Allocation Policies in Blood Transfusion

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Red Blood Cell (RBC) transfusion is an integral part of many medical treatments and surgeries. Recently, a growing body of research suggests a correlation between the age of transfused blood and adverse clinical outcomes for its recipients. Therefore, there is a need for effective and practical inventory management policies which could reduce the age of transfused RBC units without compromising their availability. In this paper, we focus on policies which determine how RBC units are allocated to patients. We study a stylized queueing model of a hospital blood bank and consider a family of threshold based allocation policies. We characterize the sojourn time distribution of RBC units in inventory and calculate the distribution of the age of transfused units, as well as the proportion of lost demand and outdates. Our analysis allows us to capture the age-availability trade-off achieved under the threshold policy. In a numerical study, we explore the factors affecting the performance of the threshold policy and investigate when it is more likely to be effective in comparison to reducing the shelf-life. We find that when reducing the shelf-life has a high impact on availability, the threshold policy is more effective. This is observed, in particular, for blood types with lower demand, for smaller hospitals, and when a lower age of transfused RBCs is required. Furthermore, using our analytical results together with models mapping the age of RBCs to the corresponding probability of adverse medical outcomes, we examine the trade-off between availability and the health outcomes achieved under the threshold policy. In particular, we demonstrate the importance of taking into account the entire distribution of the age of transfused RBCs when evaluating the health outcomes of allocation policies.

Key words: Allocation policies; age-availability trade-off; perishable inventory systems; queues.

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1. Introduction

1.1. Background and Motivation

Red Blood Cell (RBC) transfusion is an integral part of many medical treatments and surgeries; in 2011, over 13 million units of voluntarily donated RBCs were transfused to patients across the United States (NBCUS 2011). While advances in storage solutions led to an increase of RBCs shelf-life from 35 to 42 days in the late 1970s, an extensive body of recent cohort studies (e.g., Koch et al. (2008), Eikelboom et al. (2010), Offner et al. (2002), Frank et al. (2013)) suggests a range of moderate to strong correlation between receiving "older" blood and increased risk of adverse medical outcomes such as infection, morbidity, and mortality. The results are however still inconclusive (Lelubre et al. 2009). It remains for undergoing randomized controlled clinical trials such as ABLE (Lacroix 2011) to further clarify the relationship between the age of blood and patient outcomes; a question which has been referred to as "the most critical issue facing transfusion medicine" (Ness 2011).

It will take several years before the results of these clinical trials are available. Nevertheless, if it turns out that there is indeed an association between receiving older blood and increased risk of complications, the resulting increase in demand for fresher blood could have a significant impact on the supply and availability of RBC units in blood banks and hospitals (Dzik et al. 2013, Sayers and Centilli 2012). It is therefore of crucial importance to understand this impact and be ready to adapt to new regulations on the shelf-life of RBCs. While the medical community works on understanding the "storage lesion" and the efficacy of RBC transfusions, the operations research community can contribute by exploring new inventory policies. A recent commentary (Sayers and Centilli 2012) in the journal of Transfusion, specifically emphasizes the need for inventory and supply chain management policies that could contribute to reducing the age of transfused RBC units without compromising the adequacy of supply. Nevertheless, although there are a few papers investigating the impact of shortening the shelf-life of blood (e.g., Fontaine et al. 2011, Blake et al. 2012), limited attention has been given to alternative inventory policies.

In this paper, we focus on the RBC allocation policy, i.e., the policy according to which the RBC units are allocated to patients in the hospital. The impact of the allocation policy can be intuitively understood by comparing two extreme policies namely the First-in, First-Out (FIFO) and Last-In, First Out (LIFO) policies. Under the FIFO policy, which is the most commonly used in practice (Dzik et al. 2013), always the oldest available unit is allocated to a new demand for transfusion. While this favors availability by minimizing the number of outdated units, it clearly results in a higher age of transfused units comparing to the LIFO policy which always allocates the freshest unit available. This trade-off between the age and availability motivates the important question of whether by adopting a different allocation policy, the age of transfused units can be reduced to a desired level without significantly affecting the availability of units in the hospital.

We study a family of threshold based allocation policies previously introduced in the literature. The policy was originally proposed by Haijema et al. (2007) in the context of platelet inventory management, and was used to explore the "age–availability" trade–off for RBCs in a simulation study by Atkinson et al. (2012). Under the threshold policy, the oldest RBC unit that is younger than a given threshold is transfused, and if no such unit is available, the freshest available unit is allocated. To wit, the threshold policy includes both the FIFO and LIFO policies and aims to capture the age-availability tradeoff by using a controlled combination of both policies.

The main goal of our analysis is to develop a better understanding of the performance and benefits of the threshold policy as an important and practical family of allocation policies. While previously proposed in the literature, the performance of the threshold policy under different system parameters and its benefits over reducing the shelf-life under the FIFO policy are not well-understood. Furthermore, previous studies only consider the average age of transfused RBC units. However, since the relation between the age of blood and the probability of adverse outcomes could potentially be nonlinear (see Pereira 2013 for potential relationship functions), understanding the health benefit of allocation policies requires information on the resulting *distribution* of the age of transfused units. The objectives of this study are thus to (*i*) evaluate the performance of the threshold policy in terms of the resulting availability and distribution of the age of transfused RBC units; (*ii*) asses its performance for different system parameters and in comparison to reducing the shelf-life under the FIFO policy; and (*iii*) provide further insights on the age-availability trade-off and design of allocation policies.

1.2. Analytical Framework, Contributions and Summary of Results

We study a stylized queueing model of a hospital blood bank by adopting the framework of Kaspi and Perry (1983). In particular, we assume the following. Donors (supply) and demand for RBC units arrive to the hospital according to independent Poisson processes. The supply side resembles that of a hospital which locally collects blood from volunteers. Even though many hospitals use ordering policies and receive deliveries of RBCs from regional suppliers, they still face high variability in the size of deliveries. In general one expects a high variability on both the demand and supply side, which supports the Poisson assumption. For instance, in Atkinson et al. (2012) it is observed that the output of simulation is close to the empirical data when the coefficient of variation (CV) of both donations and demand is set to 1.32. The assumption is also validated using data from a regional blood center in Canada in Kopach et al. (2008). This assumption also implies that donors donate one unit of RBC, and demand is also satisfied by one unit. The latter does not always hold in reality as some patients may require multiple units of RBCs. While in practice compatible blood types are sometimes substituted for one another, for tractability we assume that the demand and supply for the inventory of each blood type is independent of other types. In addition, in accordance with current practice, we assume that RBC units can be transfused up to a fixed number of days (currently 42) after donation and are discarded after this deadline. Finally, we assume that unsatisfied demand is lost and the required RBC units are imported from an external source, as is common in practice.

These assumptions allow us to model the blood bank as a queuing system operating under the discipline corresponding to the allocation policy. Particularly, the oldest-blood-first and freshest-blood-first policies translate to FIFO and LIFO queueing disciplines, respectively. Although stylized, the model captures the main operational features of the system namely uncertainty in demand and supply and the fixed shelf-life of RBC units. It also provides us with a framework to implement allocation policies and study their outcomes.

This paper makes contributions to both queueing theory and blood inventory management literatures. On the theoretical end, we present the first exact analysis of the stochastic perishable inventory system operating under the threshold policy. Our analysis are based on a two–stage model of the system. In this model, fresh units arrive at Stage 1, but move to Stage 2 if their sojourn time exceeds the threshold value. Demand is satisfied using the oldest unit from Stage 1. If Stage 1 is empty, then the freshest unit in Stage 2 is allocated. As a result, Stage 1 is a FIFO system with supply and demand processes identical to those of the original system, while Stage 2 is a LIFO system whose supply and demand processes are respectively driven by the outdate and loss processes of Stage 1. The main complexity in the analysis of the threshold policy is that the supply and demand processes of Stage 2 depend on the state of Stage 1. Consequently, direct analysis of Stage 2 inventory without considering the dynamics of Stage 1 is not possible. Further, the sojourn time of units in Stage 2 is bounded, as the shelf-life is finite. Our main contribution is to develop a novel method to characterize the sojourn time of RBC units in Stage 2 inventory for a given threshold value.

We first assume that units have infinite shelf-life in Stage 2 and characterize the sojourn time of units under this assumption. The main idea behind our approach is to track a tagged unit in Stage 2 and decompose its sojourn time into idle and busy periods of Stage 1 which operates under the FIFO policy. Thus, the first part of our analysis is to characterize these idle and busy periods by extending the results for the FIFO system. We then construct a sequence of modified Stage 2 systems in which the sojourn time of units is limited to a finite number of Stage 1 idle and busy periods. We show that as the number of idle and busy periods considered asymptotically tends to infinity, the sojourn time of units in these modified systems converges (in distribution) to that of the units in the actual system. Finally, we employ the asymptotic analysis of the modified systems to obtain the performance measures of interest for the original system, i.e., the distribution of the age of transfused RBC units and the proportion of outdates and lost demand.

On the application end, we provide several insights on the age–availability trade–off for RBCs. Specifically, we provide insights on when the threshold policy is more likely to be effective and perform better than reducing the shelf-life under the FIFO policy. Our numerical study demonstrates that the effectiveness of the threshold policy increases as the size of the hospital and the demand size for individual blood types decreases. In contrast, for larger hospitals and blood types with higher demand, or when a small reduction in the age of transfused RBCs is required, the performance of the threshold policy is observed to be similar to that of simply reducing the shelf-life. Furthermore, we incorporate the *distribution* of the age of transfused RBCs and demonstrate that the underlying relationship between the age of transfused RBC units and the risk of adverse outcomes must be considered when choosing an allocation policy. In particular, it is observed that when comparing two different allocation policies under certain relationship functions, the one resulting in a lower expected age of transfused units could lead to a higher probability of adverse outcomes. Finally, our exact results on the trade–off between availability and the *expected* age of transfused RBCs substantiate those based on the simulation study of Atkinson et al. (2012).

1.3. Related Literature

Analyzing perishable inventory systems using queueing theory goes back to Graves (1978, 1982) and Nahmias (1980). The model considered in this paper was first introduced by Graves (1978) and further studied in Kaspi and Perry (1983). They analyze the model under the FIFO discipline and obtain performance measures of the system including the distribution of the time between outdates, and that of the number of units in the system. Many papers have since considered variations and extensions of the so called stochastic perishable inventory problem. Examples include the problem with renewal supply (Kaspi and Perry 1984), quality inspections (Perry 1999), and batch demand and donations (Goh et al. 1993). A direct application of this model to RBC inventory management is also studied in Kopach et al. (2008). A common assumption among almost all these papers is that the system operates under the FIFO policy. There are a few number of notable exceptions. In Keilson and Seidmann (1990), the authors study the LIFO policy. Parlar et al. (2011) also consider the LIFO policy and compare its performance with that of FIFO in a profit maximization setting. Goh et al. (1993) provide approximate analysis for a two-stage system where units arrive at Stage 1, but move to Stage 2 after passing an age threshold. Each stage has an independent demand stream. The issuing policy within each stage is FIFO, however, demand for Stage 2 can be satisfied using Stage 1 inventory in case of a shortage. In contrast, our system has a single demand stream that is satisfied from Stage 1 as long as there are units available. If Stage 1 is empty, demand is fulfilled from Stage 2 when possible. Also, the issuing policy within Stage 1 is FIFO, but demand is satisfied from Stage 2 in a LIFO manner. While in this case Stage 1 behaves similar to a regular FIFO system, the analysis of Stage 2 is much more involved and requires a different approach.

Our work relates to an extensive body of literature on perishable inventory systems; see Nahmias (2011), Baron (2011) and Karaesmen et al. (2011) for comprehensive reviews and Piersikalla (2004) for applications in blood supply chain management. A closely related area is the study of allocation or issuing policies in periodic inventory systems with perishable products. Pierskalla and Roach (1972) consider the problem for a finite-horizon periodic system in which inventory is categorized based on age levels. At the beginning of a period, the demand and supply for each category is observed and a decision for assigning inventory to demand is made. Under the assumption that the demand for any age level can be satisfied by the same or fresher categories, the optimality of the FIFO policy in minimizing the total lost demand and outdates is established. It is also shown that the FIFO policy minimizes the cumulative stockouts for all periods when excess demand is backlogged. The results are proved for deterministic demand and supply, and their extension to the stochastic case is discussed. Haijema et al. (2007) study inventory management of blood platelets (shelf-life of 5-7 days) with two demand streams; "young" and "any". For the first stream younger platelets are highly preferred while the second stream can be satisfied using platelets of any age. They implement the same threshold policy considered in this paper and use simulation and Markov Decision Processes to develop near optimal order-up-to replenishment policies. In a profit maximization setting Deniz et al. (2010) consider a product with a lifetime of two periods and study joint replenishment-issuing decisions.

Finally, there are a few simulation studies in the medical literature directly related to our application. Atkinson et al. (2012) develop a simulation model based on data from Stanford University Medical Center and investigate the trade–off between the expected age of transfused units and availability of units in the hospital. They observe that by adopting a policy with threshold equal to 14 days, the hospital can reduce the average age of transfused blood by 10 to 20 days while only increasing the fraction of lost demand by 0.5%. Blake et al. (2012) use a simulation model to asses the impact of shorter shelf-life on the blood supply chain of Héma-Québec. They find that a shelf-life of 28 or 21 is feasible, but will have a higher impact on smaller hospitals. A shelf life of 14 or less is observed to result in excessive increases to outdates and emergency orders for both the suppliers and hospitals of all size. They also find that high outdates and lost demand are mainly related to rarer blood types. In another simulation study, Fontaine et al. (2011) investigate the impact of reducing the shelf-life for Stanford University Medical Center. They observe a high impact on both availability and outdates in case of reducing the shelf-life to 21 or less.

1.4. Organization of the Paper

We start with a formal description of our stylized queueing system and its performance measures pertaining to the age and availability of transfused RBCs. In Section 3, we study two important members of the family of threshold policies; namely the FIFO and LIFO policies. Before presenting our main results in Section 5, we give some theoretical results for the FIFO policy in Section 4 required for our analysis of the threshold policy. Section 6 contains the results of our numerical study. Finally, in Section 7 we present our concluding remarks and a discussion of future research. Proof of Theorems 1 and 2 are given in the Appendix. The proof of all other results can be found in the Online Appendix.

2. Model Description

In this section we describe the model and formally introduce the performance measures of interest. Donated RBC units (hereinafter referred to as units) arrive at the hospital blood bank according to a Poisson process with intensity λ. It is assumed that units are "fresh", i.e., have age zero upon arrival to the inventory. However, the analysis can be easily extended to the case where all units are delayed for some constant time, e.g. for tests, before arriving at the inventory. Demand for RBC transfusions occur according to an independent Poisson process with intensity μ. Shelf-life of units in the inventory is equal to a constant γ. Demand occurring while the inventory level is at zero is lost. Otherwise, if there are available units in the inventory then one is allocated to the demand according to the policy in effect.

It is convenient to view the system as an $M/M/1+D^s$ queue in which donated units are the arrivals and each service completion corresponds to a patient receiving a transfusion. The queue has arrival rate λ, service rate μ, and its service discipline coincides with the allocation policy of the system. In addition, units have a deterministic patience until the end of service (the $+D^s$ in the Kendall notation) equal to γ, i.e., abandon the system if their sojourn time exceeds γ.

Consider the system operating under some allocation policy denoted by π, and let S^π denote the random variable associated with the steady-state sojourn time of units in inventory (or equivalently in the corresponding queueing system). Observe that S^π has a probability mass at γ and a continuous density on $(0,\gamma)$. For outdated units we have $S^\pi = \gamma$, while for each transfused unit $S^\pi \in (0,\gamma)$ is equal to the age of the unit at the time of transfusion.

We study the outdate probability q^π, and the cumulative distribution function (cdf) of the random variable associated with the age of transfused units A^π. Observe that both performance measures can be expressed in terms of the random variable S^π. The outdate probability is

$$q^\pi = P(S^\pi = \gamma), \tag{1}$$

and the cdf of A^π is given by

$$A^\pi(x) = \begin{cases} P(S^\pi \le x | S^\pi < \gamma) = P(S^\pi \le x)/(1-q^\pi), & 0 \le x < \gamma, \\ 1, & x \ge \gamma. \end{cases} \tag{2}$$

Another performance measure is the probability of a demand being lost ℓ^π which directly corresponds to the availability of units in the hospital. As discussed in Parlar et al. (2011), a simple conservation law relates this measure to the outdate probability. Under any policy π, as long as the steady-state limits exist, we have

$$\lambda(1-q^\pi) = \mu(1-\ell^\pi). \tag{3}$$

It is clear from (3) that loss and outdate probability can both be used as measures of availability when comparing policies. In particular, minimizing the outdate probability is equivalent to minimizing the loss probability and vice versa.

We close this section by mentioning that, throughout the paper, whenever we refer to the Laplace transform (LT) of a random variable we mean the LT of its probability density function (pdf) or equivalently the Laplace-Stieltjes transform (LST) of its cdf.

3. Review of the FIFO and LIFO Policies

The above model has been previously studied in the literature under FIFO and LIFO policies. Some of the methods used in the analysis, however, will be used in our evaluation of the threshold policy. Furthermore, as previously mentioned, FIFO and LIFO belong to the family of threshold policies. Therefore, we next review the relevant results and use them to obtain the performance measures of interest under these policies.

3.1. The LIFO Policy

The analysis of the LIFO policy is due to Keilson and Seidmann (1990) and Parlar et al. (2011), who study the distribution of the sojourn time of units in the inventory, S^L (where "L" is the shorthand notation for LIFO). The analysis is based on the following observations, valid under the LIFO policy. First, the sojourn time of a new unit arriving in inventory only depends on future demand and unit arrivals. Second, for any unit with sojourn time less than γ, we know that all units which arrived during the sojourn time of the unit also had sojourn times less than γ, and hence were not outdated. It follows that $S^L = \min(\tilde{S}^L, \gamma)$, where $\tilde{S}^L$ is the random variable associated with the sojourn time of units in inventory if they had infinite shelf-life. As discussed in Parlar et al. (2011) since both the demand and supply are Poisson, $\tilde{S}^L$ has the same distribution as the length of a busy period in an $M/M/1$ queue with arrival rate λ and service rate μ.

Given the above, the outdate probability is $P(S^L = \gamma) = P(\tilde{S}^L \geq \gamma)$. Parlar et al. (2011) also give an expression for the LT of the truncated busy period S^L, from which the LT of A^L can be obtained. The formula is however not computationally useful. In the following proposition we present all performance measures of interest directly in terms of the cdf of the busy period $B(x) \equiv P(\tilde{S}^L \leq x)$, which can be computed efficiently by numerically inverting its LT,

$$\int_0^\infty e^{-\theta x} B(x)dx = \frac{2\mu}{\theta\left(\lambda + \mu + \theta + \sqrt{(\lambda + \mu + \theta)^2 - 4\lambda\mu}\right)},$$

(see, e.g., Gross et al. 2008, page 102). Note that when $\lambda > \mu$, $P(\tilde{S}^L < \infty) = (\mu/\lambda) < 1$, that is $B(x)$ is improper. However, since S^L is bounded, for any positive λ and μ, $P(S^L < \infty) = 1$.

Proposition 1. *Under the LIFO policy the outdate probability is*

$$q^L = 1 - B(\gamma). \tag{4}$$

Furthermore, the cdf of the age of transfused units is

$$A^L(x) = \begin{cases} B(x)/(1 - q^L), & 0 \leq x < \gamma, \\ 1, & x \geq \gamma, \end{cases} \tag{5}$$

and the expected age of transfused units is given by

$$E[A^L] = \gamma - \frac{\int_0^\gamma B(y)dy}{1 - q^L}. \tag{6}$$

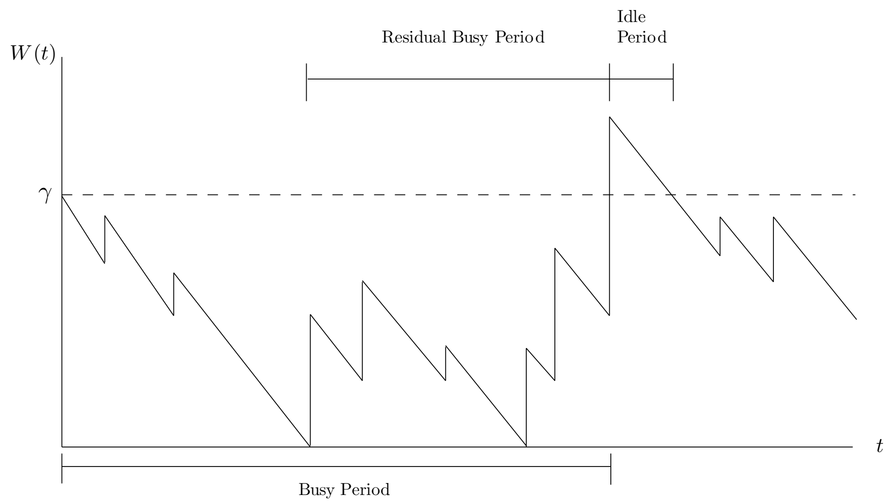

Figure 1 A typical sample path of the process W

3.2. The FIFO Policy

The stochastic perishable inventory problem under the FIFO policy was first studied in Graves (1978) and Kaspi and Perry (1983) and recently revisited in Parlar et al. (2011). The analysis is based on the so called Virtual Outdating Process (VOP) $W \equiv \{W(t); t \geq 0\}$, which returns, as a function of time, the remaining time until the next outdate if no new demands were to arrive. VOP is useful since it is a (strong) Markov process and contains important information about the state of the system. In particular, the age of the oldest unit in inventory at any time $t \geq 0$ is $\gamma - W(t)$, the event $\{W(s) > \gamma\}$ implies no inventory at time s, and $\{W(s^-) = 0\}$ indicates that a unit was outdated at time s.

W has càdlàg (right-continuous with left limits) sample paths with upward jumps. Jumps occur either when the oldest unit is allocated to a demand (when $0 < W < \gamma$, at Poisson rate μ) or is outdated (when W hits zero). In both cases, jump sizes are equal to the inter-arrival time of units to the inventory and hence exponentially distributed with rate λ. Figure 1 illustrates a sample path of process W.

Kaspi and Perry (1983) show that W has the same distribution as the virtual waiting time process of an $M/M/1+D$ queue with arrival rate μ and service rate λ, in which the idle periods are deleted and customers do not join the system if they have to wait more than γ before starting service (the $+D$ in the Kendall notation). Using this observation they obtain the steady-state distribution of W. Let f denote the steady-state pdf of W. We have (Parlar et al. 2011)

$$f(x) = \begin{cases} f(0)e^{-(\lambda-\mu)x}, & 0 < x < \gamma, \\ f(0)e^{\mu\gamma - \lambda x}, & x \geq \gamma, \end{cases} \tag{7}$$

where

$$f(0)=\begin{cases}\frac{\lambda(\lambda-\mu)}{\lambda-\mu e^{-(\lambda-\mu)\gamma}}, & \lambda\neq\mu,\\ \frac{\lambda}{1+\lambda\gamma}, & \lambda=\mu.\end{cases}$$

Having the steady-state distribution of W, we can obtain the required performance measures. Let "F" be the shorthand notation for FIFO policy. The following proposition summarizes the results.

Proposition 2. *Under the FIFO policy the outdate probability is*

$$q^F=\begin{cases}\frac{\lambda-\mu}{\lambda-\mu e^{-(\lambda-\mu)\gamma}}, & \lambda\neq\mu,\\ \frac{1}{1+\lambda\gamma}, & \lambda=\mu.\end{cases} \tag{8}$$

Furthermore, the cdf of the age of transfused units is

$$A^F(x)=\begin{cases}\frac{e^{-(\lambda-\mu)(\gamma-x)}-e^{-(\lambda-\mu)\gamma}}{1-e^{-(\lambda-\mu)\gamma}}, & \lambda\neq\mu, 0\leq x<\gamma,\\ x/\gamma, & \lambda=\mu, 0\leq x<\gamma,\\ 1, & x\geq\gamma,\end{cases} \tag{9}$$

and the expected age of transfused units is given by

$$E[A^F]=\begin{cases}\frac{(\lambda-\mu)\gamma-(1-e^{-(\lambda-\mu)\gamma})}{(\lambda-\mu)(1-e^{-(\lambda-\mu)\gamma})}, & \lambda\neq\mu,\\ \frac{\gamma}{2}, & \lambda=\mu.\end{cases} \tag{10}$$

The tractable form of the performance measures under the FIFO policy allows us to obtain simple yet useful structural results presented in the following corollary without a proof.

Corollary 1. *Under the FIFO policy:*

1. *For a fixed supply-to-demand ratio* (λ/μ), *both the outdate probability* q^F *and loss probability* ℓ^F *are strictly decreasing in* μ.
2. *For a fixed supply-to-demand ratio* (λ/μ), *as* μ *increases, the expected age of transfused units* $E[A^F]$ *increases for* $\lambda>\mu$ *and decreases for* $\lambda<\mu$.
3. *The distribution of the age of transfused units* $A^F(\cdot)$ *is strictly convex for* $\lambda>\mu$, *strictly concave for* $\lambda<\mu$ *and uniform for* $\lambda=\mu$.

We shall demonstrate and discuss some of these results in the numerical study of Section 6.

4. Additional Results for the FIFO Policy

Before turning to the analysis of the threshold policy we need some additional results for a FIFO system in which units have a shelf-life T. Consider the queueing counterpart of the system. The queue alternates between busy and idle periods. During a busy period, there are units available in inventory while during an idle period, the inventory is empty. We also define the *residual busy period* as the time interval between the epoch when a unit is outdated until the start of the next idle period. In what follows, we present the distribution of the number of units that are outdated during a (residual) busy period as well as the LT of the length of the (residual) busy period given the number of outdates. We also obtain the distribution of lost demand during an idle period as well as the LT of the length of the idle period given the number of lost demand.

We prove the results using sample path analysis of the process W. First, following Kaspi and Perry (1983), we define the stopping time

$$\tau = \inf\{t \geq 0; W(t) = 0 \text{ or } W(t) > T\},$$

on $\{W(0) > 0\}$. We shall use the notation P_x and E_x to denote conditional probability and expectation given the initial value $W(0) = x > 0$. Let us define

$$\hat{g}_x(\theta) \equiv E_x[e^{-\theta\tau}\mathbf{1}_{\{W(\tau)=0\}}], \tag{11}$$

$$\hat{h}_x(\theta) \equiv E_x[e^{-\theta\tau}\mathbf{1}_{\{W(\tau)>T\}}]. \tag{12}$$

Then for $x > T$, $\hat{g}_x(\theta) = 0$ and $\hat{h}_x(\theta) = 1$. For each $x \in (0, T]$ from Kaspi and Perry (1983) (see also Cohen 1982, page 548), we have

$$\hat{g}_x(\theta) = \frac{e^{-\alpha_1(\theta)x}(\lambda + \alpha_1(\theta))e^{-\alpha_2(\theta)T} - e^{-\alpha_2(\theta)x}(\lambda + \alpha_2(\theta))e^{-\alpha_1(\theta)T}}{(\lambda + \alpha_1(\theta))e^{-\alpha_2(\theta)T} - (\lambda + \alpha_2(\theta))e^{-\alpha_1(\theta)T}}, \tag{13}$$

$$\hat{h}_x(\theta) = \frac{(e^{-\alpha_2(\theta)x} - e^{-\alpha_1(\theta)x})(\lambda + \alpha_1(\theta))(\lambda + \alpha_2(\theta))}{\lambda((\lambda + \alpha_1(\theta))e^{-\alpha_2(\theta)T} - (\lambda + \alpha_1(\theta))e^{-\alpha_1(\theta)T})}, \tag{14}$$

where

$$\alpha_1(\theta) = (\theta + \mu - \lambda + ((\lambda + \mu + \theta)^2 - 4\lambda\mu)^{1/2})/2,$$

$$\alpha_2(\theta) = (\theta + \mu - \lambda - ((\lambda + \mu + \theta)^2 - 4\lambda\mu)^{1/2})/2.$$

From this one can obtain, given the starting point $x \in (0, T]$, the probability that W hits zero before upcrossing T, that is

$$P_x(W(\tau) = 0) = \hat{g}_x(0) = \frac{e^{-(\mu-\lambda)x} - (\lambda/\mu)e^{-(\mu-\lambda)T}}{1 - (\lambda/\mu)e^{-(\mu-\lambda)T}}, \tag{15}$$

and the probability of its complimentary event, i.e., that W upcrosses T before hitting zero,

$$P_x(W(\tau) > T) = \hat{h}_x(0) = 1 - \hat{g}_x(0) = \frac{1 - e^{-(\mu-\lambda)x}}{1 - (\lambda/\mu)e^{-(\mu-\lambda)T}}. \tag{16}$$

Next, consider W right after a jump caused by hitting zero. Let $t = 0$ be the time of the jump and note that the starting point $W(0) \in (0, \infty)$ is exponentially distributed with rate λ. Let

$$p \equiv P(W(\tau) > T),\ \hat{g}(\theta) \equiv E[e^{-\theta\tau}|W(\tau) = 0],\ \hat{h}(\theta) \equiv E[e^{-\theta\tau}|W(\tau) > T].$$

Observe that p is the probability that W upcrosses T before hitting zero, $\hat{g}(\theta)$ is the LT of the time it takes for W to hit zero given that it happens before upcrossing T, and $\hat{h}(\theta)$ is the LT of the time it takes for W to upcross T given that it occurs before hitting zero.

LEMMA 1. *For* $\lambda \neq \mu$,

$$\hat{g}(\theta) = \left(\frac{\lambda e^{-\alpha_2(\theta)T}(1-e^{-(\lambda+\alpha_1(\theta))T}) - \lambda e^{-\alpha_1(\theta)T}(1-e^{-(\lambda+\alpha_2(\theta))T})}{(\lambda+\alpha_1(\theta))e^{-\alpha_2(\theta)T} - (\lambda+\alpha_2(\theta))e^{-\alpha_1(\theta)T}}\right)/(1-p),$$
$$\hat{h}(\theta) = \left(e^{-\lambda T} + \frac{(\lambda+\alpha_1(\theta))(1-e^{-(\lambda+\alpha_2(\theta))T}) - (\lambda+\alpha_2(\theta))(1-e^{-(\lambda+\alpha_1(\theta))T})}{(\lambda+\alpha_1(\theta))e^{-\alpha_2(\theta)T} - (\lambda+\alpha_2(\theta))e^{-\alpha_1(\theta)T}}\right)/p,$$

and for $\lambda = \mu$,

$$\hat{g}(\theta) = \left(\frac{2\lambda(e^{T\delta(\theta)}-1)}{\theta(e^{T\delta(\theta)}-1)+2\lambda(e^{T\delta(\theta)}-1)+(e^{T\delta(\theta)}+1)\delta(\theta)}\right)/(1-p),$$
$$\hat{h}(\theta) = \left(\frac{2e^{(T/2)(\theta+\delta(\theta))}\delta(\theta)}{\theta(e^{T\delta(\theta)}-1)+2\lambda(e^{T\delta(\theta)}-1)+(e^{T\delta(\theta)}+1)\delta(\theta)}\right)/p,$$

where $\delta(\theta) \equiv \sqrt{\theta(\theta+4\lambda)}$ *and*

$$p = \begin{cases} e^{-\lambda T} + \frac{(1-e^{-\lambda T})-(\lambda/\mu)(1-e^{-\mu T})}{1-(\lambda/\mu)e^{-(\mu-\lambda)T}}, & \lambda \neq \mu, \\ 1/(1+\lambda T), & \lambda = \mu. \end{cases} \tag{17}$$

We proceed with the analysis of the residual busy period R. Let M be the number of outdates during the residual busy period. Also, let $\hat{r}(\theta)$ denote the LT of R, and let $\hat{r}_m(\theta)$ denote the LT of R given $M = m$.

PROPOSITION 3. *The number of units that are outdated during the residual busy period is Geometrically distributed with parameter* p *as given in* (17)*, i.e.,*

$$P(M = m) = (1-p)^m p, \tag{18}$$

for $m \geq 0$*. Moreover, the LT of the length of the residual busy period given* $M = m$ *is*

$$\hat{r}_m(\theta) = \hat{h}(\theta)\big(\hat{g}(\theta)\big)^m, \tag{19}$$

and the LT of the length of the residual busy period is given by

$$\hat{r}(\theta) = \frac{p\hat{h}(\theta)}{1-(1-p)\hat{g}(\theta)}. \tag{20}$$

We next consider full busy periods (see Figure 1 for a realization). Note that the length of the busy periods are i.i.d. random variables. Let N be the number outdates during a generic busy period denoted by Z. Also, let $\hat{z}(\theta)$ denote the LT of Z, and let $\hat{z}_n(\theta)$ denote the LT of Z given $N = n$.

Proposition 4. *The distribution of the number of units that are outdated during a busy period is given by*

$$P(N=n)=\begin{cases}\hat{h}_T(0), & n=0,\\ \hat{g}_T(0)p(1-p)^{n-1}, & n>0.\end{cases} \tag{21}$$

Moreover, the LT of the length of the busy period given $N=n$ is

$$\hat{z}_n(\theta)=\begin{cases}\frac{\hat{h}_T(\theta)}{\hat{h}_T(0)}, & n=0,\\ \frac{\hat{g}_T(\theta)}{\hat{g}_T(0)}\hat{h}(\theta)\left(\hat{g}(\theta)\right)^{n-1}, & n>0,\end{cases} \tag{22}$$

and the length of the busy period has LT given by

$$\hat{z}(\theta)=\hat{h}_T(s)+\frac{p\hat{h}(\theta)\hat{g}_T(\theta)}{1-(1-p)\hat{g}(\theta)}. \tag{23}$$

Finally, we consider the idle periods which are independent and exponentially distributed. Let L be the number of lost demand during a generic idle period I. Also let $\hat{i}(\theta)$ and $\hat{i}_l(\theta)$ denote the LT of I and the LT of I given $L=l$, respectively.

Proposition 5. *Idle periods are exponentially distributed with parameter λ, that is*

$$\hat{i}(\theta)=\frac{\lambda}{\lambda+\theta}. \tag{24}$$

Moreover, the number of lost demand during an idle period is Geometrically distributed with parameter $\lambda/(\mu+\lambda)$, i.e.,

$$P(L=l)=\left(\frac{\mu}{\mu+\lambda}\right)^l\left(\frac{\lambda}{\mu+\lambda}\right), \tag{25}$$

and the length of the idle period given the number of lost demand $L=l$ has an Erlang$(l+1,\lambda+\mu)$ distribution, that is,

$$\hat{i}_l(\theta)=\left(\frac{\lambda+\mu}{\lambda+\mu+\theta}\right)^{l+1}. \tag{26}$$

5. The Threshold Policy

We analyze the threshold policy by considering a two-stage representation of the system operating under a threshold policy with parameter $T\in(0,\gamma)$. Figure 2 depicts this two-stage representation of the system. Fresh units arrive at *Stage 1* according to a Poisson process with rate λ and stay there for a maximum of T time units after which they are transferred to *Stage 2*. Units remain in Stage 2 for up to $\gamma-T$ additional time units and are eventually outdated if their age exceeds the shelf-life γ before they are allocated. Demand for Stage 1 inventory occurs according to a Poisson process with rate μ and is satisfied according to a FIFO policy. If there are no units available in

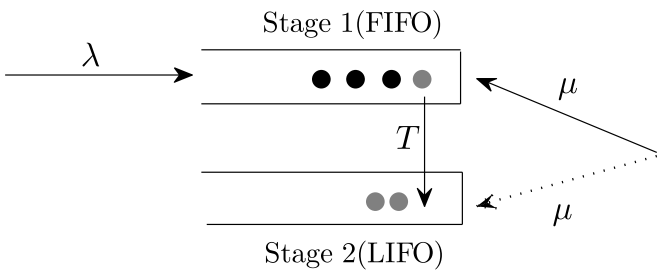

Figure 2 A two–stage representation of the system under the threshold policy

Stage 1, a unit from Stage 2 is allocated according to a LIFO policy. Demand occurring while there are no units available is lost.

Let "T" be the shorthand notation for a threshold policy with parameter T, so that S^T is the random variable representing the steady-state sojourn time of units under a threshold policy with parameter T. Let $S_1^T \in (0,T]$ denote the random variable representing the steady-state sojourn time of units in Stage 1. Also, for units which transfer to Stage 2, let $S_2^T \in (0,\gamma - T]$ denote the random variable representing their steady-state sojourn time in Stage 2. To simplify the notation we omit the superscript T from S_1^T and S_2^T. Then,

$$S^T = S_1 \mathbf{1}_{\{S_1<T\}} + (T+S_2)\mathbf{1}_{\{S_1=T\}}. \tag{27}$$

It is easy to see that Stage 1 is an independent system operating under the FIFO policy, in which units have shelf-life T. Hence, given that a unit is allocated from Stage 1, its sojourn time distribution is known through the analysis of Subsection 3.2. In particular, let q_1, ℓ_1, $A_1(\cdot)$ and $E[A_1]$, respectively, denote the outdate probability, loss probability, cdf of the age of transfused units, and the expected age of transfused units in a FIFO system where units have shelf-life T. Then all these measures can be obtained from Proposition 2 by setting $\gamma = T$.

However, both demand and arrival processes of Stage 2 depend on the state of Stage 1. During a busy period in Stage 1, demand is only satisfied from Stage 1 inventory but units may pass the threshold age T and hence move to Stage 2. During an idle period in Stage 1, demand is satisfied from Stage 2 inventory but there are no arrivals at Stage 2. To characterize the distribution of S_2 we first consider a system in which units have infinite shelf-life in Stage 2. Let $\tilde{S}_2$ denote the random variable representing the steady-state sojourn time of units in this system. Then since the allocation policy in Stage 2 is LIFO, using similar arguments to those in Section 3.1, we have $S_2 = \min(\tilde{S}_2, \gamma - T)$.

Note that a unit can arrive at Stage 2 during a busy period in Stage 1 and can be allocated to a demand during one of the idle periods of Stage 1. In general, however, since the shelf-life is infinite, to find the sojourn time of allocated units one needs to consider infinitely many such idle periods. Note that $\tilde{S}_2$ has an improper distribution whenever $\lambda q_1 > \mu\ell_1$, in which case $P(\tilde{S}_2 < \infty) = (\mu\ell_1)/(\lambda q_1)$. Our approach is based on analyzing a sequence of modified Stage 2 systems in which each unit can only be allocated during a finite number k of Stage 1 idle periods, after its arrival at Stage 2. Specifically, in the k^{th} *modified system* a unit which is not allocated by the end of the k^{th} busy period after its arrival at Stage 2 is *discarded*. We denote the random variable associated with the steady-state sojourn time of units in the k^{th} modified system by $\tilde{S}_{2,k}$ and show that as k tends to infinity, the distribution of $\tilde{S}_{2,k}$ converges to that of $\tilde{S}_2$. By analyzing the more tractable random variable $\tilde{S}_{2,k}$, we are then able to obtain the LT of $\tilde{S}_2$.

In the next subsection, we explain how the required performance measures under the threshold policy can be obtained from the distribution of $\tilde{S}_2$. In Subsection 5.2 we analyze the modified systems and use them to obtain the LT of $\tilde{S}_2$.

5.1. Obtaining the Performance Measures

We first examine the outdate probability. For a unit to be outdated it must first move to Stage 2 and then, assuming it has an infinite shelf life, spend more than $\gamma - T$ in Stage 2. Note that q_1 is the probability that a unit moves to Stage 2. Hence, the outdate probability of a policy with threshold T is given by

$$q^T = q_1 P(\tilde{S}_2 \geq \gamma - T). \tag{28}$$

To obtain the distribution of the age of transfused units A^T, we condition on whether the unit is allocated while in Stage 1 or 2. Denote these events by $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, and note that $P(\mathcal{S}_1) + P(\mathcal{S}_2) + q^T = 1$. First, clearly $P(A^T \leq x|\mathcal{S}_1) = A_1(x)$. Second, given that a unit is allocated from Stage 2, we know that its age is greater than T, and hence for $x \leq T$ we have $P(A^T \leq x|\mathcal{S}_2) = 0$. For $T < x < \gamma$,

$$P(A^T \leq x|\mathcal{S}_2) = P(\tilde{S}_2 \leq x - T)/P(\tilde{S}_2 < \gamma - T).$$

Noting that

$$P(\mathcal{S}_1) = \frac{1 - q_1}{1 - q^T},\ P(\mathcal{S}_2) = \frac{q_1 P(\tilde{S}_2 < \gamma - T)}{1 - q^T},$$

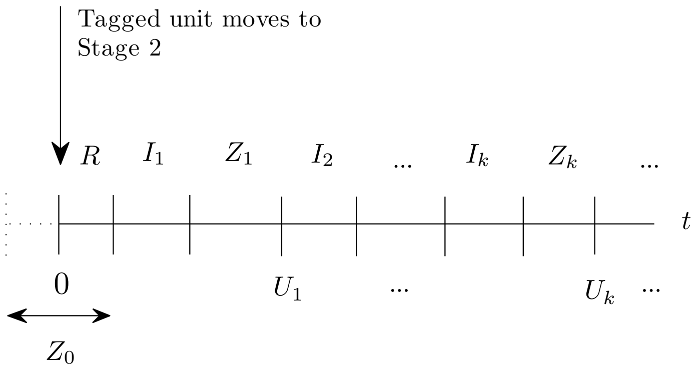

Figure 3 An illustration of random variables R, I_i, Z_i, U_k on the time line

and combining the two cases we have

$$A^T(x) \equiv P(A^T \le x) = \begin{cases} P(\mathcal{S}_1)A_1(x), & x < T, \\ P(\mathcal{S}_1) + \frac{q_1}{1-q^T} P(\tilde{S}_2 \le x - T), & T \le x < \gamma, \\ 1, & x \ge \gamma. \end{cases} \tag{29}$$

Finally, the expected age of transfused units $E[A^T]$, can be computed using

$$E[A^T] = E[A_1]P(\mathcal{S}_1) + \big(T + E[S_2|S_2 < \gamma - T]\big)P(\mathcal{S}_2),$$

where

$$E[S_2|S_2 < \gamma - T] = \gamma - T - \frac{\int_0^{\gamma-T} P(\tilde{S}_2 \le x)dx}{P(\tilde{S}_2 < \gamma - T)}.$$

For all the performance measures, $P(\tilde{S}_2 \le x)$ for $x \le \gamma - T$ can be computed by numerically inverting its LT, i.e., $E[e^{-\theta \tilde{S}_2}]/\theta$, where $E[e^{-\theta \tilde{S}_2}]$ will be given in Theorem 4 of Subsection 5.2.

5.2. Sojourn Time of Units in Stage 2

In this subsection, we obtain the LT of $\tilde{S}_2$ by considering a sequence of modified systems. Before formalizing the approach we introduce some notation. Consider a *tagged* unit which has just arrived at Stage 2. Let $Z_i, i \ge 0$ and $I_i, i \ge 1$ denote, respectively, the length of the i^{th} busy and idle period in Stage 1 followed by the arrival of the tagged unit to Stage 2. Accordingly, Z_0 corresponds to the length of the busy period during which the unit arrives at Stage 2. Note that $\{I_i; i \ge 1\}$ and $\{Z_i; i \ge 0\}$ are sequences of i.i.d. random variables having the same distribution as I and Z, respectively. Thus, we have their LTs from Propositions 4 and 5, respectively. Furthermore, the time interval between the epoch when the tagged unit moves to Stage 2 until the start of the first idle period is a residual busy period R, the LT of which is given by Proposition 3. Let $t = 0$ be the instance the tagged unit moves to Stage 2 and let $U_k, k \geq 1$, denote the time when the k^{th} busy period ends. That is, for $k \geq 1$,

$$U_k = R + \sum_{i=1}^{k} I_i + \sum_{i=1}^{k} Z_i. \tag{30}$$

Letting $X_0 = R$ and $X_i = I + Z$ for $i = 1, 2, ...$, and noting that $\{X_i; i \geq 1\}$ is an i.i.d sequence with $X_i > 0$ for all $i \geq 1$, $\{U_k, k \geq 1\}$ can be viewed as the arrival epochs of a delayed renewal process. Figure 3 presents an illustration of the corresponding renewal process on the time line.

Now recall that $\tilde{S}_{2,k}$ is the sojourn time of units in the k^{th} modified system in which a unit is discarded if it is not allocated by the end of the k^{th} busy period (or equivalently the beginning of the $(k+1)^{st}$ idle period). Therefore, for a unit that is allocated by the end of the k^{th} busy period we have $\tilde{S}_{2,k} = \tilde{S}_2$, while for a unit that is still in the system by the end of the k^{th} busy period we have $\tilde{S}_{2,k} = U_k$. Formally, for $k \geq 1$,

$$\tilde{S}_{2,k} = \tilde{S}_2 \mathbf{1}_{\{\tilde{S}_2 < U_k\}} + U_k \mathbf{1}_{\{\tilde{S}_2 \geq U_k\}}. \tag{31}$$

THEOREM 1. *Consider the sojourn time of units in Stage 2 assuming infinite shelf-life $\tilde{S}_2$, and the sojourn time of units in the k^{th} modified system $\tilde{S}_{2,k}$. We have*

$$\lim_{k \to \infty} P(\tilde{S}_{2,k} \leq x) = P(\tilde{S}_2 \leq x), \quad x \in [0, \infty),$$
$$\lim_{k \to \infty} E[e^{-\theta \tilde{S}_{2,k}}] = E[e^{-\theta \tilde{S}_2}], \quad \theta > 0.$$

The theorem implies that for sufficiently large number of idle periods, the sojourn time distribution of the units in the modified system becomes arbitrary close to that of units in the system with infinite shelf-life. It also indicates that the LT of $\tilde{S}_2$ can be obtained by first obtaining the LT of $\tilde{S}_{2,k}$ and then letting $k \to \infty$. While the result is sufficient for our analysis, in the following theorem we state a stronger convergence result for S_2, i.e., the actual sojourn time of units in Stage 2. Recall that $S_2 = \min(\tilde{S}_2, \gamma - T)$. Hence, similarly if we let

$$S_{2,k} \equiv \min(\tilde{S}_{2,k}, \gamma - T), \quad k \geq 1, \tag{32}$$

denote the truncated sojourn time of units in the k^{th} modified system, one would expect $S_{2,k}$ to converge to S_2 as k tends to infinity. Indeed, the following theorem establishes their almost sure convergence.

THEOREM 2. *Consider S_2 the actual sojourn time of units in Stage 2, and $S_{2,k}$ as defined in* (32). *We have $P(S_{2,k} \to S_2) = 1$, that is the sequence of random variables $\{S_{2,k}\}$ converges to S_2 with probability 1.*

We now turn to the analysis of the modified systems. Consider the k^{th} modified system. Note that given the number of units that are in front of the tagged unit at the beginning of any idle period, its *remaining* sojourn time is independent of the past. For the k^{th} modified system, let $\hat{\varphi}^k_{\nu,i}(\theta), 1 \le i \le k+1$ denote the LT of the remaining sojourn time of the tagged unit at the beginning of the i^{th} idle period, given that it has ν units in front of it. Then, $\hat{\varphi}^k_{\nu,1}(\theta)$ is the LT of the remaining sojourn time of the unit at the beginning of the first idle period, given that the number of units moving to Stage 2 during the residual busy period is $\nu \ge 0$. Recall that M denotes the number of outdates during the residual busy period and $\hat{r}_\nu(\theta)$ denotes the LT of the length of the residual busy period given $M=\nu$. Thus, we have

$$E[e^{-\theta \tilde{S}_{2,k}}] = \sum_{\nu=0}^{\infty} P(M=\nu)\hat{r}_\nu(\theta)\hat{\varphi}^k_{\nu,1}(\theta). \tag{33}$$

We first express $\hat{\varphi}^k_{\nu,1}(\theta)$ for $k \ge 1$ in Theorem 3, then we use (33) to find the LT of $\tilde{S}_{2,k}$. Finally, in Theorem 4 we apply Theorem 1 to obtain the LT of $\tilde{S}_2$.

The following lemma presents a recursive relation for $\hat{\varphi}^k_{\nu,i}(\theta)$, which can be used to obtain $\hat{\varphi}^k_{\nu,1}(\theta)$. We consider the tagged unit at the beginning of the i^{th} idle period given that it has $\nu \ge 0$ units in front of it. We then condition on the number of demand arrivals during the idle period. By considering two cases depending on whether the unit is allocated during the idle period or not, we are able to relate the LT of the remaining sojourn time of the tagged unit at the beginning of the i^{th} idle period to that of the unit at the beginning of the $(i+1)^{st}$ idle period.

Lemma 2. *For $1 \le i \le k$ and $\nu \ge 0$ we have*

$$\hat{\varphi}^k_{\nu,i}(\theta) = \sum_{l=0}^{\nu}\sum_{n=0}^{\infty} P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)\hat{\varphi}^k_{\nu+n-l,i+1}(\theta) + \left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}. \tag{34}$$

Note that by definition of $\hat{\varphi}^k_{\nu,i}(\theta)$ we have $\hat{\varphi}^k_{\nu,k+1}(\theta)=1$ for all $\nu \ge 0$. Thus, for a given k and starting from $i=k$ one can use Lemma 2 to recursively solve for $\hat{\varphi}^k_{\nu,1}(\theta)$. The next theorem expresses $\hat{\varphi}^k_{\nu,1}(\theta)$ as a function of k. First, define

$$c_1(\theta) \equiv \hat{h}(\theta)p\left(\frac{\mu}{\lambda+\mu+\theta}\right), \qquad c_2(\theta) \equiv (1-p)\hat{g}(\theta)\left(\frac{\mu}{\lambda+\mu+\theta}\right), \tag{35}$$

with $\hat{g}(\theta)$, $\hat{h}(\theta)$ and p given in Lemma 1, and let

$$\xi_i(\theta) \equiv \begin{cases} \hat{h}_T(\theta) + \hat{g}_T(\theta)c_1(\theta)/\left(1-c_2(\theta)\right), & i=0, \\ \xi_0(\theta) + \hat{g}_T(\theta)c_1(\theta)/\left(1-c_2(\theta)\right)^2, & i=1, \\ \hat{g}_T(\theta)c_1(\theta)\left(c_2(\theta)\right)^{i-2}/\left(1-c_2(\theta)\right)^{i+1}, & i \ge 2, \end{cases} \tag{36}$$

$$\beta_i(\theta) \equiv \begin{cases} c_1(\theta)/\left(1-c_2(\theta)\right), & i=0, \\ c_1(\theta)\left(c_2(\theta)\right)^{i-1}/\left(1-c_2(\theta)\right)^{i+1}, & i \geq 1, \end{cases} \tag{37}$$

with $\hat{h}_T(\theta)$ and $\hat{g}_T(\theta)$ given in (15) and (16) respectively. Next, define the nested sum $Y_\nu(d)$ for non-negative integers ν and d as

$$Y_\nu(d) \equiv \sum_{j_0=0}^{0} \xi_{j_0}(\theta) \sum_{j_1=0}^{1} \xi_{j_1}(\theta) \sum_{j_2=0}^{2-j_1} \xi_{j_2}(\theta) \sum_{j_3=0}^{3-j_2-j_1} \xi_{j_3}(\theta) \cdots \sum_{j_{d-1}=0}^{(d-1)-j_{d-2}-\cdots-j_1} \xi_{j_{d-1}}(\theta) \left(\frac{\mu}{\lambda+\mu+\theta}\right)^{\nu+1} \binom{\nu+1}{d-j_{d-1}-\cdots-j_1}, \tag{38}$$

where we adopt the convention that any empty sum is equal to 0 and any empty product equal to 1. Note that d is the number of sums in $Y_\nu(d)$. For $d=0$, $Y_\nu(d)$ has no sums, that is

$$Y_\nu(0) = \left(\frac{\mu}{\lambda+\mu+\theta}\right)^{\nu+1} \binom{\nu+1}{0} = \left(\frac{\mu}{\lambda+\mu+\theta}\right)^{\nu+1},$$

and for $d=1$ the expression only contains the first sum. Noting that the first sum simplifies to $\xi_0(\theta)$, we have

$$Y_\nu(1) = \xi_0(\theta) \left(\frac{\mu}{\lambda+\mu+\theta}\right)^{\nu+1} \binom{\nu+1}{1}.$$

Similarly, for $d \in \{2,3,...\}$, (38) includes the first d sums.

THEOREM 3. *The LT of the remaining sojourn of a unit at the beginning of the first idle period in the k^{th} modified system, given it has $\nu \geq 0$ units in front of it $\hat{\varphi}^k_{\nu,1}(\theta)$ is given by*

$$\hat{\varphi}^k_{\nu,1}(\theta) = \sum_{i=0}^{k-1} Y_\nu(i) \left(1 - (\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right) \left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i + \left(\hat{i}(\theta)\hat{z}(\theta)\right)^k. \tag{39}$$

Using (33) we obtain $E[e^{-\theta \tilde{S}_{2,k}}]$ and let $k \to \infty$ to obtain the LT of $\tilde{S}_2$. Define the nested sum $X(d,w)$ for positive integers d and w as

$$X(d,w) \equiv \sum_{j_1=0}^{w} \xi_{j_1}(\theta) \sum_{j_2=0}^{w+1-j_1} \xi_{j_2}(\theta) \sum_{j_3=0}^{w+2-j_2-j_1} \xi_{j_3}(\theta) \cdots \sum_{j_{d-1}=0}^{w+(d-2)-j_{d-2}-\cdots-j_1} \xi_{j_{d-1}}(\theta) \beta_{w+(d-1)-j_{d-1}-\cdots-j_1}(\theta). \tag{40}$$

Note that $X(d,w)$ contains the first $d-1$ sums, such that $X(1,w) = \beta_w(\theta)$ for all $w \in \{1,2,\ldots\}$. Moreover, $X(d,w)$ satisfies the recursive relation given by

$$X(d,w) = \sum_{i=0}^{w} \xi_i(\theta) X(d-1, w+1-i),$$

for $d \in \{2,3,...\}$, which can be used to calculate $X(i,1)$ for $i \in \{1,2,...\}$, as needed in Theorem 4 below.

THEOREM 4. *The LT of $\tilde{S}_2$ is given by*

$$E[e^{-\theta \tilde{S}_2}] = \beta_0(\theta) + \xi_0(\theta) \sum_{i=1}^{\infty} X(i,1) \left(\frac{\lambda}{\lambda + \mu + \theta} \right)^i .$$

6. Numerical Results and Observations

Let ϑ denote the age of units at the time of arrival to the inventory. Unless otherwise noted, similar to Atkinson et al. (2012) we assume that it takes 2 days to test and process the units, i.e., we use $\vartheta = 2$ in the numerical examples. However, we shall specify the threshold T with respect to the arrival of units to Stage 1. For instance, a threshold value of $T = 4$ implies that units move to Stage 2 after 6 days.

6.1. The Expected Age–Availability Trade-off

We start by investigating the trade-off between the expected age of transfused units and the loss probability as a measure for availability. Since the age of blood is measured in days, we compute the expected age of transfused units under a policy π using

$$\text{EA}(\pi) = \vartheta + \sum_{n=1}^{\gamma} n \cdot P(n-1 < A^{\pi} < n),$$

where as mentioned earlier constant ϑ is the initial age of units at the time of arrival to the inventory and $\gamma = 42 - \vartheta$ is the maximum number of days units can be held in the inventory. Note that here A^{π} corresponds to the age of units at the time of transfusion with respect to their arrival in inventory.

Figure 4 illustrates the tradeoff for $\mu = 15$ (roughly the demand for A+ blood type at Stanford University Medical Center) and different values of the supply-to-demand ratio. The qualitative features of the trade-off curve we obtain are similar to those observed in Atkinson et al. (2012). The shape of the trade-off curve depends on the supply-to-demand ratio: As the supply-to-demand ratio increases, the tradeoff curve tends to a vertical line and the loss probability decreases. When the ratio is 1.05, the curve is nearly vertical and the loss probability is very small (less than 0.01) for all threshold values. On the other hand, as the ratio decreases the trade-off curve tends to a horizontal line and the loss probability increases. When the ratio is 0.95 the curve is nearly horizontal and, regardless of the policy, the loss probability is quite high (above 0.05). For supply-to-demand ratios close to 1, a meaningful trade-off between the expected age and the loss probability is achieved. As the threshold value decreases, we observe a large decrease in the expected age and a small increase in the proportion of lost demand.

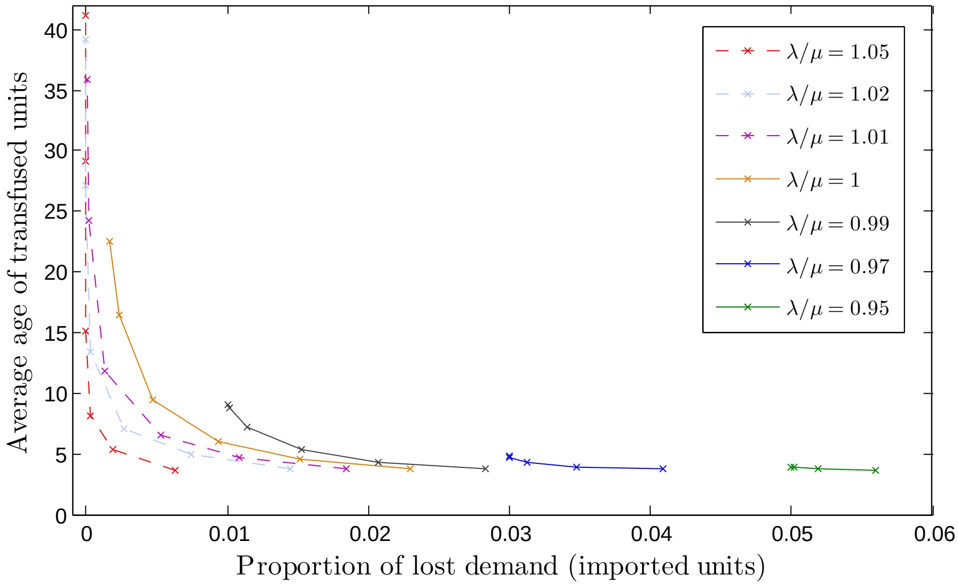

Figure 4 The trade-off between the expected age of transfused units and the loss probability. The six tick marks on each curve from left to right correspond to threshold values of 40 (FIFO), 28, 14, 7, 4, and 0 (LIFO).

6.2. Performance of the Threshold Policy

Next, we take a closer look at the performance of the threshold policy when the supply-to-demand ratio is close to 1. Tables 1 to 3 list the detailed performance of the threshold policy for systems with large ($\mu = 15$), medium ($\mu = 7$), and small ($\mu = 2$) demand sizes, respectively. The demand rates are roughly equal to those for A+, B+ and AB+ blood types in Stanford University Medical Center. For each demand rate, the performance of the policy is presented for different supply-to-demand ratios and threshold values.

As we expect from (3), for each demand size, a higher supply-to-demand ratio results in a higher outdate probability and a lower probability of loss. Moreover, the demand size plays an important role in performance of the threshold policy. For a fixed supply-to-demand ratio, regardless of the threshold value, we observe a better outcome in terms of the loss and outdate probability as the demand rate increases. Furthermore, the range of outdate and loss probabilities among the policies becomes smaller as the demand rate increases.

In Tables 1 to 3, for each policy, we also present the proportion of units which are transferred to Stage 2. Note that this value is equal to the resulting outdate probability under the FIFO policy if the shelf-life is reduced to the threshold value. Thus, by comparing this value to the outdate probability of the corresponding threshold policy we are able to compare the performance of the threshold policy with that of simply reducing the shelf-life to the threshold value.

	FIFO	$T=28$	$T=14$	$T=7$	$T=4$	LIFO
	$\lambda/\mu=0.98$					
Prop. of units transferred to Stage 2	0	0.0000	0.0003	0.0028	0.0085	1
Prop. of units allocated from Stage 2	0	0.0000	0.0000	0.0000	0.0011	1
Outdate prob. for units transferred to Stage 2	-	1.0000	1.0000	0.9961	0.8726	0.0146
Outdate prob.	0.0000	0.0000	0.0003	0.0028	0.0075	0.0146
Loss prob.	0.0200	0.0200	0.0203	0.0227	0.0273	0.0343
Mean age of transfused units	5.9	5.9	5.6	4.9	4.1	3.7
	$\lambda/\mu=1.00$					
Prop. of units transferred to Stage 2	0	0.0024	0.0047	0.0094	0.0164	1
Prop. of units allocated from Stage 2	0	0.0000	0.0000	0.0000	0.0012	1
Outdate prob. for units transferred to Stage 2	-	1.0000	1.0000	0.9986	0.9293	0.0230
Outdate prob.	0.0017	0.0024	0.0047	0.0094	0.0152	0.0230
Loss prob.	0.0017	0.0024	0.0047	0.0094	0.0152	0.0230
Mean age of transfused units	22.5	16.5	9.5	6.0	4.5	3.7
	$\lambda/\mu=1.02$					
Prop. of units transferred to Stage 2	0	0.0196	0.0199	0.0223	0.0278	1
Prop. of units allocated from Stage 2	0	0.0000	0.0000	0.0000	0.0010	1
Outdate prob. for units transferred to Stage 2	-	1.0000	1.0000	0.9996	0.9638	0.0338
Outdate prob.	0.0196	0.0196	0.0199	0.0223	0.0268	0.0338
Loss prob.	0.0000	0.0000	0.0003	0.0027	0.0074	0.0145
Mean age of transfused units	39.1	27.1	13.4	7.1	4.9	3.7

Table 1 Performance of the threshold policy for $\mu=15$ and different values of supply-to-demand ratio

We observe that the demand size and the threshold value are two key factors in determining the effectiveness of the threshold policy in comparison to reducing the shelf-life. Consider Table 1, corresponding to the system with large demand. Even for small threshold values, almost all the units transferred to Stage 2 are outdated (see outdate probability for units transferred to Stage 2), thereby effectively leading to the same outcome achieved when the shelf-life of units is reduced to the threshold value. A similar outcome is achieved for large threshold values. For example, for $T=28$, regardless of the demand size, almost no units are allocated from Stage 2. However, when the demand size is smaller (see Tables 2 and 3) and the threshold value is sufficiently small, a considerable proportion of units are allocated from Stage 2. While the proportion is small enough to keep the expected age of transfused units low, it results in a significantly smaller outdate probability when compared to reducing the shelf-life. For instance, when $\lambda=\mu=2$ and $T=7$ (see Table 3), 1.51% of units are allocated from Stage 2. The result is a lower outdate probability (5.24%) compared to that of reducing the shelf-life to $\gamma=7$ (6.67%).

Recalling that Stage 1 behaves like a FIFO system with shelf-life T, the above observation can be explained in light of Corollary 1. Keeping the supply-to-demand ratio fixed, as the demand decreases, the proportion of units transferred to Stage 2 and the proportion of demand directed to Stage 2 increase. Reducing the threshold has a similar effect. The result is that a higher proportion of units are allocated from Stage 2 inventory, which in turn, increases the effectiveness of the threshold policy over reducing the shelf-life in terms of availability.

It is important to note that the benefits of the threshold policy are less significant when the impact of reducing the shelf-life on availability is smaller. Note that reducing the shelf-life results in a larger increase in the outdate probability when the demand size is smaller. For example, reducing the shelf-life to 14 increases the outdate probability from 0.0017 to 0.0047 (by 0.0030) when $\lambda = \mu = 15$, and from 0.0123 to 0.0345 (by 0.0222) when $\lambda = \mu = 2$. Similarly, small reductions in the shelf-life (e.g., to 28) have a small effect on the outdate probability regardless of the demand size (although the effect is higher when the demand is smaller). These observations are inline with those reported in the simulation study of Blake et al. (2012).

Alternatively, one could think of using a combination of the threshold policy and reduction of the shelf-life. However, this is less likely to be effective. Consider a threshold policy with parameter T. If we reduce the shelf-life γ, Stage 1 remains unchanged but the remaining shelf-life of units in Stage 2 ($\gamma - T$) and hence the proportion of units allocated from this stage decreases. Therefore, when compared to just using a threshold policy with parameter T, the expected age will be approximately the same while the outdate probability will be higher.

In summary, a threshold policy results in almost the same expected age obtained if the shelf-life is reduced to the threshold value. However, for a sufficiently small demand size and threshold value, the threshold policy results in a lower outdate probability (and higher availability) when compared to reducing the shelf-life. Thus, our results suggests that for blood types with high demand or when a small decrease in the age of RBCs is required, the advantages of applying a threshold policy comparing to reduction of the shelf-life are minimal. At the same time, we observe a smaller effect on availability for such systems in case of a reduction in shelf-life. On the other hand, for lower demand rates or when a larger reduction in the age of RBCs is required, as the system becomes more vulnerable to the impact of reducing the shelf-life, a threshold policy is expected to maintain a suitable utilization of Stage 2 inventory and thus be effective in improving the availability. Extending this observation to hospital sizes, our model suggests a more significant benefit in using the threshold policy for smaller hospitals where reducing the shelf-life has been predicted to be more detrimental to the availability of RBC units (Blake et al. 2012); an observation which is also evident from our results.

We close this subsection with a few important remarks on the comparison of the threshold policy with shortening the shelf-life. First, although small threshold values may not seem plausible when

	FIFO	$T=28$	$T=14$	$T=7$	$T=4$	LIFO
	$\lambda/\mu=0.98$					
Prop. of units transferred to Stage 2	0	0.0004	0.0033	0.0119	0.0260	1
Prop. of units allocated from Stage 2	0	0.0000	0.0000	0.0005	0.0073	1
Outdate prob. for units transferred to Stage 2	-	1.0000	1.0000	0.9554	0.7231	0.0250
Outdate prob.	0.0001	0.0004	0.0033	0.0113	0.0188	0.0250
Loss prob.	0.0201	0.0204	0.0232	0.0311	0.0384	0.0445
Mean age of transfused units	9.5	9.1	7.4	5.4	4.5	4.1
	$\lambda/\mu=1.00$					
Prop. of units transferred to Stage 2	0	0.0051	0.0101	0.0200	0.0345	1
Prop. of units allocated from Stage 2	0	0.0000	0.0000	0.0005	0.0074	1
Outdate prob. for units transferred to Stage 2	-	1.0000	1.0000	0.9732	0.7914	0.0337
Outdate prob.	0.0036	0.0051	0.0101	0.0195	0.0273	0.0337
Loss prob.	0.0036	0.0051	0.0101	0.0195	0.0273	0.0337
Mean age of transfused units	22.5	16.5	9.5	6.0	4.7	4.1
	$\lambda/\mu=1.02$					
Prop. of units transferred to Stage 2	0	0.0200	0.0227	0.0310	0.0446	1
Prop. of units allocated from Stage 2	0	0.0000	0.0000	0.0005	0.0070	1
Outdate prob. for units transferred to Stage 2	-	1.0000	1.0000	0.9845	0.8479	0.0440
Outdate prob.	0.0197	0.0200	0.0227	0.0305	0.0378	0.0440
Loss prob.	0.0001	0.0004	0.0032	0.0112	0.0185	0.0248
Mean age of transfused units	35.5	23.9	11.6	6.6	4.8	4.1

Table 2 Performance of the threshold policy for $\mu=7$ and different values of supply-to-demand ratio

	FIFO	$T=28$	$T=14$	$T=7$	$T=4$	LIFO
	$\lambda/\mu=0.98$					
Prop. of units transferred to Stage 2	0	0.0096	0.0260	0.0583	0.1034	1
Prop. of units allocated from Stage 2	0	0.0000	0.0004	0.0247	0.0555	1
Outdate prob. for units transferred to Stage 2	-	1.0000	0.9846	0.7498	0.4903	0.0543
Outdate prob.	0.0050	0.0096	0.0255	0.0437	0.0507	0.0543
Loss prob.	0.0249	0.0294	0.0450	0.0629	0.0697	0.0732
Mean age of transfused units	16.9	13.4	8.9	6.2	5.4	5.1
	$\lambda/\mu=1.00$					
Prop. of units transferred to Stage 2	0	0.0175	0.0345	0.0667	0.1111	1
Prop. of units allocated from Stage 2	0	0.0000	0.0004	0.0151	0.0550	1
Outdate prob. for units transferred to Stage 2	-	1.0000	0.9888	0.7857	0.5345	0.0630
Outdate prob.	0.0123	0.0175	0.0341	0.0524	0.0594	0.0630
Loss prob.	0.0123	0.0175	0.0341	0.0524	0.0594	0.0630
Mean age of transfused units	22.5	16.5	9.5	6.3	5.4	5.1
	$\lambda/\mu=1.02$					
Prop. of units transferred to Stage 2	0	0.0288	0.0446	0.0757	0.1192	1
Prop. of units allocated from Stage 2	0	0.0000	0.0004	0.0147	0.0540	1
Outdate prob. for units transferred to Stage 2	-	1.0000	0.9920	0.8181	0.5774	0.0724
Outdate prob.	0.0244	0.0288	0.0442	0.0619	0.0688	0.0724
Loss prob.	0.0049	0.0094	0.0251	0.0432	0.0502	0.0538
Mean age of transfused units	27.6	19.1	10.2	6.5	5.4	5.1

Table 3 Performance of the threshold policy for $\mu=2$ and different values of supply-to-demand ratio

	FIFO	$T=4$	$T=3$	$T=2$	LIFO
Prop. of units transferred to Stage 2	0	0.0278	0.0326	0.0424	1
Prop. of units allocated from Stage 2	0	0.0006	0.0026	0.0097	1
Outdate prob. for units transferred to Stage 2	-	0.9785	0.9225	0.7802	0.4956
Outdate prob.	0.0196	0.0272	0.0301	0.0331	0.0364
Loss prob.	0.0000	0.0078	0.0107	0.0138	0.0171
Mean age of transfused units	39.1	12.9	12.2	11.7	11.6

Table 4 Performance of the threshold policy for $\mu=15, \lambda/\mu=1.02$ and $\vartheta=10$

the age of units at the time of arrival to inventory is low, in some hospitals the age of units at the time of receipt could be high (e.g., average 10.2 days reported for a hospital in Sayers and Centilli (2012)), making small threshold values relevant. In Table 4, we report the performance of the threshold policy with small threshold values for $\mu=15$ and $\lambda/\mu=1.02$ when $\vartheta=10$. We observe that when the age of units at the time of receipt is relatively high, using a policy with small thresholds could have significant benefits even when the demand size is large. Second, while we observe a low utilization of Stage 2 inventory under the assumption of stationary supply and demand, in practice, episodes of high demand or low supply could result in rapid consumption of units in Stage 2 and hence further differentiate the threshold policy from reduction of the shelf-life. Finally, as our numerical results suggest, the threshold policy provides similar or better performance when compared to reducing the shelf-life and hence can be considered as a practical alternative to it.

6.3. Going Beyond the Expected Age: Probability of Adverse Outcomes–Availability Trade-off

So far, we have focused on the expected age of transfused units when assessing the outcome of threshold policies. However, as mentioned in the introduction, the actual relationship between the age of RBCs and their health outcomes is still under investigation. Thus, although in general the proportion of units allocated from Stage 2 seems to be low and hence its effect on the average age of transfused units insignificant, one should still be cautious about the age of transfused units from Stage 2 as its health effects could be significant in the long term.

To investigate the factors that affect the distribution of the age of units transfused from Stage 2, we computed its cdf for different demand rates and supply-to-demand ratios. We observed that while the supply-to-demand ratio has little effect on the shape of the distribution, the demand size plays an important role. Figure 5 presents the cdf of the age of transfused units from Stage 2 for different supply-to-demand ratios and system sizes when $T=4$. We observe that a higher

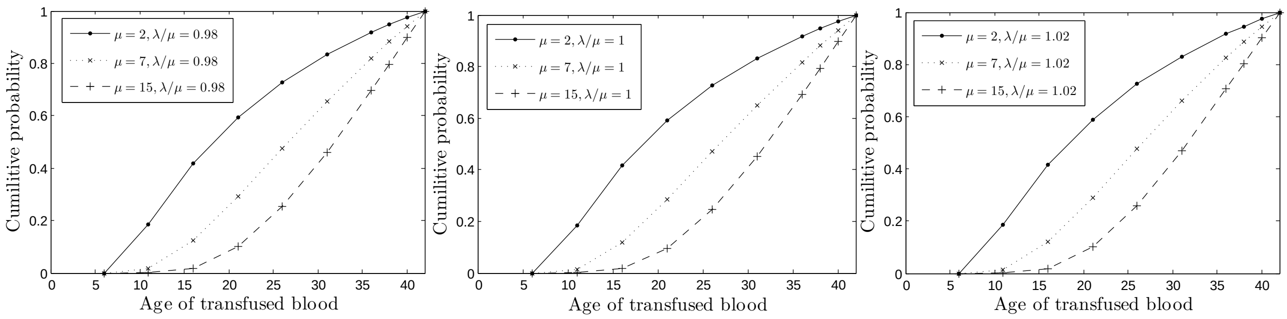

Figure 5 Distribution of the age of transfused units from Stage 2 for threshold policy with $T = 4$

demand size is associated with a higher age of transfused units from Stage 2. On the other hand, as the demand size decreases and hence a higher proportion of units are allocated from Stage 2, the age of transfused units tend to be smaller. The figure is representative and we observe a similar relationship for other threshold values not presented here.

Next, we examine the performance of the threshold policy under possible relationship functions mapping the age of RBCs to the probability of adverse outcome after the transfusion. We consider three hypothetical relationship functions suggested in Pereira (2013). The three models, which we refer to as Model 1, 2 and 3 (see Figure 6), are based on the time course of storage induced defects, reported in the medical literature. Model 3, for example, illustrates the case where units younger than 14 days have no clinically detectable effect, but after 14 days the effect rapidly increases to attain its maximum at 28 days. Given a relationship function $J(\cdot)$ and initial age of units ϑ, we estimate the Average Probability of Adverse Outcomes (APAO) under an allocation policy π as

$$\text{APAO}(\pi) = \sum_{n=1}^{\gamma} P(n-1 < A^{\pi} < n) J(n+\vartheta).$$

In figure 7, we illustrate the trade-off between APAO and proportion of lost demand achieved under different allocation policies and supply-to-demand ratios for a system with $\mu = 2$. As expected, under the linear relationship function (Model 1) the trade-off curves look similar to those in Figure 4, i.e., the curves for the lost demand versus the expected age trade-off. The results are however different for Models 2 and 3. We observe that a policy that results in a lower expected age could be associated with higher probability of adverse outcomes. Under Model 3, for instance, for threshold values 7, 4 and 0 (LIFO), both the proportion of loss demand and APAO are strictly higher when compared to those for threshold value 14. The reason is that for these threshold values, the relatively small proportion of units which are allocated from Stage 2 are associated with high probability of adverse outcomes. On the contrary, when $T = 14$ almost all units are allocated from

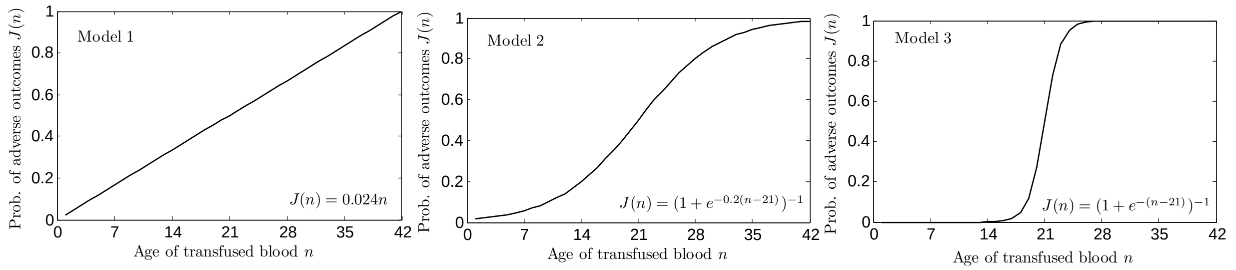

Figure 6 Three hypothetical models representing the relation between the age of transfused RBCs and probability of adverse clinical outcomes.

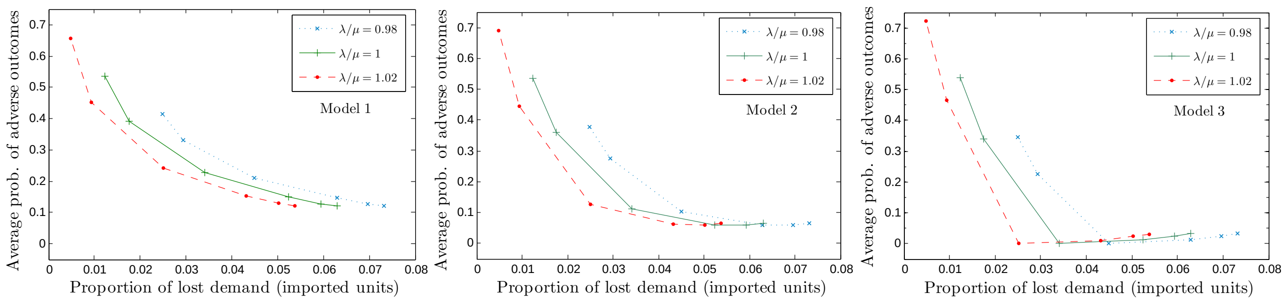

Figure 7 Trade-off between average probability of adverse outcomes and proportion of lost demand under three different relationship models for $\mu = 2$. The six tick marks on each curve from left to right correspond to threshold values of 40 (FIFO), 28, 14, 7, 4, and 0 (LIFO).

Stage 1, resulting in a smaller APAO and probability of loss compared to lower threshold values. We conclude that the underlying relationship between the age of RBCs and the corresponding probability of adverse outcomes is an important factor that must be taken into account when choosing an allocation policy. In particular, the expected age of transfused units may only be a suitable measure for assessing the outcome of allocation policies, if the relation between the probability of adverse outcomes and the age of RBCs is approximately linear.

Finally, while in the above trade-offs we used the loss probability as the measure for availability, one should notice that a low loss probability, e.g., when the supply-to-demand ratio is high, could be associated with a high outdate probability as reported in Tables 1 to 3. A high loss probability translates into a high import rate of blood for hospitals. If the imported units are of higher age, they could further increase the average probability of adverse outcomes. In this case the age of imported units must be taken into account when assessing the outcome of allocation policies.

7. Conclusions

We study a stochastic perishable inventory system operating under a family of threshold allocation policies. Our model captures the main operational features of a hospital blood bank. We provide an exact characterization of the sojourn time distribution of units for the threshold policy. This allows us to observe the age-availability trade-off by computing the distribution of the age of transfused units as well as the proportion of lost demand and outdates. Through a numerical study, we provide insights on the benefits of the threshold policy and identify important factors that should be considered when implementing any allocation policy. We observe that reducing the shelf-life, specially in large amounts, has a higher impact on availability for smaller hospitals and blood types with lower demand. We further find that in such cases the threshold policy is more effective. Therefore, we recommend the threshold policy to be considered as a viable and practical policy for reducing the age of transfused RBCs, specially where shortening the shelf-life is not feasible. Moreover, our numerical study shows that the actual benefits of allocation policies depend on the relationship between the entire distribution of the age of RBCs and the associated health outcomes. Thus, design and implementation of any allocation policy must be in light of results from the ongoing clinical trials.

Our study is the first to use an analytical approach in assessing the benefits of allocation policies in the context of blood transfusion. Accordingly, future work should investigate the effects of substitution of blood types, ordering policies, age of the units at the time of receipt by the hospital, and batch demands on the age-availability trade-off. While we expect our insights to hold under more general circumstances, empirical validation of the results would be of great value.

Finally, in this paper we focus on a simple and practical family of allocation policies. While characterizing the optimal policy is of interest, both from the theoretical and practical perspectives, it remains a challenging problem. Some initial work has been done by Sabouri et al. (2013).

Appendix A. Proof of Theorems 1 and 2.

Proof of Theorem 1. We first show that the cdf of $\tilde{S}_{2,k}$ converges to that of $\tilde{S}_2$ for all $x \in [0, \infty)$. Then, the second part follows from the continuity theorem for Laplace transforms (see Feller 1971, page 431). Let A_j denote the event that "the unit is allocated during the j^{th} idle period" and let $\bar{A}_k$ denote the event that "the unit is not allocated during any of the first k idle periods". Note that $\cup_{j=1}^{k} A_j = \{\tilde{S}_2 < U_k\}$ and $\bar{A}_k = \{\tilde{S}_2 \geq U_k\}$. Now consider the cdf of $\tilde{S}_{2,k}$. Using (31) we can write

$$
\begin{aligned}
P(\tilde{S}_{2,k} \leq x) &= \sum_{j=1}^{k} P(\tilde{S}_{2,k} \leq x | A_j) P(A_j) + P(\tilde{S}_{2,k} \leq x | \bar{A}_k) P(\bar{A}_k) \\
&= \sum_{j=1}^{k} P(\tilde{S}_2 \leq x | A_j) P(A_j) + P(U_k \leq x | \bar{A}_k) P(\bar{A}_k). \qquad (41)
\end{aligned}
$$

Letting $k \to \infty$ in (41), the last term on the RHS vanishes. To see this note that

$$\lim_{k\to\infty} P(U_k \le x|\bar{A}_k)P(\bar{A}_k) = \lim_{k\to\infty} P(U_k \le x, \bar{A}_k) \le \lim_{k\to\infty} P(U_k \le x) = 0,$$

where the last equality holds because U_k is the k^{th} renewal epoch of a delayed renewal process. Thus, for any $x \in [0,\infty)$ we have

$$\lim_{k\to\infty} P(\tilde{S}_{2,k} \le x) = \sum_{j=1}^{\infty} P(\tilde{S}_2 \le x|A_j)P(A_j) = P(\tilde{S}_2 \le x).$$

Note that since x is finite the last equality follows even if the cdf of $\tilde{S}_2$ is improper. Hence, the proof is complete. □

Proof of Theorem 2. To prove the theorem it is sufficient to show that for each $\epsilon > 0$, $\sum_{k=1}^{\infty} P(|S_{2,k} - S_2| \ge \epsilon) < \infty$. Then from the Borel-Cantelli Lemma we have $P(|S_{2,k} - S_2| \ge \epsilon \ i.o.) = 0$ (where *i.o.* stands for infinitely often) implying that $P(S_{2,k} \to S_2) = 1$ (see Billingsley 1995, page 70) as claimed. To this end, we consider the random variable $\tilde{S}_2$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and decompose the sample space Ω for $k \ge 1$ as

$$\begin{aligned}
C_{1,k} &\equiv \{\omega \in \Omega; \tilde{S}_2(\omega) \le U_k(\omega)\},\\
C_{2,k} &\equiv \{\omega \in \Omega; \gamma - T \le U_k(\omega) < \tilde{S}_2(\omega)\},\\
C_{3,k} &\equiv \{\omega \in \Omega; U_k(\omega) < \gamma - T \le \tilde{S}_2(\omega)\},\\
C_{4,k} &\equiv \{\omega \in \Omega; U_k(\omega) < \tilde{S}_2(\omega) < \gamma - T\},
\end{aligned}$$

such that $\cup_{i=1}^{4} C_{i,k} = \Omega$ for any $k \ge 1$. Note that since $S_{2,k} = \min(\tilde{S}_{2,k}, \gamma - T)$, for any $\omega \in C_{1,k} \cup C_{2,k}$ we have $S_2(\omega) = S_{2,k}(\omega)$. Thus, for each $\epsilon > 0$, $\{|S_{2,k} - S_2| \ge \epsilon\} \subseteq C_{3,k} \cup C_{4,k}$, implying that for all $k \ge 1$, $P(|S_{2,k} - S_2| \ge \epsilon) \le P(C_{3,k} \cup C_{4,k})$. Therefore,

$$\sum_{k=1}^{\infty} P(|S_{2,k} - S_2| \ge \epsilon) \le \sum_{k=1}^{\infty} P(C_{3,k} \cup C_{4,k}) \le \sum_{k=1}^{\infty} P(U_k \le \gamma - T).$$

It remains for us to show that $\sum_{k=1}^{\infty} P(U_k \le \gamma - T) < \infty$. Indeed, defining the stopping time $\sigma = \inf\{n \ge 1; U_n > \gamma - T\}$ we have

$$\sum_{k=1}^{\infty} P(U_k \le \gamma - T) = \sum_{k=1}^{\infty} P(\sigma > k) = E[\sigma] < \infty,$$

where the inequality follows from the fact that U_k is the k^{th} renewal epoch of a delayed renewal process and hence the expected time for it to pass any constant threshold is finite. □

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Online Appendix: Additional Proofs

Proof of Proposition 1. The outdate probability q^L is equal to the probability that the busy period lasts longer than γ, which gives (4). By definition $P(\tilde{S}^L \le x) = P(S^L \le x)$ for all $x < \gamma$, yielding (5) using (2). To obtain the expected value we use (5) to write

$$E[A^L] = \int_0^\infty (1 - A^L(y))dy = \int_0^\gamma (1 - A^L(y))dy = \gamma - \frac{\int_0^\gamma B(y)dy}{1 - q^L},$$

as given in (6). $\square$

Proof of Proposition 2. The outdates occur at instances when W hits zero, the long-run average rate of which is $f(0)$, and hence $q^F = f(0)/\lambda$ which yields (8). To obtain $A^F(\cdot)$, note that the remaining age of units at the time of transfusion is embedded at epochs right before jumps on a W sample path when $0 < W < \gamma$. Thus, from (7) and using the Poisson Arrival Sees Time Average (PASTA) property of the demand process, we have for $x < \gamma$,

$$\begin{aligned} A^F(x) = P(S^F \le x | S^F < \gamma) &= \lim_{t\to\infty} P(\gamma - W(t) \le x | 0 < W(t) < \gamma) \\ &= \lim_{t\to\infty} P(\gamma - x \le W(t) < \gamma)/P(0 < W(t) < \gamma) = \frac{\int_{\gamma-x}^\gamma f(y)dy}{\int_0^\gamma f(y)dy} \end{aligned}$$

which after simplification gives (9). Using (9) the derivation of (10) is straightforward. For the case where $\lambda = \mu$, the results are obtained by letting $\lambda \to \mu$ and applying L'Hôpital's rule. $\square$

Proof of Lemma 1. We first derive $\hat{h}(\theta)$. Note that with probability $e^{-\lambda T}$, we have $W(0) > T$, i.e., the first jump is greater than T and hence $\tau = 0$. Conditioning on $W(0)$ we have

$$\hat{h}(\theta) \equiv E[e^{-\theta\tau} | W(\tau) > T] = \frac{E[e^{-\theta\tau}\mathbf{1}_{\{W(\tau)>T\}}]}{P(W(\tau) > T)} = \frac{\int_0^T \hat{h}_x(\theta)\lambda e^{-\lambda x}dx + e^{-\lambda T}}{\int_0^T \hat{h}_x(0)\lambda e^{-\lambda x}dx + e^{-\lambda T}}.$$

Next, we consider $\hat{g}(\theta)$. Note that given $W(\tau) = 0$, the first jump must be some $x \le T$. Again by conditioning on $W(0)$ we get

$$\hat{g}(\theta) \equiv E[e^{-\theta\tau} | W(\tau) = 0] = \frac{E[e^{-\theta\tau}\mathbf{1}_{\{W(\tau)=0\}}]}{P(W(\tau) = 0)} = \frac{\int_0^T \hat{g}_x(\theta)\lambda e^{-\lambda x}dx}{\int_0^T \hat{g}_x(0)\lambda e^{-\lambda x}dx}.$$

Computing the integrals and noting that $P(W(\tau) = 0) + P(W(\tau) > T) = 1$, gives the results. $\square$

Proof of Proposition 3. Let $t = 0$ be the time the outdate occurs. Then W has just hit zero, causing a jump with size $W(0) \in (0, \infty)$ independent of the history of the process. If $W(0) > T$, then the residual busy period is over with $R = 0$. Otherwise, the residual busy period ends the first time W upcrosses T (see Figure 1 for a realization of R). However, before this happens, W may hit zero as new outdates may occur. Therefore, the probability that there are $M = m \ge 0$ outdates during the residual busy period is equal to the probability that W hits zero m times before upcrossing T. Note that from the strong Markov property of W, every time it hits zero the process regenerates and a new i.i.d. cycle starts. Also, the probability of hitting zero before crossing over T for each cycle is $p \equiv P(W(\tau) > T)$. Therefore, we have $P(M = m) = (1-p)^m p$. From Lemma 1, it then follows for the length of the residual busy period given the number of outdates, that

$$\hat{r}_m(\theta) \equiv E[e^{-sR}|M = m] = \hat{h}(\theta)\left(\hat{g}(\theta)\right)^m.$$

Removing the condition on M, we get the LT of the length of the busy period:

$$\hat{r}(\theta) \equiv E[e^{-\theta R}] = \sum_{m=0}^{\infty} P(M = m)E[e^{-\theta R}|M = m] = \sum_{m=0}^{\infty}(1-p)^m p\hat{h}(\theta)\left(\hat{g}(\theta)\right)^m = \frac{p\hat{h}(\theta)}{1-(1-p)\hat{g}(\theta)}.$$

□

Proof of Proposition 4. Each busy period starts with a fresh unit arriving at an empty system. Let $t = 0$ be the start of the busy period, then $W(0) = T$. The busy period ends the first time W upcrosses T. Each time W hits zero before this happens, a unit is outdated. Note that starting from level T two cases can occur: either W upcrosses T before hitting zero, with probability $P_T(W(\tau) > T) = \hat{h}_T(0)$, or it first hits zero, with probability $P_T(W(\tau) = 0) = \hat{g}_T(0) = 1 - \hat{h}_T(0)$. In the first case, the busy period ends with no outdates occurring during it, so $P(N = 0) = \hat{h}_T(0)$, and the LT of the conditional length of the busy period is

$$\hat{z}_0(\theta) \equiv E[e^{-\theta Z}|N = 0] = E_T[e^{-\theta\tau}|W(\tau) > T] = \hat{h}_T(\theta)/\hat{h}_T(0).$$

In the second case after W hits zero, by the strong Markov property of W, a new i.i.d. cycle independent of the history of the process starts and the time until W upcrosses T has the same distribution as a residual busy period. Thus, for $n > 0$ we have

$$P(N = n) = \hat{g}_T(0)P(M = n-1) = \hat{g}_T(0)p(1-p)^{n-1}.$$

That is, to have n outdates during the busy period, starting from $W(0) = T$, W must first hit zero and then before crossing level T it must hit zero $n-1$ ($n \geq 1$) additional times. It follows that

$$\hat{z}_n(\theta) \equiv E[e^{-\theta Z}|N = n] = E_T[e^{-s\tau}|W(\tau) = 0]E[e^{-\theta R}|M = n-1] = \frac{\hat{g}_T(\theta)}{\hat{g}_T(0)}\hat{r}_{n-1}(\theta).$$

Substituting from (19) we get the result for the $n > 0$ case in (22). Finally, by the same argument and again due to the strong Markov property of the process W, given $W(0) = T$ we have $Z = \tau + \mathbf{1}_{\{W(\tau)=0\}}R$. It follows that

$$\hat{z}(\theta) \equiv E[e^{-\theta Z}] = E_T[e^{-\theta\tau}\mathbf{1}_{\{W(\tau)>T\}}] + E_T[e^{-\theta(\tau+R)}\mathbf{1}_{\{W(\tau)=0\}}]$$

$$= E_T[e^{-\theta\tau}\mathbf{1}_{\{W(\tau)>T\}}] + E[e^{-\theta R}]E_T[e^{-\theta\tau}\mathbf{1}_{\{W(\tau)=0\}}]$$
$$= \hat{h}_T(\theta) + \hat{r}(\theta)\hat{g}_T(\theta). \tag{42}$$

Substituting $\hat{r}(\theta)$ from (20) we get (23) which completes the proof. □

Proof of Proposition 5. An idle period starts whenever W upcrosses T, with the length of the idle period being equal to the size of the over-shoot and hence exponentially distributed with parameter λ (see e.g., Kaspi and Perry 1983). To see (25), note that the demand and unit arrivals can be viewed as two competing Poisson processes, and hence $P(L=l)$ is the probability that the demand process with intensity μ wins l times before the arrival process does. Finally, (26) follows from the fact that the time between subsequent demand arrivals, given that the idle period has not ended, is exponentially distributed with rate $\lambda+\mu$. □

Proof of Lemma 2. Consider the tagged unit at the beginning of the i^{th} idle period with ν units in front of it. We condition on the number of demands arriving during the i^{th} idle period and consider two cases: (i) there are $L=l\le\nu$ and (ii) there are $L\ge\nu+1$ demands during the i^{th} idle period. In case (i), the unit is not allocated during the i^{th} idle period and hence will be in the system at the beginning of the $(i+1)^{st}$ idle period. Conditioning on the number of outdates during the $(i+1)^{st}$ busy period $N=n\ge 0$, the time interval between the start of the i^{th} and $(i+1)^{st}$ idle periods has LT $\hat{i}_l(\theta)\hat{z}_n(\theta)$. Also, the unit will have $\nu+n-l$ units in front of it at the beginning of the $(i+1)^{st}$ idle period and hence the LT of its remaining sojourn time at the beginning of the $(i+1)^{st}$ idle period is $\hat{\varphi}^k_{\nu+n-l,i+1}(\theta)$. It follows that the remaining sojourn time of the unit at the beginning of the i^{th} idle period given $N=n$ and $L=l\le\nu$ has LT $\hat{i}_l(\theta)\hat{z}_n(\theta)\hat{\varphi}^k_{\nu+n-l,i+1}(\theta)$. Removing the conditions on L and N, the first term on the right-hand side (RHS) follows. In case (ii), the unit is allocated during the i^{th} idle period. Thus, its remaining sojourn is equal to the time it takes until the $(\nu+1)^{st}$ demand arrival. Note that given the idle period has not ended, the time between demand arrivals are exponentially distributed with rate $\lambda+\mu$. Thus, the time until the arrival of the $(\nu+1)^{st}$ demand is Erlang distributed with parameter $(\lambda+\mu)$ and $(\nu+1)$ phases, and hence its LT is given by

$$\left(\frac{\lambda+\mu}{\mu+\lambda+\theta}\right)^{\nu+1}. \tag{43}$$

Also, from Proposition 5 the event $\{L\ge\nu+1\}$ has probability

$$\sum_{l=\nu+1}^{\infty} P(L=l) = 1 - \sum_{l=0}^{\nu}\left(\frac{\mu}{\mu+\lambda}\right)^l\left(\frac{\lambda}{\mu+\lambda}\right) = \left(\frac{\mu}{\mu+\lambda}\right)^{\nu+1},$$

which after being multiplied by (43) gives the second term on the RHS. □

Proof of Theorems 3 and 4

We need the following lemmas before presenting the proofs.

LEMMA 3. *For $\omega \geq 0$ we have*

$$\sum_{l=0}^{v} \binom{v+n-l+1}{\omega} = \sum_{\kappa=0}^{\omega} \binom{v+1}{\omega+1-\kappa} \binom{n+1}{\kappa}. \tag{44}$$

Proof. Starting from the left-hand side (LHS) we first claim that

$$\sum_{l=0}^{v} \binom{v+n-l+1}{\omega} = \binom{v+n+2}{\omega+1} - \binom{n+1}{\omega+1}, \tag{45}$$

which can be proved by induction on v. For $v=0$ (45) becomes

$$\binom{n+1}{\omega} = \binom{n+2}{\omega+1} - \binom{n+1}{\omega+1},$$

which is the Pascal's recurrence (see, e.g., Gross 2008, page 218). Now assume for some $v \geq 1$ that

$$\sum_{l=0}^{v-1} \binom{v+n-l}{\omega} = \binom{v+n+1}{\omega+1} - \binom{n+1}{\omega+1},$$

then

$$\begin{aligned}
\sum_{l=0}^{v} \binom{v+n-l+1}{\omega} &= \binom{v+n+1}{\omega} + \sum_{l=1}^{v} \binom{v+n-l+1}{\omega} = \binom{v+n+1}{\omega} + \sum_{l=0}^{v-1} \binom{v+n-l}{\omega} \\
&= \binom{v+n+1}{\omega} + \binom{v+n+1}{\omega+1} - \binom{n+1}{\omega+1} \quad \text{(induction hypothesis)} \\
&= \binom{v+n+2}{\omega+1} - \binom{n+1}{\omega+1}, \quad \text{(Pascal's recurrence)}
\end{aligned}$$

as claimed. Next, applying Vandermonde's convolution (see, e.g., Gross 2008, page 226) to the first term we get

$$\binom{v+n+2}{\omega+1} = \sum_{\kappa=0}^{\omega+1} \binom{v+1}{\omega+1-\kappa} \binom{n+1}{\kappa} = \sum_{\kappa=0}^{\omega} \binom{v+1}{\omega+1-\kappa} \binom{n+1}{\kappa} + \binom{n+1}{\omega+1},$$

which after substituting in (45) gives (44). □

LEMMA 4. *For $i \geq 0$ we have*

$$\sum_{n=0}^{\infty} \left(\frac{\mu}{\lambda+\mu+\theta}\right)^{n} P(N=n)\hat{z}_n(\theta) \binom{n+1}{i} = \xi_i(\theta), \tag{46}$$

$$\sum_{m=0}^{\infty} \left(\frac{\mu}{\lambda+\mu+\theta}\right)^{m+1} P(M=m)\hat{r}_m(\theta) \binom{m+1}{i} = \beta_i(\theta) \tag{47}$$

with $\xi_i(\theta)$ and $\beta_i(\theta)$ given in (36) and (37), respectively.

Proof. We give a proof for (47); (46) can be obtained similarly. Substituting for $P(M=l)$ and $\hat{r}_m(\theta)$ from (18) and (19) into (47), and using the definitions in (35) the LHS becomes

$$c_1(\theta)\sum_{m=0}^{\infty}(c_2(\theta))^m\binom{m+1}{i},$$

establishing (47) for $i=0$. To obtain the formula for $i\geq 1$, note that

$$\begin{aligned}\frac{i!}{(1-c_2(\theta))^{i+1}} = \frac{d^i}{d\,(c_2(\theta))^i}\sum_{m=0}^{\infty}(c_2(\theta))^m = \sum_{m=0}^{\infty}\frac{d^i}{d\,(c_2(\theta))^i}(c_2(\theta))^m &= \sum_{m=0}^{\infty}m(m-1)\cdots(m-i+1)\,(c_2(\theta))^{m-i}\\ &= \sum_{m=i}^{\infty}m(m-1)\cdots(m-i+1)\,(c_2(\theta))^{m-i}.\end{aligned}$$

Multiplying both sides by $c_2(\theta)^{i-1}/i!$ yields:

$$\frac{(c_2(\theta))^{i-1}}{(1-c_2(\theta))^{i+1}} = \sum_{m=i}^{\infty}(c_2(\theta))^{m-1}\binom{m}{i} = \sum_{m=i-1}^{\infty}(c_2(\theta))^m\binom{m+1}{i} = \sum_{m=0}^{\infty}(c_2(\theta))^m\binom{m+1}{i},$$

from which (47) follows. $\square$

LEMMA 5. *For $\nu, i \geq 0$ the following identity holds:*

$$\sum_{l=0}^{\nu}\sum_{n=0}^{\infty}P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)Y_{\nu+n-l}(i) = \left(\frac{\lambda}{\lambda+\mu+\theta}\right)Y_\nu(i+1). \tag{48}$$

Proof. Substituting for $P(L=l), \hat{i}_l(\theta)$ and $Y_{\nu+n-l}(i)$ into the LHS from (25),(26) and (38), respectively and changing the order of sums, after some simplifications we can rewrite the LHS as

$$\begin{aligned}&\left(\frac{\mu}{\lambda+\mu+\theta}\right)^{\nu+1}\left(\frac{\lambda}{\lambda+\mu+\theta}\right)\xi_0(\theta)\sum_{j_1=0}^{1}\xi_{j_1}(\theta)\sum_{j_2=0}^{2-j_1}\xi_{j_2}(\theta)\cdots\sum_{j_{i-1}=0}^{(i-1)-j_{i-2}-\cdots-j_1}\xi_{j_{i-1}}(\theta)\\ &\qquad\sum_{n=0}^{\infty}\left(\frac{\mu}{\lambda+\mu+\theta}\right)^n P(N=n)\hat{z}_n(\theta)\sum_{l=0}^{\nu}\binom{\nu+n-l+1}{i-j_{i-1}-\cdots-j_1}.\end{aligned} \tag{49}$$

Using Lemma 3 the last sum is

$$\sum_{l=0}^{\nu}\binom{\nu+n-l+1}{i-j_{i-1}-\cdots-j_1} = \sum_{j_i=0}^{i-j_{i-1}-\cdots-j_1}\binom{\nu+1}{(i+1)-j_i-j_{i-1}-\cdots-j_1}\binom{n+1}{j_i},$$

which allows us to rewrite (49) as

$$\begin{aligned}&\left(\frac{\mu}{\lambda+\mu+\theta}\right)^{\nu+1}\left(\frac{\lambda}{\lambda+\mu+\theta}\right)\xi_0(\theta)\sum_{j_1=0}^{1}\xi_{j_1}(\theta)\sum_{j_2=0}^{2-j_1}\xi_{j_2}(\theta)\cdots\sum_{j_{i-1}=0}^{(i-1)-j_{i-2}-\cdots-j_1}\xi_{j_{i-1}}(\theta)\\ &\qquad\sum_{j_i=0}^{i-j_{i-1}-\cdots-j_1}\sum_{n=0}^{\infty}\left(\frac{\mu}{\lambda+\mu+\theta}\right)^n P(N=n)\hat{z}_n(\theta)\binom{n+1}{j_i}\binom{\nu+1}{(i+1)-j_i-\cdots-j_1}.\end{aligned} \tag{50}$$

Rearranging the terms in (50) and using (46) we arrive at the RHS:

$$\left(\frac{\lambda}{\lambda+\mu+\theta}\right)\xi_0(\theta)\sum_{j_1=0}^{1}\xi_{j_1}(\theta)\sum_{j_2=0}^{2-j_1}\xi_{j_2}(\theta)\cdots\sum_{j_{i-1}=0}^{(i-1)-j_{i-2}-\cdots-j_1}\xi_{j_{i-1}}(\theta)$$
$$\sum_{j_i=0}^{i-j_{i-1}-\cdots-j_1}\xi_{j_i}(\theta)\left(\frac{\mu}{\lambda+\mu+\theta}\right)^{\nu+1}\binom{\nu+1}{(i+1)-j_i-\cdots-j_1}=\left(\frac{\lambda}{\lambda+\mu+\theta}\right)Y_\nu(i+1).$$

This completes the proof. □

Proof of Theorem 3. The proof is by induction on k. For $k=1$, noting that $\hat{\varphi}^k_{\nu,k+1}(\theta)=1$ and using (34) we have

$$\begin{aligned}\hat{\varphi}^1_{\nu,1}(\theta) &= \sum_{l=0}^{\nu}\sum_{n=0}^{\infty}P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)\hat{\varphi}^1_{\nu+n-l,2}(\theta)+\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}\\ &= \sum_{l=0}^{\nu}P(L=l)\hat{i}_l(\theta)\sum_{n=0}^{\infty}P(N=n)\hat{z}_n(\theta)+\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}. \end{aligned} \tag{51}$$

Noting that

$$\sum_{l=0}^{\nu}P(L=l)\hat{i}_l(\theta)=\sum_{l=0}^{\nu}\left(\frac{\lambda}{\mu+\lambda}\right)\left(\frac{\mu}{\mu+\lambda}\right)^l\left(\frac{\lambda+\mu}{\lambda+\mu+\theta}\right)^{l+1}=\left(\frac{\lambda}{\mu+\lambda}\right)\left(1-\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}\right), \tag{52}$$

and using (24), we can simplify (51) to obtain

$$\begin{aligned}\hat{\varphi}^1_{\nu,1}(\theta) &= \left(1-\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}\right)\hat{i}(\theta)\hat{z}(\theta)+\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}\\ &= \left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}(1-\hat{i}(\theta)\hat{z}(\theta))+\hat{i}(\theta)\hat{z}(\theta)=Y_\nu(0)(1-\hat{i}(\theta)\hat{z}(\theta))+\hat{i}(\theta)\hat{z}(\theta),\end{aligned}$$

which establishes (39) for $k=1$. No assume that (39) holds for some $k\geq 1$. From (34) we have

$$\hat{\varphi}^{k+1}_{\nu,1}(\theta)=\sum_{l=0}^{\nu}\sum_{n=0}^{\infty}P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)\hat{\varphi}^{k+1}_{\nu+n-l,2}(\theta)+\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}. \tag{53}$$

Note that by construction of the k^{th} modified system we have $\hat{\varphi}^{k+1}_{\nu+n-l,2}(\theta)=\hat{\varphi}^{k}_{\nu+n-l,1}(\theta)$. Substituting $\hat{\varphi}^{k}_{\nu+n-l,1}(\theta)$ for $\hat{\varphi}^{k+1}_{\nu+n-l,2}(\theta)$ in (53) and using (39) we get

$$\begin{aligned}\hat{\varphi}^{k+1}_{\nu,1}(\theta) &= \sum_{l=0}^{\nu}\sum_{n=0}^{\infty}P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)\\ &\quad\times\left(\sum_{i=0}^{k-1}Y_{\nu+n-l}(i)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i+(\hat{i}(\theta)\hat{z}(\theta))^k\right)+\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}\\ &= \sum_{l=0}^{\nu}\sum_{n=0}^{\infty}P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)\sum_{i=0}^{k-1}Y_{\nu+n-l}(i)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i\\ &\quad+\sum_{l=0}^{\nu}\sum_{n=0}^{\infty}P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)(\hat{i}(\theta)\hat{z}(\theta))^k+\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}.\end{aligned} \tag{54}$$

We start with the first term in (54). Rearranging the terms and applying Lemma 5 we have

$$\begin{aligned}
&\sum_{l=0}^{\nu}\sum_{n=0}^{\infty} P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)\sum_{i=0}^{k-1} Y_{\nu+n-l}(i)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i\\
&=\sum_{i=0}^{k-1}\sum_{l=0}^{\nu}\sum_{n=0}^{\infty} P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)Y_{\nu+n-l}(i)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i\\
&=\sum_{i=0}^{k-1}\left(\frac{\lambda}{\lambda+\mu+\theta}\right)Y_\nu(i+1)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i\\
&=\sum_{i=1}^{(k+1)-1} Y_\nu(i)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{(k+1)-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i. \qquad (55)
\end{aligned}$$

The second term in (54) can be evaluated by rearranging the sums and using (52):

$$\sum_{l=0}^{\nu}\sum_{n=0}^{\infty} P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)(\hat{i}(\theta)\hat{z}(\theta))^k=\left(1-\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}\right)\left(\hat{i}(\theta)\hat{z}(\theta)\right)^{k+1}. \qquad (56)$$

Finally, substituting (55) and (56) into (54) we get

$$\begin{aligned}
\hat{\varphi}_{\nu,1}^{k+1}(\theta) &= \left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}+\sum_{i=1}^{(k+1)-1} Y_\nu(i)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{(k+1)-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i+\left(1-\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}\right)\left(\hat{i}(\theta)\hat{z}(\theta)\right)^{k+1}\\
&=\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k+1}\right)+\sum_{i=1}^{(k+1)-1} Y_\nu(i)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{(k+1)-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i+\left(\hat{i}(\theta)\hat{z}(\theta)\right)^{k+1}\\
&=\sum_{i=0}^{(k+1)-1} Y_\nu(i)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{(k+1)-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i+\left(\hat{i}(\theta)\hat{z}(\theta)\right)^{k+1}.
\end{aligned}$$

This completes the proof. □

Proof of Theorem 4. Substituting $\hat{\varphi}_{\nu,1}^k(\theta)$ from (39) into (33) we have

$$\begin{aligned}
E[e^{-\theta\tilde{S}_{2,k}}] &= \sum_{\nu=0}^{\infty} P(M=\nu)\hat{r}_\nu(\theta)\left(\sum_{i=0}^{k-1} Y_\nu(i)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i+(\hat{i}(\theta)\hat{z}(\theta))^k\right)\\
&=\sum_{i=0}^{k-1}\sum_{\nu=0}^{\infty} P(M=\nu)\hat{r}_\nu(\theta)Y_\nu(i)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i+\sum_{\nu=0}^{\infty} P(M=\nu)\hat{r}_\nu(\theta)(\hat{i}(\theta)\hat{z}(\theta))^k.
\end{aligned} \qquad (57)$$

We deal with the two terms separately. First, substituting for $Y_\nu(i)$ in the first term and using Lemma 4, we have

$$\begin{aligned}
&\sum_{\nu=0}^{\infty}\sum_{i=0}^{k-1} P(M=\nu)\hat{r}_\nu(\theta)Y_\nu(i)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i\\
&=\sum_{i=0}^{k-1}\Bigg[\xi_0(\theta)\sum_{j_1=0}^{1}\xi_{j_1}(\theta)\cdots\sum_{j_{i-1}=0}^{(i-1)-j_{i-2}-\cdots-j_1}\xi_{j_{i-1}}(\theta)\sum_{\nu=0}^{\infty} P(M=\nu)\hat{r}_\nu(\theta)\left(\frac{\mu}{\lambda+\mu+\theta}\right)^{\nu+1}\binom{\nu+1}{i-j_{i-1}-\cdots-j_1}\\
&\quad\times\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i\Bigg]\\
&=\sum_{\nu=0}^{\infty} P(M=\nu)\hat{r}_\nu(\theta)\left(\frac{\mu}{\lambda+\mu+\theta}\right)^{\nu+1}\left(1-(\hat{i}(\theta)\hat{z}(\theta))^k\right)
\end{aligned}$$

$$+\sum_{i=1}^{k-1}\left[\xi_0(\theta)\sum_{j_1=0}^{1}\xi_{j_1}(\theta)\cdots\sum_{j_{i-1}=0}^{(i-1)-j_{i-2}-\cdots-j_1}\xi_{j_{i-1}}(\theta)\sum_{\nu=0}^{\infty}P(M=\nu)\hat{r}_\nu(\theta)\left(\frac{\mu}{\lambda+\mu+\theta}\right)^{\nu+1}\binom{\nu+1}{i-j_{i-1}-\cdots-j_1}\right.$$
$$\left.\times\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i\right]$$
$$=\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k}\right)\beta_0(\theta)+$$
$$+\sum_{i=1}^{k-1}\left[\xi_0(\theta)\sum_{j_1=0}^{1}\xi_{j_1}(\theta)\cdots\sum_{j_{i-1}=0}^{(i-1)-j_{i-2}-\cdots-j_1}\xi_{j_{i-1}}(\theta)\beta_{i-j_{i-1}-\cdots-j_1}(\theta)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i\right]. \quad (58)$$

Using the definition of $X(d,w)$ in (40) for $w=1$, (58) becomes

$$\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k}\right)\beta_0(\theta)+\xi_0(\theta)\sum_{i=1}^{k-1}X(i,1)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i. \quad (59)$$

Next, the second term in (57) is simply

$$\sum_{\nu=0}^{\infty}P(M=\nu)\hat{r}_\nu(\theta)(\hat{i}(\theta)\hat{z}(\theta))^k=(\hat{i}(\theta)\hat{z}(\theta))^k\hat{r}(\theta). \quad (60)$$

Substituting (59) and (60) back into (57) we have

$$E[e^{-\theta\tilde{S}_{2,k}}]=(\hat{i}(\theta)\hat{z}(\theta))^k\hat{r}(\theta)+\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k}\right)\beta_0(\theta)$$
$$+\xi_0(\theta)\sum_{i=1}^{k-1}X(i,1)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i \quad (61)$$
$$=(\hat{i}(\theta)\hat{z}(\theta))^k\hat{r}(\theta)+\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k}\right)\beta_0(\theta)$$
$$+\xi_0(\theta)\sum_{i=1}^{k-1}X(i,1)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i-\xi_0(\theta)\sum_{i=1}^{k-1}X(i,1)(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i. \quad (62)$$

Letting $k\to\infty$ in (62), the first term goes to 0 and the second term converges to $\beta_0(\theta)$. Therefore, to get the final result it remains to show that for all $\theta>0$ the last term converges to 0 as $k\to\infty$. To this end, consider $E[e^{-\theta\tilde{S}_{2,k}}]$ in (61) and note that it converges if and only if the sequence $\{F_k\}_{k\geq 2}$ with $F_k\equiv\sum_{i=1}^{k-1}X(i,1)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i$, is convergent. However, from Theorem 1 we know that $E[e^{-\theta\tilde{S}_{2,k}}]$ converges and therefore $\{F_k\}$ is indeed convergent. Thus, we have $|F_{k+1}-F_k|\to 0$ as $k\to\infty$. Now observe that

$$F_{k+1}-F_k=\sum_{i=1}^{k}X(i,1)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k+1-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i-\sum_{i=1}^{k-1}X(i,1)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i$$
$$=X(k,1)\left(1-\hat{i}(\theta)\hat{z}(\theta)\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^k+\left(1-\hat{i}(\theta)\hat{z}(\theta)\right)\sum_{i=1}^{k-1}X(i,1)(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i$$
$$=\left(1-\hat{i}(\theta)\hat{z}(\theta)\right)\sum_{i=1}^{k}X(i,1)(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i,$$

and hence, $\lim_{k\to\infty}\sum_{i=1}^{k}X(i,1)(\hat{i}(\theta)\hat{z}(\theta))^{k-i}(\frac{\lambda}{\lambda+\mu+\theta})^i=0$, for all $\theta>0$. Finally, noting that for $k\geq 2$

$$\sum_{i=1}^{k}X(i,1)(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i\geq\sum_{i=1}^{k-1}X(i,1)(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i,$$

we can conclude that the last term in (62) vanishes as $k\to\infty$, which completes the proof. □