$. Moreover, $\mathbb{E}D < C$.*

The restriction $\mathbb{E}D < C$ is, of course, necessary to ensure that the inventory process can be governed in a way that it remains stable and the long-run average costs remain finite. See Federgruen and Zipkin [13] for a more detailed discussion in the special case where no salvage opportunities exist and no fixed inventory adjustment costs are incurred.

Assumption 2 in Huh et al. [21] requires us to limit the type of expected holding and backlogging cost functions that may be used:

Assumption 6. $\mathcal{L}(y) = \mathbb{E}h((y - D^{l+1})^+) + \mathbb{E}p((D^{l+1} - y)^+)$, *where $h(\cdot)$ and $p(\cdot)$ are bounded from below and above by affine functions, i.e., strictly positive constants $\underline{h}, \bar{h}, \underline{p}, \bar{p}$ exist with*

$$\underline{h} \le \frac{h(z') - h(z'')}{z' - z''} \le \bar{h}, \quad \underline{p} \le \frac{p(z') - p(z'')}{z' - z''} \le \bar{p}$$

*for any pair of distinct nonnegative numbers $z'$ and $z''$.*

The holding and backlogging cost structure in Assumption 6 is the commonly used structure, both in the literature and in practice. However, some models allow for $h(\cdot)$ and $p(\cdot)$ that grow *superlinearly*, but are bounded by a polynomial function of a higher degree, as in Assumption 2. This generalization will be discussed in Section 8.

To prove Theorem 4, it therefore suffices to be shown that Condition (SC) in Huh et al. [21] is satisfied. We need some additional notation. Let

$X_t^0 =$ the inventory level at the beginnning of period $t$ after any inventory adjustments initiated $L$ periods earlier
$X_t^l = X_t^0 +$ inventory adjustments to take effect within the next $l$ periods, $l = 1, \ldots, L-1$,
$X_t^L = y_t = X_t^0 +$ all inventory adjustments to take effect within the next $L$ periods.

A function $g : \mathbb{R}^n \to \mathbb{R}$ is a *symmetrically linearly bounded above* (SLBA) function if there exist positive scalers $\zeta$ and $\rho$ such that $g(\mathbf{x}) \le \zeta + \rho\|\mathbf{x}\|$ with $\|\mathbf{x}\|$ the 1-norm of $\mathbf{x}$. A function $g : \mathbb{R}^n \to \mathbb{R}$ is a *symmetrically quadratically bounded above* (SQBA) function if there exist positive scalers $\zeta, \rho$ and $\xi$ such that $g(\mathbf{x}) \le \zeta + \rho\|\mathbf{x}\| + \xi(\|\mathbf{x}\|)^2$.

Condition (SC) Let $\mathbf{X} = (X^0, X^1, \ldots, X^{L-1})$ be an arbitrary vector of inventory levels in any given period. There exist constants $\bar{M} \ge 0$ and $\underline{M} \le 0$ satisfying the following.

(a) Let $\mathbf{X}' = (X'^0, X'^1, \ldots, X'^{L-1})$ denote an inventory vector identical to $\mathbf{X}$ except for *one* component, say $l \in \{0, \ldots, L-1\}$. There exists a a real-valued function $\eta^l(\mathbf{X}, \mathbf{X}')$ with the following properties.

  (i) For any Markov policy $\delta$, there exists a Markov policy $\delta'$ such that for all $N \geq 1$:

$$J^N(\delta', \mathbf{X}') \leq J^N(\delta, \mathbf{X}) + \eta^l(\mathbf{X}, \mathbf{X}'),$$

  where $J^N(\delta, \mathbf{X})$ $[J^N(\delta', \mathbf{X}')]$ denotes the expected total costs over a planning horizon of $N$ periods when starting with the inventory vector $\mathbf{X}$ $[\mathbf{X}]$ and following policy $\delta$ $[\delta']$.

  (ii) If $X^l \geq \bar{M}$ and $X'^l = I^l - 1$, then $\eta^l(\mathbf{X}, \mathbf{X}') \leq 0$.

  (iii) If $X^l < I'^l \leq \underline{M}$, then $\eta^l(\mathbf{X}, \mathbf{X}') \leq 0$.

  (iv) If $I^l = 0$, then $\eta^l(\mathbf{X}, \mathbf{X}')$ is a SQBA function of $\mathbf{X}'^l$.

(b) Let $X^L$ be such that $y = X^L > \max\{\bar{M}, X^{L-1}\}$ and let $\delta$ be any Markov policy. Then, there exists an action $X'^L = y'$ such that $X^{L-1} \leq X'^L \leq \max\{\bar{M}, X^{L-1}\}$ and a policy $\delta'$ such that for any $N \geq 1$,

$$J^N(X'^L, \delta', \mathbf{X}) \leq J^N(X^L, \delta, \mathbf{X}),$$

where $J^N(X^L, \delta, \mathbf{X})$ $[J^N(X'^L, \delta', \mathbf{X})]$ denotes the expected total costs over a planning horizon of $N$ periods when the initial inventory vector is $\mathbf{X}$ and the initial inventory position is set to $X^L$ $[X'^L]$.

The following Lemma shows that Condition (SC) is, indeed, satisfied. Together with Assumption 6 this provides the proof for Theorem 4.

Lemma 4. *Condition (SC) holds under Assumptions 1–6.*

*Proof.* See the Appendix.

As pointed out in Huh et al. [21], the preservation property establishes that, for any discount factor $0 < \alpha < 1$, a discounted cost optimal stationary policy exists and that this policy inherits the structural properties established in Theorem 3. As far as the long-run average cost policy $\delta^*$ is concerned, the preservation property "however, is, in itself, insufficient to show that $\delta^*$ inherits the structural properties" in Theorem 3. However, the proof of the long-run average cost policy $\delta^*$ sharing these properties can be complicated, with similar arguments as those employed in Section 5 of Huh et al. [21] for the inventory models addressed there.

**7. Easily implementable heuristics: numerical examples** The structure of the optimal policy may be too complex for implementation, in several managerial settings. This applies, in particular, to the most general model where there may be intervals on which the order-up-to or salvage-down-to quantity is given by general non-linear functions $\{B_n(\cdot), S_n(\cdot)\}$. One recommendation is to replace these functions by a linear (or possibly piecewise linear) function, far more easily understood and accepted.

More specifically, it is easily verified that in any period $n = N, \ldots, 1$, values $L_n < U_n$ exist such that $y_n^*(x_n) = C_n$ for all $x_n < L_n$, and $y_n^*(x_n) = x - C_n^v$ for all $x > U_n$. Procurement models are typically solved on a rolling-horizon basis and only the policy rule pertaining to the first period, period $N$, needs to be implemented. In case the functions $\{B_N(\cdot)\}$ and $\{S_N(\cdot)\}$ have nonlinear components, replace, on $[L_N, U_N]$, $y_N^*(\cdot)$ by $\tilde{y}_N(\cdot)$ as follows: on any interval $[\underline{x}, \bar{x}]$ in which the optimal policy prescribes an order [salvage quantity], throughout, replace the curve corresponding with $\{y_N^*(\cdot)\}$ by the line connecting $(\underline{x}, y_N^*(\underline{x}))$ and $(\bar{x}, y_N^*(\bar{x}))$. On all other intervals, maintain the policy rule $y_N^*(\cdot)$ without any modifications.

As mentioned, the second and fourth interval in Table 2 and Figure 2 vanish when the functions $A_1(\cdot)$ and $A_2(\cdot)$ have at most one root. In all of our numerical experience, this is always the case, reducing the policy structure to that in Table 3 and Figure 3. Moreover, in all of our numerical experience dealing with unimodal demand distributions, the complexity of a nonlinear $B(\cdot)$ or $S(\cdot)$ function never arises, so that the above suggestions for a simplified policy structure never applied, because the structure of the exact optimal policy $\{y_n^*(\cdot)\}$ is already of the desired, simple (piecewise linear) form. The possibility of nonlinear $B(\cdot)$ functions was exemplified by Gallego and Scheller-Wolf [18] dealing with the special case of our model, where salvaging is not an option. The authors identified one such instance by entertaining an artificial demand distribution with $\{1,6,7\}$ as its three-point support such that $\mathbb{P}[D=1]=\mathbb{P}[D=7]=0.15$ and $\mathbb{P}[D=6]=0.7$.

We illustrate our results with a set of 13 instances obtained by the systematic variation of 7 key parameters in the model. All instances use stationary data and demand distributions. All demand distributions are Normals truncated at zero. The 13 instances share the parameters $h=1, \mathbb{E}D=5, K_v=2, C_v=10, c_v=1.3$. The remaining parameters are specified in Table 6.

Table 6. Parameter setting for numerical studies

| Scenario | $K$ | $C$ | $c$ | $\alpha$ | $l$ | $p$ | $\sigma$ |
|---|---|---|---|---|---|---|---|
| base case | 2 | 10 | 3 | 1 | 2 | 5 | 2 |
| high fixed ordering cost | **10** | 10 | 3 | 1 | 2 | 5 | 2 |
| low fixed ordering cost | **0** | 10 | 3 | 1 | 2 | 5 | 2 |
| large order capacity | 2 | **20** | 3 | 1 | 2 | 5 | 2 |
| small order capacity | 2 | **2** | 3 | 1 | 2 | 5 | 2 |
| high unit ordering cost | 2 | 10 | **20** | 1 | 2 | 5 | 2 |
| low unit ordering cost | 2 | 10 | **1.5** | 1 | 2 | 5 | 2 |
| small $\alpha$ | 2 | 10 | 3 | **0.7** | 2 | 5 | 2 |
| long lead time | 2 | 10 | 3 | 1 | **5** | 5 | 2 |
| zero lead time | 2 | 10 | 3 | 1 | **0** | 5 | 2 |
| high service level | 2 | 10 | 3 | 1 | 2 | **49** | 2 |
| volatile demand | 2 | 10 | 3 | 1 | 2 | 5 | **5** |
| stable demand | 2 | 10 | 3 | 1 | 2 | 5 | **0.5** |

The base case example is illustrated by Figure 4, in which we display the function $y_N^*(\cdot)$ on the left panel and the value functions $f_N(\cdot), f_N^1(\cdot)$ and $f_N^2(\cdot)$ on the right panel. For $x<9$, it optimal to place a maximum size order; for $9 \le x < 16$, it is optimal to order up to the level 19. For $16 \le x \le 32$ it is optimal to stay put, and for $32 < x < 38$, it is optimal to salvage down to the level 28. Finally for $x \ge 38$ it is optimal to initiate a maximum salvage quantity. Parallel figures for the remaining 12 instances are contained in the online appendix. Note that $y_N^*(\cdot)$ is piecewise linear in all instances so that the suggested policy simplifications do not need to be undertaken.

**8. Concluding remarks.** This paper analyzes a general periodic review inventory planning model that allows for the simultaneous treatment of three prevalent complicating factors: (a) bilateral inventory adjustments, (b) capacity limits for such adjustments, and (c) fixed costs for any such adjustments. Prior literature has addressed only subsets of these complications. We characterize the structure of an optimal policy, both for finite and infinite horizon models. We also show that earlier structural results can be obtained as corollaries of our general theory. The analyses are enabled by the identification of a new convexity property that generalizes all existing ones, as in Table 1.

It is of interest to generalize our results further. Specific directions include combined inventory control and pricing models, i.e., allowing the demand distribution to be endogenously controlled, for example by the dynamic selection of a price level. This would generalize the work of Federgruen and Heching [10] and Chen and Simchi-Levi [3, 4] which fail to allow for inventory reductions or capacity limits.

We are also confident that some of the technical restrictions can be relaxed, for example Assumption 6. Assumption 2 ensures that the $\mathcal{L}_n(\cdot)$ functions are polynomially bounded. It should be

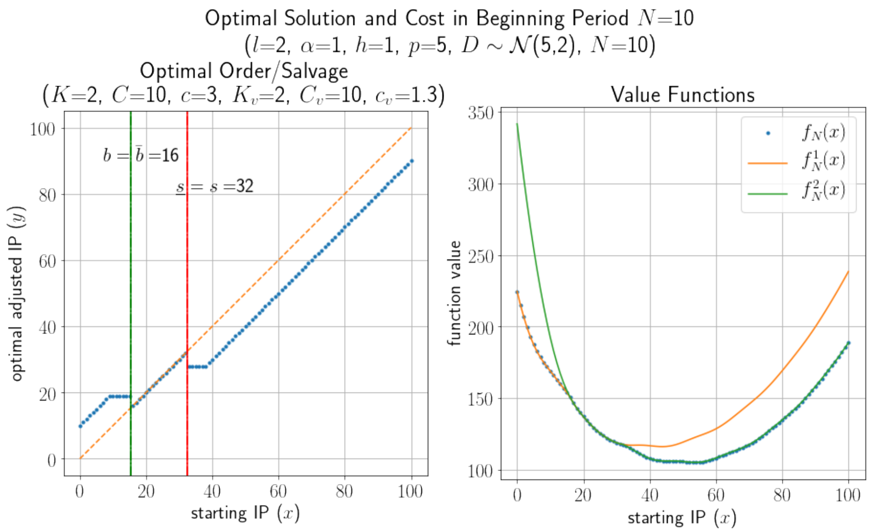


FIGURE 4. A numerical example of optimal policy and value functions

possible to eliminate Assumption 6 by generalizing Condition (SC) in Huh et al. [21] to allow for cost differentials $\eta(\cdot,\cdot)$ that are "*symmetrically polynomially bounded above*".

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## Appendix. Proofs

The proof of Proposition 4 is rather involved and lengthy and requires us to first demonstrate the preservation result in the special case where $\beta_1=\beta_2=0$, as per the following Lemma.

LEMMA 5 **(Preservation Property)**. *If a function $g(\cdot)$ is strongly $(C_1K_1, C_2K_2)$-convex, then*

$$\begin{aligned} f_1(x) &= \min_{y\in[x,x+C_1']}\{K_1\delta(y-x)+g(y)\},\\ f_2(x) &= \min_{y\in[x-C_2',x]}\{K_2\delta(x-y)+g(y)\},\\ f(x) &= \min\{f_1(x), f_2(x)\}. \end{aligned}$$

*are also strongly $(C_1K_1, C_2K_2)$-convex for any $C_1'\ge C_1, C_2'\ge C_2$.*

*Proof of Lemma 5* (I) First, we show $f_1(\cdot)\in SC_{C_1K_1,C_2K_2}$. Let

$$\Delta_1 = K_1 + f_1(x+a) - f_1(x) - \frac{a}{b}\Big(f_1(y) - f_1(y-b) - K_2\Big). \tag{34}$$

It suffices to show that $\Delta_1\ge 0$ for $y\le x, a\in[0,C_1]$ and $b\in(0,C_2]$. To this end, we consider the following four different cases for the pair of values $f_1(x+a)$ and $f_1(y-b)$.

(a) $\underline{f_1(x+a)=g(x+a) \text{ and } f_1(y-b)=g(y-b)}$. In this case, we have

$$\begin{aligned} \Delta_1 &= K_1 + g(x+a) - f_1(x) - \frac{a}{b}\Big(f_1(y) - g(y-b) - K_2\Big)\\ &\ge K_1 + g(x+a) - g(x) - \frac{a}{b}\Big(g(y) - g(y-b) - K_2\Big) \ge 0, \end{aligned}$$

where the first inequality follows from the definition of $f_1(\cdot)$, and the second inequality follows from the strong $(C_1K_1, C_2K_2)$-convexity of $g(\cdot)$.

(b) $f_1(x+a)=g(x+a)$ and $f_1(y-b)=g(y-b+u)+K_1$ with some $u\in[0,C_1']$. In this case, we have

$$\Delta_1 = K_1 + g(x+a) - f_1(x) - \frac{a}{b}\Big(f_1(y) - g(y-b+u) - K_1 - K_2\Big).$$

Based on the value of $f_1(y)$ we consider the following two subcases:

(b.1) $f_1(y)\le g(y-b+u)+K_1+K_2$. Since $a\in[0,C_1]\subset[0,C_1']$ we know $f_1(x)\le g(x+a)+K_1$, hence

$$\Delta_1 \ge K_1 + g(x+a) - f_1(x) \ge K_1 + g(x+a) - (g(x+a)+K_1) \ge 0.$$

(b.2) $f_1(y) > g(y-b+u)+K_1+K_2$. Knowing that $0\le u\le C_1'$, first we show $u<b$. Obviously, this is true if $C_1'<b$. Otherwise consider $b\le C_1'$, suppose on the contrary that $b\le u\le C_1'$, then $y\le y-b+u\le y+C_1'$ and by the definition of $f_1(\cdot)$ we have

$$f_1(y)\le g(y-b+u)+K_1,$$

which together with the subcase assumption $f_1(y)>g(y-b+u)+K_1+K_2$ implies that $K_2<0$, contradicting the fact that $K_2\ge 0$. Hence we have shown that $0\le u<b$, which implies $b-u\in(0,C_2]$. Therefore

$$\begin{aligned}\Delta_1 &\ge K_1 + g(x+a) - f_1(x) - \frac{a}{b-u}\Big(f_1(y) - g(y-b+u) - K_1 - K_2\Big)\\ &\ge K_1 + g(x+a) - g(x) - \frac{a}{b-u}\Big(g(y) - g(y-b+u) - K_2\Big) \ge 0,\end{aligned}$$

where the second inequality follows from $f_1(x)\le g(x)$ and $f_1(y)\le g(y)+K_1$ by the definition of $f_1(\cdot)$, and the last inequality follows from the strong $(C_1K_1,C_2K_2)$-convexity of $g(\cdot)$.

(c) $f_1(x+a)=g(x+a+u)+K_1$ with some $u\in[0,C_1']$ and $f_1(y-b)=g(y-b)$. Since $u\in[0,C_1']$, we have $f_1(x)\le g(x+u)+K_1$ and therefore

$$\begin{aligned}\Delta_1 &= K_1 + g(x+a+u) + K_1 - f_1(x) - \frac{a}{b}\Big(f_1(y) - g(y-b) - K_2\Big)\\ &\ge K_1 + g(x+u+a) - g(x+u) - \frac{a}{b}\Big(g(y) - g(y-b) - K_2\Big) \ge 0,\end{aligned}$$

where the last inequality follows from the strong $(C_1K_1,C_2K_2)$-convexity of $g(\cdot)$.

(d) $f_1(x+a)=g(x+a+u)+K_1$ and $f_1(y-b)=g(y-b+w)+K_1$ with some $u,w\in[0,C_1']$. In this case, $\Delta_1$ defined by (34) can be written as

$$\Delta_1 = K_1 + g(x+a+u) + K_1 - f_1(x) - \frac{a}{b}\Big(f_1(y) - g(y-b+w) - K_1 - K_2\Big). \tag{35}$$

Based on the value of $f_1(y)$ we consider the following two subcases:

(d.1) $f_1(y)\le g(y-b+w)+K_1+K_2$. In this case if $a+u\le C_1'$, we know $f_1(x)\le g(x+a+u)+K_1$ and hence by (35) we have

$$\begin{aligned}\Delta_1 &\ge K_1 + g(x+a+u) + K_1 - f_1(x)\\ &\ge K_1 + g(x+a+u) + K_1 - (g(x+a+u)+K_1) = K_1 \ge 0.\end{aligned}$$

If $a+u>C_1'$, again by (35) we have

$$\begin{aligned}\Delta_1 &\ge K_1 + g(x+a+u) + K_1 - (g(x+C_1')+K_1) - \frac{a}{b}\Big(f_1(y) - g(y-b+w) - K_1 - K_2\Big)\\ &\ge K_1 + g(x+a+u) - g(x+C_1') - \frac{a+u-C_1'}{b}\Big(f_1(y) - g(y-b+w) - K_1 - K_2\Big)\\ &\ge K_1 + g(x+a+u) - g(x+C_1') - \frac{a+u-C_1'}{b}\Big(g(y+w) - g(y+w-b) - K_2\Big) \ge 0,\end{aligned}$$

where the first inequality is from the definition of $f_1(\cdot)$, the second inequality follows from the fact that $0 < a+u-C_1' \le a$ and the case assumption $f_1(y) \le g(y-b+w)+K_1+K_2$, the third inequality is again implied by the definition of $f_1(\cdot)$ such that $f_1(y) \le g(y+w)+K_1$, and the last inequality follows from the strong $(C_1K_1, C_2K_2)$-convexity of $g(\cdot)$ noticing that $0 < a+u-C_1' \le a \le C_1$ and $x+C_1' \ge y+w$.

(d.2) $\underline{f_1(y) > g(y-b+w)+K_1+K_2}$. With the same proof as in (b.2) we can show that $0 \le w < b$, implying $b-w \in (0, C_2]$, hence using $u \in [0, C_1']$ we have,

$$\begin{aligned}
\Delta_1 &\ge K_1 + g(x+a+u) + K_1 - f_1(x) - \frac{a}{b-w}\Big(f_1(y) - g(y-b+w) - K_1 - K_2\Big) \\
&\ge K_1 + g(x+a+u) + K_1 - (g(x+u)+K_1) - \frac{a}{b-w}\Big(g(y) - g(y-b+w) - K_2\Big) \\
&\ge K_1 + g(x+u+a) - g(x+u) - \frac{a}{b-w}\Big(g(y) - g(y-b+w) - K_2\Big) \ge 0,
\end{aligned}$$

where the first inequality follows from $\frac{a}{b} \le \frac{a}{b-w}$, since $0 < b-w \le b$, the second inequality follows from the definition of $f_1(\cdot)$ such that $f_1(x) \le g(x+u)+K_1$ and $f_1(y) \le g(y)+K_1$, and the last inequality follows from the strong $(C_1K_1, C_2K_2)$-convexity of $g(\cdot)$.

Combining (a)-(d) we have shown that $f_1(x) \in SC_{C_1K_1, C_2K_2}$.

(II) Next we prove that $f_2(x) \in SC_{C_1K_1, C_2K_2}$. We first re-denote $f_1$ and $f_2$ more precisely as

$$f^g_{1,CK}(x) = \min_{y\in[x,x+C]}\{K\delta(y-x)+g(y)\}, \quad f^g_{2,CK}(x) = \min_{y\in[x-C,x]}\{K\delta(x-y)+g(y)\},$$

where both $f_1$ and $f_2$ are functions of $g, C, K$, and $x$. In part (I) we have essentially proved that $g(\cdot) \in SC_{C_1K_1,C_2K_2}$ implies $f^g_{1,C_1'K_1}(\cdot) \in SC_{C_1K_1,C_2K_2}$ for $C_1' \ge C_1$, and in this part we want to show that $g(\cdot) \in SC_{C_1K_1,C_2K_2}$ also implies $f^g_{2,C_2'K_2}(\cdot) \in SC_{C_1K_1,C_2K_2}$ for $C_2' \ge C_2$.

Applying Lemma 1 (i), if $g(x) \in SC_{C_1K_1,C_2K_2}$, then $h(x) := g(-x) \in SC_{C_2K_2,C_1K_1}$, and hence by part (I) we know $f^h_{1,C_2'K_2}(x) \in SC_{C_2K_2,C_1K_1}$. We can make further manipulations as

$$f^h_{1,C_2'K_2}(x) = \min_{y\in[x,x+C_2']}\{K_2\delta(y-x)+h(y)\} = \min_{-y\in[-x-C_2',-x]}\{K_2\delta(-x-(-y))+h(y)\}.$$

Transforming variable $y' = -y$, we get

$$f^h_{1,C_2'K_2}(x) = \min_{y'\in[-x-C_2',-x]}\{K_2\delta(-x-y')+h(-y')\} = \min_{y'\in[-x-C_2',-x]}\{K_2\delta(-x-y')+g(y')\} = f^g_{2,C_2'K_2}(-x),$$

implying that $f^g_{2,C_2'K_2}(-x) \in SC_{C_2K_2,C_1K_1}$. Applying Lemma 1 (i) again we see $f^g_{2,C_2'K_2}(x) \in SC_{C_1K_1,C_2K_2}$, confirming the strong $(C_1K_1, C_2K_2)$-convexity of $f_2(x)$.

(III) Finally we show $f(x) \in SC_{C_1K_1,C_2K_2}$. Let

$$\Delta = K_1 + f(x+a) - f(x) - \frac{a}{b}\Big(f(y) - f(y-b) - K_2\Big). \tag{36}$$

We consider the following four different cases for the pair of values $f(x+a)$ and $f(y-b)$ to show that $\Delta \ge 0$ for all $y \le x, a \in [0, C_1]$ and $b \in (0, C_2]$. Notice that the definition of $f(x)$ implies $f(x) \le f_1(x)$ and $f(x) \le f_2(x)$.

(a) $\underline{f(x+a) = f_1(x+a) \text{ and } f(y-b) = f_1(y-b)}$. We have

$$\begin{aligned}
\Delta &= K_1 + f_1(x+a) - f(x) - \frac{a}{b}\Big(f(y) - f_1(y-b) - K_2\Big) \\
&\ge K_1 + f_1(x+a) - f_1(x) - \frac{a}{b}\Big(f_1(y) - f_1(y-b) - K_2\Big) \ge 0,
\end{aligned}$$

where the last inequality follows directly from the strong $(C_1K_1, C_2K_2)$-convexity of $f_1(x)$.

(b) $\underline{f(x+a)=f_1(x+a) \text{ and } f(y-b)=f_2(y-b)}$. We can rewrite (36) as

$$\Delta = K_1 + f_1(x+a) - f(x) - \frac{a}{b}\Big(f(y) - f_2(y-b) - K_2\Big). \tag{37}$$

Per definition, $f_2(y-b)=g(y-b)$ or $f_2(y-b)=g(y-b-u)+K_2$ with some $u\in(0,C_2']$. We consider these two subcases:

(b.1) $\underline{f_2(y-b)=g(y-b)}$. Since $f_1(y-b)\le g(y-b)$ by the definition, we have $f(y-b)=f_1(y-b)$ and this subcase becomes case (a) and $\Delta\ge 0$ follows.

(b.2) $\underline{f_2(y-b)=g(y-b-u)+K_2 \text{ for some } u\in(0,C_2']}$. Then (37) becomes

$$\begin{aligned}\Delta &= K_1 + f_1(x+a) - f(x) - \frac{a}{b}\Big(f(y) - g(y-b-u) - K_2 - K_2\Big)\\ &\ge K_1 + f_1(x+a) - f(x) - \frac{a}{b}\Big(g(y-u) - g(y-u-b) - K_2\Big),\end{aligned}$$

where the inequality follows from $f(y)\le f_2(y)\le g(y-u)+K_2$, by the definitions of $f$ and $f_2$. Now if $f_1(x+a)=g(x+a)$, we have

$$\begin{aligned}\Delta &\ge K_1 + g(x+a) - f(x) - \frac{a}{b}\Big(g(y-u) - g(y-u-b) - K_2\Big)\\ &\ge K_1 + g(x+a) - g(x) - \frac{a}{b}\Big(g(y-u) - g(y-u-b) - K_2\Big) \ge 0,\end{aligned}$$

where the second inequality follows from $f(x)\le g(x)$ and the last inequality follows from the strong $(C_1K_1, C_2K_2)$-convexity of $g(x)$. Otherwise if $f_1(x+a)=g(x+a+w)+K_1$ for some $w\in(0,C_1']$, we have

$$\begin{aligned}\Delta &\ge K_1 + g(x+a+w) + K_1 - f(x) - \frac{a}{b}\Big(g(y-u) - g(y-u-b) - K_2\Big)\\ &\ge K_1 + g(x+a+w) + K_1 - f_1(x) - \frac{a}{b}\Big(g(y-u) - g(y-u-b) - K_2\Big)\\ &\ge K_1 + g(x+a+w) + K_1 - (g(x+w)+K_1) - \frac{a}{b}\Big(g(y-u) - g(y-u-b) - K_2\Big)\\ &\ge K_1 + g(x+w+a) - g(x+w) - \frac{a}{b}\Big(g(y-u) - g(y-u-b) - K_2\Big) \ge 0,\end{aligned}$$

where the second and third inequalities follow from $f(x)\le f_1(x)$ and $f_1(x)\le g(x+w)+K_1$ with $w\in(0,C_1']$, respectively, and the last inequality follows from the strong $(C_1K_1, C_2K_2)$-convexity of $g(x)$.

(c) $\underline{f(x+a)=f_2(x+a) \text{ and } f(y-b)=f_2(y-b)}$. The proof is analogous to case (a).

(d) $\underline{f(x+a)=f_2(x+a) \text{ and } f(y-b)=f_1(y-b)}$. We can rewrite (36) as

$$\Delta = K_1 + f_2(x+a) - f(x) - \frac{a}{b}\Big(f(y) - f_1(y-b) - K_2\Big). \tag{38}$$

By its definition, $f_1(y-b)=g(y-b)$ or $f_1(y-b)=g(y-b+w)+K_1$ with some $w\in(0,C_1']$. We consider these two subcases:

(d.1) $\underline{f_1(y-b)=g(y-b)}$. Since $f_2(y-b)\le g(y-b)$, per definition, we have $f(y-b)=f_2(y-b)$ and this subcase becomes case (c) and $\Delta\ge 0$ follows.

(d.2) $\underline{f_1(y-b)=g(y-b+w)+K_1 \text{ with some } w\in(0,C_1']}$. Then (38) becomes

$$\Delta = K_1 + f_2(x+a) - f(x) - \frac{a}{b}\Big(f(y) - g(y-b+w) - K_1 - K_2\Big). \tag{39}$$

We first show that $f(y)\le g(y-b+w)+K_1+K_2$ always holds in this subcase. To this end, note that $b\in(0,C_2]$ and $w\in(0,C_1']$, if $w\ge b$, then $w-b\in[0,C_1')$ and hence

$$f(y)\le f_1(y). = \inf_{u\in[y,y+C_1']}\{K_1\delta(u-y)+g(u)\} \le g(y+(w-b))+K_1 \le g(y-b+w)+K_1+K_2,$$

otherwise if $w < b$, then $b - w \in (0, C_2)$ and hence

$$f(y) \le f_2(y) = \inf_{u \in [y - C_2', y]} \{K_2 \delta(y-u) + g(u)\} \le g(y - (b-w)) + K_2 \le g(y - b + w) + K_1 + K_2.$$

We have thus proved that given $b \in (0, C_2]$ and $w \in (0, C_1']$,

$$f(y) \le g(y - b + w) + K_1 + K_2. \tag{40}$$

Similarly we can prove that given $a \in [0, C_1]$ and $u \in (0, C_2']$,

$$f(x) \le g(x + a - u) + K_1 + K_2. \tag{41}$$

Next we consider the possible values of $f_2(x+a)$. If $f_2(x+a) = g(x+a)$, then by (39) and (40),

$$\Delta = K_1 + g(x+a) - f(x) - \frac{a}{b}\Big(f(y) - g(y-b+w) - K_1 - K_2\Big) \ge K_1 + g(x+a) - f_1(x) \ge 0,$$

where the last inequality follows from the definition of $f_1(x)$. On the other hand if $f_2(x + a) = g(x + a - u) + K_2$ with some $u \in (0, C_2']$, then by (39), (40) and (41)

$$\Delta = K_1 + g(x+a-u) + K_2 - f(x) - \frac{a}{b}\Big(f(y) - g(y-b+w) - K_1 - K_2\Big) \ge 0.$$

Consequently, combining (a)-(d) we have proved that $f(x) \in SC_{C_1K_1, C_2K_2}$. The proof of this proposition is also completed. □

*Proof of Proposition 4.* We first prove that $g_0(\cdot)$ is continuous. Note that

$$g_0(x) = \min\left\{\min_{x \le y \le x + C_1}\{K_1 + \beta_1(y-x) + g(y)\}, \min_{x - C_2 \le y \le x}\{K_2 + \beta_2(y-x) + g(y)\}, g(x)\right\}.$$

Thus, continuity of $g_0(\cdot)$ follows by showing that $\min_{x \le y \le x+C_1}\{K_1 + \beta_1(y-x) + g(y)\}$ is continuous in $x$ and $\min_{x - C_2 \le y \le x}\{K_2 + \beta_2(y-x) + g(y)\}$ is continuous in $x$. Both continuity results follow from Berge's Maximum Theorem result, since the minimands are continuous functions and the feasible sets are continuous correspondences of $x$, see e.g. Theorem 9.14 in Sundaram [33].

It is not hard to see $g(\cdot) \in SC_{C_1K_1, C_2K_2} \Rightarrow g_1(\cdot), g_2(\cdot) \in SC_{C_1K_1, C_2K_2}$: Lemma 1 (iii) shows that $g(y) \in SC_{C_1K_1, C_2K_2} \Rightarrow g(y) + \beta_1 y \in SC_{C_1K_1, C_2K_2}$ for any $\beta_1$; then by Lemma 5, $g_1(x) + \beta_1 x = \min_{y \in [x, x + C_1']}\{K_1\delta(y-x) + \beta_1 y + g(y)\} \in SC_{C_1K_1, C_2K_2}$, and hence $g_1(x) \in SC_{C_1K_1, C_2K_2}$ using Lemma 1 (iii) again. Similarly we can show $g_2(x) \in SC_{C_1K_1, C_2K_2}$.

Note that if $\beta_1 \ne \beta_2$ we cannot directly apply Lemma 5 to claim strong $(C_1K_1, C_2K_2)$-convexity of $g_0(\cdot)$.

For any $x \ge y$, $u \in [0, C_1]$ and $t \in (0, C_2]$, we need to show that

$$0 \le \Delta = K_1 + g_0(x+u) - g_0(x) - \frac{u}{t}\Big(g_0(y) - g_0(y-t) - K_2\Big). \tag{42}$$

As is in the proof of Lemma 5 (III), we consider the following four cases for the pair of values $g_0(x+u)$ and $g_0(y-t)$:

(a) $g_0(x+u) = g_1(x+u)$, $g_0(y-t) = g_1(y-t)$;
(b) $g_0(x+u) = g_1(x+u)$, $g_0(y-t) = g_2(y-t)$;
(c) $g_0(x+u) = g_2(x+u)$, $g_0(y-t) = g_2(y-t)$;
(d) $g_0(x+u) = g_2(x+u)$, $g_0(y-t) = g_1(y-t)$.

First note that both $g_1(\cdot), g_2(\cdot) \in SC_{C_1K_1,C_2K_2}$, therefore case (a) and (c) can be easily proved in the same way as (a) and (c) of Lemma 5 (III), respectively.

For case (b), given $g_0(x+u) = g_1(x+u) \le g_2(x+u)$, if $g_1(x+u) = g_2(x+u)$, this becomes case (c). Otherwise, $g_1(x+u) < g_2(x+u)$; then, by Corollary 1 (ii), we have $g_1(y-t) \le g_2(y-t)$ since $y-t \le x+u$. Thus, $g_0(y-t) = g_1(y-t) = g_2(y-t)$, so that case (a) applies.

Thus, only case (d) remains to be proven. Notice that if $g_2(x+u) = g(x+u)$, then the relations

$$g_2(x+u) = g_0(x+u) \le g_1(x+u) \le g(x+u) = g_2(x+u)$$

implies $g_0(x+u) = g_1(x+u) = g_2(x+u)$, so that case (a) applies. Similarly if $g_1(y-t) = g(y-t)$, we can deduct that $g_0(y-t) = g_1(y-t) = g_2(y-t)$ and case (c) applies. Therefore we only need to consider the distinct situations where

$$g_1(y-t) = \widetilde{g}_1(y-t) = K_1 + g(B(y-t)) + \beta_1(B(y-t) - y + t) < g(y-t), \tag{43}$$
$$g_2(x+u) = \widetilde{g}_2(x+u) = K_2 + g(S(x+u)) + \beta_2(S(x+u) - x - u) < g(x+u), \tag{44}$$

where $B(\cdot)$ and $S(\cdot)$ are defined by (27) and (28) with $C_1$ and $C_2$ replaced by $C_1'$ and $C_2'$, respectively. For notational simplicity, we henceforth denote $B(y-t)$ and $S(x+u)$ by $\widetilde{B}$ and $\widetilde{S}$, respectively. Noticing that $\widetilde{B} \in (y-t, y-t+C_1']$, $\widetilde{S} \in [x+u-C_2', x+u)$ and $u \in [0, C_1]$, $t \in (0, C_2]$, it is easy to see that

$$g_0(y) \le \begin{cases} g_1(y) \le K_1 + \beta_1(\widetilde{B} - y) + g(\widetilde{B}), & \text{if } y \le \widetilde{B}, \text{ (since } \widetilde{B} \le y - t + C_1' < y + C_1') \\ g_2(y) \le K_2 + \beta_2(\widetilde{B} - y) + g(\widetilde{B}), & \text{if } y > \widetilde{B}; \text{ (since } \widetilde{B} > y - t \ge y - C_2') \end{cases} \tag{45}$$

$$g_0(x) \le \begin{cases} g_1(x) \le K_1 + \beta_1(\widetilde{S} - x) + g(\widetilde{S}), & \text{if } x < \widetilde{S}, \text{ (since } \widetilde{S} < x + u \le x + C_1') \\ g_2(x) \le K_2 + \beta_2(\widetilde{S} - x) + g(\widetilde{S}), & \text{if } x \ge \widetilde{S}. \text{ (since } \widetilde{S} \ge x + u - C_2' \ge x - C_2') \end{cases} \tag{46}$$

We therefore distinguish among the 4 cases determined by the relative position of $y$ vis-à-vis $\widetilde{B}$ and $x$ vis-à-vis $\widetilde{S}$.

(a) $\underline{y \le \widetilde{B}, x < \widetilde{S}}$. In this case, by (45) and (43) we have

$$g_0(y) \le K_1 + \beta_1(\widetilde{B} - y) + g(\widetilde{B}) = g_1(y-t) - \beta_1 t. \tag{47}$$

Taking (44), (46) and (47) into (42), we get

$$\begin{aligned} \Delta &= K_1 + g_2(x+u) - g_0(x) - \frac{u}{t}\Big(g_0(y) - g_1(y-t) - K_2\Big) \\ &\ge K_1 + K_2 + g(\widetilde{S}) + \beta_2(\widetilde{S} - x - u) - K_1 - \beta_1(\widetilde{S} - x) - g(\widetilde{S}) \\ &\quad - \frac{u}{t}\Big(g_1(y-t) - \beta_1 t - g_1(y-t) - K_2\Big) \\ &\ge K_2 - \beta_1 u + \frac{u}{t}\Big(\beta_1 t + K_2\Big) = \Big(1 + \frac{u}{t}\Big)K_2 \ge 0, \end{aligned}$$

where the second inequality follows from

$$\beta_2(\widetilde{S} - x - u) - \beta_1(\widetilde{S} - x) = (\beta_2 - \beta_1)(\widetilde{S} - x - u) - \beta_1 u \ge -\beta_1 u$$

by $\widetilde{S} < x + u$ and the assumption $\beta_1 \ge \beta_2$.

(b) $\underline{y \le \widetilde{B}, x \ge \widetilde{S}}$. In this case, taking (44), (46) and (47) into (42) we have

$$\begin{aligned} \Delta &= K_1 + g_2(x+u) - g_0(x) - \frac{u}{t}\Big(g_0(y) - g_1(y-t) - K_2\Big) \\ &\ge K_1 + K_2 + g(\widetilde{S}) + \beta_2(\widetilde{S} - x - u) - K_2 - \beta_2(\widetilde{S} - x) - g(\widetilde{S}) \\ &\quad - \frac{u}{t}\Big(g_1(y-t) - \beta_1 t - g_1(y-t) - K_2\Big) \\ &= K_1 + \frac{u}{t}K_2 + (\beta_1 - \beta_2)u \ge 0. \end{aligned}$$

(c) $\underline{y > \widetilde{B}, x \geq \widetilde{S}}$. In this case, taking (43)–(46) into (42) we have

$$\begin{aligned}
\Delta &= K_1 + g_2(x+u) - g_0(x) - \frac{u}{t}\Big(g_0(y) - g_1(y-t) - K_2\Big) \\
&\geq K_1 + K_2 + g(\widetilde{S}) + \beta_2(\widetilde{S} - x - u) - K_2 - \beta_2(\widetilde{S} - x) - g(\widetilde{S}) \\
&\quad - \frac{u}{t}\Big(K_2 + \beta_2(\widetilde{B} - y) + g(\widetilde{B}) - K_1 - g(\widetilde{B}) - \beta_1(\widetilde{B} - y + t) - K_2\Big) \\
&\geq K_1 - \beta_2 u + \frac{u}{t}\Big(\beta_2 t + K_1\Big) = \Big(1 + \frac{u}{t}\Big)K_1 \geq 0,
\end{aligned}$$

where the third inequality follows from

$$\beta_2(\widetilde{B} - y) - \beta_1(\widetilde{B} - y + t) = (\beta_2 - \beta_1)(\widetilde{B} - y + t) - \beta_2 t \leq -\beta_2 t$$

by $\widetilde{B} > y - t$ and the assumption $\beta_1 \geq \beta_2$.

(d) $\underline{y > \widetilde{B}, x < \widetilde{S}}$. Note that in this case we must have $u \in (0, C_1]$, since if $u = 0$ there cannot be $x < \widetilde{S}$ by $\widetilde{S} \in [x + u - C_2', x + u)$. It then follows that

$$g_0(x) \leq g_2(x) \leq K_2 + \beta_2(\widetilde{S} - u - x) + g(\widetilde{S} - u), \text{ (since } x - C_2' \leq \widetilde{S} - u < x) \tag{48}$$
$$g_0(y) \leq g_1(y) \leq K_1 + \beta_1(\widetilde{B} + t - y) + g(\widetilde{B} + t). \text{ (since } y < \widetilde{B} + t \leq y + C_1') \tag{49}$$

Depending on the order of $\widetilde{B} + t$ and $\widetilde{S} - u$, we consider the following two situations:

(d.1) $\underline{\widetilde{B} + t \leq \widetilde{S} - u}$. The following ranking applies:

$$y - t < \widetilde{B} < y < \widetilde{B} + t \leq \widetilde{S} - u < x < \widetilde{S} < x + u,$$

where the first inequality follows from $\widetilde{B} = B(y - t) > y - t$ and the last inequality from $\widetilde{S} = S(x + u) < x + u$.

Taking (43), (44) and (48), (49) into (42) we get

$$\begin{aligned}
\Delta &= K_1 + g_2(x+u) - g_0(x) - \frac{u}{t}\Big(g_0(y) - g_1(y-t) - K_2\Big) \\
&\geq K_1 + K_2 + g(\widetilde{S}) + \beta_2(\widetilde{S} - x - u) - K_2 - \beta_2(\widetilde{S} - u - x) - g(\widetilde{S} - u) \\
&\quad - \frac{u}{t}\Big(K_1 + \beta_1(\widetilde{B} + t - y) + g(\widetilde{B} + t) - K_1 - g(\widetilde{B}) - \beta_1(\widetilde{B} - y + t) - K_2\Big) \\
&= K_1 + g(\widetilde{S}) - g(\widetilde{S} - u) - \frac{u}{t}\Big(g(\widetilde{B} + t) - g(\widetilde{B}) - K_2\Big) \geq 0,
\end{aligned}$$

where the last inequality follows from the definition of strong $(C_1K_1, C_2K_2)$-convexity of $g(\cdot)$ with $x = \widetilde{S} - u$, $y = \widetilde{B} + t$ and $u \in [0, C_1]$ and $t \in (0, C_2]$.

(d.2) $\underline{\widetilde{B} + t > \widetilde{S} - u}$. Now the following rankings apply:

$$y - t < \widetilde{B} < y < \widetilde{B} + t, \quad \widetilde{S} - u < \widetilde{B} + t, \quad \widetilde{S} - u < x < \widetilde{S} < x + u. \tag{50}$$

Note that (48) and (49) still hold.

Using (43) and (44), (42) can be written as

$$\begin{aligned}
\Delta &= K_1 + g_2(x+u) - g_0(x) - \frac{u}{t}\Big(g_0(y) - g_1(y-t) - K_2\Big) \\
&= K_1 + K_2 + g(\widetilde{S}) + \beta_2(\widetilde{S} - x - u) - g_0(x) \\
&\quad - \frac{u}{t}\Big(g_0(y) - K_1 - g(\widetilde{B}) - \beta_1(\widetilde{B} - y + t) - K_2\Big).
\end{aligned}$$

Having mentioned that $u > 0$ in this case, $\Delta \geq 0$ is equivalent to

$$\frac{g_0(y) + \beta_1 y - g(\widetilde{B}) - \beta_1 \widetilde{B} - K_1 - K_2}{t} - \frac{g(\widetilde{S}) + \beta_2 \widetilde{S} - g_0(x) - \beta_2 x + K_1 + K_2}{u} \leq \beta_1 - \beta_2. \tag{51}$$

Conditioning on the signs of the two numerators on the left hand side of (51), three subcases need to be considered:

(i) $\underline{g_0(y)+\beta_1 y-g(\widetilde{B})-\beta_1\widetilde{B}-K_1-K_2\leq 0}$. Then using (46),

$$\begin{aligned}
&\frac{g_0(y)+\beta_1 y-g(\widetilde{B})-\beta_1\widetilde{B}-K_1-K_2}{t}-\frac{g(\widetilde{S})+\beta_2\widetilde{S}-g_0(x)-\beta_2 x+K_1+K_2}{u}\\
&\quad\leq -\frac{g(\widetilde{S})+\beta_2\widetilde{S}-g_0(x)-\beta_2 x+K_1+K_2}{u}\\
&\quad\leq -\frac{g(\widetilde{S})+\beta_2\widetilde{S}-[K_1+g(\widetilde{S})+\beta_1(\widetilde{S}-x)]-\beta_2 x+K_1+K_2}{u}\\
&\quad= \frac{(\beta_1-\beta_2)(\widetilde{S}-x-u)-K_2}{u}+(\beta_1-\beta_2)\leq\beta_1-\beta_2,
\end{aligned}$$

where the last inequality follows from $\widetilde{S}<x+u$ and the assumption $\beta_1\geq\beta_2$.

(ii) $\underline{g(\widetilde{S})+\beta_2\widetilde{S}-g_0(x)-\beta_2 x+K_1+K_2\geq 0}$. Then using (45),

$$\begin{aligned}
&\frac{g_0(y)+\beta_1 y-g(\widetilde{B})-\beta_1\widetilde{B}-K_1-K_2}{t}-\frac{g(\widetilde{S})+\beta_2\widetilde{S}-g_0(x)-\beta_2 x+K_1+K_2}{u}\\
&\quad\leq \frac{g_0(y)+\beta_1 y-g(\widetilde{B})-\beta_1\widetilde{B}-K_1-K_2}{t}\\
&\quad\leq \frac{[K_2+g(\widetilde{B})+\beta_2(\widetilde{B}-y)]+\beta_1 y-g(\widetilde{B})-\beta_1\widetilde{B}-K_1-K_2}{t}\\
&\quad= \frac{(\beta_1-\beta_2)(y-t-\widetilde{B})-K_1}{t}+(\beta_1-\beta_2)\leq\beta_1-\beta_2,
\end{aligned}$$

where the last inequality follows from $\widetilde{B}>y-t$ and the assumption $\beta_1\geq\beta_2$.

(iii) $\underline{g_0(y)+\beta_1 y-g(\widetilde{B})-\beta_1\widetilde{B}-K_1-K_2>0 \text{ and } g(\widetilde{S})+\beta_2\widetilde{S}-g_0(x)-\beta_2 x+K_1+K_2<0.}$ Before proving (51) we first show that, in view of (50), there exist $t_0$ and $u_0$ with $0<y-\widetilde{B}\leq t_0\leq t$ and $0<\widetilde{S}-x\leq u_0\leq u$ such that

$$g(y-t_0)=K_1+\beta_1(\widetilde{B}-y+t_0)+g(\widetilde{B}), \tag{52}$$
$$g(x+u_0)=K_2+\beta_2(\widetilde{S}-x-u_0)+g(\widetilde{S}). \tag{53}$$

For $v\in[y-\widetilde{B},t]$, let

$$h(v)=g(y-v)-[K_1+\beta_1(\widetilde{B}-y+v)+g(\widetilde{B})],$$

which is a continuous function. Then, since

$$\begin{aligned}
h(y-\widetilde{B})&=g(\widetilde{B})-[K_1+\beta_1\cdot 0+g(\widetilde{B})]=-K_1\leq 0,\\
h(t)&=g(y-t)-[K_1+\beta_1(\widetilde{B}-y+t)+g(\widetilde{B})]=g(y-t)-\widetilde{g}_1(y-t)\geq 0,
\end{aligned}$$

by the mean value theorem there exists $t_0\in[y-\widetilde{B},t]$ such that $h(t_0)=0$, i.e., $g(y-t_0)=K_1+\beta_1(\widetilde{B}-y+t_0)+g(\widetilde{B})$. Similarly we can show the existence of a value $u_0$ satisfying $0<\widetilde{S}-x\leq u_0\leq u$ and (53).

Next we proceed to prove (51). We have

$$\begin{aligned}
&\frac{g_0(y)+\beta_1 y-g(\widetilde{B})-\beta_1\widetilde{B}-K_1-K_2}{t}-\frac{g(\widetilde{S})+\beta_2\widetilde{S}-g_0(x)-\beta_2 x+K_1+K_2}{u}\\
&\quad\leq \frac{g_0(y)+\beta_1 y-g(\widetilde{B})-\beta_1\widetilde{B}-K_1-K_2}{t_0}-\frac{g(\widetilde{S})+\beta_2\widetilde{S}-g_0(x)-\beta_2 x+K_1+K_2}{u_0}\\
&\quad= \frac{g_0(y)-[K_1+\beta_1(\widetilde{B}-y+t_0)+g(\widetilde{B})]-K_2+\beta_1 t_0}{t_0}\\
&\qquad -\frac{K_1+[K_2+\beta_2(\widetilde{S}-x-u_0)+g(\widetilde{S})]-g_0(x)+\beta_2 u_0}{u_0}\\
&\quad= \frac{g_0(y)-g(y-t_0)-K_2}{t_0}-\frac{K_1+g(x+u_0)-g_0(x)}{u_0}+\beta_1-\beta_2\leq\beta_1-\beta_2,
\end{aligned}$$

where the first inequality follows from the conditions specifying case (iii); the last inequality follows from the strong $(C_1K_1, C_2K_2)$-convexity of $g(\cdot)$ and the fact that $g_0(\cdot) \le g(\cdot)$, specifically,

$$\begin{aligned} K_1 + g(x+u_0) - g_0(x) - \frac{u_0}{t_0}\Big(g_0(y) - g(y-t_0) - K_2\Big) \\ \ge K_1 + g(x+u_0) - g(x) - \frac{u_0}{t_0}\Big(g(y) - g(y-t_0) - K_2\Big) \ge 0, \end{aligned}$$

which implies (noticing $t_0$ and $u_0$ are both positive)

$$\frac{g_0(y) - g(y-t_0) - K_2}{t_0} - \frac{K_1 + g(x+u_0) - g_0(x)}{u_0} \le 0.$$

□

*Proof of Lemma 4.* The proof is analogous to that in Sections C.1 and C.2 in the electronic companion of Huh et al. [21]. The proof is given in three parts:

Condition SC (a): $X_1'^l > X_1^l$: We compare two inventory level vectors $\mathbf{X}_1$ and $\mathbf{X}_1'$. Assume that these two vectors are identical except $X_1'^l > X_1^l$ form some $l \in \{0, 1, \ldots, L-1\}$. Let $\Delta = X_1'^l - X_1^l$. We will then prove parts (i), (iii) and (iv) of Condition (SC) (a). (Part (ii) is not applicable since we consider the $X_1'^l > X_1^l$ case in in this part.)

For any Markov policy $\delta$, let $\delta'$ be the following policy. If $l < L-1$, then $X_1'^{L-1} = X_1^{L-1}$. Let the $\delta'$ policy order or salvage the same quantity as the $\delta$ policy in every period. We call this the "mimic" policy of $\delta$. If $l = L-1$, then the $\delta'$ policy initiates the same salvage batches as the $\delta$ policy, but does not order anything for the first $\Delta$ units ordered by the $\delta$ policy, and then matches $\delta$'s orders unit-by-unit. Recall $u_t = X_t^L - X_t^{L-1}$ is the number of units ordered by $\delta$ in period $t \ge 1$. Then, the order quantity $u_t'$ of the $\delta'$ policy is given by, for $t \ge 1$,

$$u_t' = \begin{cases} [\sum_{t'=1}^{t} u_{t'} - \Delta]^+ & \text{if } \sum_{t'=1}^{t-1} u_{t'} < \Delta \\ u_t & \text{otherwise} \end{cases}.$$

Note that $u_t'$ is a feasible inventory adjustment quantity. In every period $t \ge 1$: let $t^*$ denote the first period in which $u_{t^*}' > 0$. Then $u_t' = 0$ or $u_t' = u_t$ for all $t < t^*$ and $u_t' = u_t$ for all $t > t^*$, both feasible. Moreover,

$$0 \le u_{t^*}' = \sum_{t'=1}^{t^*} u_t' - \Delta \le \sum_{t'=1}^{t^*} u_t' - \sum_{t'=1}^{t^*-1} u_t' = u_{t^*},$$

hence feasible as well, where the inequality follows from $\sum_{t'=1}^{t^*-1} u_t' \le \Delta$ by the definition of $t^*$. We say $\delta'$ is a "wait-and-mimic" policy of $\delta$.

The remainder of the proof is analogous to that in Huh et al. [21].

Condition SC (a): $X_1'^l < X_1^l$: The proof is is analogous to that in Huh et al. [21] with the following adaptation: In case $l < L-1$, let $\delta'$, the mimic policy of $\delta$, order and salvage the same quantity as the $\delta$ policy in every period. For the case where $l = L-1$, the policy $\delta'$ mimics $\delta$ for the first $T_0$ periods, i.e., it orders and salvages the same quantity as policy $\delta$; thereafter, the specification of $\delta'$ is identical to that in Huh et al. [21].

Condition SC (b): Analogous to the proof in Huh et al. [21]. □

## Appendix. Online companion for numerical examples

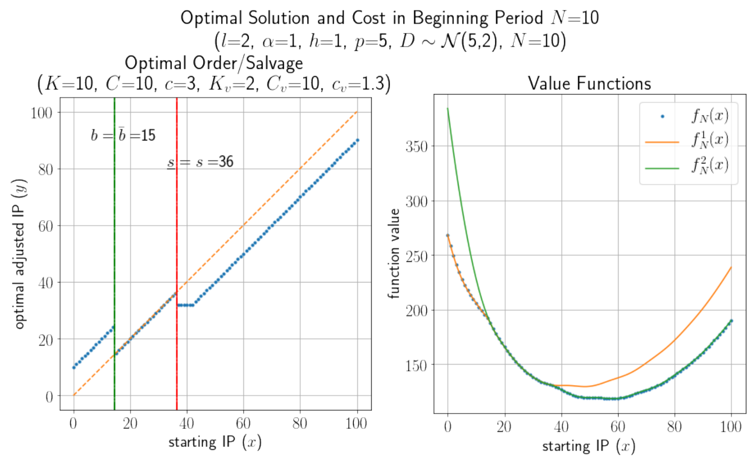


FIGURE 5. Numerical example: high fixed ordering cost (big $K$)

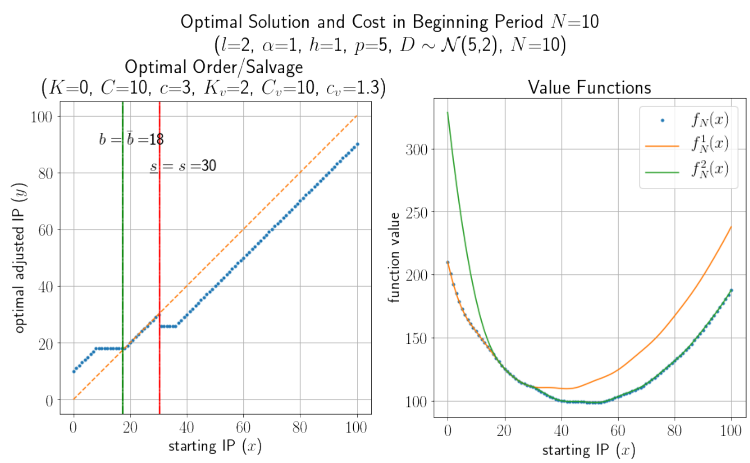


FIGURE 6. Numerical example: low fixed ordering cost (small $K$)

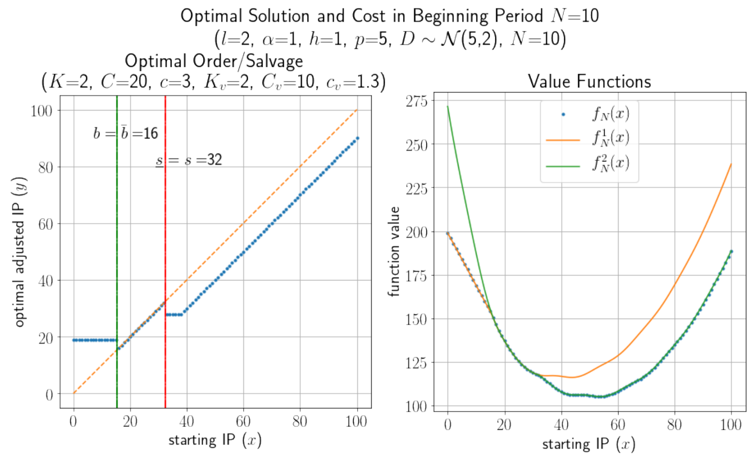


FIGURE 7. Numerical example: large ordering capacity (big $C$)

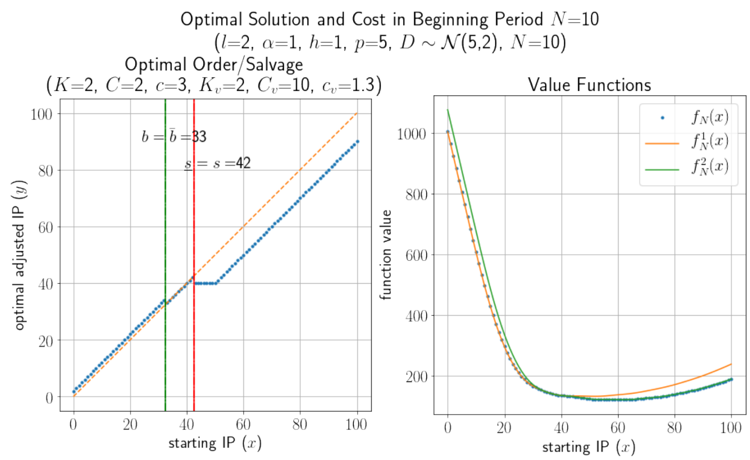


FIGURE 8. Numerical example: small ordering capacity (small $C$)

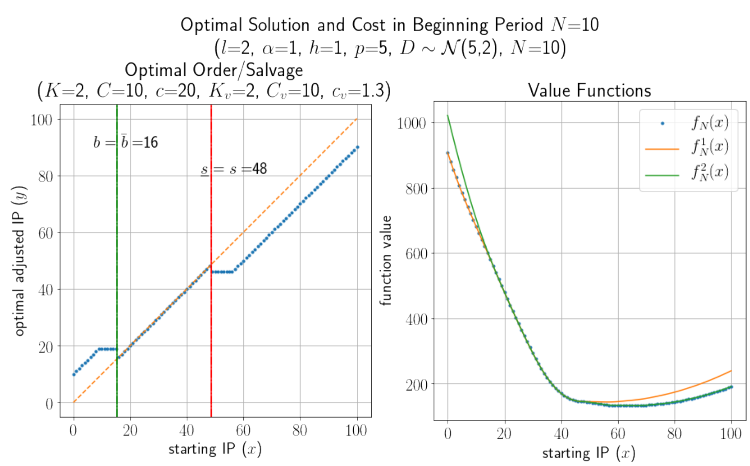


FIGURE 9. Numerical example: high unit cost (big $c$)

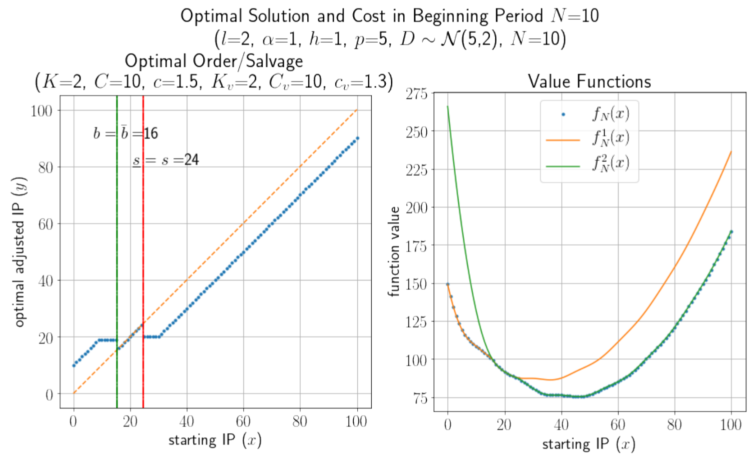


Figure 10. Numerical example: low unit cost (small $c$)

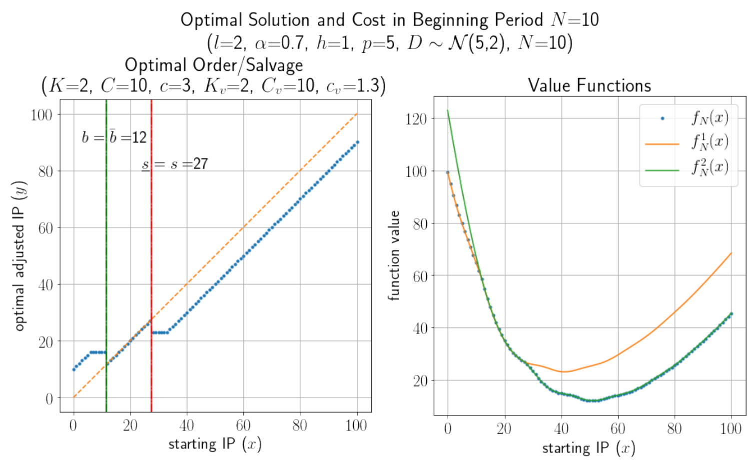


Figure 11. Numerical example: small $\alpha$

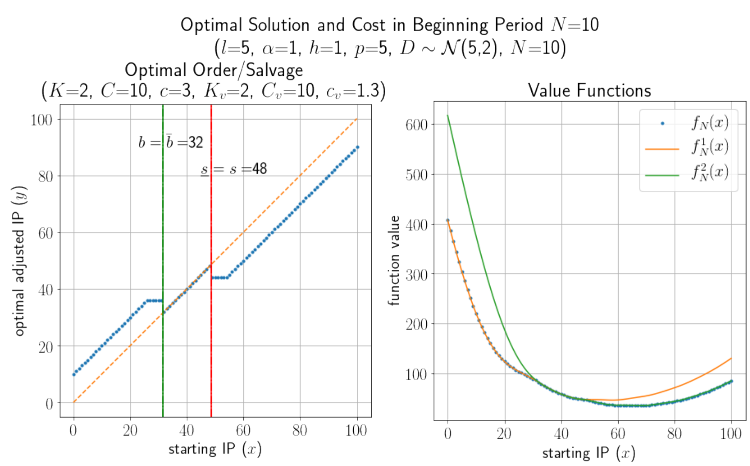


Figure 12. Numerical example: long lead time (big $l$)

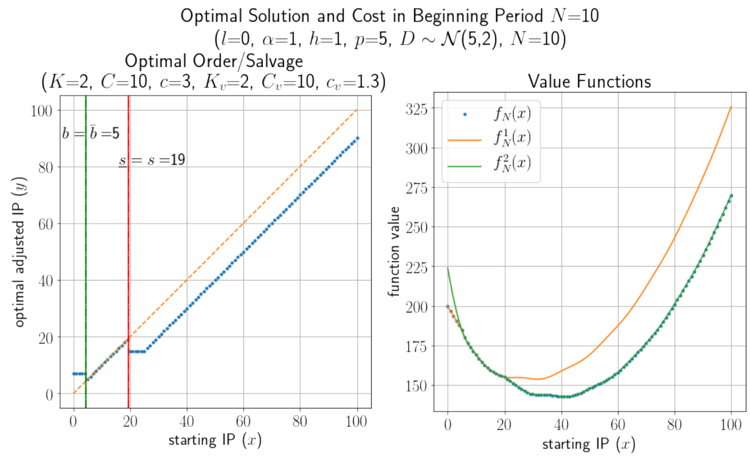


FIGURE 13. Numerical example: zero lead time ($l = 0$)

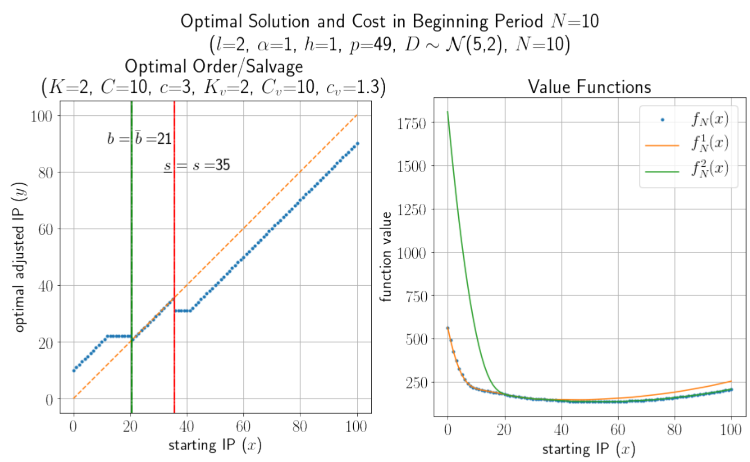


FIGURE 14. Numerical example: high service level (big $p$)

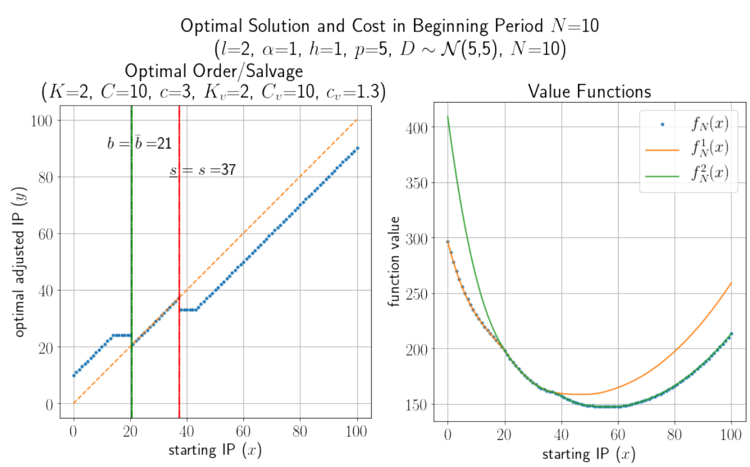


FIGURE 15. Numerical example: volatile demand (big $\sigma$)

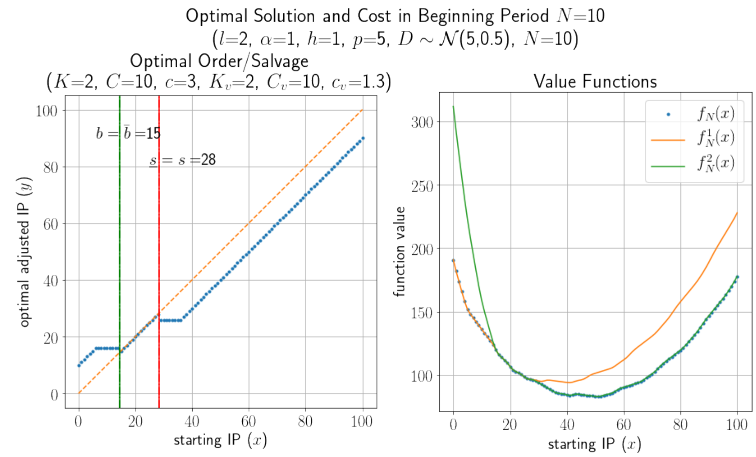


FIGURE 16. Numerical example: stable demand (small $\sigma$)