**Submitted to *Operations Research***
**manuscript OPRE-2006-10-394.R2**

# Dynamic Pricing of Perishable Assets under Competition

Guillermo Gallego
IEOR Department, Columbia University, New York, NY 10027, gmg2@columbia.edu

Ming Hu
Rotman School of Management, University of Toronto, Toronto, ON M5S3E6, Canada, ming.hu@rotman.utoronto.ca

Existing revenue management solutions employed by legacy carriers, which are based on allocating capacity to pre-defined fare classes, are increasingly inadequate to compete against low-cost carriers that impose few or no fare restrictions and make their fares widely available through the Internet. To avoid revenue erosion due to inadequate solutions, legacy carriers need to develop systems that take into account consumer purchasing behavior which itself depends on the fares available at the time of purchase. This requires a choice-based, multi-player, game theoretic formulation of dynamically pricing perishable capacities over finite horizons. Here we present such a formulation as a stochastic game in continuous time. Since this problem is generally intractable, we provide sufficient conditions for the existence of open-loop and closed-loop Nash equilibria of the corresponding differential game resulting from an affine functional approximation approach to the stochastic problem. We show that efficiently-computable pricing heuristics based on these open-loop and closed-loop policies are asymptotic equilibria in an appropriate sense for the stochastic game under various information assumptions.

*Key words*: Oligopoly; dynamic pricing; finite horizon; Nash equilibrium; supermodular game; revenue management.

*Subject classifications*: Transportation: yield management, pricing. Dynamic programming/optimal control: intensity control. Games/group decisions: noncooperative, stochastic, differential.

*Area of review*: Revenue management.

*History*: Submitted Oct 2006; 1st revision May 2008; 2nd revision Nov 2009.

## 1. Introduction

### 1.1. Motivation

Current revenue management (RM) practice, in spite of its success and popularity, is based on assumptions that may no longer hold true in real world situations. One of the critical flaws is that current RM (or yield management) models are designed under the assumption that demands for different fare classes are independent random variables with a pricing team designing fares and another team allocating capacity to fare classes. This flaw is exacerbated by low cost carriers offering fares with few or no restrictions and by Internet-enabled price transparency (Cooper et al. 2006). Most of the literature in RM deals with the issue of capacity allocation. Pricing decisions are kept at a more strategic level but to our knowledge there is little science behind current pricing practices. Pricing and capacity allocation decisions might likely be separated because of difficulties collecting competitors' prices and the complexity of analyzing competitive models.

While carriers and solution providers agree that new solutions are needed to stem revenue erosion, there is no agreement as to what needs to be done. Some solution providers have proposed an incremental approach that keeps pricing and capacity allocation separate, acknowledging that demand among fares are dependent random variables with most of the demand going to the lowest open fare. This line of research will likely provide some respite and extend the life of traditional RM models. The success of such systems will depend on the extent to which the pricing team is able to select the right prices in a competitive environment and the extent to which the demand forecasts are sensitive to competitive pricing.

At the other end of the spectrum, RM researchers and practitioners are trying to integrate pricing and capacity allocation into a single system that takes into account pricing and quality attributes of the products available to customers at the time of purchase. The challenge, of course, is the availability of data and the complexity of solving such systems. Recent development of search engine technology makes competitors' prices instantaneously available to consumers, and can also be fed into competitive pricing models as input data. Indeed, online travel sites such as Expedia, Hotwire, Orbitz, Priceline, Kayak and Travelocity gather information and list flight, hotel, car rental and cruise fares almost in real time among competitors across the travel industries. Moreover, price search engines such as Google Product provide real-time product prices both online and in local stores. On the technical side there have been advances in demand modeling that take into account customer choice behavior (see Talluri and van Ryzin 2004 for the single leg case and Gallego et al.

2004, van Ryzin and Liu 2008 for the network case). While these demand models can be readily extended to competitive settings, the problem of finding optimal or near-optimal controls under competition is far from trivial.

### 1.2. Contribution

We formulate a non-zero-sum non-cooperative dynamic pricing game and address the problem of integrating pricing and capacity allocation into a single framework where multiple capacity providers compete to sell their own fixed initial capacities of differentiated perishable items over the same finite sales horizon without replenishment opportunities. The arrival rate of customers is time-varying; demand for each firm is modeled as a non-homogeneous Poisson process with rate dependent on the arrival rate and prices offered by all firms. This stochastic game can be viewed as extending the static Bertrand-Edgeworth-Chamberlin competition (price competition of differentiated products with capacity constraints, see Vives 1999, Section 6.5) with zero costs to a more general environment with intertemporal pricing flexibility and demand uncertainty.

Previous work in RM on demand choice models without competition assumes that prices of available products are given exogenously. This is consistent with current RM practice of separating pricing decisions from capacity allocation decisions. Presumably, one of the reasons to separate these two decisions is that the group responsible for pricing is looking into competitive issues in determining fares. These fares are then passed to the capacity allocation group. While it is possible to analyze capacity allocation with fare restriction under competition, the computational burden of solving the resulting problem is daunting even for the single leg problem. Moreover, this does not resolve the problem of setting prices. Our model, in contrast, integrates pricing and capacity allocation and makes it easier to analyze the competition even in a network setting. In practice, pricing teams can take our solution a step further and design a menu of fares that span and synthesize solutions to our competitive model under different instances of the demand model.

We show that the stochastic game reduces to a deterministic differential game if each firm approximates its value function by a quasi-static affine function. We fully characterize the open-loop

Nash equilibria (OLNE) to the differential game. We demonstrate that if revenue rate functions have a supermodular nature the best-response tatônnement scheme is efficient in computing a fixed-pricing OLNE. This implies that firms can be divided into two categories, those with ample capacity and those with scarce capacity. Firms with ample capacity use a pricing policy to maximize their revenue rates and firms with limited capacity use their relatively higher market clearing prices. We show that the open-loop policy cannot benefit from feedback in the differential game, and that it is in fact a closed-loop Nash equilibrium (CLNE) of the differential game. We discuss how the open-loop policy can be used as a heuristic for the stochastic game with asymptotically good properties. In practice, we expect firms to apply the open-loop heuristic in a rolling horizon fashion, and this will result in time-varying prices as capacities will evolve stochastically and will be almost surely different from those predicted by the differential game.

### 1.3. Literature Review

There is a growing body of literature on RM in the context of competition. Depending on the chosen decision variables, RM is categorized as either quantity-based or price-based. Netessine and Shumsky (2005) examine one-shot quantity-based games of booking limit control under both horizontal competition and vertical competition. Talluri (2003) studies a dynamic quantity-based RM model in a duopoly where each firm sells differentiated products and makes an available offer set from a pre-determined fare menu. Oligopoly pricing, common in the economics and marketing literature, is gaining traction within the RM community. Granot et al. (2007) analyze a multi-period duopoly pricing game where homogeneous perishable goods are sold to impatient consumers who visit only one of the retailers in each period. Levin et al. (2009) present a unified stochastic dynamic pricing game of multiple firms where differentiated goods are sold to finite segments of strategic customers who may time their purchases. Though we do not consider consumer strategic behavior, we allow for more general demand structures.

Our paper formulates the competitive dynamic pricing game of selling perishable assets as an intensity control game with demands modeled as non-homogeneous Poisson processes. This approach complements previous works on revenue management and oligopoly pricing. In the terminology of game theory, Perakis and Sood (2006) consider a finite-horizon discrete-time stochastic game; Bernstein and Federgruen (2004) consider an infinite-horizon discrete-time stochastic game. With regard to modeling demand uncertainty in periodic-review models, Perakis and Sood (2006) assume that an uncertainty factor contained within an uncertainty set is associated with the demand; Bernstein and Federgruen (2004) assume that the demand for each period is of a multiplicative form; Federgruen and Heching (1999) assume an additive form of demand in their numerical study.

From the perspective of methodology, there appears to be at least two research streams in pricing under competition. One stream characterizes the market equilibrium by the methodology of quasi-variational inequalities (QVI). Perakis and Sood (2006) address a multi-period discrete-time competitive dynamic pricing model of a single asset with demand uncertainty, and use ideas from robust optimization and variational inequalities. Nguyen and Perakis (2005) extend the single-asset model to a multi-product competitive pricing game. Mookherjee and Friesz (2008) consider a combined pricing, resource allocation, overbooking RM problem over networks as well as under competition. This line of research deals with a periodic review formulation and aims to design efficient algorithms to compute market equilibirium prices arising from the joint variational inequality.

The other line of research falls under the framework of the supermodular game (Topkis 1979, Milgrom and Roberts 1990). This line of research derives sufficient conditions of equilibrium's existence and uniqueness by verifying supermodularity and "diagonal dominance" conditions, respectively. Bernstein and Federgruen (2003, 2004, 2005) have a series of papers studying games of joint pricing and inventory control in the interface of RM and supply chain management. Gallego et al. (2006) study an oligopolistic price competition game with general attraction demand functions and convex costs, and prove a linear convergence to the equilibrium of a simultaneous discrete tatônnement scheme.

Two papers related to our setting are Lin and Sibdari (2009) and Xu and Hopp (2006). The first authors prove the existence of a pure-strategy Nash equilibrium in a discrete-time stochastic game.

This model can be viewed as the discrete-time counterpart to the continuous-time stochastic game considered in this paper. The main difference, apart from the choice of how to model time, is that we focus on the near-optimality of simple heuristics derived from the corresponding differential game in light of the fact that the stochastic game is intractable. Similar to our paper in a continuous-review setting, the latter authors study a dynamic pricing problem under oligopolistic competition with one-shot initial inventory replenishment. The authors establish a weak perfect Bayesian equilibrium of the price and inventory replenishment game. There are several significant differences between the present paper and that of Xu and Hopp. Most significantly, the authors formulate a stochastic differential game with a continuous state space and obtain a cooperative fixed-pricing equilibrium strategy. We formulate a stochastic game with a discrete state space and solve its corresponding deterministic differential game for simple feedback-type heuristics. Second, the authors assume a correlated demand structure in which customers are modeled as an atomic flow according to a geometric Brownian motion with a quasilinear utility function. We consider a demand structure in which customers arrive according to a non-homogeneous Poisson process with a more general utility function though demand correlation is not considered. Third, the authors assume perfect competition with homogeneous products (Vives 1999, Chapter 5) while we consider imperfect competition with differentiated products (Vives 1999, Chapter 6).

We characterize equilibria to the differential game and study their asymptotic behavior in the stochastic game. The deterministic differential game theory has been successfully applied to marketing and economics, especially in the area of dynamic pricing. Eliashberg and Jeuland (1986) characterize the nature of the dynamic equilibrium prices that prevail during the competitive period in which a monopolist encounters a new firm entry. Gaimon (1989) derives both OLNE and CLNE for a differential game where two competing firms choose prices and production capacity when new technology reduces firms' operating costs. Mukhopadhyay and Kouvelis (1997) propose a differential game formulation to analyze a duopoly with firms competing on quality and price, and derive an OLNE and CLNE under a linear feedback law.

The remainder of this paper is organized as follows. Section 2 describes our modeling assumptions and formulations of the stochastic game and its differential counterpart. Section 3 studies the OLNE and CLNE for the differential game. Section 4 provides links between the stochastic and differential game, and proves the asymptotic equilibrium behavior of feedback-type heuristics. Section 5 considers commonly-used demand structures as examples and illustrates with numerical experiments. Section 6 offers concluding remarks and points out directions of future research. Most proofs are relegated to the appendix.

## 2. The Model

### 2.1. Notation and Assumptions

$\mathcal{I} := \{1, 2, \ldots, m\}$ denotes the set of firms (players) in the market. Any entry in all vectors is assumed to be in $\mathbb{R}_+ := [0, +\infty)$. $x_i$ denotes the $i$th component of vector $\vec{x}$, and $\vec{x}_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$ is the vector of components other than $i$. $\vec{e}_i$ denotes a vector with the $i^{th}$ element 1 and all other elements 0's. $\mathcal{A}\backslash\mathcal{B}$ denotes the set difference between the sets $\mathcal{A}$ and $\mathcal{B}$. *LHS* and *RHS* are shorthand for *left-hand side* and *right-hand side*, respectively. A function is said to be *increasing* (*decreasing*) when it is nondecreasing (nonincreasing).

We consider a market of firms competing in selling substitutable perishable assets, where demand is influenced by prices across the market. At time $t = 0$, each firm $i$ has an initial capacity of $c_i \in \mathbb{N}$ units. Let $\mathcal{X} := \times_i [0, c_i]$. All firms have the same sales horizon of length $T > 0$. We count the time forwards. We use $t$ for the elapsed time and $s := T - t$ for the remaining time.

At any time $0 \leqslant t \leqslant T$, the vector of demand intensities $\vec{\xi}(t)$ is determined as a multiple of the current customer arrival rate $\lambda(t)$ and a consumer choice probability function $\vec{d}(\vec{p}(t)) \in [0, 1]$ that is influenced by the current market price vector $\vec{p}(t)$, namely, the demand intensity function can be written as

$$\vec{\xi}(\vec{p}(t)) := \lambda(t)\vec{d}(\vec{p}(t)).$$

We denote the cumulative customer arrival by $\Lambda(t) := \int_0^t \lambda(v)\, dv$ and the revenue rate function for any firm $i$ by $r_i(\vec{p}) := p_i d_i(\vec{p})$. We assume that all firms perceive the same demand intensity function. At this stage, we make minimal assumptions regarding the shape of the demand intensity function for any firm $i$.

ASSUMPTION 1 (DIFFERENTIABILITY). *$d_i(\vec{p})$ is continuously differentiable in $\vec{p}$.*

ASSUMPTION 2 (DOWNWARD-SLOPING). $\dfrac{\partial d_i(\vec{p})}{\partial p_i} < 0$.

ASSUMPTION 3 (SUBSTITUTES). $\dfrac{\partial d_i(\vec{p})}{\partial p_j} \geqslant 0,\ \forall j \neq i.$

Assumptions 2-3 are fairly standard in the oligopoly pricing literature. Assumption 2 expresses the usual assumption of strict downward-sloping. Assumption 3 states that assets provided by firms are substitutes. Under Assumptions 1-3, we have the following immediate result by equivalent properties of quasi-convexity for a single-variable function (Boyd and Vandenberghe 2004, Section 3.4).

LEMMA 1 (QUASI-LINEARITY OF DEMAND). *$d_i(\vec{p})$ is quasi-linear in $p_j$ for all $j$.*

ASSUMPTION 4 (PSEUDO-CONCAVITY OF REVENUE). *$r_i(\vec{p})$ is pseudo-concave in $p_i$.*

As a reminder, a function $f$ is pseudo-convex on a non-empty open set $X$ if for any $x, y \in X$, $(y-x)^{\mathrm{T}} \nabla_x f(x) \geqslant 0 \Rightarrow f(y) \geqslant f(x)$, where $\nabla_x$ is the gradient operator. $f$ is pseudo-concave if and only if $-f$ is pseudo-convex. By Mangasarian (1987), the Karush-Kuhn-Tucker (KKT) conditions for a nonlinear maximization problem are *sufficient* for optimality when the objective function is pseudo-concave and the *LHS*'s of non-positive constraints are quasi-convex. Assumption 4 is satisfied by most commonly-used demand functions such as the MultiNomial Logit (MNL) and linear demand functions (see Examples 1 and 2).

We further make the following assumptions on each firm's strategy space.

ASSUMPTION 5 (COMPACT STRATEGY SPACE IN DIFFERENTIAL GAME). *$p_i$ is chosen from a closed interval $\mathcal{P}_i := [0, p_i^{\max}] \subset \mathbb{R}_+$ when firm $i$ has positive on-hand inventory.*

ASSUMPTION 6 (SLATER'S CONDITION). *There exists $\vec{p} \in \mathcal{P} := \times_i \mathcal{P}_i$ such that $\vec{0} < \vec{d}(\vec{p}) < \vec{c}$.*

Assumption 6 is the Slater constraint qualification that is needed to ensure the KKT differential conditions for nonlinear optimization problems with constraints $\vec{0} \leqslant \vec{d}(\vec{p}) \leqslant \vec{c}$ are *necessary* for optimality. Assumption 6 can be ensured if $\vec{p}^{\max}$ is chosen high enough.

In the literature of dynamic pricing, a *null* price is commonly assumed to exist as a convenient mechanism to shut down stochastic demand when inventory reaches zero. We assume

Assumption 7 (Null Price Option in Stochastic Game). *A null price option $p_i^{null}(\vec{p}_{-i})$ dependent on competitors' price $\vec{p}_{-i}$ is available to any firm $i$ when its inventory drops to zero.*

Moreover, we assume for any set $S \subseteq \mathcal{I}$ of firms with positive inventory, there exists a continuous demand intensity function $\vec{d}^S(\vec{p}) : \mathbb{R}_+^{|S|} \mapsto \mathbb{R}_+^{|S|}$ among them.

Finally, without loss of generality, we assume that the salvage value of the asset at the end of horizon is zero and that all other costs are sunk. We can always transform a problem with positive salvage cost $q_i$ for firm $i$ to a zero-salvage-cost case by changing variables $p_i \leftarrow p_i - q_i$ in the demand intensity function.

### 2.2. Formulation of the Continuous-Time Stochastic Game

We consider a finite-horizon, multi-player, non-zero-sum, non-cooperative stochastic game. This formulation can be viewed as a game version of the optimal dynamic pricing problem considered in Gallego and van Ryzin (1994). The firms control the demand intensity by adjusting price. In the stochastic game, the demand intensity is stochastic. More specifically, demand for the product is assumed to be a non-homogeneous Poisson process with Markovian intensities. At time $0 \leqslant t \leqslant T$, firm $i$ applies its own non-anticipating price $p_i(t)$. Let $N_i(t)$ denote the number of items sold up to time $t$ for firm $i$. Mathematically, $N_i(t)$ is a controlled point process with instantaneous intensity $\xi_i(t)$ to be a function of the joint price vector $\vec{p}(t)$. A demand for any firm $i$ is realized at time $t$ if $dN_i(t) = 1$. We denote the joint Markovian allowable pricing policy space by $\mathcal{U}$, where any joint allowable pricing policy $\vec{u} = \left\{\vec{p}(t, \vec{n}(t)), 0 \leqslant t \leqslant T \mid p_i(t, \vec{n}(t)) \in \mathcal{P}_i \cup \{p_i^{null}(\vec{p}_{-i})\} \text{ for all } i\right\}$ satisfies that $\int_0^T dN_i(t) \leqslant c_i$ almost sure (a.s.) for all $i$. By the Markovian property of $\mathcal{U}$, we mean that the price policy $u_i$ offered by firm $i$ is a function of the elapsed time and current joint inventory level; that is, $\vec{p}(t,\vec{n}(t)) = \vec{p}\,(t, C_1 - N_1(t), C_2 - N_2(t), \ldots, C_m - N_m(t))\,, 0 \leqslant t \leqslant T$. In terms of game theory, we want to analyze feedback strategies. This requires that we have a somehow restrictive information structure:

Definition 1 (Strong Information Structure). All firms have perfect knowledge about each other's inventory levels at any time.

This assumption used to be unrealistic but now inventory information in real time is revealed in some way as almost all online travel agencies and major airlines offer a feature of previewing seat availability from their websites. We also consider two weaker information structures:

Definition 2 (Weak Information Structure). All firms only know the initial joint inventory level.

Definition 3 (Weak Information Structure with Observable Prices). All firms know the initial joint inventory level and can observe competitors' pricing instantaneously at any time.

Given pricing policy $\vec{u} \in \mathcal{U}$, we denote the expected profit for any firm $i$ by

$$J_i(\vec{u}) = E\left[\int_0^T p_i(t)\, dN_i(t)\right].$$

The goal of any firm $i$ is to maximize its total expected profit over the sales horizon. A joint pricing policy $\vec{u}^* \in \mathcal{U}$ constitutes a Nash equilibrium if, whenever any firm modifies its policy away from the equilibrium, its own payoff will not increase. More precisely, $\vec{u}^*$ is called a Markovian equilibrium strategy if $J_i\left(u_i, \vec{u}^*_{-i}\right) \leqslant J_i\left(\vec{u}^*\right)$ for all $i$ and $(u_i, \vec{u}^*_{-i}) \in \mathcal{U}$. By extending Brémaud (1980, Theorem VII.T1) to the context of a stochastic game, one can rigorously justify that the following set of Hamilton-Jacobi-Bellman (HJB) equations is a sufficient condition for the Markovian equilibrium strategy.

Proposition 1 (Stochastic Game). *If functions $V_i(s,\vec{n}) : [0,T] \times \mathbb{Z}^m \cap \mathcal{X} \mapsto \mathbb{R}_+$ for all $i$ satisfy the following set of HJB equations simultaneously*

$$-\frac{\partial V_i(T-t,\vec{n})}{\partial t} = \lambda(t) \max_{p_i} \left\{ r_i(\vec{p}) - \vec{d}(\vec{p})^{\mathrm{T}} \nabla \vec{V}_i(T-t,\vec{n}) \right\}, \tag{1}$$

*where* $\nabla\vec{V}_i(s,\vec{n}) := (\Delta V_{i,1}(s,\vec{n}), \Delta V_{i,2}(s,\vec{n}), \ldots, \Delta V_{i,m}(s,\vec{n}))$ *and* $\Delta V_{i,j}(s,\vec{n}) := V_i(s,\vec{n}) - V_i(s,\vec{n} - \vec{e}_j)$*, with boundary conditions for all* $i$*,* $V_i(0,\vec{n}) = 0$ *and* $V_i(s,\vec{n}) = 0$ *if* $n_i = 0$ *for all* $s \in [0,T]$*, then* $V_i(T-t,\vec{n})$ *for all* $i$ *are the equilibrium value-to-go functions of a Markovian equilibrium strategy* $\vec{u}^* = \{\vec{p}^*(t,\vec{n}), (t,\vec{n}) \in [0,T] \times \mathbb{Z}^m \cap \mathcal{X}\} \in \mathcal{U}$ *such that* $p_i^*(t,\vec{n})$ *achieves the maximum in the HJB equation (1) for any firm* $i$ *at any* $(t,\vec{n})$*.*

Little is known about the existence of Nash equilibrium in a general stochastic game and the best one may hope for is an $\epsilon$-Nash equilibrium. Vieille (2000) proves that two-player non-zero-sum stochastic games with a finite number of states always have approximate equilibria ($\epsilon$-Nash equilibrium for all $\epsilon > 0$); nevertheless, the general existence problem remains an open question. For a discrete-time version of our stochastic game, Lin and Sibdari (2009, Theorem 3.1) prove the existence of Nash equilibrium by backwards induction using a theorem by Debreu (Vives 1999, Theorem 2.1; Debreu 1952). In this paper we focus on the asymptotically optimal heuristics suggested by solving the corresponding differential game, which is formulated in the next section.

### 2.3. Formulation of the Deterministic Differential Game

We formulate the following deterministic version of our stochastic game, where the demand is a deterministic fluid. We consider both weak and strong information structure in the differential game, which correspond to two different solution schemes: open-loop strategies and closed-loop strategies. Let us denote by $\vec{x}(t)$ the joint inventory level at time $t$, which is a continuous quantity in the differential game.

Definition 4 (Open-Loop Strategy). An open-loop strategy for firm $i$ is a time path $p_i(t)$, $0 \leqslant t \leqslant T$ such that given the initial joint inventory level, it assigns a control for every time $t$. The set of all joint allowable open-loop strategies is denoted by $\mathcal{U}^o$.

Definition 5 (Closed-Loop Strategy). A closed-loop memoryless (closed-loop, hereafter) strategy for firm $i$ is a decision rule $p_i(t,\vec{x}(t))$, $0 \leqslant t \leqslant T$ such that given the initial joint inventory level, it observes the current joint inventory level $\vec{x}(t)$ and assigns a control for every time $t$. The set of all joint allowable closed-loop strategies is denoted by $\mathcal{U}^c$.

In an open-loop strategy, firms make an irreversible commitment to a future course of action. The formulation of the open-loop strategy has taken into account the future competitive environment but remains unaltered once the game starts. Alternatively, closed-loop strategies capture the feedback reaction of competitors to the firm's chosen course of action. An open-loop strategy only needs the weak information structure while a closed-loop strategy requires the strong information structure. The strong information structure and weak information structure with observable prices are equivalent for the differential game since in the deterministic model, any firm can accurately compute the instant capacities of other firms by monitoring competitors' prices.

Given pricing policy $\vec{u}\in\mathcal{U}^o$ ($\mathcal{U}^c$), we denote the total profit for any firm $i$ by

$$J_i^d(\vec{u}) = \int_0^T \lambda(t)\, r_i(\vec{p}(t))\, dt.$$

The vector of capacities evolves according to a kinematic equation: for any $i$,

$$\dot{x_i}(t) = -\lambda(t)\, d_i(\vec{p}(t)),\ 0 \leqslant t \leqslant T, \quad x_i(0) = c_i,$$

with the nonnegative state constraint

$$x_j(t) = c_j - \int_0^t \lambda(s) d_j(\vec{p}(s))\, ds \geqslant 0,\ \forall\, j,\ 0 \leqslant t \leqslant T. \tag{2}$$

Lemma 2 (Nonnegativity of State). *The set of constraints $\{x_i(t) \geqslant 0,\ 0 \leqslant t \leqslant T\}$ on nonnegative state at any time is equivalent to the single constraint $x_i(T) \geqslant 0$ on nonnegative state at the end of the horizon.*

By Lemma 2, our differential game with constraints on state variables reduces to a differential game with inequality constraints on the terminal state only.

Each firm's problem is to maximize its own total revenue. The definitions of Nash equilibrium for open-loop and closed-loop strategies follow immediately, such that OLNE (resp. CLNE) is an $m$-tuple of open-loop (resp. closed-loop) strategies $\vec{u}^* \in \mathcal{U}^o$ ($\mathcal{U}^c$) satisfying $J_i^d\left(u_i, \vec{u}^*_{-i}\right) \leqslant J_i^d\left(\vec{u}^*\right)$ for all $i$ and $(u_i, \vec{u}^*_{-i}) \in \mathcal{U}^o$ ($\mathcal{U}^c$). Since the set of all possible decision rules $p_i(t, \vec{x}(t))$ contains all of the open-loop strategies $p_i(t)$, we can conclude that $\mathcal{U}^o \subset \mathcal{U}^c$. As shown in Fershtman (1987, Lemma 2.1), an OLNE is a special case of CLNE in a differential game.

Notice that both OLNE and CLNE are initial-condition $(T, \vec{c})$ dependent in general. Open-loop and closed-loop strategies (even in terms of control time paths) are generally different in a non-zero-sum differential game. We will show, however, that we can write fixed-pricing OLNE to our differential game in feedback form; they are indeed non-degenerate CLNE that generate the same equilibrium state trajectory and control path as their open-loop counterparts. This exception is due to the special structure of our differential game that the objective functions and the *RHS*'s of kinematic equations are state-independent.

## 3. Nash Equilibrium of the Differential Game

### 3.1. Open-Loop Nash Equilibrium (OLNE)

Introducing absolutely continuous costate variable $\vec{\mu}_i(t)$, $0 \leqslant t \leqslant T$, we define for all firm $i$ the Hamiltonians $H_i : [0,T] \times X \times \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}$ by

$$H_i\left(t, \vec{x}, \vec{p}(t), \vec{\mu}_i(t)\right) = \lambda(t)[r_i(\vec{p}(t)) - \sum_j \mu_{ij}(t) d_j(\vec{p}(t))], \quad 0 \leqslant t \leqslant T. \tag{3}$$

With some regularity condition (Assumption 6), any OLNE $\vec{p}(t) = \vec{p}^*(t), 0 \leqslant t \leqslant T$, its corresponding costate trajectory $\vec{\mu}_i(t), 0 \leqslant t \leqslant T$ for all $i$ and equilibrium state trajectory $\vec{x}(t), 0 \leqslant t \leqslant T$ need to satisfy the following set of *necessary* conditions (namely, the Pontryagin Maximum Principle, see Dockner et al. 2000, Theorem 4.2 and Sethi and Thompson 2005, Section 3.1):

$$p_i(t) \text{ maximizes }_{p_i(t) \in \mathcal{P}_i \cup \{p_i^{null}(\vec{p}_{-i}(t))\}} H_i, \text{ for almost all } 0 \leqslant t \leqslant T \text{ and all } i, \tag{4}$$

$$-\frac{\partial \mu_{ij}(t)}{\partial t} = \frac{\partial H_i}{\partial x_j}, \text{ for almost all } 0 \leqslant t \leqslant T \text{ and all } i, j, \tag{5}$$

$$\dot{x}_i(t) = -\lambda(t) d_i(\vec{p}(t)), \ 0 \leqslant t \leqslant T, \quad x_i(0) = c_i, \text{ for all } i, \tag{6}$$

$$\mu_{ij}(T) x_j(T) = 0, \quad \mu_{ij}(T), x_j(T) \geqslant 0, \text{ for all } i, j. \tag{7}$$

By Lemma 2, we have the constrained end-point $x_i(T) \geqslant 0$ for all $i$, resulting in the last condition. As the transversality condition, condition (7) essentially is the KKT complementary slackness condition for the end-point: If $x_j(T) > 0$ at equilibrium, then $x_j(t) > 0$ for all $0 \leqslant t \leqslant T$ and thus the condition $\mu_{ij}(t) = 0$ applies for all $0 \leqslant t \leqslant T$ and any $i$; if $\mu_{ij}(T) > 0$ for some $i$ at equilibrium, then we have $x_j(T) = 0$.

Since the Hamiltonians do not explicitly depend on the state dynamics, condition (5) reduces to

$$-\frac{\partial \mu_{ij}(t)}{\partial t} = 0, \text{ for almost all } 0 \leqslant t \leqslant T \text{ and all } i, j.$$

By Royden (1988, Lemma 5.13), all absolutely continuous costate trajectories $\mu_{ij}(t)$ for all $i, j$ with derivative equal to zero almost everywhere must be constant.

Lemma 3 (Constant Costate Trajectory). *For any OLNE, its corresponding costate trajectories must be constant.*

Hence we drop the time argument and denote the costate variable by $\mu_{ij}$ for all $i, j$, which has the interpretation of the shadow value of player $j$'s state variable $x_j$ to player $i$. Though the shadow values $\mu_{ij}$, for all $i, j$ are independent of time along the state trajectory of equilibrium, they may be dependent on the initial state $\vec{c}$.

By Assumption 4 and Lemma 1, a Pontryagin point, namely a solution satisfying the set of necessary conditions (4)-(7), is in fact a maximum point. This follows because a Pontryagin point is a KKT point of each firm's control problem when expressed as a nonlinear programming problem (Craven 1998, Section 5.10 and Mangasarian 1965). Thus the set of conditions (4)-(7) is also a *sufficient* condition for an OLNE. By Lemma 3 and rewriting conditions (6) and (7), we have the following characterization of an OLNE.

Proposition 2 (Open-Loop Nash Equilibrium). *The open-loop policy $\{\vec{p}(t) : 0 \leqslant t \leqslant T\}$ is an OLNE if and only if there exists a set of costate variables $\{\mu_{ij}\}$ such that $\vec{d}(\vec{p}(t)) \geqslant \vec{0}$ for all $0 \leqslant t \leqslant T$ and the following set of conditions is satisfied:*

$$p_i(t) \text{ maximizes }_{p_i(t) \in \mathcal{P}_i \cup \{p_i^{null}(\vec{p}_{-i}(t))\}} \left\{ r_i(\vec{p}(t)) - \sum_j \mu_{ij} d_j(\vec{p}(t)) \right\}, \text{ for almost all } 0 \leqslant t \leqslant T \text{ and all } i, \tag{8}$$

$$\mu_{ij} \left[ c_j - \int_0^T \lambda(t) d_j(\vec{p}(t))\, dt \right] = 0, \text{ for all } i, j. \tag{9}$$

We shall consider a special kind of OLNE, a fixed-pricing policy $\vec{p}(t) = \vec{p}^*$ for all $0 \leqslant t \leqslant T$, that can be guaranteed under fairly general assumptions. By Proposition 2, if there exists $\vec{p} = \vec{p}^*$ together with a set of co-state variables $\{\mu_{ij}\}$ satisfying $\vec{d}(\vec{p}) \geqslant \vec{0}$ and the set of conditions

$$p_i \text{ maximizes }_{p_i \in \mathcal{P}_i} \left\{ r_i(\vec{p}) - \sum_j \mu_{ij} d_j(\vec{p}) \right\}, \text{ for all } i, \tag{10}$$

$$\mu_{ij} \left[ c_j - \Lambda(T) d_j(\vec{p}) \right] = 0, \text{ for all } i, j, \tag{11}$$

we obtain $\vec{p}(t) = \vec{p}^*, 0 \leqslant t \leqslant T$ as an OLNE. Restricted within the domain of a fixed-pricing policy, it is not optimal for any firm to use a null price all the time. Hence we can ignore the option of null price in (10) and confine the joint fixed-pricing policy $\vec{p}$ within the compact set $\mathcal{P}$ when searching for the desired fixed-pricing OLNE. The set of conditions (10)-(11) can be viewed as the KKT conditions (a necessary and sufficient condition for equilibrium by Assumptions 2-4) for the following one-shot Bertrand-Edgeworth-Chamberlin game (P0) with zero marginal cost: given competitors' prices $\vec{p}_{-i}$, each player $i$ is to simultaneously solve

$$\begin{array}{lll} (\text{P0}_i) & \max\limits_{p_i \in \mathcal{P}_i} & r_i(\vec{p}) \\ & \text{s.t.} & 0 \leqslant d_j(\vec{p}) \leqslant c_j / \Lambda(T), \text{ for all } j. \end{array}$$

The following result follows immediately.

Lemma 4 (Constrained Static Game). *Any equilibrium to game (P0) is a fixed-pricing OLNE to the differential game.*

Next we take two approaches to analyze game (P0) and identify fixed-pricing OLNE for the differential game.

**3.1.1. Generalized Nash Game Approach** Game (P0) is a generalized Nash game (Ichiishi 1983, Rosen 1965), where each player's strategy set is dependent on the competitors' strategies, in contrast to a Nash game where the joint strategy set is full Cartesian products of individual strategy sets. A set of strategies within the constrained joint strategy set is a generalized Nash equilibrium if no player can do better by unilaterally changing his or her strategy to one that maintains the joint constraints satisfied. We have the following result on the existence of a generalized Nash equilibrium to game (P0), and equivalently that of the OLNE in the differential game by Lemma 4.

Theorem 1 (Existence of OLNE via Concavity). *There exists an OLNE in the differential game.*

*Proof of Theorem 1.* By Lemma 1, the set $\{p_i(\vec{p}_{-i}) \mid 0 \leqslant d_j(\vec{p}) \leqslant c_j/\Lambda(T)\}$ is convex for all $j$. By Assumptions 5 and 6, the strategy set $\{p_i(\vec{p}_{-i}) \mid \vec{0} \leqslant \vec{d}(\vec{p}) \leqslant \vec{c}/\Lambda(T)\} \cap \mathcal{P}_i$ for any firm $i$ given competitors' strategy $\vec{p}_{-i}$ is nonempty, compact and convex. By Assumption 4, the existence result for game (P0) thus for the differential game follows by Ichiishi (1983, Theorem 4.3.1). □

The equilibrium existence result of the generalized Nash game (P0) can be obtained under general assumptions on the demand structure and strategy space, though computing a generalized Nash equilibrium remains a challenging task up to date. However, with one additional assumption on the strategy space we can obtain even richer existence results and more effective computation mechanism by focusing on an unconstrained Nash game that provides a subset of Nash equilibria for game (P0).

**3.1.2. Nash Game Approach** We can relate the problem of solving for a generalized Nash equilibrium of game (P0) to the following game (P1) with relaxed constraints: given competitors' prices $\vec{p}_{-i}$, each player $i$ is to simultaneously solve

$$\begin{array}{lll} (\mathrm{P1}_i) & \max\limits_{p_i \in \mathcal{P}_i} & r_i(\vec{p}) \\ & \text{s.t.} & 0 \leqslant d_i(\vec{p}) \leqslant c_i/\Lambda(T). \end{array}$$

In contrast with game (P0), the firms in game (P1) have bounded rationality and ignore the capacity constraints of competitors in their best responses. From the perspective of dual variables, game (P1) is to set $\mu_{ij} = 0$ for $j \neq i$ and all $i$ in the KKT conditions (10)-(11) of game (P0).

Lemma 5 (Constrained Static Game of Bounded Rationality). *Any generalized Nash equilibrium to game (P1) is one to game (P0).*

We want to impose one more fairly reasonable assumption on the joint strategy set $\mathcal{P}$ which is the Cartesian products of each firm's strategy set $\mathcal{P}_i$. This assumption allows us to relax the generalized Nash game (P1) to an auxiliary Nash game (P2) with the strategy space as a compact lattice, which provides a subset of equilibria for the original generalized Nash game (P0). We can then resort to the existing framework of quasi-concave or supermodular game for equilibrium analysis and computation. As a technical treatment, we fix a small $0 < \epsilon < 1$.

Assumption 8 (Achievable Market-Clearing Price). *For any* $(s, \vec{x}) \in (0, T] \times \times_i [\epsilon, c_i]$ *and any* $\vec{p}_{-i} \in \mathcal{P}_{-i}$, *there exists*

$$p_i^0(s, x_i, \vec{p}_{-i}) := \inf\{p_i \geqslant 0 \mid d_i(\vec{p}) \leqslant x_i/\Lambda(s)\}$$

*and* $p_i^0(T, \epsilon, \vec{p}_{-i}^{\max}) \leqslant p_i^{\max}$. *Furthermore,* $\vec{d}(\vec{p}) \geqslant \vec{0}$ *for any* $\vec{p} \in \mathcal{P}$.

By Assumptions 2 and 3, we have the following result with the proof relegated to the appendix.

Lemma 6 (Monotonicity of Market-Clearing Price). $p_i^0(s, x_i, \vec{p}_{-i})$ *is increasing in* $\vec{p}_{-i}$ *and decreasing in* $x_i/\Lambda(s)$.

Remark 1. Combining Lemma 6 and Assumption 8, we have $p_i^0(s, x_i, \vec{p}_{-i}) \leqslant p_i^0(T, \epsilon, \vec{p}_{-i}^{\max}) \leqslant p_i^{\max}$ for any $(s, \vec{x}) \in (0, T] \times \times_i [\epsilon, c_i]$ and any $\vec{p}_{-i} \in \mathcal{P}_{-i}$. This basically says that there exists some price within each firm's compact feasible price interval such that it can sell no more than its capacity regardless of prices the other firms choose. Assumption 8 is not overly restrictive. In reality, no matter how high competitors' prices are, there always exists a finite price for a firm such that the demand can be almost turned off. Technically, suppose $\vec{p}^{\max}$ is a solution to the system $d_i(\vec{p}) = \epsilon/\Lambda(T)$ for all $i$; we have Assumption 8 hold since $p_i^0(T, \epsilon, \vec{p}_{-i}^{\max}) = p_i^{\max}$ for all $i$. Therefore to make the assumption as general as possible, we can select $\vec{p}^{\max}$ as any reasonably large solution to the system $d_i(\vec{p}) = \epsilon/\Lambda(T)$ for all $i$. As an another technical remark, the reason we bound the initial inventory away from zero is that when the inventory as a continuous quantity approaches to zero it is possible that we may lose compactness needed in equilibrium analysis for the set $\mathcal{P}$ satisfying Assumption 8. This technical treatment is not overly restrictive since in the stochastic game inventory levels are non-negative integers; in the differential game, we will see that along the equilibrium trajectory, the ratios of inventories to the remaining time never approach to zero. Furthermore, to ensure the demand is always well-defined, we assume that $\vec{d}(\vec{p}) \geqslant \vec{0}$ for any $\vec{p} \in \mathcal{P}$. For example, if the demand function is in a linear form as $\vec{d}(\vec{p}) = \vec{a} - B\vec{p}$, where $\vec{a} > \vec{0}$ and $B$ is a diagonally dominant matrix with diagonal entries positive and off-diagonal entries non-positive; for such a system, if $\vec{p}^{\max}$ is selected as a solution to the system $d_i(\vec{p}) = \epsilon/\Lambda(T) > 0$ for all $i$, $\vec{d}(\vec{p})$ is always non-negative for all $\vec{p} \in \mathcal{P}$.

Now let us define the following unconstrained auxiliary pricing game (P2): given competitors' prices $\vec{p}_{-i} \in \mathcal{P}_{-i}$, each player $i$ is to simultaneously solve

$$(\mathrm{P2}_i) \qquad \max_{p_i \in \mathcal{P}_i} \ \pi_i(\vec{p}) := \max_{p_i \in \mathcal{P}_i} \ \min\{r_i(\vec{p}), p_i c_i/\Lambda(T)\}.$$

Lemma 7 (Unconstrained Static Game). *Under the additional Assumption 8, game (P2) is equivalent to game (P1).*

By Lemma 4, 5 and 7, we can focus on the auxiliary Nash game (P2) for the existence of Nash equilibrium in the original differential game. Before we resort to the existing framework of analyzing a Nash game, we show the following structural properties shared between the revenue rate functions and the objective functions in game (P2).

Lemma 8 (Structural Equivalence). *(i) $r_i(\vec{p})$ is quasi-concave in $p_i$ $\Leftrightarrow$ $\pi_i(\vec{p})$ is quasi-concave in $p_i$ for all $c_i$; (ii) $r_i(\vec{p})$ is (log-)supermodular in $\vec{p}$ $\Leftrightarrow$ $\pi_i(\vec{p})$ is (log-)supermodular in $\vec{p}$ for all $c_i$.*

Having Lemma 8, by Vives (1999, Theorem 2.1 (Debreu) and Theorem 2.5 (Topkis)) we can establish the existence result for the OLNE in the differential game, under general assumptions on the revenue rate functions which are satisfied by most commonly-used demand structures (see Examples 1 and 2).

Theorem 2 (Existence of OLNE via Supermodularity). *Under the additional Assumption 8, that $r_i(\vec{p})$ is (log-)supermodular in $\vec{p}$ for all $i$ can replace Assumption 4 to guarantee there exists an OLNE in the differential game.*

We can further ensure the uniqueness of Nash equilibrium to game (P2) by some "diagonal dominance" condition (Vives 1999, Section 2.5) under the framework of either quasi-concave game or supermodular game. The supermodular framework provides additional advantage in computing the equilibrium by the tatônnement best-response scheme.

**3.1.3. Open-Loop Nash Equilibrium as a Function of the Initial Condition** In order to write an OLNE in feedback form, we want to establish a one-to-one mapping between an initial condition and its corresponding OLNE. If we have the uniqueness result for game (P2), there naturally exists such a one-to-one correspondence. Then we can write the fixed-pricing OLNE $\vec{p}^*$ for a game (P2) with the remaining sales horizon $s = T - t$ and initial joint capacity $\vec{x}$ as a function of the initial condition $(s, \vec{x}) \in (0, T] \times \bigtimes_i [\epsilon, c_i]$ and denote the mapping by $\vec{p}^o(s, \vec{x})$. There are other ways to specify the mapping $\vec{p}^o(s, \vec{x})$ even if the equilibrium to game (P2) is not unique. For example, suppose the revenue rate functions are (log-)supermodular, then there exists a lattice of equilibria in game (P2). By Assumption 3, $\pi_i(\vec{p})$ is increasing in $\vec{p}_{-i}$, thus the firms all prefer the largest equilibrium in the lattice (Bernstein and Federgruen 2005, Theorem 2), so we can still establish the one-to-one correspondence by designating the mapping as to the largest equilibrium in such a supermodular game. The mapping can also be made for any fixed-pricing OLNE suggested by game (P0) as long as we are consistent in selecting the set of co-state variables for each initial condition and in designating the mapping if there are multiple equilibria for the same set of co-state variables. The following proposition summarizes structural results of this mapping.

Proposition 3 (Open-Loop Mapping). *(i) For any $(s, \vec{x})$, $(s', \vec{x}')$ such that $\vec{x}/\Lambda(s) = \vec{x}'/\Lambda(s')$, we have $\vec{p}^o(s, \vec{x}) = \vec{p}^o(s', \vec{x}')$;*

*(ii) If $d_i(\vec{p})$ is twice continuously differentiable for all $i$, then $\vec{p}^o(s, \vec{x})$ is continuous almost everywhere;*

*(iii) If $r_i(\vec{p})$ is (log-)supermodular in $\vec{p}$ for all $i$, then $\vec{p}^o(s, \vec{x})$ is decreasing in $\vec{x}/\Lambda(s)$.*

Part (i) of Proposition 3 essentially argues that an initial condition $(s, \vec{c})$ plays a role in determining the equilibrium outcome only in terms of the run-out rates $\{c_i/\Lambda(s),\, i \in \mathcal{I}\}$. Part (ii) implies the robustness of an equilibrium in the sense that small perturbations of the initial condition due to possible data inaccuracy do not significantly change the equilibrium. Part (iii) shows that the equilibrium prices of all firms move downward as the initial capacity of some firm increases and/or the sales horizon shortens ceteris paribus. In light of part (iii), we observe that the increasing monotonicity of value functions in inventory, ingrained in folklore in monopoly RM models, may break down for oligopoly RM models: if $r_i(\vec{p})$ is locally increasing in $p_i$, together with Assumption 3, we have $r_i(\vec{p})$ is locally increasing in $\vec{p}$. Combining two monotone functions, profit functions $\pi_i(\vec{p}^o(s,\vec{x})) = r_i(\vec{p}^o(s,\vec{x}))$ for all $i$ in the differential game under the fixed-pricing OLNE $\vec{p}^o(s,\vec{x})$ are possible to be locally decreasing in $\vec{x}$.

Recall that $\vec{p}^o(T,\vec{c})$ is a solution to conditions (10)-(11) that characterize the fixed-pricing OLNE when the initial condition is $(T,\vec{c})$. Realizing that condition (11) can be written as

$$\mu_{ij}\left[(c_j - \Lambda(t)d_j(\vec{p})) - \Lambda(T-t)d_j(\vec{p})\right] = 0, \text{ for all } i,j,$$

we see $\vec{p}^o(T,\vec{c})$ is also a solution to conditions (10)-(11) when the initial condition is $\left(T-t, \vec{c} - \Lambda(t)\vec{d}(\vec{p}^o(T,\vec{c}))\right)$ for all $0 \leqslant t \leqslant T$. The the following result follows immediately.

Proposition 4 (Trajectory of OLNE). *$\vec{p}^o(T,\vec{c}) = \vec{p}^o\left(T-t, \vec{c} - \Lambda(t)\vec{d}(\vec{p}^o(T,\vec{c}))\right)$ for all $0 \leqslant t \leqslant T$.*

### 3.2. Closed-Loop Nash Equilibrium (CLNE)

Though the CLNE under the strong information structure is generally hard to solve for the differential game, we observe that the fixed-pricing OLNE $\vec{p}^*(t) = \vec{p}^o(T,\vec{c})$ for $0 \leqslant t \leqslant T$ specified in the previous section can also be described in a feedback form as:

$$\vec{p}^c(t,\vec{x}) = \begin{cases} \vec{p}^o(T-t,\vec{x}), & \text{for } (t,\vec{x}) \in (0,T] \times \bigtimes_i [\epsilon, c_i], \\ \vec{p}^o(T-t,\vec{\epsilon}), & \text{for } (t,\vec{x}) \in (0,T] \times \bigtimes_i [0,\epsilon). \end{cases} \tag{12}$$

The separate treatment for $(t,\vec{x}) \in (0,T] \times \bigtimes_i [0,\epsilon)$ is consistent with Remark 1 to ensure $\vec{p}^c(t,\vec{x})$ is well-defined within the compact set $\mathcal{P}$. By Proposition 4, the equilibrium state trajectory of the differential game under the closed-loop strategy $\vec{p}^c(t,\vec{x})$ self-enforcingly evolves according to $\vec{x}(t) = \vec{c} - \Lambda(t)\vec{d}(\vec{p}^o(T,\vec{c}))$ for all $t$ while yielding $\vec{p}^c(t,\vec{x}(t)) = \vec{p}^o(T,\vec{c})$ for all $t$ along the state trajectory. Therefore the specified closed-loop strategy generates identical equilibrium trajectory of control and state as the fixed pricing open-loop strategy. Furthermore, it is indeed a CLNE.

Proposition 5 (Existence of CLNE via Construction). *The closed-loop strategy $\vec{p}^c(t,\vec{x})$ is a CLNE of the differential game.*

*Proof of Proposition 5.* Under the guaranteed constraint qualification, the necessary condition for the CLNE is the set of conditions (4)-(7) with condition (5) replaced by

$$-\frac{\partial \mu_{ij}}{\partial t} = \frac{\partial H_i}{\partial x_j} + \sum_{k\neq i} \frac{\partial H_i}{\partial p_k}\frac{\partial p_k^*}{\partial x_j},\ 0 \leqslant t \leqslant T,\ \text{for all}\ i,j. \tag{13}$$

Along the equilibrium state trajectory $\vec{x}(t) = \vec{c} - \Lambda(t)\vec{d}(\vec{p}^o(T,\vec{c}))$, the equilibrium price path $\vec{p}^*(t) = \vec{p}^o(T,\vec{c}), 0 \leqslant t \leqslant T$ is state-invariant. Thus $\partial p_k^*/\partial x_j = 0$ for all $k, j$. Recall that the OLNE $\vec{p}^o(T,\vec{c})$ together with its costate trajectory $\vec{\mu}_i(t) = \vec{\mu}_i$ for all $i$ and state trajectory $\vec{x}(t) = \vec{c} - \Lambda(t)\vec{d}(\vec{p}^o(T,\vec{c}))$ satisfy conditions (4)-(7). We see that the closed-loop strategy $\vec{p}^c(t,\vec{x}), 0 \leqslant t \leqslant T$ together with the same trajectory of costate variables and state as the OLNE indeed satisfies the set of necessary conditions for the CLNE – conditions (4)-(7) with condition (5) replaced by condition (13). By Assumption 4 and Lemma 1, we obtain the desired result since the set of necessary conditions is also sufficient: by extending Sethi and Thompson (2005, Theorem 3.1) to our game and realizing that $H_i(t,\vec{x},\vec{p},\vec{\mu}_i) = \lambda(t)[r_i(\vec{p}) - \sum_j \mu_{ij} d_j(\vec{p})]$ is state-independent with $r_i(\vec{p})$ pseudo-concave in $p_i$, $d_j(\vec{p})$, $\forall j$, quasi-convex in $p_i$ and the *LHS* function of terminal constraint $\vec{x}(T) \geqslant 0$ linear (thus quasi-concave) in $\vec{x}$. $\square$

As a minimum requirement, the CLNE $\vec{p}^c(t,\vec{x})$ with any fixed $0 < \epsilon < 1$ is time consistent (Dockner et al. 2000, Theorem 4.3). Furthermore, if we can strengthen Assumption 8 such that it is satisfied for $\epsilon = 0$ then the CLNE $\vec{p}^c(t,\vec{x})$ with $\epsilon = 0$ will be indeed subgame perfect so that the strategy represents optimal behavior not only along the equilibrium state trajectory but also anywhere off this trajectory (Dockner et al. 2000, Section 4.3).

## 4. Links to the Stochastic Game

### 4.1. The Differential Game as an Affine Functional Approximation

It is well known that the HJB equation can be equivalently stated as an optimization problem (see Adelman 2007 for an application in discrete-time RM). The equivalent problem for the set of HJB equations (1) of the continuous-time stochastic game is that any firm $i$ simultaneously solves its own optimization problem given competitors' strategy $\vec{p}_{-i}(t,\vec{n})$:

$$\begin{aligned} \min_{V_i(\cdot,\cdot)} \quad & V_i(T,\vec{c}) \\ \text{s.t.} \quad & -\frac{\partial V_i(T-t,\vec{n})}{\partial t} \geqslant \lambda(t)\left\{r_i(\vec{p}(t,\vec{n})) - \vec{d}(\vec{p}(t,\vec{n}))^{\mathrm{T}} \nabla \vec{V}_i(T-t,\vec{n})\right\}, \\ & \forall\, p_i(t,\vec{n}) \in \mathcal{P}_i \cup \{p_i^{null}(\vec{p}_{-i}(t,\vec{n}))\}, (t,\vec{n}) \in [0,T] \times \mathbb{Z}^m \cap \mathcal{X}. \end{aligned}$$

We now approximate the value functions $V_i(s,\vec{n})$ for all $i$ by the following quasistatic affine functions:

$$V_i(s,\vec{n}) \backsimeq W_i(s,\vec{n}) := \int_{T-s}^{T} \theta_i(t)\, dt + \vec{n}^{\mathrm{T}} \vec{w}_i, \quad \vec{w}_i \geqslant \vec{0}. \tag{14}$$

Realizing that $-\partial W_i(T-t,\vec{n})/\partial t = \theta_i(t)$ and $\nabla \vec{W}_i(T-t,\vec{n}) = \vec{w}_i$, we can approximate the minimization problem for any firm $i$ as the following continuous-time nonlinear programming problem:

$$\begin{aligned} (\mathrm{D}_i) \qquad \min_{\theta_i(t), t\in[0,T], \vec{w}_i \geqslant \vec{0}} \quad & \int_0^T \theta_i(t)\, dt + \vec{c}^{\mathrm{T}} \vec{w}_i \\ \text{s.t.} \quad & \theta_i(t) \geqslant \lambda(t)\left\{r_i(\vec{p}(t)) - \vec{d}(\vec{p}(t))^{\mathrm{T}} \vec{w}_i\right\}, \quad \forall\, p_i(t) \in \mathcal{P}_i \cup \{p_i^{null}(\vec{p}_{-i}(t))\}, t \in [0,T]. \end{aligned}$$

Since $(\mathrm{D}_i)$ is a minimization problem, it is optimal to set

$$\theta_i(t) = \lambda(t) \max_{p_i(t) \in \mathcal{P}_i \cup \{p_i^{null}(\vec{p}_{-i}(t))\}} \{r_i(\vec{p}(t)) - \vec{d}(\vec{p}(t))^{\mathrm{T}} \vec{w}_i\}$$

for all $t \in [0,T]$ in the objective function. Then the objective of any firm $i$ becomes

$$\min_{\vec{w}_i \geqslant \vec{0}} \left\{ \max_{p_i(t) \in \mathcal{P}_i \cup \{p_i^{null}(\vec{p}_{-i}(t))\}, t\in[0,T]} \int_0^T \lambda(t) r_i(\vec{p}(t))\, dt + \left(\vec{c} - \int_0^T \lambda(t) \vec{d}(\vec{p}(t))\, dt\right)^{\mathrm{T}} \vec{w}_i \right\},$$

which is the maximization problem with capacity constraints (2) in the differential game dualized by the vector $\vec{w}_i$. Strong duality holds here since this continuous-time maximization primal problem has pseudo-concave objective function and quasi-convex constraints, and both primal and dual are feasible (Zalmai 1985).

Proposition 6 (Approximation to Stochastic Game). *The stochastic game reduces to the deterministic differential game when each firm uses a quasistatic affine function (14) to approximate its own value function.*

When the above strong duality holds, first-order approximated capacity values $\vec{w}_i$ (dual variables) of all firms to any firm $i$ are exactly equal to constant costate variables (shadow prices) $\vec{\mu}_i$ in the differential game when an OLNE is considered (see conditions (8) and (9)). The capacity value of any competitor to the firm itself is nonnegative due to the substitutability between the differentiated products: intuitively, when a firm has limited capacity compared to ample capacity, it has incentive to increase its own price; in the competitive market of strategic complements this pressure of lifting price is passed to all competitors in reaching an equilibrium.

### 4.2. Closed-Loop Heuristics as Asymptotic Equilibrium

The differential game can be analyzed for suggesting tractable heuristics to the original non-zero-sum stochastic game. In this section, we demonstrate three heuristics under three information structures and show that they are asymptotic equilibrium for the stochastic game in an appropriate sense.

**4.2.1. Weak Information Structure** Under the weak information structure, only initial joint inventory level is public information. Given such limited source of information, the OLNE that does not require observing the evolving state sustains as an $\epsilon$-Nash equilibrium in a relative sense asymptotically when supply and demand are proportionally scaled up, in particular, when the initial capacities $\vec{c}$ and the cumulative customer arrival $\Lambda(T)$ are scaled up but with their ratios $\vec{c}/\Lambda(T)$ fixed.

Definition 6 ($\epsilon$-Nash Equilibrium). For any $\epsilon > 0$, $\vec{u}^* \in \mathcal{U}$ is called an $\epsilon$-Nash equilibrium of the stochastic game if $J_i\left(u_i, \vec{u}^*_{-i}\right)/J_i\left(\vec{u}^*\right) \leqslant 1+\epsilon$ for all $u_i \in \mathcal{U}_i$ and all $i$.

Remark 2. Similar to the concept considered in Bernstein and Federgruen (2003), $\epsilon$ here refers to a *relative* amount by which a firm's profit can be improved by a unilateral deviation from the equilibrium.

When $m=1$, there is no competition but monopoly. Let us first review bounds on the performance of the fixed-pricing policy suggested by the differential problem in the monopolist stochastic pricing problem, then proceed to use it to prove our asymptotic optimality result in the oligopoly case. In the monopolist stochastic problem, given a pricing policy $p \in \mathcal{U}$, an initial stock $n>0$ and a sales horizon $s>0$, we denote the expected revenue by $J^p(s,n) := E\left[\int_0^s p_v\, dN_v\right]$, where $J^p(0,n)=0$ for $\forall\, n \in \mathbb{N} \cup \{0\}$ and $J^p(s,0)=0$ for $\forall\, s \in \mathbb{R}_+$. The optimal total expected revenue generated over the selling horizon for the stochastic problem is denoted by $J^s(s,n)$, i.e., $J^s(s,n) := \sup_{p \in \mathcal{U}} J^p(s,n)$. The maximal total revenue generated over the selling horizon in the deterministic differential problem is denoted by $J^d(s,n)$. Let $J^f(s,n)$ denote the total expected revenue for the stochastic problem under the fixed-pricing heuristic suggested by solving the differential problem.

Lemma 9 (Bounds in Monopoly RM Problem).

$$J^d(s,n)\left(1-\frac{1}{2\sqrt{\min\{n,\Lambda(s)d(p^*)\}}}\right) \leqslant J^f(s,n) \leqslant J^s(s,n) \leqslant J^d(s,n)\,. \tag{15}$$

Summarizing performance bounds from Gallego and van Ryzin (1994), Lemma 9 suggests that the simple fixed-pricing heuristic $p^*$ results in revenues that are at least proportional to the upper bound. Moreover, the proportional factor approaches one as both the supply and the demand increase. The asymptotic optimality of the fixed-pricing heuristic for the monopolistic problem is not hard to obtain by noting that when $\min\{n,\Lambda(s)d(p^*)\}$ is large enough, $J^f(s,n)$ and $J^s(s,n)$ are sufficiently close. For the oligopolist stochastic game, the problem for an individual firm to maximize its own expected total revenue while all other firms use a fixed-pricing policy is nothing but a monopolist optimal dynamic pricing problem since the demand intensity function for this firm is time invariant. Realizing this, we are ready to extend asymptotic optimality of the fixed-pricing policy in the monopolistic problem to $\epsilon$-Nash equilibrium of the fixed-pricing policy in the multi-player stochastic game.

Theorem 3 (OPNE as $\epsilon$-Nash Equilibrium). *Consider the stochastic game. For any $\epsilon>0$, the fixed-pricing equilibrium $\vec{p}^{\,o}(T,\vec{c})$ is an $\epsilon$-Nash equilibrium of the stochastic game if the initial condition satisfies* $\max\{c_i, \Lambda(T)d_i(\vec{p}^{\,o}(T,\vec{c}))\} \geqslant ((1+\epsilon)/2\epsilon)^2$ *for all $i$.*

**4.2.2. Strong Information Structure** Under the strong information structure, the inventory level of any firm in real time is public information. The closed-loop strategy $\vec{p}^c(t,\vec{x})$ in the differential game provides a heuristic in feedback form. By extending Maglaras and Meissner (2006) to the game context, we will show that this heuristic is a Nash equilibrium asymptotically in a limiting regime as demand and supply grow proportionally large. Specifically, using $k$ as an index, we consider a sequence of stochastic games with customer arrival $\lambda^k(t) = k\lambda(t)$, $\forall\, t \in [0,T]$ and initial joint inventory $\vec{c}^k = k\vec{c}$, and let $k$ increase to infinity; hereafter, a superscript $k$ denotes quantities that scale with $k$.

Definition 7 (Asymptotic Nash Equilibrium). $\vec{u}^* \in \mathcal{U}$ is called an asymptotic Nash equilibrium of the stochastic game if $\vec{u}^*$ is a Nash equilibrium of the limiting game as $k$ increases to infinity.

In this section and the following one, we need to make an assumption on the continuity of the mapping $\vec{p}^o(t,\vec{x})$, which Proposition 3 part (ii) provides a sufficient condition to guarantee.

Assumption 9 (Continuity of Open-Loop Mapping). *$\vec{p}^o(t,\vec{x})$ (thus $\vec{p}^c(t,\vec{x})$) is continuous almost everywhere.*

The cumulative demand for firm $i$ up to time $t$ is $N_i(A_i^k(t))$, where $A_i^k(t) = \int_0^t k\,\lambda(v) d_i(v)\,dv$, where $d_i(v) = d_i(\vec{p}^c(v, \vec{X}^k(v)/k))$, and $X_i^k(v) = kc_i - N_i(A_i^k(v))$ for all $i$ denotes the remaining inventory for firm $i$ under the heuristic. Note that $A_i^k(0) = 0$, $A_i^k(t)$ is nondecreasing and $A_i^k(t) - A_i^k(v) \leqslant (\Lambda(t) - \Lambda(v))kd_i^{\max}$, where $d_i^{\max} := \max_{\vec{p}\in\mathcal{P}} d_i(\vec{p})$. This implies that the family of process $\{A_i^k(t)/k\}$ for all $i$ is equicontinuous, and therefore relatively compact. By the Ascoli-Arzelá Theorem, the sequence $\{A_i^k(t)/k\}$ has a converging subsequence, say $\{k_m\}$, such that $A_i^{k_m}(t)/k_m \to \bar{A}_i(t)$ for all $i$: for $i = 1$, there exists a converging subsequence $\{k_1\}$, such that $A_1^{k_1}(t)/k_1 \to \bar{A}_1(t)$; for $i = 2$, along sequence $\{k_1\}$, there exists a converging subsequence of $\{k_2\}$, such that $A_1^{k_2}(t)/k_2 \to \bar{A}_1(t)$ and $A_2^{k_2}(t)/k_2 \to \bar{A}_2(t)$; we can repeat the process until we have a subsequence $\{k_m\}$ satisfying the desired property. Recall that the functional strong law of large numbers for the Poisson process asserts that $N_i(kt)/k \to t$, a.s. uniformly in $t \in [0,T]$ as $k \to \infty$. Along the subsequence $\{k_m\}$ we get that $N_i(A_i^k(t))/k$ converges to $\bar{A}_i(t)$ for all $i$, and therefore that $\bar{X}_i^k(t) := X_i^k(t)/k$ converges to a limit $\bar{x}_i(t)$ for all $i$; the two converging results hold a.s. uniformly in $t \in [0,T]$. Using the continuity of $\vec{d}(\vec{p})$ and $\vec{p}^c(t,\vec{x})$, by Dai and Williams (1995, Lemma 2.4), we get that as $k_m \to \infty$, for all $i$,

$$\frac{1}{k_m} A_i^{k_m}(t) = \int_0^t \lambda(v) d_i \left( \vec{p}^c \left( v, \frac{\vec{X}^{k_m}(v)}{k_m} \right) \right) dv \to \int_0^t \lambda(v) d_i \left( \vec{p}^c \left( v, \vec{\bar{x}}(v) \right) \right) dv, \text{ a.s. uniformly in } t \in [0,T].$$

Thus we get that as $k_m \to \infty$, for all $i$,

$$\bar{X}_i^{k_m}(t) = c_i - \frac{1}{k_m} N_i(A_i^{k_m}(t)) \to c_i - \int_0^t \lambda(v) d_i \left( \vec{p}^c \left( v, \vec{\bar{x}}(v) \right) \right) dv = c_i - \Lambda(t) d_i(\vec{p}^o(T,\vec{c})),$$

a.s. uniformly in $t \in [0,T]$, which shows that the limiting trajectories do not depend on the selection of the converging subsequence $\{k_m\}$. By Assumption 9, we have as $k_m \to \infty$,

$$\vec{p}^c(t, \vec{\bar{X}}^{k_m}(t)) \to \vec{p}^c \left( t, \vec{c} - \Lambda(t) \vec{d}(\vec{p}^o(T,\vec{c})) \right) = \vec{p}^o(T,\vec{c}), \text{ a.s. uniformly in } t \in [0,T].$$

Again by Dai and Williams (1995, Lemma 2.4), the revenue extracted under the closed-loop strategy $\vec{p}^c(t,\vec{x})$ after normalization is, for all $i$, as $k_m \to \infty$,

$$\frac{1}{k_m} J_i \left( \vec{p}^c(t, \vec{\bar{X}}^{k_m}(t)), 0 \leqslant t \leqslant T \right) = \frac{1}{k_m} \int_0^T p_i^c(t, \vec{\bar{X}}^{k_m}(t)) dN_i(A_i^{k_m}(t)) \to \Lambda(T) r_i(\vec{p}^o(T,\vec{c})), \text{ a.s.}$$

For any closed-loop strategy $\vec{p'}(t,\vec{x}), 0 \leqslant t \leqslant T$ that firms decide to follow in the stochastic game, $\vec{p'}(t, \vec{X}^k(t)/k), 0 \leqslant t \leqslant T$ is implemented in the scaled system. By the same argument, we can show that $\vec{\bar{X}}(t) = \vec{X}^k(t)/k$ converges to a limit $\vec{\bar{x}}'(t)$ a.s. uniformly in $t \in [0,T]$; hence $\vec{p'}(t, \vec{\bar{X}}(t))$ converges to $\vec{p'}(t, \vec{\bar{x}}'(t))$ a.s. uniformly in $t \in [0,T]$, which is exactly the control path generated by the same closed-loop strategy $\vec{p'}(t,\vec{x})$ in the differential game. Realizing this correspondence, we see that by applying a CLNE suggested by the differential game in the stochastic game, such a strategy sustains as a Nash equilibrium in the limiting regime of the scaled stochastic games.

PROPOSITION 7 (CLNE AS ASYMPTOTIC NASH EQUILIBRIUM). *Suppose that demand and capacity of the stochastic game are scaled according to $\lambda^k(t) = k\lambda(t),\ \forall t \in [0,T]$ and $\vec{c}^k = k\vec{c}$. Under the additional Assumption 9, the heuristic $\vec{p}^c(t,\vec{x})$ is an asymptotic Nash equilibrium in the limiting regime of the sequence of scaled stochastic games as $k \to \infty$.*

**4.2.3. Weak Information Structure with Observable Price** Strong information structure is needed when analyzing closed-loop strategies. In reality, the weak information structure with observable price history can be a more reasonable assumption (Xu and Hopp 2006). We assume all firms know the initial inventory levels and can observe the instantaneous prices of competitors, so that every firm can estimate the remaining inventory levels of its competitors. We define the market price history up to time $t$ as $H(t) = \sigma\{\vec{p}(v), 0 \leqslant v \leqslant t\}$ and $h(t)$ as a realization of $H(t)$. We propose the following feedback-type heuristic: for all $i$,

$$p_i^h(t, x_i, h(t)) = p_i^c\left(t, x_i, \vec{c}_{-i} - \int_0^t \lambda(v)\vec{d}_{-i}\left(\vec{p}(v)\right) dv\right), \; 0 \leqslant t \leqslant T. \tag{16}$$

Since the information structure considered in this section may be the closest to the reality, we believe the proposed heuristic and its asymptotic optimal behavior are of particular interest to the practice.

Similarly to the analysis in the previous section, we can show that the family of cumulative demand-rate process $\{A_i^k(t)/k\}$ for all $i$ under the heuristic $\vec{p}^h$ has a converging subsequence $\{k_m^h\}$ on which the mean state process with demand uncertainty $\vec{\bar{X}}^h(t)$ converges almost sure to a deterministic state process $\vec{\bar{x}}^h(t)$ uniformly in $t \in [0, T]$. Along the deterministic limiting trajectory $\vec{\bar{x}}^h(t)$, $x_j^h(t) = c_j - \int_0^t \lambda(v) d_j(\vec{p}(v))\, dv$ for all $j \neq i$, in other words, the price history accurately reveals the competitors' inventory information. It is easy to see the heuristic $\vec{p}^h$ is equivalent to the heuristic $\vec{p}^c$ in the fluid-scale limiting regime in that they generate the same trajectory of control and state. Similar arguments to the previous section lead to the following result.

Proposition 8 (Closed-Loop Heuristic as Asymptotic Nash Equilibrium). *Suppose that demand and capacity of the stochastic game are scaled according to $\lambda^k(t) = k\lambda(t)$, $\forall\, t \in [0, T]$ and $\vec{c}^k = k\vec{c}$. Under the additional Assumption 9, the heuristic $\vec{p}^h(t, \vec{x}, h(t)) = \left(p_i^h(t, x_i, h(t))\right)_i$ is an asymptotic Nash equilibrium in the limiting regime of the sequence of scaled stochastic games as $k \to \infty$.*

## 5. Examples

### 5.1. Demand Structures

We consider several of the most frequently-used classes of demand functions and demonstrate conditions for the existence of the OLNE to the differential game.

Example 1 (General Attraction Models). In the attraction models, customers choose each firm with probability proportional to its attraction value. Specifically, we have the following customer choice probability functions: for all $i$,

$$d_i(\vec{p}) = \frac{a_i(p_i)}{\sum_{j=0}^{m} a_j(p_j)}, \tag{17}$$

where $a_i(p_i) > 0$ is the attraction value for firm $i$ and $a_0 = a_0(p_0) \geqslant 0$ is interpreted as the fixed value of the no-purchase option.

Lemma 10 (Sufficient Condition of Pseudo-Convexity). *If a twice continuously differentiable function $f:\mathbb{R}\to\mathbb{R}$ satisfies $f'(x)=0 \Rightarrow f''(x)>0$, then $f$ is pseudo-convex, i.e., for any $x_1$ and $x_2$, $(x_1-x_2)f'(x_2) \geqslant 0 \Rightarrow f(x_1) \geqslant f(x_2)$.*

We have the following structural results on the general attraction demand functions. For simplicity, let $a_i := a_i(p_i), a_i' := \partial a_i(p_i)/\partial p_i$ and $a_i'' := \partial^2 a_i(p_i)/\partial {p_i}^2$. We assume $a_i$ is twice continuously differentiable.

Proposition 9 (Structures of Attraction Models). *(i) if $a_i' \leqslant 0$ for all $i$, $d_i(\vec{p})$ is quasi-linear in $p_j$ for all $j$;*

*(ii) if $2a_i' - a_i a_i''/a_i' > (resp. <) 0$, $r_i(\vec{p})$ is pseudo-convex (resp. pseudo-concave) in $p_i$.*

The MNL model assumes $a_i(p_i) = \beta_i \exp\{-\alpha_i p_i\}$, $\alpha_i, \beta_i > 0$ for all $i$. As an immediate result, we have the following corollary.

Corollary 1. *For the MNL model, $d_i(\vec{p})$ is quasi-linear in $p_j$ for all $j$ and $r_i(\vec{p})$ is pseudo-concave in $p_i$.*

Thus by Theorem 1, there exists an OLNE in the differential game for the MNL model. We can also obtain the equilibrium existence result for the general attraction models by the Nash game approach and the framework of supermodular game.

Proposition 10 (Existence of OLNE for Attraction Models). *For any initial condition,*

*(i) if $a_i' \leqslant 0$ for all $i$, there exists an OLNE;*

*(ii) furthermore, if $a_i$ is log-concave in $p_i$ and for any $\vec{p} \in \mathcal{P}$ and all $i$,*

$$\frac{a_i}{\sum_{j \neq i} a_j} \leqslant \frac{a_i'}{\sum_{j \neq i} a_j'}, \tag{18}$$

*there exists a unique Nash equilibrium to game (P2).*

Remark 3. Note that a weaker condition

$$\frac{a_i}{\sum_j a_j} < \frac{a_i'}{\sum_{j \neq i} a_j'}$$

as Equation (11) provided in Bernstein and Federgruen (2004) is not sufficient to guarantee the uniqueness of a Nash equilibrium to game (P2). Condition (18) is satisfied by the MNL model with constant price sensitivity $\alpha_i = \alpha > 0$, $\forall\, i$, $\beta_i > 0, \forall\, i$ and $a_0 \geqslant 0$.

Example 2 (The Linear Model). The customer choice probability function has the form of $d_i(\vec{p}) = a_i - b_i p_i + \sum_{j \neq i} c_{ij} p_j$, $a_i, b_i > 0$, $c_{ij} \geqslant 0, j \neq i$ for all $i$. Then it is easy to check that $d_i(\vec{p})$ is quasi-linear in $p_j$ for all $j$, $r_i(\vec{p})$ is strictly concave in $p_i$ and supermodular in $\vec{p}$. Under the additional condition of diagonal dominance that $b_i > \sum_{j \neq i} c_{ij}$ for all $i$, there exists a unique Nash equilibrium to game (P2) and a unique generalized Nash equilibrium to game (P1) for any set of costate variables.

### 5.2. Numerical Examples

Using the tatônnement scheme as a computational tool for the fixed-pricing OLNE, we are empowered to study the convergence rate of the scheme, the effects of irrationality of competitors and the performance of proposed heuristics for the stochastic game.

Example 3 (OLNE for MNL). We consider price competition among $m = 5$ firms and use a tatônnement scheme for the MNL model. Table 1 summarizes the equilibrium price vector $\vec{p}^*$ and demand vector $\vec{d}(\vec{p}^*)$ under different initial conditions from all firms with abundant inventories to all firms with limited inventories, provided no-purchase value $a_0 = 0.25$, $(\alpha_1, \alpha_2, \cdots, \alpha_5) = (.5, .75, 1, 1.25, 1.5)$, $(\beta_1, \beta_2, \cdots, \beta_5) = (.5, .75, 1, 1.25, 1.5)$. Notice that for this example when $c_i/\Lambda(T)$ decreases for some firm $i$ while the initial conditions are fixed for other firms, the market equilibrium $\vec{p}^*$ moves up. This is consistent with the supermodularity; firm $i$ tends to price higher with relatively less inventories and other firms in the market react by increasing their prices accordingly to stay at equilibrium.

Figure 1 illustrates the linear convergence of tatônnement scheme for the MNL model with various no-purchase value $a_0$, provided the initial condition $\vec{c}/\Lambda(T) = (0.09, 0.5, 0.17, 0.5, 0.24)$ and starting point $\vec{p}^0 = (4, 3, 3, 3, 3)$. We see that with a larger no-purchase value, the linear convergence is faster. □

| $\vec{c}/\Lambda(T)$ | $\vec{p}^*$ | $\vec{d}(\vec{p}^*)$ |
|---|---|---|
| (.50,.50,.50,.50,.50) | (2.21,1.55,1.21,1.01,.87) | (.10,.14,.17,.21,.24) |
| (.09,.50,.50,.50,.50) | (2.38,1.55,1.21,1.01,.88) | (.09,.14,.18,.21,.24) |
| (.09,.13,.50,.50,.50) | (2.41,1.66,1.22,1.01,.88) | (.09,.13,.18,.21,.24) |
| (.09,.13,.17,.50,.50) | (2.44,1.68,1.28,1.02,.88) | (.09,.13,.17,.21,.24) |
| (.09,.13,.17,.21,.50) | (2.46,1.69,1.29,1.04,.88) | (.09,.13,.17,.21,.25) |
| (.09,.13,.17,.21,.24) | (2.54,1.74,1.33,1.07,.92) | (.09,.13,.17,.21,.24) |

**Table 1** Open-Loop Equilibrium for the MNL with Various Initial Conditions

Example 4 (Irrationality). An important question in games is what happens if some firm deviates from the equilibrium. To address this question we focus on the differential game and define rational firms as those who always use their best response functions. *Irrational* firms are those who use less sophisticated strategies such as price matching, pricing to maximize revenue rates (even when capacities are low) or use market clearing prices (even when capacities are high). Rational firms can still reach a Nash equilibrium among themselves that is a best response to all other firms, including irrational firms. Following the competitive setting of Example 3, we numerically

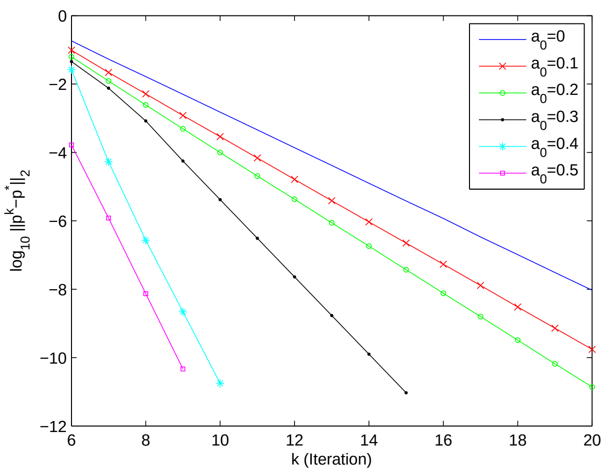


**Figure 1** Linear Convergence of Tatônnement for the MNL with Various No-Purchase Values

investigate the effects that irrational firms have on their own revenues as well as those of other firms. We use the same parameters for the MNL model as in Example 3.

Table 2 specifies various scenarios with irrational firms that ignore the capacity effect and always try to maximize their revenue rates. There is a severe loss for any irrational firm, compared to the revenue it could make by using best-response strategy. The extent of the loss is roughly proportional to the capacity level; with less capacity, the firm suffers more by pricing irrationally. When irrational firms with limited capacities keep their prices low, they have two effects on their competitors: first, before irrational firms run out of stock, their relatively low prices hurt competitors with more price-sensitive competitors taking larger losses; second, after irrational firms run out of stock, competitors can increase prices and improve profits. Price increase after some irrational firm runs out of capacity is consistent with the supermodularity since the firm is forced to increase its price to a null price. The combined effect of the two depends on which one is dominant. Low revenue-rate maximizing prices of irrational firms can mislead their competitors into thinking that they have abundant capacities. In spite of the second effect, rational firms with limited capacities suffer more than rational firms with abundant capacities because when they can increase prices after some irrational firms run out of capacity, they likely have already consumed most of their limited capacities at a relatively low price due to the misleading information.

Table 3 shows that there is a loss for both irrational firms and their rational competitors when irrational firms use the market clearing price when they have abundant capacities. In Table 4, we consider a situation where an irrational firm selects a price uniformly between the revenue-rate maximizing price and the market clearing price, and we demonstrate by simulation the phenomenon that rational firms may come ahead after some irrational firms run out of capacity before the end of selling horizon. We use an initial condition $\vec{c}/\Lambda(T) = (.20, .20, .17, .20, .23)$ for this simulation, where firms 3, 4 and 5 have limited capacities. The table suggests that using a randomized policy results in loss for the irrational firms and may cause gains for strategic competitors using best-response pricing.

Not surprisingly, irrational firms consistently lose revenues relative to playing with their equilibrium strategies. More interestingly, the losses incurred by irrational firms themselves are consistently larger than the losses inflicted on other firms. We find that rational firms with more price-sensitive demands suffer larger losses when irrational firms with limited capacities maximize their revenue rates instead of using their (higher) optimal market clearing prices. However, by pricing low, irrational firms run out of capacity before the end of the horizon, allowing rational firms to respond with higher prices afterwards. In some cases, rational firms may come out ahead as a result of irrational firms' pricing low relative to their capacities. When irrational firms with abundant capacities use their market clearing prices instead of their (higher) optimal revenue-rate maximizing prices, all rational firms suffer with those who have more price-sensitive demands suffering the most. In this case, irrational firms do not run out of capacity until the end of the horizon so rational firms have no chance of recovering from the inflicted losses. The negative effect on rational firms is larger when irrational firms have abundant capacities but use market clearing prices instead of their optimal revenue-rate maximizing prices. When irrational firms have limited capacities but use revenue-rate maximizing prices instead of optimal market clearing prices, the losses to rational firms are much smaller and can in fact be negative. When irrational firms select random prices between their market clearing and revenue-rate maximizing prices, the effect on rational firms can be either positive or negative. Rational firms with price-sensitive demands are more likely to suffer, but rational firms with higher capacity have more chances to recover and come out ahead after some irrational firms run out of capacity. □

| Irrational Set | $\vec{c}/\Lambda(T)$ | $J^d/\Lambda(T)$ | Run-out Set |
|---|---|---|---|
| $\varnothing$ | (.05,.50,.50,.50,.50) | (.1831,.2267,.2246,.2223,.2196) | $\{1\}$ |
| $\{1\}$ | (.05,.50,.50,.50,.50) | (60%,100%,100%,100%,100%) | $\varnothing$ |
| $\varnothing$ | (.05,.06,.50,.50,.50) | (.1961,.1748,.2544,.2510,.2474) | $\{1,2\}$ |
| $\{1\}$ | (.05,.06,.50,.50,.50) | (57%,100%,100%,100%,100%) | $\{2\}$ |
| $\{2\}$ | (.05,.06,.50,.50,.50) | (100%,54%,100%,100%,100%) | $\{1\}$ |
| $\{1,2\}$ | (.05,.06,.50,.50,.50) | (53%,53%,101%,101%,101%) | $\varnothing$ |
| $\varnothing$ | (.05,.06,.08,.50,.50) | (.2181,.1924,.1924,.3072,.3011) | $\{1,2,3\}$ |
| $\{1\}$ | (.05,.06,.08,.50,.50) | (52%,100%,100%,100%,100%) | $\{2,3\}$ |
| $\{2\}$ | (.05,.06,.08,.50,.50) | (100%,50%,100%,100%,100%) | $\{1,3\}$ |
| $\{3\}$ | (.05,.06,.08,.50,.50) | (100%,100%,52%,100%,100%) | $\{1,2\}$ |
| $\{1,2\}$ | (.05,.06,.08,.50,.50) | (49%,49%,81%,104%,104%) | $\{3\}$ |
| $\{1,3\}$ | (.05,.06,.08,.50,.50) | (49%,80%,51%,103%,103%) | $\{2\}$ |
| $\{2,3\}$ | (.05,.06,.08,.50,.50) | (81%,49%,50%,103%,103%) | $\{1\}$ |
| $\{1,2,3\}$ | (.05,.06,.08,.50,.50) | (47%,48%,49%,102%,102%) | $\varnothing$ |

**Table 2** Effects on Payoffs if Irrational Firms Maximize Their Revenue Rates

| Irrational Set | $\vec{c}/\Lambda(T)$ | $J^d/\Lambda(T)$ | Run-out Set |
|---|---|---|---|
| $\{\varnothing\}$ | (.11,.15,.19,.22,.25) | (.2144,.2131,.2114,.2093,.2071) | $\{\varnothing\}$ |
| $\{1\}$ | (.11,.15,.19,.22,.25) | (99%,98%,98%,98%,98%) | $\{1\}$ |
| $\{1,2\}$ | (.11,.15,.19,.22,.25) | (96%,97%,96%,96%,96%) | $\{1,2\}$ |
| $\{1,2,3\}$ | (.11,.15,.19,.22,.25) | (91%,93%,93%,92%,92%) | $\{1,2,3\}$ |
| $\{1,2,3,4\}$ | (.11,.15,.19,.22,.25) | (82%,84%,85%,86%,84%) | $\{1,2,3,4\}$ |
| $\{1,2,3,4,5\}$ | (.11,.15,.19,.22,.25) | (39%,44%,47%,50%,53%) | $\{1,2,3,4,5\}$ |

**Table 3** Effects on Payoffs if Irrational Firms Use Market Clearing Prices

| Irrational Set | Distributions of Price | $E(J^d/\Lambda(T))$ |
|---|---|---|
| $\varnothing$ | N/A | (.2259,.2244,.2194,.2162,.2137) |
| $\{4\}$ | $p_4 \sim Unif(.8,1.5)$ | (120%,109%,96%,88%,96%) |
| $\{5\}$ | $p_5 \sim Unif(.8,1.5)$ | (121%,113%,103%,102%,93%) |

**Table 4** Effects on Payoffs if Irrational Firms Use Randomly Selected Prices

Example 5 (Performance of Heuristics in Stochastic Game). We use the same MNL parameters as in Example 3. In Table 5, we numerically verify the asymptotic optimality of the three heuristics discussed in Section 4.2.

In light of Theorem 3, given any initial condition $(T, \vec{c})$, the relative difference in performance between the fixed-pricing heuristic and the equilibrium for any firm is no more than $1/(2\sqrt{\max\{c_i, \Lambda(T) d_i^*(\vec{p}_{-i}^*)\}} - 1)$, which is very small for reasonable size of sales horizon and initial capacities. When $T = 100$, $\vec{n} > 30\vec{e}$, $\lambda(t) = 1$, the relative difference in performance of the fixed-pricing heuristic, based on a simulation with 100,000 sample paths, is within 1% of the deterministic upper bound, and thus within 1% of the stochastic Nash equilibrium. We notice that less price-sensitive firm becomes nearly optimal for smaller initial capacity; when $\vec{c} > 17\vec{e}$, the relative difference in performance of the fixed-pricing heuristic for firm 1 is within 1% of its value at the stochastic Nash equilibrium. These results suggest that even for problems of moderate size, the fixed-pricing heuristic performs quite well. They also suggest that if demand functions are public information and prices can be set freely, then one should not see great benefits from the highly dynamic pricing practices for moderate to large sized problems.

To test the performance of feedback-type heuristics $\vec{p}^c$ under the strong information structure and $\vec{p}^h$ under the weak information structure with observable price history, we solve for a discrete-time version of the stochastic game by backward induction (Lin and Sibdari 2009) as a benchmark. We observe that when $T = 100$, $\vec{n} > 40\vec{e}$, $\lambda(t) = 1$, the relative difference in performance of any of the two heuristics to the equilibrium, based on a simulation with 100,000 sample paths, is within 9% of each firm's equilibrium payoff. We further notice that neither one of the two heuristics outperforms the other, which indicates that both of the following two aspects on information under competition play a role: 1) obscure signaling of price information about inventories may soften direct competition and 2) accurate information about competitors' inventory levels can help firms sharpen their best-responses. $\square$

## 6. Conclusions

We believe that appropriately applied dynamic pricing can "ration" capacity more profitably than by limiting supply in a fare-menu-based RM system. This is due to the following two facts. First, dynamic pricing takes into account competitors' prices in real time, resulting in more accurate

| $\vec{c}$ | $\vec{p}^*(T,\vec{c})$ | $J^f(T,\vec{c})/J^d(T,\vec{c})$ |
|---|---|---|
| $1\vec{e}$ | (10.49,7.54,5.94,4.93,4.23) | (63.2%,63.4%,63.2%,63.3%,63.2%) |
| $5\vec{e}$ | (6.80,5.08,4.09,3.45,3.00) | (82.3%,82.6%,82.6%,82.4%,82.4%) |
| $10\vec{e}$ | (4.61,3.61,3.00,2.58,2.27) | (87.5%,87.5%,87.5%,87.4%,87.6%) |
| $15\vec{e}$ | (2.41,2.15,1.90,1.70,1.54) | (89.7%,89.9%,89.8%,89.7%,89.8%) |
| $20\vec{e}$ | (2.23,1.57,1.23,1.11,1.05) | (99.8%,98.7%,93.7%,91.2%,91.1%) |
| $25\vec{e}$ | (2.21,1.55,1.21,1.01,0.87) | (100%,99.9%,99.5%,97.8%,94.2%) |
| $30\vec{e}$ | (2.21,1.55,1.21,1.01,0.87) | (100%,100%,100%,99.8%,99.0%) |
| $37\vec{e}$ | (2.21,1.55,1.21,1.01,0.87) | (100%, 100%, 100%, 100%, 100%) |

**Table 5** Asymptotic Optimality of the Fixed-Pricing Heuristic, where $T = 100$, $\vec{e} = (1,1,1,1,1)$

sales forecasts. Second, dynamic pricing is not limited to a pre-specified menu of fares. By basing prices on current competitors' prices and by having more liberty to price a product, dynamic pricing has the potential to be more effective than capacity allocation with a given fare structure. We have shown how a range of capacity pricing problems under competition can be formulated as intensity control games and analyzed by considering the corresponding differential game. It is encouraging that the existence of the OLNE in feedback form can be established for most of the commonly-used demand intensity functions under natural assumptions on the parameters. Not only is the equilibrium easy to compute by adaptive learning algorithms, but the use of heuristics suggested by the feedback-type solutions for the stochastic game is provably asymptotically optimal and appears to work well in our numerical experiments. We strongly believe that this class of competitive capacity pricing models should be a topic of intense interest to managers in a wide range of industries.

On the methodological level, this paper relates the stochastic game to its deterministic counterpart. First, we show that a quasi-static affine approximation approach to the stochastic game results in the deterministic differential game. Second, we identify OLNE and CLNE for the differential game, which are asymptotic equilibria in the stochastic game. No doubt other variants of the competitive RM problem can be attacked using the same approach.

Many authors prefer a discrete-time formulation because it keeps the problem in the realm of stochastic dynamic programming rather than stochastic optimal control; the techniques of the former are better known than those of the latter. Researchers may also argue that for computational purposes, one needs to discretize time and therefore it is better to do it before formulating the problem rather than after. While some of these concerns are valid, we have strong arguments to support the continuous-time formulation. First, the formulation is more natural since the customer arrival process occurs in continuous time. Second, some results and insights become clear in the continuous-time formulation and are hard to see in the discrete-time formulation. For example, the fixed-pricing heuristic derives naturally from the continuous-time formulation but it is not at all obvious from the discrete-time formulation. Third, the solution to the continuous-time formulation consists of the smooth pasting of solutions to differential equations whereas the solution to the discrete-time problem requires an optimization step at each time period. Because of this, the discrete-time formulation requires a smaller mesh than the continuous-time formulation to achieve the same level of accuracy (Feng and Gallego 2000). Moreover, the mesh of the discrete-time formulation also requires that the probability of more than one event happening is negligible.

We show that under the assumption that all firms agree on the same time-homogeneous choice probability function, the specified CLNE has a fixed pricing trajectory for each firm in the differential game. This result contradicts neither the basic concept of dynamic pricing nor the common perception of changing prices in the market for the following reasons. The solution to the stochastic game does not generate a fixed-pricing policy. The closed-loop heuristic suggested by the differential game is an asymptotic equilibrium for the stochastic game. This policy is sensitive to the demand model, to the inventory information and to the rationality of competitors. In practice our model suggests firms should price according to their best-response functions that are calibrated to their best knowledge and information in a rolling horizon fashion. The firms are supposed to constantly re-solve their best-response problems after updating their information about joint inventory level, market prices and the demand function. The resulting pricing strategy is most likely to be time-varying as 1) market joint capacities might not decrease linearly due to stochasticity neglected in the deterministic problem; 2) competitors change their prices; and 3) the demand function is updated over time. Moreover, we emphasize that the closed-loop heuristics that come out of the paper is the result of an equilibrium analysis of best-response correspondences. In practice, airline companies rely too much on price matching policies which can result in very poor performance compared to best responses or even heuristics that approximate best responses to price changes.

## Acknowledgments

The authors thank Professor Awi Federgruen, Garud Iyengar, Costis Maglaras, Steven Kou, Benny Mantin and seminar participants at the McCombs School of Business, the Fuqua School of Business and the Columbia Business School for their helpful comments.

### Appendix. Proofs.

*Proof of Lemma 2.* Suppose $x_i(t) \geqslant 0,\ 0 \leqslant t \leqslant T$, then in particular at the end of the sales horizon, we should be left with nonnegative inventory, i.e., $x_i(T) \geqslant 0$. On the contrary, suppose $x_i(T) \geqslant 0$. By the demand nonnegativity, it must be the case that for all $0 \leqslant t \leqslant T$, $x_i(t) = c_i - \int_0^t \lambda(v) d_i(\vec{p}(v))\, dv \geqslant c_i - \int_0^T \lambda(v) d_i(\vec{p}(v))\, dv = x_i(T) \geqslant 0$. □

*Proof of Lemma 5.* Suppose $\vec{p}^*$ is a Nash equilibrium of game (P1), namely, given $\vec{p}^*_{-i}$ for all $i$, $p^*_i$ maximizes $r_i(p_i, \vec{p}^*_{-i})$ subject to $0 \leqslant d_i(p_i, \vec{p}^*_{-i}) \leqslant c_i/\Lambda(T)$. Thus we must have $0 \leqslant d_j(\vec{p}^*) \leqslant c_j/\Lambda(T)$ for all $j$. Therefore given $\vec{p}^*_{-i}$ for all $i$, $p^*_i$ also maximizes $r_i(p_i, \vec{p}^*_{-i})$ subject to $0 \leqslant d_j(p_i, \vec{p}^*_{-i}) \leqslant c_j/\Lambda(T)$ for all $j$, namely, $\vec{p}^*$ is a generalized Nash equilibrium of game (P0). □

*Proof of Lemma 6.* Suppose for some firm $i$, $p^0_i(s, x_i, \vec{p}^1_{-i}) < p^0_i(s, x_i, \vec{p}^2_{-i})$ for some $\vec{p}^1_{-i} \geqslant \vec{p}^2_{-i}$. This leads to $x_i/\Lambda(s) = d_i(p^0_i(\vec{p}^1_{-i}), \vec{p}^1_{-i}) \geqslant d_i(p^0_i(\vec{p}^1_{-i}), \vec{p}^2_{-i}) > d_i(p^0_i(\vec{p}^2_{-i}), \vec{p}^2_{-i}) = x_i/\Lambda(s)$, which is a contradiction. The first inequality is due to Assumption 3 and the second inequality is due to Assumption 2. □

*Proof of Lemma 7.* For any given $\vec{p}_{-i} \in \mathcal{P}_{-i}$, let us consider firm $i$'s problem (P2$_i$). By Assumption 8, $p^0_i(T, c_i, \vec{p}_{-i})$ exists and $0 \leqslant p^0_i(T, c_i, \vec{p}_{-i}) \leqslant p^{\max}_i$. If $p^0_i(T, c_i, \vec{p}_{-i}) > 0$, by Assumption 2, for any $0 \leqslant p_i \leqslant p^0_i(T, c_i, \vec{p}_{-i})$, we have $d_i(\vec{p}) \geqslant c_i/\Lambda(T)$ so the objective function of (P2$_i$) is $c_i p_i$ which is increasing in $p_i$; otherwise $p^0_i(T, c_i, \vec{p}_{-i}) = 0$, the objective function of (P2$_i$) is zero for for any $0 \leqslant p_i \leqslant p^0_i(T, c_i, \vec{p}_{-i})$. In either case, it is never optimal for firm $i$ to choose any price below $p^0_i(T, c_i, \vec{p}_{-i})$: due to the continuous strategy set of each firm and continuity of the demand function, it is at least better to choose the price at $p_i^0(T, c_i, \vec{p}_{-i})$ than a price below. Thus we can restrict the strategy set of firm $i$ to an interval $[p_i^0(T, c_i, \vec{p}_{-i}), p_i^{\max}]$, where capacity constraint $\Lambda(T)d_i(\vec{p}) \leqslant c_i$ is satisfied by Assumptions 2 and 8. For any $p_i \geqslant p_i^0(T, c_i, \vec{p}_{-i})$, the objective function of (P2$_i$) is $r_i(\vec{p})T$. Hence, for any given $\vec{p}_{-i} \in \mathcal{P}_{-i}$, firm $i$'s problem (P2$_i$) is equivalent to (P1$_i$). $\square$

*Proof of Lemma 8.* Without loss of generality, we can assume $\Lambda(T) = 1$. The $\Leftarrow$ direction for both parts is trivial: we can select $\bar{c}_i$ such that $d_i(\vec{p}) \leqslant \bar{c}_i$ for all $\vec{p} \in \mathcal{P}$, then the results immediately follow by realizing $\pi_i(\vec{p}) = \min\{r_i(\vec{p}), \bar{c}_i p_i\} = r_i(\vec{p})$.

Let us consider the $\Rightarrow$ direction. (i) The result follows immediately by realizing that quasi-concavity is preserved under minimization. (ii) Suppose $r_i(\vec{p})$ is supermodular or log-supermodular in $\vec{p}$. We want to show that $\pi_i(\vec{p})$ is supermodular or log-supermodular in $\vec{p}$, respectively. By the definition of supermodularity, we need to show that $\pi_i(p_i', \vec{p}_{-i}) - \pi_i(p_i, \vec{p}_{-i}) \leqslant \pi_i(p_i', \vec{p'}_{-i}) - \pi_i(p_i, \vec{p'}_{-i})$ or $\log \pi_i(p_i', \vec{p}_{-i}) - \log \pi_i(p_i, \vec{p}_{-i}) \leqslant \log \pi_i(p_i', \vec{p'}_{-i}) - \log \pi_i(p_i, \vec{p'}_{-i})$ respectively for any fixed $p_i \leqslant p_i'$ and $\vec{p}_{-i} \leqslant \vec{p'}_{-i}$. For notational ease, we drop subindices and denote $p_i$, $p_i'$, $\vec{p}_{-i}$ and $\vec{p'}_{-i}$ by $p$, $p'$, $\vec{q}$ and $\vec{q'}$ respectively. Let us define $D_i(\vec{p}) = \log(\min\{d_i(\vec{p}), c_i\})$. Realizing $\log p_i$ and $\log p_i'$ are modular terms, we show case by case $\pi_i(p', \vec{q}) - \pi_i(p, \vec{q}) \leqslant \pi_i(p', \vec{q'}) - \pi_i(p, \vec{q'})$ or $D_i(p', \vec{q}) - D_i(p, \vec{q}) \leqslant D_i(p', \vec{q'}) - D_i(p, \vec{q'})$ respectively conditional on whether demands are capacitated. The monotonicity of the demand function $d_i(\cdot, \cdot)$ in the joint strategy space can be illustrated in Figure 2: by Assumption 2, $d_i(p, \vec{q}) \geqslant d_i(p', \vec{q})$ and $d_i(p, \vec{q'}) \geqslant d_i(p', \vec{q'})$; by Assumption 3, $d_i(p, \vec{q}) \leqslant d_i(p, \vec{q'})$ and $d_i(p', \vec{q}) \leqslant d_i(p', \vec{q'})$. Hence, the relationship between demands and the capacity $c_i$ reduces to the following 6 cases.

Case 1. $d_i(p', \vec{q}) \geqslant c_i$. We have $d_i(p, \vec{q'}) \geqslant d_i(p, \vec{q}) \geqslant d_i(p', \vec{q}) \geqslant c_i$ and $d_i(p', \vec{q'}) \geqslant d_i(p', \vec{q}) \geqslant c_i$. Then $\pi_i(p', \vec{q}) - \pi_i(p, \vec{q}) = (p' - p) \cdot c_i = \pi_i(p', \vec{q'}) - \pi_i(p, \vec{q'})$, or $D_i(p', \vec{q}) - D_i(p, \vec{q}) = \log(c_i) - \log(c_i) = D_i(p', \vec{q'}) - D_i(p, \vec{q'})$, respectively.

Case 2. $d_i(p, \vec{q'}) \leqslant c_i$. We have $d_i(p', \vec{q}) \leqslant d_i(p, \vec{q}) \leqslant d_i(p, \vec{q'}) \leqslant c_i$ and $d_i(p', \vec{q'}) \leqslant d_i(p, \vec{q'}) \leqslant c_i$. Then by the (log-)supermodularity of $r_i(\cdot, \cdot)$, we have $\pi_i(p', \vec{q}) - \pi_i(p, \vec{q}) = r_i(p', \vec{q}) - r_i(p, \vec{q}) \leqslant r_i(p', \vec{q'}) - r_i(p, \vec{q'}) = \pi_i(p', \vec{q'}) - \pi_i(p, \vec{q'})$, or $D_i(p', \vec{q}) - D_i(p, \vec{q}) = \log d_i(p', \vec{q}) - \log d_i(p, \vec{q}) \leqslant \log d_i(p', \vec{q'}) - \log d_i(p, \vec{q'}) = D_i(p', \vec{q'}) - D_i(p, \vec{q'})$, respectively.

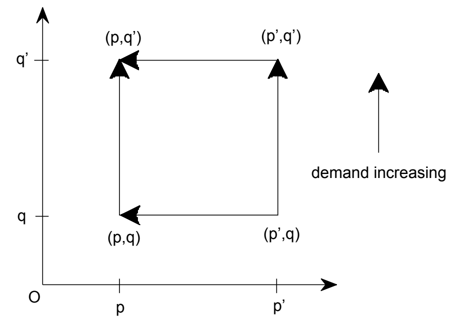


**Figure 2** Monotonicity of the demand function on the lattice

Case 3. $d_i(p', \vec{q}) < c_i$, $d_i(p, \vec{q'}) > c_i$.

Case 3.1. $d_i(p, \vec{q}) \leqslant c_i$, $d_i(p', \vec{q'}) \leqslant c_i$. Then $\pi_i(p', \vec{q}) - \pi_i(p, \vec{q}) = r_i(p', \vec{q}) - r_i(p, \vec{q}) \leqslant r_i(p', \vec{q'}) - r_i(p, \vec{q'}) = r_i(p', \vec{q'}) - p \cdot d_i(p, \vec{q'}) \leqslant r_i(p', \vec{q'}) - p \cdot c_i = \pi_i(p', \vec{q'}) - \pi_i(p, \vec{q'})$, or $D_i(p', \vec{q}) - D_i(p, \vec{q}) = \log d_i(p', \vec{q}) - \log d_i(p, \vec{q}) \leqslant \log d_i(p', \vec{q'}) - \log d_i(p, \vec{q'}) \leqslant \log d_i(p', \vec{q'}) - \log c_i = D_i(p', \vec{q'}) - D_i(p, \vec{q'})$, respectively, where the first inequality in each case is due to the (log-)supermodularity of $r_i(\cdot, \cdot)$.

Case 3.2. $d_i(p, \vec{q}) \leqslant c_i$, $d_i(p', \vec{q'}) > c_i$. Then $d_i(p', \vec{q}) - c_i \leqslant d_i(p, \vec{q}) - c_i < 0$. Hence, $p' \cdot [d_i(p', \vec{q}) - c_i] \leqslant p \cdot [d_i(p, \vec{q}) - c_i]$, which justifies the following inequality $\pi_i(p', \vec{q}) - \pi_i(p, \vec{q}) = p' \cdot d_i(p', \vec{q}) - p \cdot d_i(p, \vec{q}) \leqslant p' \cdot c_i - p \cdot c_i = \pi_i(p', \vec{q'}) - \pi_i(p, \vec{q'})$, or $D_i(p', \vec{q}) - D_i(p, \vec{q}) = \log d_i(p', \vec{q}) - \log d_i(p, \vec{q}) \leqslant 0 = \log(c_i) - \log(c_i) = D_i(p', \vec{q'}) - D_i(p, \vec{q'})$, respectively.

Case 3.3. $d_i(p, \vec{q}) > c_i$, $d_i(p', \vec{q'}) \leqslant c_i$. Then $\pi_i(p', \vec{q}) - \pi_i(p, \vec{q}) = p' \cdot d_i(p', \vec{q}) - p \cdot c_i \leqslant p' \cdot d_i(p', \vec{q'}) - p \cdot c_i = \pi_i(p', \vec{q'}) - \pi_i(p, \vec{q'})$, or $D_i(p', \vec{q}) - D_i(p, \vec{q}) = \log d_i(p', \vec{q}) - \log(c_i) \leqslant \log d_i(p', \vec{q'}) - \log(c_i) = D_i(p', \vec{q'}) - D_i(p, \vec{q'})$, respectively.

Case 3.4. $d_i(p, \vec{q}) > c_i$, $d_i(p', \vec{q'}) > c_i$. Then $\pi_i(p', \vec{q}) - \pi_i(p, \vec{q}) = p' \cdot d_i(p', \vec{q}) - p \cdot c_i \leqslant p' \cdot c_i - p \cdot c_i = \pi_i(p', \vec{q'}) - \pi_i(p, \vec{q'})$, or $D_i(p', \vec{q}) - D_i(p, \vec{q}) = \log d_i(p', \vec{q}) - \log(c_i) \leqslant \log(c_i) - \log(c_i) = D_i(p', \vec{q'}) - D_i(p, \vec{q'})$, respectively. □

*Proof of Proposition 3.* (i) From the objective functions $\pi_i(\vec{p})$ for all $i$, it is easy to see only $c_i/\Lambda(T)$ for all $i$ matters for the equilibrium.

(ii) Since the demand function $d_i(\vec{p})$ (hence $r_i(\vec{p})$) for all $i$ is twice continuously differentiable, $p_i^*(\vec{p}_{-i})$ that maximizes revenue rate $r_i(\vec{p})$ for all $i$ is differentiable. Thus $\min\{p_i^*(\vec{p}_{-i}), p_i^{\max}\}$ for all $i$

is differentiable almost everywhere. Under the stipulations, there exists a unique Nash equilibrium for game (P2) that can be characterized by the following set of equations: for some set $S \subseteq \mathcal{I}$,

$$\begin{cases} d_i(\vec{p}) = x_i/\Lambda(s), & i \in S, \\ p_i = \min\{p_i^*(\vec{p}_{-i}), p_i^{\max}\}, & i \in \mathcal{I} \backslash S. \end{cases} \tag{19}$$

According to elements in set $S$, the initial condition space $(0, T] \times \bigtimes_i [\epsilon, c_i]$ of $(s, \vec{x})$ can be at most divided into $2^m$ subspaces. By the Inverse Function Theorem, the solution $\vec{p}^o(s, \vec{x})$ to the system of equations (19) is differentiable almost everywhere within the interior of each subspace, and is continuous almost everywhere on the boundary of any two subspaces.

(iii) The convergence of the tatônnement scheme guaranteed by the stipulations under the framework of supermodular game provides an intuitive way of proving the monotone property of the OLNE in its initial condition. Due to the uniqueness of a Nash equilibrium given its initial condition, the tatônnement scheme is globally stable by Vives (1999, Corollary to Theorem 2.10): from *any* starting point the best-response tatônnement scheme converges. Without loss of generality, we consider two initial conditions $(s, \vec{x})$ and $(s', \vec{x}')$ such that $x_i/\Lambda(s) < x_i'/\Lambda(s')$ for some $i$ and $x_j/\Lambda(s) = x_j'/\Lambda(s')$ for all $j \neq i$. Starting from the equilibrium price vector $\vec{p}^o(s, \vec{x})$ under the initial condition $(s, \vec{x})$, we can use the tatônnement scheme to compute the equilibrium price vector $\vec{p}^o(s', \vec{x}')$ under the initial condition $(s', \vec{x}')$. By the definition of equilibrium, we have $p_i^o(s, \vec{x}) = \max\{p_i^0(s, x_i, \vec{p}_{-i}^o(s, \vec{x})), \min\{p_i^*(\vec{p}_{-i}^o(s, \vec{x})), p_i^{\max}\}\}$ for all $i$. Since $p_i^0(s, x_i, \vec{p}_{-i})$ is decreasing in $x_i/\Lambda(s)$, $p_i^o(s, \vec{x}) \geqslant \max\{p_i^0(s', x_i', \vec{p}_{-i}^o(s, \vec{x})), \min\{p_i^*(\vec{p}_{-i}^o(s, \vec{x})), p_i^{\max})\}\}$ for all $i$. Thus the tatônnement scheme starting at $\vec{p}^o(s, \vec{x})$ for the game with initial condition $(s', \vec{x}')$ would converge monotonically downwards to $\vec{p}^o(s', \vec{x}')$. $\square$

*Proof of Theorem 3.* With competitors using fixed-pricing policy in the stochastic game, any firm just faces the monopolist stochastic dynamic pricing problem discussed in Gallego and van Ryzin (1994). Since $\vec{p}^*$ is the OLNE of the differential game, for firm $i$, $p_i^*$ is the fixed-pricing heuristic suggested by the monopolist differential pricing problem given competitors' strategies as $\vec{p}_{-i}^*$. By Lemma 9, we have for an initial condition $(s, \vec{n})$,

$$J_i^d(\vec{p}^*)\left(1 - \frac{1}{2\sqrt{\min\{c_i, \Lambda(s) d_i^*(\vec{p}_{-i}^*)\}}}\right) \leqslant J_i(\vec{p}^*) \leqslant J_i^*(\vec{p}_{-i}^*) \leqslant J_i^d(\vec{p}^*), \tag{20}$$

where $J_i^*(\vec{p}_{-i}^*) = \sup_{(p_i, \vec{p}_{-i}^*) \in \mathcal{U}} J_i(p_i, \vec{p}_{-i}^*)$ is the maximal revenue for firm $i$ in the stochastic game given competitors' strategy $\vec{p}_{-i}^*$. For any $\epsilon_1 = \epsilon/(1+\epsilon)$, if the initial condition satisfies $\max\{c_i, \Lambda(s) d_i^*(\vec{p}_{-i}^*)\} \geqslant 1/(4{\epsilon_1}^2)$, $\forall\, i$, by inequality (20), $J_i(\vec{p}^*) \geqslant J_i^d(\vec{p}^*)\,(1-\epsilon_1) \geqslant J_i^*(\vec{p}_{-i}^*)(1-\epsilon_1)$. Thus the result follows. □

*Proof of Lemma 10.* For each $x_0$ with $f'(x_0) = 0$, we have $f''(x_0) > 0$. This means that whenever the function $f'$ reaches the value 0, it is strictly increasing. Therefore it can reach the value 0 at most once. If $f'$ does not reach the value 0 at all, then $f$ is either strictly decreasing or strictly increasing, and therefore pseudo-convex: if $f$ is strictly decreasing, then $(x_1 - x_2) f'(x_2) \geqslant 0 \Rightarrow x_1 \leqslant x_2 \Rightarrow f(x_1) \geqslant f(x_2)$; if $f$ is strictly increasing, then $(x_1 - x_2) f'(x_2) \geqslant 0 \Rightarrow x_1 \geqslant x_2 \Rightarrow f(x_1) \geqslant f(x_2)$. Otherwise $f'$ must reach the value 0 exactly once, say at $x_0$. Since $f''(x_0) > 0$, it follows that $f'(x) < 0$ for $x < x_0$, and $f'(x) > 0$ for $x > x_0$. Therefore $f$ is pseudo-convex: if $x_2 = x_0$, we always have $f(x_1) \geqslant f(x_2) = f(x_0)$ for any $x_1$; if $x_2 < x_0$, then $(x_1 - x_2) f'(x_2) \geqslant 0 \Rightarrow x_1 \leqslant x_2 \Rightarrow f(x_1) \geqslant f(x_2)$; and if $x_2 > x_0$, then $(x_1 - x_2) f'(x_2) \geqslant 0 \Rightarrow x_1 \geqslant x_2 \Rightarrow f(x_1) \geqslant f(x_2)$. □

*Proof of Proposition 9.* (i) Taking the first order derivative of $d_i(\vec{p})$ with respect to $p_i$,

$$\frac{\partial d_i}{\partial p_i} = \frac{a_i' \sum_{j \neq i} a_j}{(\sum_j a_j)^2} \leqslant 0.$$

Taking the first order derivative of $d_i(\vec{p})$ with respect to $p_j$,

$$\frac{\partial d_i}{\partial p_j} = -\frac{a_i a_j'}{(\sum_j a_j)^2} \geqslant 0.$$

The result follows by Lemma 1.

(ii) Taking the first order derivative of $r_i(\vec{p})$ with respect to $p_i$,

$$\frac{\partial r_i}{\partial p_i} = d_i + p_i \frac{\partial d_i}{\partial p_i} = \frac{a_i}{\sum_j a_j} + p_i \frac{a_i' \sum_{j \neq i} a_j}{(\sum_j a_j)^2}.$$

Taking the second order derivative of $r_i(\vec{p})$ with respect to $p_i$,

$$\frac{\partial^2 r_i}{\partial {p_i}^2} = 2\frac{\partial d_i}{\partial p_i} + p_i \frac{\partial^2 d_i}{\partial {p_i}^2} = 2\frac{a_i' \sum_{j \neq i} a_j}{(\sum_j a_j)^2} + p_i \frac{a_i' \sum_{j \neq i} a_j}{(\sum_j a_j)^2} \left( \frac{a_i''}{a_i'} - \frac{2a_i'}{\sum_j a_j} \right).$$

Whenever $\partial r_i/\partial p_i = 0$, $p_i a_i' \sum_{j\neq i} a_j/(\sum_j a_j)^2 = -a_i/\sum_j a_j$, thus

$$\frac{\partial^2 r_i}{\partial {p_i}^2} = 2\frac{a_i' \sum_{j\neq i} a_j}{(\sum_j a_j)^2} - \frac{a_i}{\sum_j a_j}\left(\frac{a_i''}{a_i'} - \frac{2a_i'}{\sum_j a_j}\right) = \frac{2a_i' - a_i a_i''/a_i'}{\sum_j a_j} > (<) 0,$$

since $2a_i' - a_i a_i''/a_i' >$ (resp. $<$)0. By Lemma 10, $r_i(\vec{p})$ is pseudo-convex (resp. pseudo-concave) in $p_i$. ▫

*Proof of Proposition 10.* (a) We want to show that the revenue rate functions $r_i(\vec{p})$, for all $i$ are log-supermodular in $\vec{p}$. Since $\tilde{r}_i(\vec{p}) := \log r_i(\vec{p}) = \log p_i + \log d_i(\vec{p}) = \log p_i + \tilde{d}_i(\vec{p})$ and $r_i(\vec{p})$ is twice continuous differentiable, we only need to check $\partial^2 \tilde{r}_i(\vec{p})/\partial p_i \partial p_j = \partial^2 \tilde{d}_i(\vec{p})/\partial p_i \partial p_j \geqslant 0$. By Bernstein and Federgruen (2004, Lemma 2), we have $\partial \tilde{d}_i/\partial p_i = (1/d_i)(\partial d_i/\partial p_i) = (\partial \tilde{a}_i/\partial p_i)(1 - d_i)$, thus by $\partial a_i/\partial p_i \leqslant 0$, $\partial^2 \tilde{d}_i/\partial p_i \partial p_j = (\partial \tilde{a}_i/\partial p_i)(\partial \tilde{a}_j/\partial p_j) d_i d_j \geqslant 0$. Therefore game (P2) is log-supermodular, hence there exists a Nash equilibrium. (b) We want to verify the diagonal dominance condition. Since $\partial \tilde{r}_i/\partial p_i = 1/p_i + \partial \tilde{d}_i/\partial p_i$,

$$\frac{\partial^2 \tilde{r}_i}{\partial {p_i}^2} = -\frac{1}{{p_i}^2} + \frac{\partial^2 \tilde{a}_i}{\partial {p_i}^2}(1-d_i) - (\frac{\partial \tilde{a}_i}{\partial p_i})^2 d_i(1-d_i) < -(\frac{\partial \tilde{a}_i}{\partial p_i})^2 d_i(1-d_i)\,, \tag{21}$$

since $\tilde{a}_i$ is concave in $p_i$. By (18), we have

$$(1-d_i)\frac{\partial \tilde{a}_i}{\partial p_i} = (1-d_i)\frac{a_i'}{a_i} < (1-d_i)\frac{\sum_{j\neq i} a_j'}{\sum_{j\neq i} a_j} = \frac{\sum_{j\neq i} a_j}{\sum_j a_j}\frac{\sum_{j\neq i} a_j'}{\sum_{j\neq i} a_j} = \sum_{j\neq i}\frac{\partial \tilde{a}_j}{\partial p_j} d_j\,. \tag{22}$$

Hence, combining inequalities (21) and (22), we get

$$-\frac{\partial^2 \tilde{r}_i}{\partial {p_i}^2} > (\frac{\partial \tilde{a}_i}{\partial p_i})^2 d_i(1-d_i) > \frac{\partial \tilde{a}_i}{\partial p_i} d_i \sum_{j\neq i}\frac{\partial \tilde{a}_j}{\partial p_j} d_j = \sum_{j\neq i}\frac{\partial^2 \tilde{d}_i}{\partial p_i \partial p_j} = \sum_{j\neq i}\frac{\partial^2 \tilde{r}_i}{\partial p_i \partial p_j}\,, \quad \forall\, i\,.$$

The result follows. By taking a close look at inequalities (21) and (22), we can relax "$<$" in assumption (18) to "$\leqslant$" and the result still holds if inequality (21) holds strictly. ▫

## References

Adelman, D. 2007. Dynamic bid prices in revenue management. *Oper. Res.* **55**(4) 647–661.

Bernstein, F., A. Federgruen. 2003. Pricing and replenishment strategies in a distribution system with competing retailers. *Oper. Res.* **51**(3) 409–426.

Bernstein, F., A. Federgruen. 2004. A general equilibrium model for industries with price and service competition. *Oper. Res.* **52**(6) 868–886.

Bernstein, F., A. Federgruen. 2005. Decentralized supply chains with competing retailers under demand uncertainty. *Management Sci.* **51**(1) 18–29.

Boyd, S., L. Vandenberghe. 2004. *Convex Optimization*. Cambridge University Press.

Brémaud, P. 1980. *Point Processes and Queues, Martingale Dynamics*. Springer-Verlag, New York.

Cooper, W., T. Homem de Mello, A. J. Kleywegt. 2006. Models of the spiral-down effect in revenue management. *Oper. Res.* **54**(5) 968–987.

Craven, B. D. 1998. *Control and optimization*. 1st ed. Chapman & Hall/CRC.

Dai, J. G., R. J. Williams. 1995. Existence and uniqueness of semimartingale reflecting brownian motions in convex. *Theory Probab. Appl.* **40**(1) 1–40.

Debreu, G. 1952. A social equilibrium existence theorem. *Proceedings of the National Academy of Sciences* **38** 886–893.

Dockner, E., S. Jorgensen, N. Van Long, G. Sorger. 2000. *Differential Games in Economics and Management Science*. Cambridge University Press, New York.

Eliashberg, J., A. P. Jeuland. 1986. The impact of competitive entry in a developing market upon dynamic pricing strategies. *Marketing Sci.* **5**(1) 20–36.

Federgruen, A., A. Heching. 1999. Combined pricing and inventory control under uncertainty. *Oper. Res.* **47**(3) 454–475.

Feng, Y., G. Gallego. 2000. Perishable asset revenue management with markovian time dependent demand intensities. *Management Sci.* **46**(7) 941–956.

Fershtman, C. 1987. Identification of classes of differential games for which the open loop is a degenerate feedback nash equilibrium. *J. Optim. Theory Appl.* **55**(2) 217–231.

Gaimon, C. 1989. Dynamic game results of the acquisition of new technology. *Oper. Res.* **37**(3) 410–425.

Gallego, G., W. T. Huh, W. Kang, R. Phillips. 2006. Price competition with the attraction demand model: Existence of unique equilibrium and its stability. *Manufacturing Service Oper. Management* **8**(4) 359–375.

Gallego, G., G. Iyengar, R. Phillips, A. Dubey. 2004. Managing flexible products on a network. Tech. Rep. CORC TR-2004-01, Columbia University.

Gallego, G., G. van Ryzin. 1994. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Sci.* **40**(8) 999–1020.

Granot, D., F. Granot, B. Mantin. 2007. A dynamic pricing model under duopoly competition. Working paper, Sauder School of Business, UBC, Canada.

Ichiishi, T. 1983. *Game Theory for Economic Analysis*. Academic Press, New York.

Levin, Y., J. McGill, M. Nediak. 2009. Dynamic pricing in the presence of strategic consumers and oligopolistic competition. *Management Sci.* **55**(1) 32–46.

Lin, K. Y., S. Sibdari. 2009. Dynamic price competition with discrete customer choices. *Eur. J. Oper. Res.* **197**(3) 969–980.

Maglaras, C., J. Meissner. 2006. Dynamic pricing strategies for multiproduct revenue management problems. *Manufacturing Service Oper. Management* **8**(2) 136–148.

Mangasarian, O. L. 1965. Pseudo-convex functions. *J. Soc. Indus. and Appl. Math. Ser. A* **3**(2) 281–290.

Mangasarian, O. L. 1987. *Nonlinear Programming*. Society for Industrial Mathematics.

Milgrom, P., J. Roberts. 1990. Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica* **58**(6) 1255–1277.

Mookherjee, R., T. L. Friesz. 2008. Pricing, allocation and overbooking in dynamic service network competition when demand is uncertain. *Production and Oper. Management* **17**(4) 1–20.

Mukhopadhyay, S. K., P. Kouvelis. 1997. A differential game theoretic model for duopolistic competition on design quality. *Oper. Res.* **45**(6) 886–893.

Netessine, S., R. A. Shumsky. 2005. Revenue management games: Horizontal and vertical competition. *Management Sci.* **51**(5) 813–831.

Nguyen, T., G. Perakis. 2005. Robust competitive multi-period pricing for multiple products with fixed capacity. Working paper, MIT, Cambridge, MA.

Perakis, G., A. Sood. 2006. Competitive multi-period pricing for perishable products: A robust optimization approach. *Math. Programming* **107**(1-2) 295–335.

Rosen, J. B. 1965. Existence and uniqueness of equilibrium points for concave $n$-person games. *Econometrica* **33**(3) 520–534.

Royden, H. L. 1988. *Real Analysis*. 3rd ed. Prentice Hall.

Sethi, S. P., G. L. Thompson. 2005. *Optimal Control Theory: Applications to Management Science and Economics*. 2nd ed. Springer.

Talluri, K. 2003. On equilibria in duopolies with finite strategy spaces. Tech. Report WP-701, Universitat Pompeu Fabra.

Talluri, K., G. van Ryzin. 2004. Revenue management unde a discrete choice model of consumer behavior. *Management Sci.* **50**(1) 15–33.

Topkis, D. M. 1979. Equilibrium points in nonzero-sum $n$-person submodular games. *SIAM J. Control Optim.* **17**(6) 773–787.

van Ryzin, G., Q. Liu. 2008. On the choice-based linear programming model for network revenue management. *Manufacturing Service Oper. Management* **10**(2) 288–310.

Vieille, N. 2000. Two-player stochastic games i: A reduction; two-player stochastic games ii: The case of recursive games; small perturbations and stochastic games. *Israel J. of Math.* **119**(1) 55–91;92–126;127–142.

Vives, X. 1999. *Oligopoly Pricing: Old Ideas and New Tools*. MIT Press, Cambridge, MA.

Xu, X., W. J. Hopp. 2006. A monopolistic and oligopolistic stochastic flow revenue management model. *Oper. Res.* **54**(6) 1098–1109.

Zalmai, G. J. 1985. Sufficient optimality conditions in continuous-time nonlinear programming. *J. Math. Anal. Applic.* **111**(1) 130–147.