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RESEARCH ARTICLE

Maximal or Minimal Differentiation in a Hotelling Market? A Fresh Perspective

Amit Pazgal[1] · **David Soberman**[2] · **Raphael Thomadsen**[3]

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Abstract A perplexing problem in spatial modelling—going back to Hotelling's linear market—is whether firms will cluster together or separate themselves. Maximal differentiation is the prevailing equilibrium when travel costs are quadratic and minimal differentiation results when price competition is limited. The reality for most markets is that the force that draws firms together (maximize demand) and the force that causes them to separate (avoid price competition) are both present. In many cases, this makes the characterization of an equilibrium difficult. The vast majority of research using the Hotelling model is based on the assumption that all potential consumers buy, yet the reality of many markets is that there are some consumers who seriously consider not buying. When allowing for the possibility that some consumers would consider not buying from either firm, we are able to identify equilibrium locations for firms that first choose locations and then prices in a Hotelling market with linear travel costs. Following the discussion above, we consider ranges of consumers' willingness to pay for the products relative to the outside good such that the market is not necessarily covered for all location choices. The analysis demonstrates the existence of a pure-strategy location equilibrium, supported by a pure-strategy pricing equilibrium, where firms are moderately differentiated and the market is covered.

Keywords Differentiation · Positioning · Price competition · Pure-strategy equilibrium

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Amit Pazgal, David Soberman and Raphael Thomadsen contributed equally to this work.

✉ Raphael Thomadsen
thomadsen@wustl.edu

1 Rice University, Houston, TX, USA

2 Rotman School of Management, University of Toronto, Toronto, ON, Canada

3 Olin Business School, Washington University in St. Louis, One Brookings Dr., Knight Hall 404, St. Louis, MO 63130, USA

1 Introduction

A central question in competitive analysis is the extent to which firms choose to differentiate or gravitate towards the center of a market. A seminal paper by Hotelling [6] on this issue demonstrates a tendency towards *minimal differentiation* where firms locate centrally to maximize market share. The problem with Hotelling's insight is that it does not hold when firms compete in prices after choosing locations. In fact, with linear travel costs, it is a challenge to even determine equilibrium location choices when firms choose prices competitively in a second stage.[1]

One solution to this problem is proposed by d'Aspremont et al. [1]. In that model, the authors solve for optimal locations by assuming that travel costs are quadratic as opposed to linear. In contrast to the Hotelling's model, the d'Aspremont et al. model generates a prediction of *maximum differentiation*. The use of quadratic travel costs ensures the existence of a pure-strategy pricing equilibrium for all possible locations. However, quadratic travel costs may not be reasonable for many industries; for example, papers studying geographic travel costs typically find that marginal travel costs decrease with greater amounts of travel (e.g., Davis [3]).

[1] Eaton and Lipsey [4] highlight the difficulties of identifying pure strategy location equilibria in the context of the Hotelling market.

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In sum, the literature does not provide a clear answer to the central question of whether firms will differentiate or cluster together in a market. Our goal is to provide new insight regarding this question by highlighting an aspect of markets that has been ignored. It turns out that the answer to the question depends on the willingness to pay for a product in the market (or, equivalently, the potential net benefit that a consumer realizes by participating in the market). In the majority of the papers that address the question of "positioning," consumers are assumed to have such high utility for the product that the no-purchase option is never considered. For example, Osborne and Pitchik [9] solve for equilibrium locations when consumers have linear travel costs and individual rationality is always strictly satisfied. They find that the firms locate at moderate distances from each other and engage in a mixed-strategy pricing equilibrium. Thus, the Osborne and Pitchik model does suggest a compromise between minimal and maximal differentiation, but only in a context where firms employ mixed pricing strategies and every consumer buys from one of the firms.

The starting points for our analysis are the following: (a) there are many markets where linear travel costs represent a better fit between a product and a consumer's needs than quadratic costs; (b) in many markets, stable (or pure strategy) prices are observed when firms choose locations before prices, and (c) many markets are characterized by a positive fraction of consumers that seriously consider not buying from either firm. In fact, we contend that the conditions of point c are more common than the conditions analyzed by the vast majority of research on this topic. In most markets, marketing campaigns are designed to generate demand not only from consumers who are contemplating the purchase of competitive products but also from consumers considering not buying anything at all. Even in mature markets, Kotler and Keller [8] note that there may be many potential users who are on the borderline between participating in the market and not buying at all.

In a nutshell, our analysis shows that firms balance the tendencies of minimal and maximal differentiation when the presence (or choice) of an outside good has a significant effect on the positioning and pricing decisions of firms. Similar to Osborne and Pitchik [9], firms are observed to choose compromise locations; however, this result obtains in a market where firms employ *pure strategies* in pricing.

Specifically, the firms choose locations and prices such that the consumers at the edge of the market are indifferent between buying and not buying. Moreover, the locations chosen by the firms are unique: were firms to adjust locations marginally, prices in the second stage would adjust to cover the entire market but profits would be strictly lower. Quite simply, the optimal locations balance the force that moves firms apart with the force that brings firms together. Locations near the center of the market lead to lower profits because consumers near the edge of the market would not buy and either firm could increase its profits by moving towards the unserved consumers and raising prices. Conversely, firms do not locate too close to the edge of the market, because that would lead to losing too many consumers in the middle of the market.

2 The Model

Our model generally follows the standard Hotelling framework. Consumers of measure 1 are uniformly spread along a market on the interval [0,1]. Consumers can consume at most one unit of one product; the utility consumer i obtains from consuming product j is $U_{ij}=V-p_j-d_{ij}$, where p_j is the price of product j and d_{ij} is the linear distance between consumer i and product j. Consumers may also choose not to consume any products, in which case the consumers earn U_{i0} which is normalized to zero.

There are two firms which have constant marginal costs of production (normalized to zero without loss of generality). The game consists of two periods. In period 1, firms simultaneously choose locations for their products along the [0,1] interval. In period 2, firms simultaneously choose prices.

Consumers will buy whichever product delivers the greatest utility, as long as the utility of the product is greater than zero. In order to guarantee the existence of an equilibrium, we assume that if both firms deliver the same (non-negative) utility, then the consumer buys from the firm that is located closest to their location.[2] This assumption ensures that there exists a (possibly mixed-strategy) Nash equilibrium in prices for any sets of firm locations (see Theorem 5 of Dasgupta and Maskin [2]). Osborne and Pitchik [9] use equivalent assumptions to ensure the existence of an equilibrium in their paper using a very similar model.

Rather than solve for prices for all potential sets of locations, we solve for the subgame perfect equilibrium by proposing an equilibrium and solving for the second-stage prices that would arise from any potential deviation.

3 The Result

Given the model structure described above, the following proposition describes the equilibrium in the first stage of the game where firms choose location along the market and the second-stage prices that support it. The proof of the proposition is provided in the Appendix.

Proposition There exists an equilibrium where the locations and equilibrium prices for firms are as follows[3]:

[2] This is important for the consumers located near the edge of the market. If a consumer is located exactly halfway in between the firms and they earn equal utility from both firms, one can set up any arbitrary rule, such as purchasing from each firm with a 50 % probability.

[3] The location of A is normalized to be less than the location of B. There is another equilibrium where the locations of the firms are switched.

If $V \in \left[\frac{3}{4}, \frac{7}{8}\right]$, firm A (B) locates distance of $V - {}^{1}/{}_{2}$ internally from point 0 (1), and both firms set price=½.

If $V \in \left[\frac{1}{2}, \frac{3}{4}\right]$, firm A (B) locates distance of ${}^{1}/{}_{4}$ internally from point 0 (1), and both firms set price=$V-¼$.

If $V < {}^{1}/{}_{2}$, both firms locate such that each operates as a separate monopoly setting a price of $V/2$ selling to a market of size V.

In contrast to cases where V is sufficiently high such that the outside option does not affect firm decision, the pure-strategy location equilibrium is supported by pure-strategy prices when V is lower. When $V \in \left[\frac{1}{2}, \frac{7}{8}\right]$, firms choose locations such that in the pricing subgame, consumers at 0 and 1 obtain exactly zero utility. For locations where the extreme consumers obtain zero surplus, prices are set a kink point in the demand curve and not according to first-order conditions. Note that for these values of V, there are sets of locations where the firms would set prices according to first-order conditions (locations where the firms are closer together). Accordingly, this result is based on the optimal choice of locations by the firms. The model reflects two counteracting forces that affect firm decisions. The first is the incentive to differentiate (or move towards the edge of the market) to ensure that the market is covered and reduce the intensity of price competition. The second is the incentive to locate centrally to obtain as large a share of the market as possible.

A key challenge in proving the proposition is the requirement to demonstrate that profits cannot increase for a firm that deviates to a location that would lead to a mixed-strategy pricing equilibrium. While we cannot explicitly solve that pricing equilibrium, we make use of a boundary argument to show that the profits earned in equilibrium are higher than the profits earned from a deviation that triggers a mixed-strategy pricing equilibrium.

We should also note that when $V < {}^{1}/{}_{2}$, both firms charge ${}^{V}/{}_{2}$ and can operate as separate monopolies leaving a measureable fraction of potential consumers unserved.

4 Discussion

In this section, we provide an outline of the proof. Suppose that $V \in \left(\frac{3}{4}, \frac{7}{8}\right)$. The proposed equilibrium locations are that firm A locates a distance $a = V - {}^{1}/{}_{2}$ internally from point 0, and firm B locates a distance $b = V - {}^{1}/{}_{2}$ internally from point 1. Prices are ${}^{1}/{}_{2}$, and each firm earns a profit of ${}^{1}/{}_{4}$. It is straightforward to confirm that this price is indeed an equilibrium price. Small price reductions do not increase profits because there is no market expansion at the edge of the market to offset the thinner margins in the middle of the market, while increases in prices lead to losing customers both in the center and the edge of the market. A discrete drop in price to $2V - {}^{3}/{}_{2}$ would allow a firm to capture the entire market but $2V - {}^{3}/{}_{2}$ is strictly less than ¼ in the $\left(\frac{3}{4}, \frac{7}{8}\right)$ interval.

The issue is therefore whether either firm has an incentive to deviate from these locations. Because of symmetry, we only need to consider whether firm A has a profitable deviation.

There will be no local deviation. To see this, we note that the binding kink point in the demand curve remains at the edge of the market if A changes its location to another location $\tilde{a}$ such that $\max\left(V - {}^{2}/{}_{3}, 1-V\right) \leq \tilde{a} \leq \min\left(V - {}^{2}/{}_{5}, V + \sqrt{2}(1-V) - 1\right)$. In such a case, firm A charges $V-\tilde{a}$, while firm B charges $V - b = {}^{1}/{}_{2}$. Firm A's profits are then $\pi^{\tilde{A}} = (V-a)\left({}^{1}/{}_{2} + a - b\right) = (V-a)\left({}^{1}/{}_{2} + a - \left(V - {}^{1}/{}_{2}\right)\right) = (V-a)\,(1-(V-a))$. Setting $\frac{\partial \pi^{\tilde{A}}}{\partial a} = 0$ yields $a = V - {}^{1}/{}_{2}$.

Because the equilibrium locations satisfy first-order conditions, the only potential profitable deviations we must consider are locations with a different price equilibrium. These entail three types of deviations.

First, firm A could deviate towards 0 to location $\tilde{a} < \max\left(V - {}^{2}/{}_{3}, 1-V\right)$. Then, in the second period, firm B will charge the same price, $V - b = {}^{1}/{}_{2}$, and A will charge either (1) $V-(1-\tilde{a}-2b)=3V-2+\tilde{a}$ (kink point where consumers at a point between firms A and B get 0 utility when $p_B=V-b$) or (2) set the price such that all of the firm's consumers get positive surplus. We present the reasoning in the Appendix to explain why neither of these possibilities leads to an increase in profits for firm A.

Second, firm A could move to a location $\min\left(V - {}^{2}/{}_{5}, V + \sqrt{2(1-V)} - 1\right) < \tilde{a} < 1-b$, towards firm B. In such a case, there are two possibilities to consider. First, firm A might locate where the pricing strategy entails a pure-strategy pricing strategy where the consumers located at 0 choose the outside option (because lowering the prices such that these consumers still choose to consume cuts margins too much), or firm A might locate so close to firm B that the only pricing equilibrium is a mixed-strategy equilibrium. We address both of these cases in the Appendix. The case where firms price according to a pure-strategy equilibrium can be solved directly. However, we cannot solve the mixed-strategy price equilibrium directly. Nevertheless, in the Appendix, we show that it is still possible to calculate bounds on the profits that firm A would earn in any mixed-strategy equilibrium, and these profits must be less than ${}^{1}/{}_{4}$.

Finally, firm A could deviate to a location between firm B and 1. While it might seem to be quite intuitive that deviating to a location on the short end of the market would lead to competition, this scenario again can lead to mixed-strategy equilibria. Thus, we again use a bounding argument to show that such an equilibrium is not optimal.

The logic for the case where $V \in \left(\frac{1}{2}, \frac{3}{4}\right)$ is very similar, although as we note in the Appendix, some of the math differs slightly. The case where $V < \frac{1}{2}$ is obvious upon inspection, although a formal proof can be supplied upon request.

5 Conclusion

In summary, the question of whether firms will position their products to ensure high levels of differentiation or whether they will gravitate closer to the median preference has been widely researched in the literature. Moreover, this question applies to many markets, from those that are literally characterized by location-based competition (e.g., retailing) to contexts where the dimension of competition is entirely perceptual (e.g., politics). The reality of most markets is one in which firms are observed to compromise and a moderately differentiated equilibrium appears to be reached. Nevertheless in the world of theory, it has been difficult to reflect this reality with an adequate model because the identification of a stable equilibrium is challenging with the standard assumptions of (a) linear travel costs, (b) two competing firms, and (c) location and price being chosen sequentially. We argue that part of this challenge has been self-created by the seemingly innocuous assumption that all consumers buy the product.

In several cases, this assumption is indeed harmless. For example, when the objective is to study either the role of informative advertising [5] or price and service competition [7] in a differentiated market, the assumption that all consumers buy in a Hotelling setup is indeed innocuous.[4] However, when the objective is to study the positioning decisions of firms, the assumption is anything but innocuous. Not only does it lead to significant problems analytically, it may not be justified when we think back to reality: in most product categories, there are at least some consumers who seriously consider not buying any product. With this more realistic starting point, our analysis allows us to show that there exists a pure-strategy location equilibrium, supported by a pure-strategy pricing equilibrium, where firms are moderately differentiated and the market is covered. So even though some consumers consider not buying in the category; in equilibrium, all consumers ultimately purchase the product. However, the mere possibility of a "no buy" option makes the firms change their location and pricing strategies, yielding an equilibrium in which firms compromise between the force that causes them to differentiate (reduce competition) and the force that draws them together (maximize demand). Further, we find that this equilibrium is stable in a context where travel costs are linear.

[4] Said differently, the focus is on situations where the individual rationality constraint does not affect the solution.

Proof that None of the Location Deviations Listed in Section 4 Lead to Increased Profits

Suppose firm A deviates to a location $\tilde{a} < \max\left(V - {}^2/_3, 1 - V\right)$. Then, in the second period, firm B will charge the same price, $V - b = {}^1/_2$, and A will charge either (1) $V - (1 - \tilde{a} - 2b) = 3V - 2 + \tilde{a}$ (kink point where consumers at a point between firms A and B get 0 utility when $p_B = V - b$) or (2) set a price such that all of the firm's consumers get positive surplus.

If A charges (1) $p_A = 3V - 2 + \tilde{a}$, there are two cases. If $\tilde{a} \geq 1 - V$, the market will not be covered, and A's profits will be $2(2 - 2V - \tilde{a})(3V - 2 + \tilde{a})$. This is maximized at $\tilde{a} = 2 - \frac{5V}{2}$, but at that point, $\tilde{a} < 1 - V$, so we can consider the case where $\tilde{a} = 1 - V$. At that point, profits will be $-4V^2 + 6V - 2 \leq {}^1/_4$. If $\tilde{a} < 1 - V$, then, profits will be $2(1 - V)(3V - 2 + \tilde{a})$. This is increasing in $\tilde{a}$, so it is maximized at $\tilde{a} = 1 - V$, where it again will be $-4V^2 + 6V - 2 \leq {}^1/_4$.[5] Alternatively, if the firm prices according to option (2), then, firm A's profits are $\pi^A = p_A\left(\frac{1 + a - b - p_A + p_B}{2}\right)$. The first-order conditions for A reveal that $p_A = \frac{3}{4} + \frac{\tilde{a} - V + p_B}{2} \geq 1 - \frac{V}{2} + \frac{\tilde{a}}{2}$, where the inequality holds because $p_B \geq V - b = {}^1/_2$. This latter inequality holds because the corresponding condition for B is that if consumers at 1 get a positive surplus, B would raise its prices until either consumers at 1 get zero surplus or $p_B = \frac{1}{4} + \frac{V - \tilde{a} + p_A}{2} > \frac{1}{2} + \frac{p_A}{2} > \frac{1}{2}$. Given this, $p_A = \frac{3}{4} + \frac{\tilde{a} - V + p_B}{2} \geq 1 - \frac{V}{2} + \frac{\tilde{a}}{2} > V - \tilde{a}$ whenever $\tilde{a} > V - {}^2/_3$, so we need only to consider the case where $\tilde{a} < V - {}^2/_3$. In such a case, the profits for A will be $\pi^A = \frac{(2 + \tilde{a} - V)^2}{8} < \frac{1}{4}$ in the relevant range.

We next consider deviations towards the competing firm into a region where the second-stage pricing game involves another pricing equilibrium. There are two potential types of pricing strategies that we could consider: (a) a pure strategy where consumers located at 0 choose the outside option (which can happen when $V < {}^{41}/_{50}$) or (b) a mixed-strategy. If firm A moves to a location where the consumers at 0 choose the outside option and the firms play a pure-strategy pricing equilibrium, then, firm A's profits are $\pi^{\tilde{A}} = p_A\left((V - p_A) + \left(\frac{1 + \tilde{a} - b - p_A + p_B}{2} - \tilde{a}\right)\right)$. Setting $\frac{\partial \pi^{\tilde{A}}}{\partial a} = 0$ yields $p_A = \frac{1 - \tilde{a} - b + p_B + 2V}{6}$ and profits of $\pi^{\tilde{A}} = \frac{(1 - \tilde{a} - b + p_B + 2V)^2}{24}$. Note that because $b = V - \frac{1}{2}$ and $p_B = \frac{1}{2}$ (see above), $\tilde{a} \geq V - \frac{2}{5}$ or else consumers at 0 would obtain a positive utility from firm

[5] These inequalities are actually strict because the only case where the equality would hold is for $V = 3/4$, but then $1 - V$ represents the equilibrium location.

A. Plugging these three conditions back into the profit function reveals that $\pi^{\tilde{A}} \leq \frac{6}{25} < \frac{1}{4}$, so such a deviation is not profitable.[6]

Finally, A might move so close to B that there is a mixed-strategy pricing equilibrium. Note that the set of reasonable prices to charge is bounded by V. Consider the highest prices charged by each firm in any mixed-strategy equilibrium. First, the highest prices for each firm must be within $1-\tilde{a}-b$ of each other or the firm with the higher maximum price would sell nothing at that price. Thus, the amount that firm A could sell at its maximum price, p^A_{Max}, is bounded from above by min $\left\{\left(V-p^A_{\text{Max}}\right)+\left(\frac{1+\tilde{a}-b-p^A_{\text{Max}}+p^B_{\text{Max}}}{2}-\tilde{a}\right), \frac{1+\tilde{a}-b-p^A_{\text{Max}}+p^B_{\text{Max}}}{2}\right\}$. Firm B faces a similar set of conditions. Note that $p^B_{\text{Max}} \leq {}^1/_2$ whenever $p^B_{\text{Max}} \geq p^A_{\text{Max}}$. To see this, note that if $p^B_{\text{Max}} > {}^1/_2$ then, the profit for firm B whenever it charges p^B_{Max} is $\pi^{\tilde{B}} = p^B_{\text{Max}}\left[\left(V-p^B_{\text{Max}}\right)+\left(\frac{1-\tilde{a}-b-p^B_{\text{Max}}+E\left(p_A|p_A>p^B_{\text{Max}}-(1-\tilde{a}-b)\right)}{2}\right)\right]$ $\Pr\left(p_A > p^B_{\text{Max}}-(1-\tilde{a}-b)\right)$.

$$\frac{\partial \pi^{\tilde{B}}}{\partial p^B_{\text{Max}}} = \frac{\left(3-2\tilde{a}+2E\left(p_A|p_A>p^B_{\text{Max}}-\left(1-\tilde{a}-b\right)\right)-12p^B_{\text{Max}}+2V\right)}{4}\cdot \Pr\left(p_A > p^B_{\text{Max}}-\left(1-\tilde{a}-b\right)\right) +$$
$$p^B_{\text{Max}}\left[\left(V-p^B_{\text{Max}}\right)+\left(\frac{1-\tilde{a}+b-p^B_{Max}+E\left(p_A|p_A>p^B_{\text{Max}}-\left(1-\tilde{a}-b\right)\right)}{2}-b\right)\right]\cdot\frac{d\Pr\left(p_A>p^B_{\text{Max}}-\left(1-\tilde{a}-b\right)\right)}{dp^B_{\text{Max}}},$$

where we substituted $b = V-\frac{1}{2}$. Both terms are negative when $p^B_{\text{Max}} \geq \frac{1}{2}$ because $p^B_{\text{Max}} \geq p^A_{\text{Max}}$ ensures that the first term is negative.

We can then calculate an upper limit for firm A's profits in any mixed-strategy equilibrium. Suppose $p^A_{\text{Max}} = V-\tilde{a} < {}^1/_2$. A's profits are then $p^A_{\text{Max}} \times \frac{1+\tilde{a}-b-p^A_{\text{Max}}+E\left(p_B|p_B>p^A_{\text{Max}}-(1-\tilde{a}-b)\right)}{2} \cdot \Pr\left(p_B > p^A_{\text{Max}}-(1-\tilde{a}-b)\right) < p^A_{\text{Max}}\frac{2+\tilde{a}-V-p^A_{\text{Max}}}{2}$. This is maximized when $p^A_{\text{Max}} = \frac{2+\tilde{a}-V}{2}$, but because we only consider the case where $p^A_{\text{Max}} \leq (V-\tilde{a})$, plugging in $p^A_{\text{Max}} = (V-\tilde{a})$ reveals that profits will never be above $(V-\tilde{a})\ (1-\ (V-\tilde{a})) < {}^1/_4$, where the inequality holds because the maximum of $x(1\text{-}x)$ is 1/4. Suppose instead $p^A_{\text{Max}} > V-\tilde{a}$. Then, profits are less than $p^A_{\text{Max}}\left[\left(V-p^A_{\text{Max}}\right)+\left(\frac{1+\tilde{a}-b-p^A_{\text{Max}}+p^B_{\text{Max}}}{2}-\tilde{a}\right)\right] < p^A_{\text{Max}}\frac{2-\tilde{a}-3p^A_{\text{Max}}+V}{2}$, where we substituted $b = V-\frac{1}{2}$ and $p^B_{\text{Max}} \leq {}^1/_2$. $p^A_{\text{Max}}\frac{2-\tilde{a}-3p^A_{\text{Max}}+V}{2}$ is maximized when $p^A_{\text{Max}} = \frac{2+V-\tilde{a}}{6}$, which is only greater than $V-\tilde{a}$ when $(V-\tilde{a}) < {}^2/_5$. Plugging $p^A_{\text{Max}} = \frac{2+V-\tilde{a}}{6}$ into $p^A_{\text{Max}}\frac{2-\tilde{a}-3p^A_{\text{Max}}+V}{2}$ yields $\frac{(2+V-\tilde{a})^2}{24} \leq \frac{6}{25} < \frac{1}{4}$. If $(V-\tilde{a}) > {}^2/_5$, we can plug in $p_{\text{Max}}{}^A = V-\tilde{a}$ *into* $p^A_{\text{Max}}\frac{2-\tilde{a}-3p^A_{\text{Max}}+V}{2}$, which gives a value of $(V-\tilde{a})\ (1-\ (V-\tilde{a})) < {}^1/_4$.

We can rule out that A would deviate to the other side of B. Note that because the most profits A can earn at any price is $2p_A(V-p_A)$, we need not consider deviations where A would charge a price above $\frac{V}{2}+\frac{\sqrt{4V^2-2}}{4}$, which is the highest price where $2p_A(V-p_A) > \frac{1}{4}$. Looking at B's incentives, at the highest price it would ever charge in a mixed- or pure-strategy equilibrium, it would earn a profit of $p^B_{\text{Max}}\left(V-p^B_{\text{Max}}+\frac{d+p^A_{\text{Max}}-p^B_{\text{Max}}}{2}\right)$, where d is the distance between A and B. Thus, the most B would ever potentially charge, plugging in $p^A_{\text{Max}} = \frac{V}{2}+\frac{\sqrt{4V^2-2}}{4}$, is $p^B_{\text{Max}} = \frac{4d+10V+\sqrt{4V^2-2}}{24}$. Given that, A's profits are bounded to be below $p_A\left(V-\frac{1}{2}-\frac{d+pa-pb}{2}\right)$. The price that maximizes this upper bound for A is $p_A = \frac{58V+\sqrt{4V^2-2}-24-20d}{48}$, and the bound on profits is $\frac{\left(58V+\sqrt{4V^2-2}-24-20d\right)^2}{4608}$. Given the bounds on V and d, this is increasing in V and decreasing in d. Evaluating this upper bound at $V = \frac{7}{8}$ and $d=0$ reveals that A's profits are less than $\frac{\left(107+\sqrt{17}\right)^2}{73728} \approx 0.17$. Thus, there is no profit-increasing deviation to the right of B.

The case where $V \in \left(\frac{1}{2}, \frac{3}{4}\right)$ follows a similar logic. Firms locate at a distance ¼ from the endpoints, charge $p = \left(V-\frac{1}{4}\right)$, and earn profits of ${}^V/_2 - {}^1/_8$. One can easily confirm that these prices form an equilibrium given the locations. Confirming that deviating to another location is not optimal follows the same logic as the case where $V \in \left(\frac{3}{4}, \frac{7}{8}\right)$ for most of the analysis. However, when one considers deviations that involve the mixed-strategy equilibria, we need to adjust the consideration for the case where $p^A_{\text{Max}} \leq (V-\tilde{a})$. We instead note that the profits from this price are less than min$[(V-\tilde{a})(1-(V-\tilde{a})), 2(V-\tilde{a})\tilde{a}]$, which is always less than the profits earned by the firm of ${}^V/_2 - {}^1/_8$ whenever $\tilde{a} > {}^3/_8$. Deviations to $\tilde{a} < {}^3/_8$ will not involve mixed-strategy pricing equilibria.

[6] Alternatively, note that profits are decreasing in $\tilde{a}$, but once $\tilde{a}$ becomes small enough that consumers at 0 make a purchase, the previous first-order condition on locations comes into effect, so profits must be smaller.

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