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A dynamic quality ladder model with entry and exit: Exploring the equilibrium correspondence using the homotopy method

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Abstract This paper explores the equilibrium correspondence of a dynamic quality ladder model with entry and exit using the homotopy method. This method is ideally suited for systematically investigating the economic phenomena that arise as one moves through the parameter space and is especially useful in games that have multiple equilibria. We briefly discuss the theory of the homotopy method and its application to dynamic stochastic games. We then present three main findings: First, the more costly and/or less beneficial it is to achieve or maintain a given quality level, the more a leader invests in striving to induce the follower to give up; the more quickly the follower does so; and the more asymmetric is the industry structure that arises. Second, the possibility of entry and exit gives rise to predatory and limit investment. Third, we illustrate and discuss the multiple equilibria that arise in the quality ladder model, highlighting the presence of entry and exit as a source of multiplicity.

Keywords Quality ladder model · Dynamic oligopoly · Homotopy method

JEL Classification L13 · C63 · C73

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1 Introduction

Pakes and McGuire (1994) develop a dynamic quality ladder model in the Markov perfect equilibrium framework of Ericson and Pakes (1995). In the Pakes and McGuire (1994) model, forward-looking oligopolistic firms compete with each other in the product market and through their investment, entry, and exit decisions. By investing in the present a firm hopes to increase the quality of its product—and ultimately its profits from product market competition—in the future. Investment, entry, and exit decisions are thus both dynamic and strategic.

The Pakes and McGuire (1994) model has been widely used as a template for dynamic models of investment. It has been adapted to study mergers (Gowrisankaran 1999; Gowrisankaran and Holmes 2004); capacity accumulation (Besanko and Doraszelski 2004; Besanko et al. 2010c); competitive convergence (Langohr 2004); advertising (Doraszelski and Markovich 2007; Dubé et al. 2005); network effects (Markovich 2008; Markovich and Moenius 2009; Chen et al. 2009); research joint ventures (Song 2010); durable goods (Goettler and Gordon 2011); investment in both vertical and horizontal product differentiation (Narajabad and Watson 2011); spillovers (Laincz and Rodrigues 2008); and the timing of version releases (Borkovsky 2010). The Pakes and McGuire (1994) model has also been used to benchmark algorithms for computing Markov perfect equilibria in the Ericson and Pakes (1995) framework.[1]

Although widely used and adapted, the Pakes and McGuire (1994) model has never been thoroughly investigated. First, Pakes and McGuire (1994) compute equilibria for just two parameterizations, thus leaving the parameter space largely unexplored. Second, Pakes and McGuire (1994) do not characterize equilibrium behavior and instead focus on the effects of different institutional arrangements on market structure and welfare. Given the model's prominence, we feel it is important to better understand the types of equilibrium behavior that arise and the ways in which behavior changes as one moves through the parameter space.

In this paper we use the homotopy method to undertake a thorough exploration of the equilibrium correspondence of a version of the Pakes and McGuire (1994) model with at most two firms. The homotopy method was first applied to dynamic stochastic games by Besanko et al. (2010b). It is a type of path-following method. Starting from a single equilibrium that has already been computed, it traces out an entire path in the equilibrium correspondence by varying one or more selected parameters of the model. The homotopy method is thus ideally suited to investigating the economic phenomena that arise as one moves through the parameter space.

[1] See Pakes and McGuire (2001), Ferris et al. (2007), Doraszelski and Judd (2011), Weintraub et al. (2010), Borkovsky et al. (2010), Farias et al. (2010), and Santos (2009).

We find that a change in parameterization that increases (decreases) the cost (benefit) of achieving or maintaining any given product quality yields more asymmetric industry structures in the short and long run. The cost is tied to the rate of depreciation and the effectiveness of investment, and the benefit is tied to the discount factor. Consider an increase in the rate of depreciation: A higher rate of depreciation makes it more costly for a firm to achieve or maintain any given quality level for its product. Therefore, it makes it more costly for the follower to catch up with the leader and thus stifles the follower's incentive to invest. Accordingly, the leadership position becomes more secure. It follows that each firm strives to be the first to gain a lead over its rival and, thereafter, to induce its rival to cease investing and perhaps even exit, so that it can ultimately achieve industry dominance. We also find that increasing the scrap value yields a more asymmetric industry structure because a higher scrap value makes exit more attractive and therefore it is easier for a leader to induce a follower to exit. However, a sufficiently high scrap value makes the industry structure less asymmetric because it induces potential entrants to engage in *opportunistic entry*, which entails entering primarily in order to exit soon after and collect a high scrap value.

We also find that the possibility of entry and exit in the Pakes and McGuire (1994) model gives rise to predatory and limit investment. This finding suggests that such behaviors are quite pervasive in the Ericson and Pakes (1995) framework, especially since the Pakes and McGuire (1994) model is arguably the simplest model in this framework that one can devise. Snider (2009) studies predation in the airline industry by structurally estimating a model of capacity accumulation. He argues that cost asymmetries amongst firms give rise to predatory investment. Similarly, much of the earlier literature (e.g., Milgrom and Roberts 1982; Fudenberg and Tirole 1986) finds that predation occurs in the face of asymmetric information and/or amongst asymmetric firms. In contrast, we see predation occur in a complete information setting amongst symmetric firms.

A second and equally important advantage of the homotopy method is that it allows us to systematically search for multiple equilibria. Multiple equilibria have long been a concern in the Ericson and Pakes (1995) framework. They are problematic for at least two reasons. First, most structural estimation methods such as Aguirregabiria and Mira (2007), Bajari et al. (2007), Pakes et al. (2007), and Pesendorfer and Schmidt-Dengler (2008) depend on the assumption that the same equilibrium is being played in all geographic markets and/or time periods. While this assumption is trivially satisfied if the equilibrium is unique, it is potentially restrictive in the presence of multiplicity. Second, it is difficult to draw conclusions from policy experiments if there are multiple equilibria, as one cannot determine which one arises in each market and/or time period after a change in policy. It is therefore important to characterize the set of equilibria in order to bound the range of outcomes that may be produced by the change in policy.

While the Pakes and McGuire (1994) algorithm has been used most often to solve for Markov perfect equilibria in the Ericson and Pakes (1995) framework, it cannot be used to systematically search for multiple equilibria. One can only take the trial-and-error approach of starting the algorithm from different points in the hope that it converges to different equilibria. The Pakes and McGuire (1994) algorithm also suffers from a more severe problem: when there are multiple equilibria, it is unable to compute a substantial fraction of them (Besanko et al. 2010b).

The homotopy method is an important step towards resolving these issues, as it allows us to systematically search for multiple equilibria and to compute equilibria that cannot be computed by the Pakes and McGuire (1994) algorithm. The homotopy method traces out an entire path in the equilibrium correspondence by varying one or more selected parameters of the model. If this path bends back on itself, then the homotopy method has identified multiple equilibria. The homotopy method is guaranteed to find all equilibria on a path it traverses and, therefore, to find all multiple equilibria that arise in this manner. However, since multiple equilibria for a given parameterization do not necessarily lie on the same path, the homotopy method is not guaranteed to find all equilibria.

We find several instances of multiple equilibria, in contrast to Pakes and McGuire's (1994) conclusion that "[w]e have computed several of our examples ... from different initial conditions, and we have always converged to the same fixed point, so nonuniqueness does not seem to be a problem with the simple functional forms we are currently using" (p. 570). In a companion paper (Borkovsky et al. 2010), we have explored the equilibrium correspondence of a quality ladder model that does not allow for entry and exit. Interestingly, in the current model multiple equilibria arise for parameterizations for which we did not find multiple equilibria in the model that does not allow for entry and exit. This suggests that entry and exit can by themselves be a source of multiplicity in the Ericson and Pakes (1995) framework.

The paper proceeds as follows. In Section 2, we present the Pakes and McGuire (1994) model. In Section 3, we briefly discuss the theory of the homotopy method as well as HOMPACK90, a suite of Fortran90 routines developed by Watson et al. (1997) that implements this method. We then explain how we use HOMPACK90 to compute equilibria of the quality ladder model. Section 4 describes the types of equilibrium behavior that can arise and the implied industry dynamics. In Section 5, we show that entry and exit can give rise to predatory and limit investment. In Section 6, we describe the instances of multiple equilibria that we have uncovered. Section 7 concludes.

2 Quality ladder model

The description of the model is abridged; please see Pakes and McGuire (1994) for details. We restrict attention to a version of the model with at most two firms. To allow for entry and exit in a way that guarantees the existence of an equilibrium, we follow Doraszelski and Satterthwaite (2010) and assume that setup costs and scrap values are privately observed random variables.

Firms and states Firm $n \in \{1, 2\}$ is described by its state $\omega_n \in \{0, 1, \ldots, M\}$. States $1, \ldots, M$ describe the product quality of a firm that is active in the product market, i.e., an incumbent firm, while state 0 identifies a firm as being inactive, i.e., a potential entrant. We model exit as a transition from state $\omega_n \neq 0$ to $\omega'_n = 0$ and entry as a transition from state $\omega_n = 0$ to state $\omega'_n \neq 0$. The vector of firms' states is $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \{0, \ldots, M\}^2$ and we use $\boldsymbol{\omega}^{[2]}$ to denote the vector (ω_2, ω_1) obtained by interchanging firms' states.

Timing In each period the sequence of events is as follows:

1. Each incumbent firm learns its scrap value and decides on exit and investment. Each potential entrant learns its setup cost and decides on entry.
2. Incumbent firms compete in the product market.
3. Exit and entry decisions are implemented.
4. The investment decisions of the remaining incumbents are carried out and their uncertain outcomes are realized. A common industry-wide depreciation shock affecting incumbents and entrants is realized.

We first describe the static model of product market competition and then turn to investment, entry, and exit dynamics.

Product market competition The product market is characterized by price competition with vertically differentiated products. There is a continuum of consumers. Each consumer purchases at most one unit of one product. The utility a consumer derives from purchasing from firm n is $g(\omega_n) - p_n + \epsilon_n$, where

$$g(\omega_n) = \begin{cases} -\infty & \text{if} \quad \omega_n = 0, \\ \omega_n & \text{if} \quad 1 \leq \omega_n \leq \omega^*, \\ \omega^* + \ln\left(2 - \exp\left(\omega^* - \omega_n\right)\right) & \text{if} \quad \omega^* < \omega_n \leq M, \end{cases} \tag{1}$$

maps the quality of the product into the consumer's valuation of it, p_n is the price, and ϵ_n represents the consumer's idiosyncratic preference for product n.[2] By setting $g(0) = -\infty$, we ensure that potential entrants have zero demand and thus do not compete in the product market. There is an outside alternative, product 0, which has utility ϵ_0. Assuming that the idiosyncratic preferences

[2] Although Pakes and McGuire (1994) state that they set $g(\cdot)$ as in (1) with $\omega^* = 12$, inspection of their C code (see also Pakes et al. 1993) shows that the results they present are in fact computed setting

$$g(\omega_n) = \begin{cases} -\infty & \text{if} \quad \omega_n = 0, \\ 3\omega_n - 4 & \text{if} \quad 1 \leq \omega_n \leq 5, \\ 12 + \ln\left(2 - \exp\left(16 - 3\omega_n\right)\right) & \text{if} \quad 5 < \omega_n \leq M. \end{cases}$$

We opt for the $g(\cdot)$ function in (1) because it yields a much richer set of equilibrium behaviors.

$(\epsilon_0, \epsilon_1, \epsilon_2)$ are independently and identically type 1 extreme value distributed, the demand for incumbent firm n's product is

$$D_n(\boldsymbol{p}; \boldsymbol{\omega}) = m \frac{\exp\left(g(\omega_n) - p_n\right)}{1 + \sum_{j=1}^{2} \exp\left(g(\omega_j) - p_j\right)}, \tag{2}$$

where $\boldsymbol{p} = (p_1, p_2)$ is the vector of prices and $m > 0$ is the size of the market (the measure of consumers).

Incumbent firm n chooses the price p_n of its product to maximize profits. Hence, its profits in state $\boldsymbol{\omega}$ are

$$\pi_n(\boldsymbol{\omega}) = \max_{p_n} D_n\left(p_n, p_{-n}(\boldsymbol{\omega}); \boldsymbol{\omega}\right)\left(p_n - c\right),$$

where $p_{-n}(\boldsymbol{\omega})$ is the price charged by the rival and $c \geq 0$ is the marginal cost of production. Given a state $\boldsymbol{\omega}$, there exists a unique Nash equilibrium of the product market game (Caplin and Nalebuff 1991). It is found easily by numerically solving the system of first-order conditions corresponding to incumbent firms' profit-maximization problems. Because product market competition does not directly affect state-to-state transitions, $\pi_n(\boldsymbol{\omega})$ can be computed before the Markov perfect equilibria of the dynamic stochastic game are computed. This allows us to treat $\pi_n(\boldsymbol{\omega})$ as a primitive in what follows.

Incumbent firms Suppose first that firm n is an incumbent firm, i.e., $\omega_n \neq 0$. We assume that at the beginning of each period each incumbent firm draws a random scrap value from a symmetric triangular distribution $F(\cdot)$ with support $[\bar{\phi} - \epsilon, \bar{\phi} + \epsilon]$. Scrap values are independently and identically distributed across firms and periods. Incumbent firm n learns its scrap value ϕ_n prior to making its exit and investment decisions, but the scrap values of its rivals remain unknown to it. If the scrap value is above a threshold $\tilde{\phi}_n$, then incumbent firm n exits the industry and perishes; otherwise it remains in the industry. This decision rule can be represented either with the cutoff scrap value $\tilde{\phi}_n$ itself or with the probability $\xi_n \in [0, 1]$ that incumbent firm n remains in the industry in state $\boldsymbol{\omega}$ because $\xi_n = \int 1(\phi_n \leq \tilde{\phi}_n) dF(\phi_n) = F(\tilde{\phi}_n)$, where $1(\cdot)$ is the indicator function, is equivalent to $\tilde{\phi}_n = F^{-1}(\xi_n)$.

If it remains in the industry, then the state of incumbent firm n in the next period is determined by the stochastic outcomes of its investment decision and an industry-wide depreciation shock that stems from an increase in the quality of the outside alternative. In particular, its state evolves according to the law of motion

$$\omega_n' = \omega_n + \tau_n - \eta,$$

where $\tau_n \in \{0, 1\}$ is a random variable governed by incumbent firm n's investment $x_n \geq 0$ and $\eta \in \{0, 1\}$ is an industry-wide depreciation shock. If $\tau_n = 1$, the investment is successful and the quality of incumbent firm n increases by one level. The probability of success is $\frac{\alpha x_n}{1+\alpha x_n}$, where $\alpha > 0$ is a measure of the effectiveness of investment. If $\eta = 1$, the industry is hit by a depreciation shock and the qualities of all products decrease by one level; this happens with probability $\delta \in [0, 1]$.

Potential entrants Suppose next that firm n is a potential entrant, i.e., $\omega_n = 0$. We assume that at the beginning of each period each potential entrant draws a random setup cost from a symmetric triangular distribution $F^e(\cdot)$ with support $[\bar{\phi}^e - \epsilon, \bar{\phi}^e + \epsilon]$. Like scrap values, setup costs are observed privately and are independently and identically distributed across firms and periods. If the setup cost is below a threshold $\tilde{\phi}^e_n$, then potential entrant n enters the industry; otherwise it perishes. This decision rule can be represented with the probability $\xi_n \in [0, 1]$ that potential entrant n enters in the industry.

Upon entry, potential entrant n undergoes a setup period. At the end of this period (i.e., at the beginning at the next period) it becomes incumbent firm n and its state is

$$\omega'_n = \omega^e - \eta,$$

where ω^e is an exogenously given initial product quality.

Value and policy functions Define $V_n(\boldsymbol{\omega})$ to be the expected net present value of firm n's cash flows if the industry is currently in state $\boldsymbol{\omega}$. Incumbent firm n's value function is $\boldsymbol{V}_n : \{1, \ldots, M\} \times \{0, \ldots, M\} \to \mathbb{R}$, and its policy functions $\boldsymbol{\xi}_n : \{1, \ldots, M\} \times \{0, \ldots, M\} \to [0, 1]$ and $\boldsymbol{x}_n : \{1, \ldots, M\} \times \{0, \ldots, M\} \to [0, \infty)$ specify the probability that it remains in the industry and its investment in state $\boldsymbol{\omega}$. Potential entrant n's value function is $\boldsymbol{V}_n : \{0\} \times \{0, \ldots, M\} \to \mathbb{R}$, and its policy function $\boldsymbol{\xi}_n : \{0\} \times \{0, \ldots, M\} \to [0, 1]$ specifies the probability that it enters the industry in state $\boldsymbol{\omega}$.

Bellman equation and optimality conditions Suppose first that firm n is an incumbent firm, i.e., $\omega_n \neq 0$. The value function $\boldsymbol{V}_n : \{1, \ldots, M\} \times \{0, \ldots, M\} \to \mathbb{R}$ is implicitly defined by the Bellman equation

$$\begin{aligned} V_n(\boldsymbol{\omega}) = \max_{\xi_n \in [0,1], x_n \geq 0} & \pi_n(\boldsymbol{\omega}) + (1 - \xi_n) \mathrm{E}\left\{\phi_n | \phi_n \geq F^{-1}(\xi_n)\right\} \\ & + \xi_n \left\{ -x_n + \beta \left(\frac{\alpha x_n}{1 + \alpha x_n} W^1_n(\boldsymbol{\omega}) + \frac{1}{1 + \alpha x_n} W^0_n(\boldsymbol{\omega}) \right) \right\}, \end{aligned} \tag{3}$$

where $\beta \in (0, 1)$ is the discount factor. Instead of the unconditional expectation $\mathrm{E}(\phi_n)$, an optimizing incumbent cares about the expectation of the scrap value conditional on collecting it:

$$\mathrm{E}\left\{\phi_n | \phi_n \geq F^{-1}(\xi_n)\right\} = \begin{cases} \bar{\phi} & \text{if} \quad T_n = -1, \\ \bar{\phi} + \epsilon \left(\dfrac{1 - 3{T_n}^2 - 2{T_n}^3}{3\left(2 - (1 + T_n)^2\right)} \right) & \text{if} \ -1 < T_n < 0, \\ \bar{\phi} + \epsilon \left(\dfrac{1 - 3{T_n}^2 + 2{T_n}^3}{3\left(1 - {T_n}^2\right)} \right) & \text{if} \ \ 0 \leq T_n < 1, \\ \bar{\phi} + \epsilon & \text{if} \quad T_n = 1, \end{cases}$$

where $T_n = \frac{1}{\epsilon}\left[F^{-1}(\xi_n) - \bar{\phi}\right] \in [-1, 1]$. $W_n^{\tau_n}(\boldsymbol{\omega})$ is the expectation of incumbent firm n's value function conditional on an investment success ($\tau_n = 1$) or failure ($\tau_n = 0$), respectively, as given by

$$
\begin{aligned}
W_n^{\tau_n}(\boldsymbol{\omega}) = & \sum_{\eta\in\{0,1\}} \delta^{\eta}(1-\delta)^{1-\eta} \\
& \times\Big[1(\omega_{-n}=0)\xi_{-n}(\boldsymbol{\omega})V_n\Big(\max\{\min\{\omega_n+\tau_n-\eta, M\}, 1\}, \omega^e-\eta\Big) \\
& + 1(\omega_{-n}>0)\Big[\xi_{-n}(\boldsymbol{\omega})\sum_{\tau_{-n}\in\{0,1\}}\left(\tfrac{\alpha x_{-n}(\boldsymbol{\omega})}{1+\alpha x_{-n}(\boldsymbol{\omega})}\right)^{\tau_{-n}}\left(\tfrac{1}{1+\alpha x_{-n}(\boldsymbol{\omega})}\right)^{1-\tau_{-n}} \\
& \times V_n\Big(\max\{\min\{\omega_n+\tau_n-\eta, M\}, 1\}, \\
& \max\{\min\{\omega_{-n}+\tau_{-n}-\eta, M\}, 1\}\Big)\Big] \\
& + (1-\xi_{-n}(\boldsymbol{\omega}))V_n\Big(\max\{\min\{\omega_n+\tau_n-\eta, M\}, 1\}, 0\Big)\Big],
\end{aligned}
\tag{4}
$$

where $x_{-n}(\boldsymbol{\omega})$ is the investment of the rival in state $\boldsymbol{\omega}$ and $\xi_{-n}(\boldsymbol{\omega})$ is the probability that a rival entrant (incumbent) enters (remains in) the industry in state $\boldsymbol{\omega}$. The min and max operators merely enforce the bounds of the state space.

Solving the maximization problem on the right-hand side of the Bellman equation (3) and using the fact that $(1-\xi_n)\mathrm{E}\left\{\phi_n|\phi_n \geq F^{-1}(\xi_n)\right\} = \int_{\phi_n\geq F^{-1}(\xi_n)} \phi_n dF(\phi_n)$, we obtain the first-order condition for $\xi_n(\boldsymbol{\omega})$:

$$
-F^{-1}(\xi_n(\boldsymbol{\omega})) + \left\{-x_n + \beta\left(\frac{\alpha x_n}{1+\alpha x_n}W_n^1(\boldsymbol{\omega}) + \frac{1}{1+\alpha x_n}W_n^0(\boldsymbol{\omega})\right)\right\} = 0. \tag{5}
$$

We further obtain the complementary slackness condition for $x_n(\boldsymbol{\omega})$:

$$
\begin{aligned}
-1 + \beta\frac{\alpha}{(1+\alpha x_n(\boldsymbol{\omega}))^2}\left(W_n^1(\boldsymbol{\omega}) - W_n^0(\boldsymbol{\omega})\right) &\leq 0, \\
x_n(\boldsymbol{\omega})\left(-1 + \beta\frac{\alpha}{(1+\alpha x_n(\boldsymbol{\omega}))^2}\left(W_n^1(\boldsymbol{\omega}) - W_n^0(\boldsymbol{\omega})\right)\right) &= 0, \\
x_n(\boldsymbol{\omega}) &\geq 0.
\end{aligned}
\tag{6}
$$

The first-order condition (5) and complementary slackness condition (6) are both necessary and sufficient.

Suppose next that firm n is a potential entrant, i.e., $\omega_n = 0$. The value function $\boldsymbol{V}_n : \{0\} \times \{0, \ldots, M\} \to \mathbb{R}$ is implicitly defined by

$$
V_n(\boldsymbol{\omega}) = \max_{\xi_n\in[0,1]} \xi_n\left\{-\mathrm{E}\left\{\phi_n^e|\phi_n^e \leq F^{e-1}(\xi_n)\right\} + \beta W_n^e(\boldsymbol{\omega})\right\}. \tag{7}
$$

Instead of the unconditional expectation $\mathrm{E}(\phi_n^e)$, an optimizing potential entrant cares about the expectation of the setup cost conditional on entering:

$$\mathrm{E}\left\{\phi_n^e | \phi_n^e \leq F^{e-1}(\xi_n)\right\} = \begin{cases} \bar{\phi}^e - \epsilon & \text{if} \quad T_n^e = -1, \\ \bar{\phi}^e + \epsilon\left(\dfrac{-1 + 3T_n^{e2} + 2T_n^{e3}}{3((1+T_n^e)^2)}\right) & \text{if } -1 < T_n^e < 0, \\ \bar{\phi}^e + \epsilon\left(\dfrac{-1 + 3T_n^{e2} - 2T_n^{e3}}{3(2-(1-T_n^e)^2)}\right) & \text{if } \ 0 \leq T_n^e < 1, \\ \bar{\phi}^e & \text{if} \quad T_n^e = 1, \end{cases}$$

where $T_n^e = \frac{1}{\epsilon}\left[F^{e-1}(\xi_n) - \bar{\phi}^e\right] \in [-1, 1]$. $W_n^e(\boldsymbol{\omega})$ is the expectation of potential entrant n's value function as given by

$$\begin{aligned} W_n^e(\boldsymbol{\omega}) = \sum_{\eta \in \{0,1\}} \delta^\eta (1-\delta)^{1-\eta} \Bigg[& 1(\omega_{-n} = 0)\xi_{-n}(\boldsymbol{\omega}) V_n(\omega^e - \eta, \omega^e - \eta) \\ & + 1(\omega_{-n} > 0) \Bigg[\xi_{-n}(\boldsymbol{\omega}) \sum_{\nu_{-n} \in \{0,1\}} \left(\frac{\alpha x_{-n}(\boldsymbol{\omega})}{1 + \alpha x_{-n}(\boldsymbol{\omega})}\right)^{\nu_{-n}} \left(\frac{1}{1 + \alpha x_{-n}(\boldsymbol{\omega})}\right)^{1-\nu_{-n}} \\ & \times V_n\Big(\omega^e - \eta, \max\left\{\min\left\{\omega_{-n} + \nu_{-n} - \eta, M\right\}, 1\right\}\Big) \Bigg] \\ & + (1 - \xi_n(\boldsymbol{\omega})) V_n(\omega^e - \eta, 0) \Bigg]. \end{aligned} \tag{8}$$

Using the fact that $-\xi_n \mathrm{E}\left\{\phi_n^e | \phi_n^e \leq F^{e-1}(\xi_n)\right\} = -\int_{\phi_n^e \leq F^{e-1}(\xi_n)} \phi_n^e dF^e(\phi_n^e)$, we obtain the first-order condition for $\xi_n(\boldsymbol{\omega})$,

$$-F^{-1}(\xi_n(\boldsymbol{\omega})) + \beta W_n^e(\boldsymbol{\omega}) = 0, \tag{9}$$

which is both necessary and sufficient.

Equilibrium We restrict attention to symmetric Markov perfect equilibria in pure strategies. Theorem 1 in Doraszelski and Satterthwaite (2010) establishes that such an equilibrium always exists. In a symmetric equilibrium, the investment decision taken by firm 2 in state $\boldsymbol{\omega}$ is identical to the investment decision taken by firm 1 in state $\boldsymbol{\omega}^{[2]}$, i.e., $x_2(\boldsymbol{\omega}) = x_1(\boldsymbol{\omega}^{[2]})$, and similarly for the entry/exit decisions and the value functions. It therefore suffices to determine the value and policy functions of firm 1, and we define $V(\boldsymbol{\omega}) = V_1(\boldsymbol{\omega})$, $\xi(\boldsymbol{\omega}) = \xi_1(\boldsymbol{\omega})$, and $x(\boldsymbol{\omega}) = x_1(\boldsymbol{\omega})$ for each state $\boldsymbol{\omega}$. Similarly, we define $W^{\tau_1}(\boldsymbol{\omega}) = W_1^{\tau_1}(\boldsymbol{\omega})$ and $W^e(\boldsymbol{\omega}) = W_1^e(\boldsymbol{\omega})$ for each state $\boldsymbol{\omega}$. Solving for an equilibrium for a particular parameterization of the model amounts to finding a value function $\boldsymbol{V}(\cdot)$ and policy functions $\boldsymbol{\xi}(\cdot)$ and $\boldsymbol{x}(\cdot)$ that satisfy the Bellman equations (3) and (7) and the optimality conditions (5), (6), and (9).

3 Computation

Our objective is to compute equilibria of the model using the homotopy method. In Section 3.1, we present a brief description of the homotopy method. In Section 3.2, we discuss HOMPACK90, a suite of Fortran90 routines that implements this method. See Borkovsky et al. (2010) for more in depth descriptions of the homotopy method and HOMPACK90. In Section 3.3, we explain how we apply the homotopy method to the quality ladder model.

3.1 Homotopy method

The homotopy method is a tool for solving systems of non-linear equations. There are two types of homotopy methods: The *artificial homotopy method* is used to obtain a solution to a system of equations for a particular parameterization (see Chapter 1 of Zangwill and Garcia 1981). The *natural parameter homotopy method* traces out an entire path of solutions by varying one or more parameters of interest. We use the latter to explore the equilibrium correspondence that maps parameters into equilibria in a systematic fashion. Hereafter, we use "the homotopy method" to refer to the natural parameter homotopy method.

The equilibrium conditions depend on the parameterization of the model. Making this dependence explicit, the equilibrium conditions can be written as the system of equations

$$\boldsymbol{H}(\boldsymbol{z}, \lambda) = \boldsymbol{0},$$

where $\boldsymbol{H} : \mathbb{R}^{N+1} \to \mathbb{R}^N$, $\boldsymbol{z} \in \mathbb{R}^N$ is the vector of the unknown values and policies, $\boldsymbol{0} \in \mathbb{R}^N$ is a vector of zeros, and $\lambda \in [0, 1]$ is the so-called homotopy parameter. We use boldface to distinguish between vectors and scalars. Depending on the application at hand, the homotopy parameter maps into one or more of the parameters of the model. The object of interest is the equilibrium correspondence

$$\boldsymbol{H}^{-1} = \{(\boldsymbol{z}, \lambda) | \boldsymbol{H}(\boldsymbol{z}, \lambda) = \boldsymbol{0}\}.$$

The homotopy method traces out an entire path of equilibria in $\boldsymbol{H}^{-1}$. We use a simple example to illustrate how this is done.

Example Let $N = 1$ and consider the equation $H(z, \lambda) = 0$ that relates a variable z with a parameter λ, where

$$H(z, \lambda) = z^3 - z + 6 - 12\lambda.$$

Here we do not use boldface for z and 0 since they are scalars. The set of solutions is $H^{-1} = \{(z, \lambda) | H(z, \lambda) = 0\}$ and is shown in Fig. 1. Inspecting Fig. 1, one can easily see that multiple solutions arise whenever the graph bends back on itself, as it does at points B and C. For example, at $\lambda = 0.5$ there are three solutions, namely $z = -1$, $z = 0$, and $z = 1$. Thus the mapping from λ to z is a correspondence, not a function.

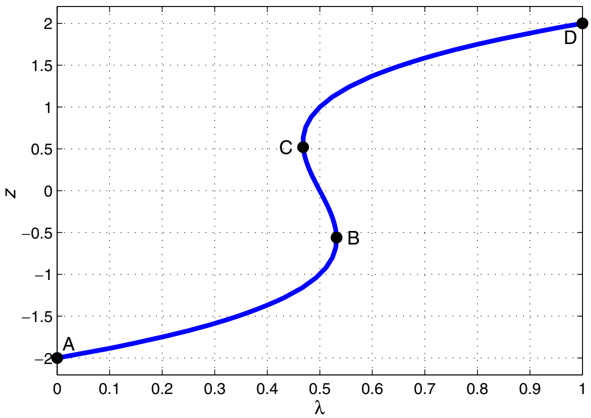


Fig. 1 Example

The homotopy method constructs a parametric path $(z(s), \lambda(s)) \in H^{-1}$. The points on this path are indexed by the auxiliary variable s that increases or decreases monotonically as we move along the path. To construct the parametric path, we proceed as follows. As $(z(s), \lambda(s)) \in H^{-1}$, it follows that $H(z(s), \lambda(s)) = 0$ for all s. Totally differentiating with respect to s yields the condition for remaining on the path:

$$\frac{\partial H(z(s), \lambda(s))}{\partial z} z'(s) + \frac{\partial H(z(s), \lambda(s))}{\partial \lambda} \lambda'(s) = 0.$$

As this is one differential equation in two unknowns, $z'(s)$ and $\lambda'(s)$, it has many solutions; however, they all describe the same path. One obvious solution is

$$z'(s) = \frac{\partial H(z(s), \lambda(s))}{\partial \lambda} = -12, \tag{10}$$

$$\lambda'(s) = -\frac{\partial H(z(s), \lambda(s))}{\partial z} = -3z^2 + 1. \tag{11}$$

The so-called basic differential equations (BDE) (10) and (11) and the initial condition $H(-2, 0) = 0$ (point A in Fig. 1) describe the parametric path $(z(s), \lambda(s)) \in H^{-1}$ given by

$$z\,(s) = -12s - 2, \tag{12}$$

$$\lambda\,(s) = -144s^3 - 72s^2 - 11s. \tag{13}$$

As s decreases monotonically from 0 to $-1/3$, Eqs. (12) and (13) trace out the set of solutions shown in Fig. 1.

While this simple example allows for an analytic solution to the BDE, most real-world problems do not; therefore, numerical methods are used. Given an initial condition for $\lambda = 0$, a homotopy algorithm proceeds along the solution path in discrete steps until it reaches $\lambda = 1$. At each step, the algorithm uses the BDE to determine the direction in (z, λ) space in which to proceed.

Regularity and differentiability requirements A central condition in the mathematical literature on the homotopy method is that the Jacobian of $\boldsymbol{H}$ must have full rank at all points on the solution path. If so, the homotopy is called regular. The other major requirement of the homotopy method is that $\boldsymbol{H}$ is differentiable. If $\boldsymbol{H}$ is regular and twice continuously differentiable, then $\boldsymbol{H}^{-1}$ consists only of continuously differentiable paths that can be easily traversed by a homotopy algorithm.

The differentiability requirement can often be satisfied by a judicious choice of functional forms.[3] In contrast, within the context of dynamic stochastic games in the Ericson and Pakes (1995) framework, it is often very difficult to verify analytically that regularity holds because the Jacobian of the system of equations tends to be intractable.

3.2 HOMPACK90 software package

HOMPACK90 is a suite of Fortran90 routines that implements the homotopy method.[4] In order to use HOMPACK90, first, the user must provide Fortran90 code that returns $\boldsymbol{H}(\boldsymbol{z}, \lambda)$ at any given point $(\boldsymbol{z}, \lambda)$. Second, the user must provide a routine that returns the Jacobian of $\boldsymbol{H}$ at any given point $(\boldsymbol{z}, \lambda)$. Many applications yield Jacobians with relatively few non-zero elements. HOMPACK90 allows the user to store such a sparse Jacobian using a sparse-matrix storage format that can substantially decrease computation time. In order to use this format, however, the user must specify the "sparsity structure" of the Jacobian, i.e., the row and column indices of potentially non-zero elements. The Jacobian can be computed either numerically (see, e.g., Chapter 7 of Judd 1998) or analytically. We compute the Jacobian analytically using ADIFOR, a program developed by Bischof et al. (1996). Third, the user must provide an initial condition in the form of a solution to the system of equations for the particular parameterization associated with $\lambda = 0$. In some cases, if the parameterization associated with $\lambda = 0$ is trivial, the solution can be computed analytically. More generally, it can be computed numerically using a number of approaches such as Gaussian methods including (but not limited to) the Pakes and McGuire (1994) algorithm, other nonlinear solvers (see Ferris et al. 2007), and artificial homotopies, which can also be implemented using HOMPACK90.

[3] In Section 2, we assume that scrap values and setup costs are drawn from triangular distributions; the resulting cumulative distribution functions are once but not twice continuously differentiable. In Eq. (21), we set $k = 2$, which yields an equation that is once but not twice continuously differentiable. Despite these violations of the differentiability requirement, we did not encounter any problems. If a problem is encountered in another application, we suggest using Beta(l, l) distributions with $l \geq 3$ instead of triangular distributions and setting $k \geq 3$.

[4] There are other software packages that implement the homotopy method. Some depend on—and exploit—the particular structure of the system of equations, e.g., with the freely-available Gambit (McKelvey et al. 2006) and PHCpack (Verschelde 1999) software packages, one can use the homotopy method to obtain solutions to polynomial systems.

3.3 Application to quality ladder model

The homotopy parameter λ maps into one or more parameters of the model; as λ varies between 0 and 1, the homotopy algorithm traces out an entire path in the equilibrium correspondence $\boldsymbol{H}^{-1}$ by varying these parameters. Before we can construct the system of equations $\boldsymbol{H}$, we must specify the function that maps λ into a parameterization. We allow the homotopy algorithm to vary β, α, δ, $\bar{\phi}$ and $\bar{\phi}^e$ by mapping λ into these parameters as follows:

$$\begin{bmatrix} \beta(\lambda) \\ \alpha(\lambda) \\ \delta(\lambda) \\ \bar{\phi}(\lambda) \\ \bar{\phi}^e(\lambda) \end{bmatrix} = \begin{bmatrix} \beta^{\text{start}} \\ \alpha^{\text{start}} \\ \delta^{\text{start}} \\ \bar{\phi}^{\text{start}} \\ \bar{\phi}^{e^{\text{start}}} \end{bmatrix} + \lambda \begin{bmatrix} \beta^{\text{end}} - \beta^{\text{start}} \\ \alpha^{\text{end}} - \alpha^{\text{start}} \\ \delta^{\text{end}} - \delta^{\text{start}} \\ \bar{\phi}^{\text{end}} - \bar{\phi}^{\text{start}} \\ \bar{\phi}^{e^{\text{end}}} - \bar{\phi}^{e^{\text{start}}} \end{bmatrix} \tag{14}$$

For example, if $\delta^{\text{start}} = 0$ and $\delta^{\text{end}} = 1$ while $\beta^{\text{start}} = \beta^{\text{end}}$, $\alpha^{\text{start}} = \alpha^{\text{end}}$, $\bar{\phi}^{\text{start}} = \bar{\phi}^{\text{end}}$, and $\bar{\phi}^{e^{\text{start}}} = \bar{\phi}^{e^{\text{end}}}$, then the homotopy algorithm traces out the equilibrium correspondence from $\delta(0) = 0$ to $\delta(1) = 1$, holding all other parameter values fixed. Setting different starting and ending values for one or more parameters allows us to explore the equilibrium correspondence by moving through the parameter space in various directions.[5]

The homotopy method operates on a system of equations. However, given the non-negativity constraint on investment, the problem that an incumbent firm has to solve gives rise to a complementary slackness condition, a combination of equalities and inequalities, rather than a first-order condition, an equation. Fortunately, the complementary slackness condition can be reformulated as a system of equations that is continuously differentiable to an arbitrary degree (Zangwill and Garcia 1981, pp. 65–68). Using the fact that we focus on symmetric equilibria in order to eliminate firm indices and multiplying through by $(1 + \alpha x(\boldsymbol{\omega}))^2$ to simplify the expressions that arise in what follows, the complementary slackness condition (6) can be restated as

$$\begin{aligned} -(1 + \alpha x(\boldsymbol{\omega}))^2 + \beta\alpha \left(W^1(\boldsymbol{\omega}) - W^0(\boldsymbol{\omega})\right) &\leq 0, \\ x(\boldsymbol{\omega}) \left(-(1 + \alpha x(\boldsymbol{\omega})^2 + \beta\alpha \left(W^1(\boldsymbol{\omega}) - W^0(\boldsymbol{\omega})\right)\right) &= 0, \\ x(\boldsymbol{\omega}) &\geq 0. \end{aligned} \tag{15}$$

Introduce another scalar variable $\zeta(\boldsymbol{\omega})$ and consider the system of equations

$$-(1 + \alpha x(\boldsymbol{\omega}))^2 + \beta\alpha \left(W^1(\boldsymbol{\omega}) - W^0(\boldsymbol{\omega})\right) + [\max\{0, \zeta(\boldsymbol{\omega})\}]^k = 0, \tag{16}$$

$$-x(\boldsymbol{\omega}) + [\max\{0, -\zeta(\boldsymbol{\omega})\}]^k = 0, \tag{17}$$

[5] For the sake of simplicity, we suppress the dependence of β, α, δ, $\bar{\phi}$ and $\bar{\phi}^e$ on λ in what follows.

where $k \in \mathbb{N}$. It is easy to see that the system of Eqs. (16) and (17) is equivalent to the complementary slackness condition (15).[6] This system is $(k-1)$ times continuously differentiable with respect to $\zeta(\boldsymbol{\omega})$. Hence, by choosing k large enough, we can satisfy the differentiability requirement of the homotopy method. The terms $[\max\{0, \zeta(\boldsymbol{\omega})\}]^k$ and $[\max\{0, -\zeta(\boldsymbol{\omega})\}]^k$ serve as slack variables that ensure that the inequalities in (15) are satisfied and the fact that $[\max\{0, \zeta(\boldsymbol{\omega})\}]^k [\max\{0, -\zeta(\boldsymbol{\omega})\}]^k = 0$ ensures that the equality in (15) holds.

We could now proceed to construct the system of equations $\boldsymbol{H}$ using Eqs. (16) and (17), the incumbent firm's Bellman equation (3) and the first-order condition for $\xi(\boldsymbol{\omega})$ in (5) for $\boldsymbol{\omega} \in \{1, \ldots, M\} \times \{0, \ldots, M\}$, and the potential entrant's first-order condition for $\xi(\boldsymbol{\omega})$ in (9) for $\boldsymbol{\omega} \in \{0\} \times \{0, \ldots, M\}$.[7] This would yield a system of $(M+1)(4M+1)$ equations in the $(M+1)(4M+1)$ unknowns $V(\boldsymbol{\omega})$, $x(\boldsymbol{\omega})$ and $\zeta(\boldsymbol{\omega})$ for $\boldsymbol{\omega} \in \{1, \ldots, M\} \times \{0, \ldots, M\}$ and $\xi(\boldsymbol{\omega})$ for $\boldsymbol{\omega} \in \{0, ..., M\}^2$. However, two problems arise: First, because we have added the slack variables, this system of equations is relatively large. This leads to increased memory requirements and computation time. Second, this system of equations yields an extremely sparse Jacobian, and we have found that excessive sparsity tends to cause HOMPACK90's sparse linear equation solver to fail; this is discussed further in Borkovsky et al. (2010).

We address these problems by solving Eq. (17) for $x(\boldsymbol{\omega})$ and then substituting

$$x(\boldsymbol{\omega}) = [\max\{0, -\zeta(\boldsymbol{\omega})\}]^k \tag{18}$$

into Eqs. (3), (5), (9), and (16). This reduces the system of $(M+1)(4M+1)$ equations in $(M+1)(4M+1)$ unknowns to a system of $(M+1)(3M+1)$ equations in $(M+1)(3M+1)$ unknowns and eliminates excessive sparsity.

Define the vector of unknowns in equilibrium as

$$\begin{aligned}\boldsymbol{z} = [&V(1,0), V(2,0), \ldots, V(M,0), V(1,1), \ldots, V(M,M),\\ &\xi(0,0), ..., \xi(M,M), \zeta(1,0), \ldots, \zeta(M,M)].\end{aligned}$$

[6] From Eqs. 16 and 17 it follows that

$$\zeta(\omega) = \begin{cases} [(1+\alpha x(\boldsymbol{\omega}))^2 + \beta\alpha\left(W^1(\boldsymbol{\omega}) - W^0(\boldsymbol{\omega})\right)]^{1/k} & \text{if } -(1+\alpha x(\boldsymbol{\omega}))^2 + \beta\alpha\left(W^1(\boldsymbol{\omega}) - W^0(\boldsymbol{\omega})\right) < 0, \\ -[x(\boldsymbol{\omega})]^{1/k} & \text{if } x(\boldsymbol{\omega}) > 0, \\ 0 & \text{if } -(1+\alpha x(\boldsymbol{\omega}))^2 + \beta\alpha\left(W^1(\boldsymbol{\omega}) - W^0(\boldsymbol{\omega})\right) \\ & \quad = x(\boldsymbol{\omega}) = 0. \end{cases}$$

The claim now follows from the fact that $\max\{0, -\zeta(\omega)\}\max\{0, \zeta(\omega)\} = 0$.

[7] To be precise, we would substitute the entry/exit policy $\xi(\omega)$ for ξ_n and the investment policy $x(\omega)$ for x_n in (3), and we would remove the max operators. We need not include the potential entrant's Bellman equation (7) in the system of equations $\boldsymbol{H}$ because $V(\boldsymbol{\omega})$ for $\boldsymbol{\omega} \in \{0\} \times \{0, 1, \ldots, M\}$ does not enter any of the equations in Section 2 aside from (7) where it is defined. This is because an incumbent firm that exits perishes; it does not become a potential entrant.

We can now construct the system of equations $\boldsymbol{H}$ as

$$H^1_{\boldsymbol{\omega}}(\boldsymbol{z},\lambda) = -V(\boldsymbol{\omega}) + \pi_1(\boldsymbol{\omega}) + (1-\xi(\boldsymbol{\omega}))\mathrm{E}\left\{\phi_n|\phi_n \geq F^{-1}(\xi(\boldsymbol{\omega}))\right\}$$
$$+\,\xi(\boldsymbol{\omega})\left\{-x(\boldsymbol{\omega}) + \beta\left(\frac{\alpha x(\boldsymbol{\omega})}{1+\alpha x(\boldsymbol{\omega})}W^1(\boldsymbol{\omega}) + \frac{1}{1+\alpha x(\boldsymbol{\omega})}W^0(\boldsymbol{\omega})\right)\right\} = 0, \tag{19}$$

$$H^2_{\boldsymbol{\omega}}(\boldsymbol{z},\lambda) = -F^{-1}(\xi(\boldsymbol{\omega}))$$
$$+\left\{-x(\boldsymbol{\omega}) + \beta\left(\frac{\alpha x(\boldsymbol{\omega})}{1+\alpha x(\boldsymbol{\omega})}W^1(\boldsymbol{\omega}) + \frac{1}{1+\alpha x(\boldsymbol{\omega})}W^0(\boldsymbol{\omega})\right)\right\} = 0, \tag{20}$$

$$H^3_{\boldsymbol{\omega}}(\boldsymbol{z},\lambda) = -(1+\alpha x(\boldsymbol{\omega}))^2 + \beta\alpha\left(W^1(\boldsymbol{\omega}) - W^0(\boldsymbol{\omega})\right) + [\max\{0,\zeta(\boldsymbol{\omega})\}]^k = 0 \tag{21}$$

for states $\boldsymbol{\omega} \in \{1,\ldots,M\} \times \{0,\ldots,M\}$, and

$$H^2_{\boldsymbol{\omega}}(\boldsymbol{z},\lambda) = -F^{-1}(\xi(\boldsymbol{\omega})) + \beta W^e(\boldsymbol{\omega}) = 0 \tag{22}$$

for states $\boldsymbol{\omega} \in \{0\} \times \{0,\ldots,M\}$, where we substitute for $W^{\tau_1}(\boldsymbol{\omega})$ using the definition in (4), for $W^e(\boldsymbol{\omega})$ using the definition in (8), and for $x(\boldsymbol{\omega})$ using the definition in (18). Equations (19), (20), (21), and (22) are used to construct the system of equations, while Eqs. (4), (8), and (18) are simply definitional shorthands. The collection of Eqs. (19), (20), and (21) for states $\boldsymbol{\omega} \in \{1,\ldots,M\} \times \{0,\ldots,M\}$, and (22) for states $\boldsymbol{\omega} \in \{0\} \times \{0,\ldots,M\}$ can be written more compactly as

$$\boldsymbol{H}(\boldsymbol{z},\lambda) = \begin{bmatrix} H^1_{(1,0)}(\boldsymbol{z},\lambda) \\ H^1_{(2,0)}(\boldsymbol{z},\lambda) \\ \vdots \\ H^3_{(M,M)}(\boldsymbol{z},\lambda) \end{bmatrix} = \boldsymbol{0},$$

where $\boldsymbol{0} \in \mathbb{R}^{(M+1)(3M+1)}$ is a vector of zeros. Any solution to this system of $(M+1)(3M+1)$ equations in $(M+1)(3M+1)$ unknowns, $\boldsymbol{z} \in \mathbb{R}^{(M+1)(3M+1)}$, is a symmetric equilibrium in pure strategies (for a given value of $\lambda \in [0,1]$). The equilibrium investment decision $x(\boldsymbol{\omega})$ in state $\boldsymbol{\omega}$ is recovered by substituting the equilibrium slack variable $\zeta(\boldsymbol{\omega})$ into definition (18).

Parameterization The baseline parameterization is presented in Table 1. Unless stated otherwise, we set parameters equal to these values. This parameterization is identical to the baseline parameterization in Pakes and McGuire (1994) except that we assume higher setup costs and scrap values. The reason is that we are interested in studying an industry that can support up to two active firms, while Pakes and McGuire (1994) study an industry that can support up to six active firms. We compute equilibria for a wide range of parameterizations

Table 1 Parameter values

Parameter	M	m	c	ω^*	β	α	δ	$\bar{\phi}$	$\bar{\phi}^e$	ϵ	ω^e
Value	18	5	5	12	0.925	3	0.7	1	3	1	4

by allowing the homotopy algorithm to vary several parameters of the model (see Eq. (14)).

Code A set of code that allows the user to compute equilibria of the quality ladder model is available on the authors' homepages. It includes (i) Matlab code that implements the Pakes and McGuire (1994) algorithm that we use to compute an initial condition for the homotopy algorithm; (ii) Fortran90 code that includes HOMPACK90 and the implementation of the quality ladder model; and (iii) additional Matlab code that analyzes the output of the homotopy algorithm. More detailed information is included within the code itself.

4 Equilibrium behavior and industry dynamics

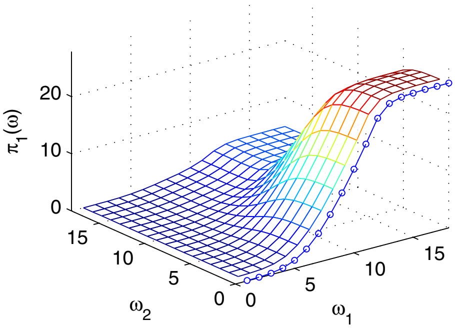
Equilibrium behavior is driven by the benefits and costs of product quality. The benefits of product quality stem from the product market; a higher product quality yields a higher price and a higher market share and, accordingly, higher profits. We begin by examining firm 1's profit function $\pi_1(\boldsymbol{\omega})$ in Fig. 2 more closely.[8] The profit function of an incumbent monopolist is plotted over the subset of the state space $\{1, \ldots, 18\} \times \{0\}$ and that of an incumbent duopolist is graphed over $\{1, \ldots, 18\} \times \{1, \ldots, 18\}$. If an incumbent duopolist has a higher (lower) quality product than its rival, we refer to it as the *leader (follower)*. In Fig. 2, the profit function is relatively flat for the follower ($\omega_1 < \omega_2$) and relatively steep for the leader ($\omega_1 > \omega_2$); hence, a follower can increase its profit relatively little by increasing its product quality, but the leader can increase its profit significantly. This is because according to the demand function (2), an increase in the leader's product quality enhances its demand (until decreasing returns to quality set in at $\omega^* = 12$) more than an increase in the follower's product quality enhances its demand.

Product quality is costly in the sense that an incumbent firm must invest in order to maintain or enhance it. One parameter that affects this cost is the rate of depreciation. As δ increases, an incumbent firm needs to invest more in order to offset the higher rate at which its product quality decreases.

To see how the benefit and cost of product quality affect equilibrium behavior, we present equilibria for $\delta \in \{0.3, 0.5, 0.6., 0.7\}$. The equilibrium investment and entry/exit policy functions are graphed in Fig. 3. The investment and exit policy functions of an incumbent monopolist are graphed over the subset of the state space $\{1, \ldots, 18\} \times \{0\}$ and those of an incumbent duopolist are graphed over $\{1, \ldots, 18\} \times \{1, \ldots, 18\}$. The entry policy function of a potential entrant facing an incumbent monopolist is graphed over the subset of the state space $\{0\} \times \{1, \ldots, 18\}$ and that of a potential entrant facing an empty industry is graphed over state $(0, 0)$.

[8] As firms are symmetric, $\pi_2(\boldsymbol{\omega})=\pi_1(\boldsymbol{\omega}^{[2]})$.

Fig. 2 Profit function $\pi_1(\omega)$. (○ = incumbent monopolist)



For $\delta = 0.3$, a follower always invests and never exits. For $\delta \in \{0.5, 0.6, 0.7\}$, a follower that falls sufficiently far behind ceases to invest and exits with positive probability. A follower in the subset of the state space that lies along the ω_2 axis has little incentive to invest because the profit function is quite flat (see Fig. 2). Furthermore, the follower determines that it is too costly to invest in *catching up* with the leader. Not surprisingly, the higher the rate of depreciation, the smaller the lead required to induce the follower to *give up*. The subset of the state space in which the follower ceases to invest does not necessarily coincide with the subset of the state space (also along the ω_2 axis) in which it exits with positive probability; this depends on the parameterization. However, increasing the rate of depreciation causes both of these subsets to grow as they do in Fig. 3.

The leader exploits these incentives by striving to move the industry into the subset of the state space in which the follower gives up. This can be seen in the policy functions for $\delta \in \{0.6, 0.7\}$; the leader invests heavily in the states adjacent to the subset in which the follower gives up. Once in this subset, the leader best responds to the follower's zero investment and imminent exit by significantly decreasing its investment.

To explore the implications of the equilibrium behavior for the dynamics of the industry, both in the short run and in the long run, we compute the transient distribution over states in period t, $\boldsymbol{\mu}^t(\cdot)$, starting from state $(\omega^e, \omega^e) = (4, 4)$ in period 0. This tells us how likely each possible industry structure is in period t, given that both firms began with the exogenous initial product quality. Figure 4 displays the transient distributions in periods 10 and 1,000. For $\delta = 0.3$, the industry structure is symmetric. For $\delta = 0.5$, even though a follower that falls sufficiently far behind ceases to invest and exits with positive probability, the industry structure is likewise symmetric because it is extremely unlikely that a follower ever falls sufficiently far behind. For $\delta \in \{0.6, 0.7\}$, the industry structure becomes asymmetric; the leader is very likely to induce the follower to give up and thus to ultimately become an incumbent monopolist.

We of course cannot present plots of policy functions and transient distributions for each equilibrium computed in our thorough exploration of the

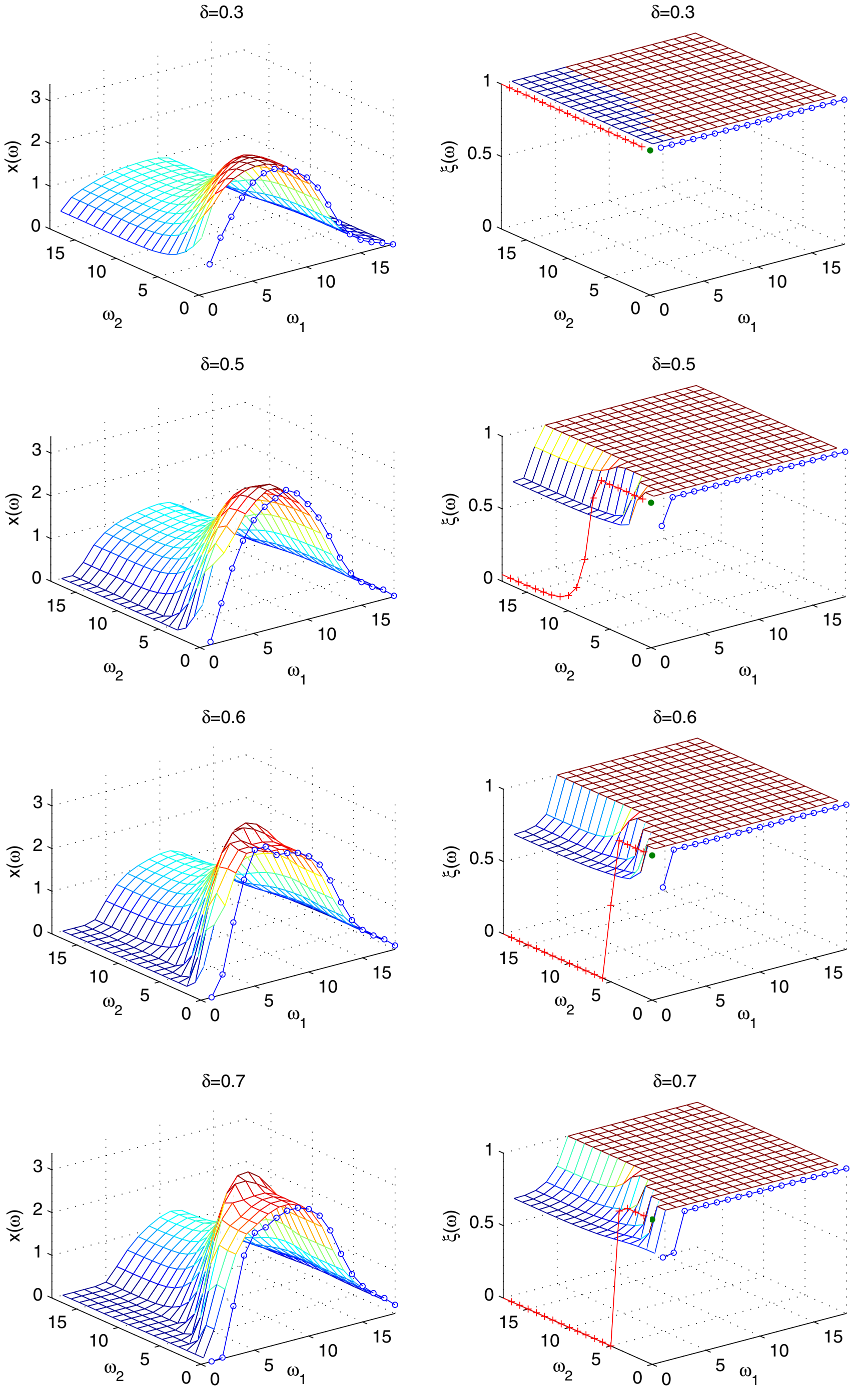


Fig. 3 Policy functions $x(\omega)$ (*left panels*) and $\xi(\omega)$ (*right panels*). (○ = incumbent monopolist; + = one potential entrant; ● = two potential entrants)

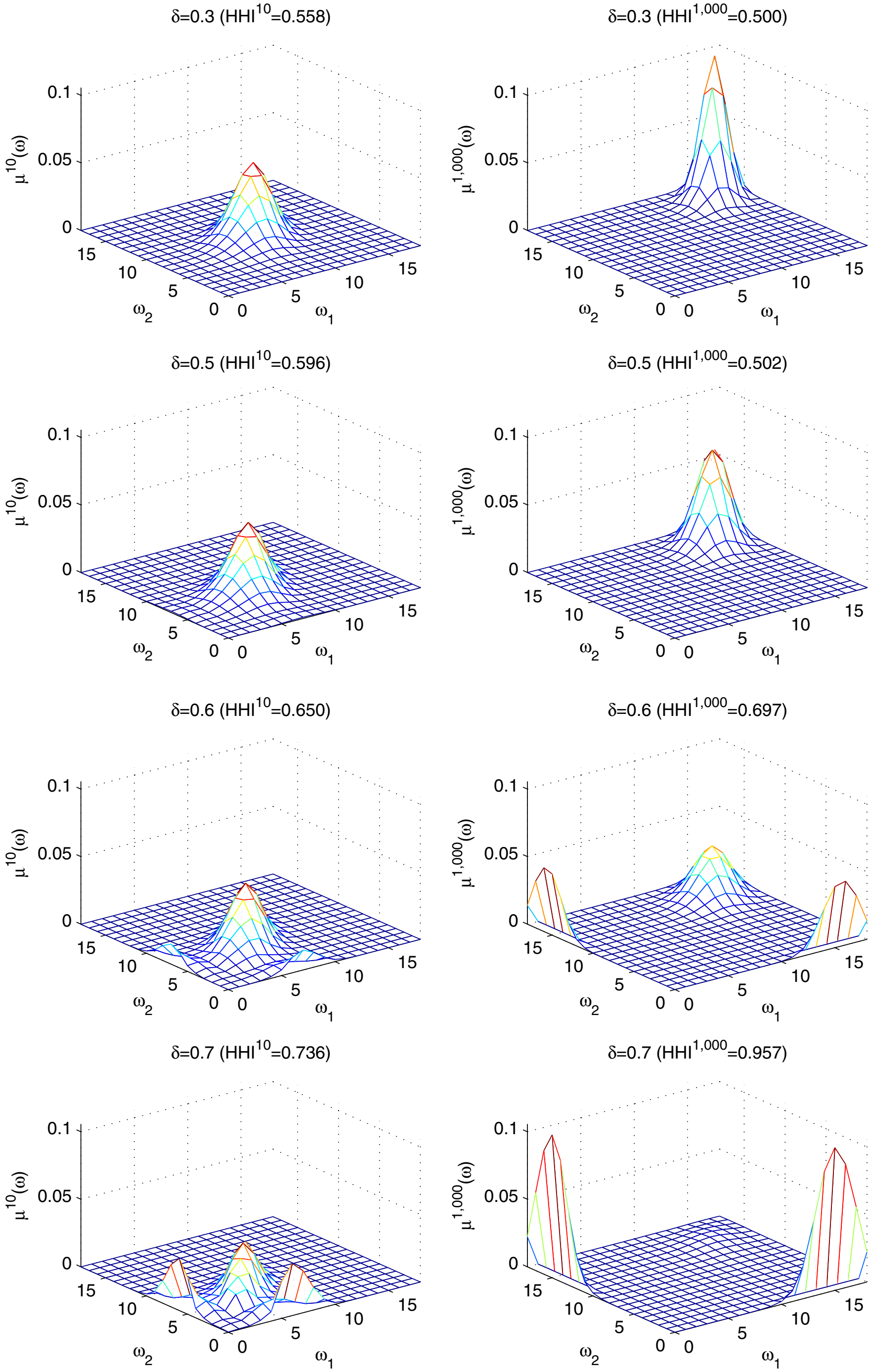


Fig. 4 Transient distributions over states in periods 10 (*left panels*) and 1,000 (*right panels*) given initial state (4,4)

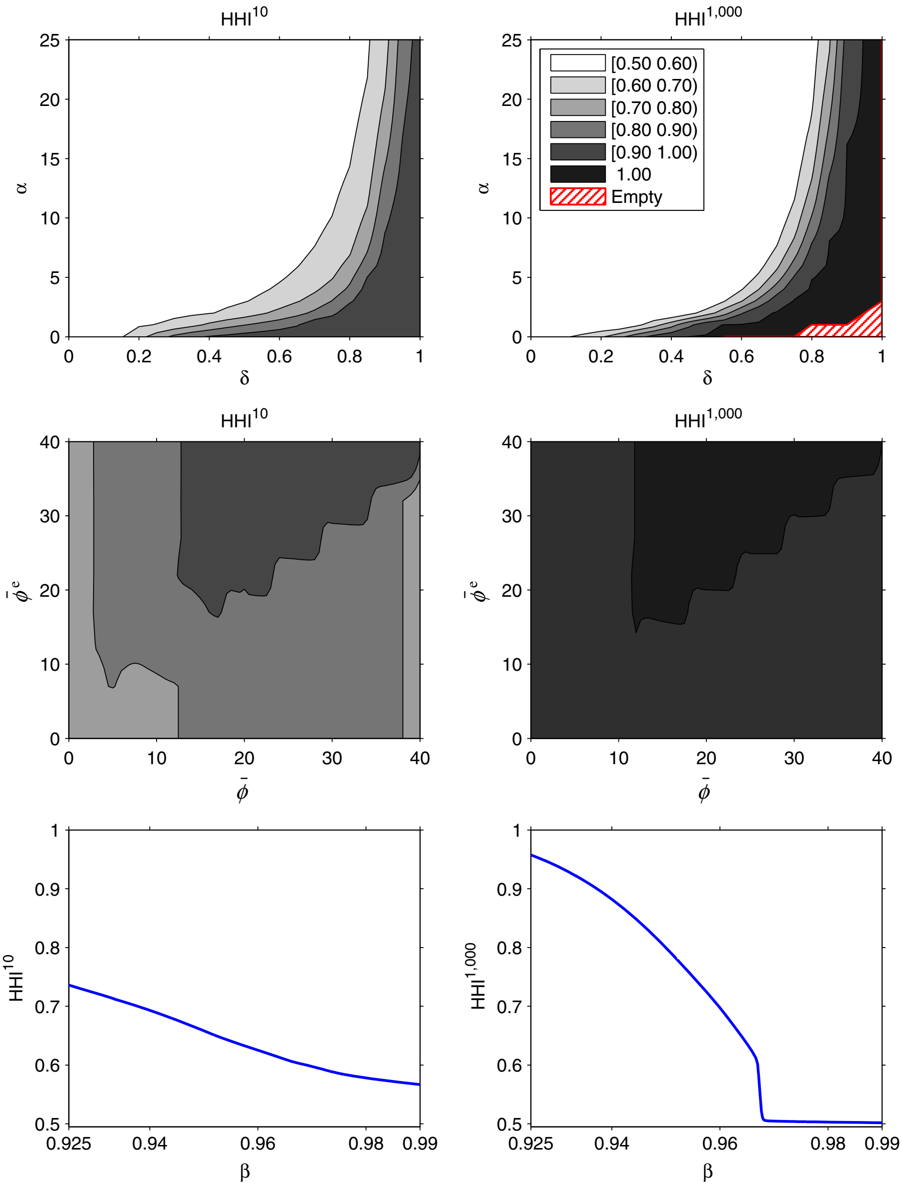


Fig. 5 Expected Herfindahl indexes in periods 10 (*left panels*) and 1,000 (*right panels*)

equilibrium correspondence. We therefore summarize an equilibrium using the expected Herfindahl indexes in period t:

$$HHI^t = \sum_{\boldsymbol{\omega} \in \{0,\ldots,M\}^2/(0,0)} \frac{\left(D_1(\mathbf{p}(\boldsymbol{\omega}); \boldsymbol{\omega})\right)^2 + \left(D_2(\mathbf{p}(\boldsymbol{\omega}); \boldsymbol{\omega})\right)^2}{\left(D_1(\mathbf{p}(\boldsymbol{\omega}); \boldsymbol{\omega}) + D_2(\mathbf{p}(\boldsymbol{\omega}); \boldsymbol{\omega})\right)^2} \frac{\mu^t(\boldsymbol{\omega})}{1 - \mu^t(0,0)}.$$

We condition on the event that the industry is not empty because the Herfindahl index is not defined for an empty industry. A more asymmetric industry structure is reflected by a higher expected Herfindahl index, with $HHI^t = 0.5$ corresponding to a symmetric duopoly and $HHI^t = 1$ corresponding to complete monopolization. Figure 5 presents HHI^{10} and $HHI^{1,000}$ for $(\delta, \alpha) \in [0, 1] \times [0, 25]$, $(\bar{\phi}, \bar{\phi}^e) \in [0, 40] \times [0, 40]$, and $\beta \in [0.925, 0.99]$.[9]

Decreasing the effectiveness of investment or the discount factor causes equilibrium behavior and industry dynamics to change in a similar way as an increase in the rate of depreciation. This is reflected in the top and bottom panels of Fig. 5. More generally, a change that increases (decreases) the cost (benefit) of achieving or maintaining any given product quality yields a more asymmetric industry structure in the short and long run. Finally, in the top-right panel, we see that for a sufficiently high rate of depreciation and a sufficiently low effectiveness of investment, the industry is empty in the long run simply because it is too costly to invest in quality.

In the middle panels of Fig. 5, we see that the expected Herfindahl indexes are increasing in the setup cost. When $\bar{\phi} > \bar{\phi}^e - 2$ (roughly below the diagonal), a potential entrant enters primarily in order to exit soon after and collect a scrap value that exceeds the setup cost. This *opportunistic entry* makes the industry structure more symmetric because a leader that would otherwise be an incumbent monopolist instead faces a low quality follower. We also see that as the scrap value increases, the expected Herfindahl indexes first increase and then decrease. The increase occurs because a higher scrap value makes exit more attractive. The decrease occurs because of opportunistic entry.

5 Predatory and limit investment

In this section, we explore the effects of entry and exit on equilibrium behavior in more detail. In particular, we discuss predatory and limit investment. Predatory and limit investment are most pronounced when an incumbent firm has an incentive to induce exit and prevent entry, respectively. These behaviors are less apparent (but present) in the equilibria presented in the previous section because a follower that falls sufficiently far behind is priced out of the market. We can see this by comparing the profit function of a monopolist to the profit function of a duopolist facing a rival in state 1; the maximum absolute difference between the functions is 0.028 (in states 13–18), and the maximum relative difference is 0.66% (in state 1). To an incumbent firm it makes little difference whether it faces a potential entrant or an incumbent firm with a very

[9] For parameterizations with multiple equilibria, we average the expected Herfindahl index across the equilibria. As discussed further in Section 6, the multiple equilibria have virtually identical expected Herfindahl indexes.

low quality product, thereby dulling the incentives to induce exit and prevent entry.

To get an unobstructed view of predatory and limit investment, we ensure that the follower is not priced out of the market by increasing the vertical intercept and decreasing the slope of the function that maps product quality into the consumer's valuation of it. To this end, we replace $g(\cdot)$ as defined in (1) with

$$g(\omega_n) = \begin{cases} -\infty & \text{if} \quad \omega_n = 0 \\ 6 + \frac{1}{2}\omega_n & \text{if} \quad 1 \leq \omega_n \leq \omega^*, \\ 6 + \frac{1}{2}\omega^* + \ln(2 - \exp(\omega^* - \omega_n)) & \text{if} \quad \omega^* < \omega_n \leq M. \end{cases}$$

Now, the difference between the profit function of a monopolist and the profit function of a duopolist facing a rival in state 1 varies from 1.331 (in state 1) to 3.701 (in state 18) in absolute terms and from 17.10% (in state 18) to 53.25% (in state 1) in relative terms. It follows that an incumbent firm has a very strong incentive to become a monopolist as opposed to a duopolist, no matter how dominant a duopolist it can be. Having increased the follower's market share, we must also increase the setup costs and scrap values; otherwise, incumbent firms never exit and potential entrants always enter.

Predatory investment We study predatory investment by comparing firms' policies in two scenarios: in the baseline scenario, the scrap value is moderate so that exit is possible but not certain ($\bar{\phi} = 20$); the counterfactual scenario differs from the baseline scenario only in that exit never occurs ($\bar{\phi} = -\infty$). To generate a clean example of predatory investment, we set the setup cost to be high enough so that entry never occurs ($\bar{\phi}^e = \infty$) in both scenarios. Figure 6 presents the policy functions $\boldsymbol{x}(\cdot)$ and $\boldsymbol{\xi}(\cdot)$ for the baseline scenario in the top panels and the policy functions $\boldsymbol{x}^{\mathrm{CFP}}(\cdot)$ and $\boldsymbol{\xi}^{\mathrm{CFP}}(\cdot)$ for the counterfactual scenario in the middle panels. By comparing the investment policy functions, we see how the opportunity to induce exit and become a perpetual monopolist affects the leader's incentives.

The difference between investment policy functions in the bottom panel of Fig. 6 exhibits a pronounced ridge that is adjacent to the subset of the state space in which the follower ceases investing and exits with positive probability. Hence, in the baseline scenario, the leader invests significantly more than in the counterfactual scenario once it gains a small lead and is in a position to induce the follower to give up. This additional investment may be considered predatory. Ordover and Willig (1981) and Cabral and Riordan (1997), for example, define an action as predatory if it is optimal when taking into consideration its effect on the likelihood that a rival exits, but suboptimal otherwise.

We devise a predatory investment summary statistic that, for a given equilibrium, reflects the expected net present value of the additional investment undertaken by incumbent duopolists because of the opportunity to induce

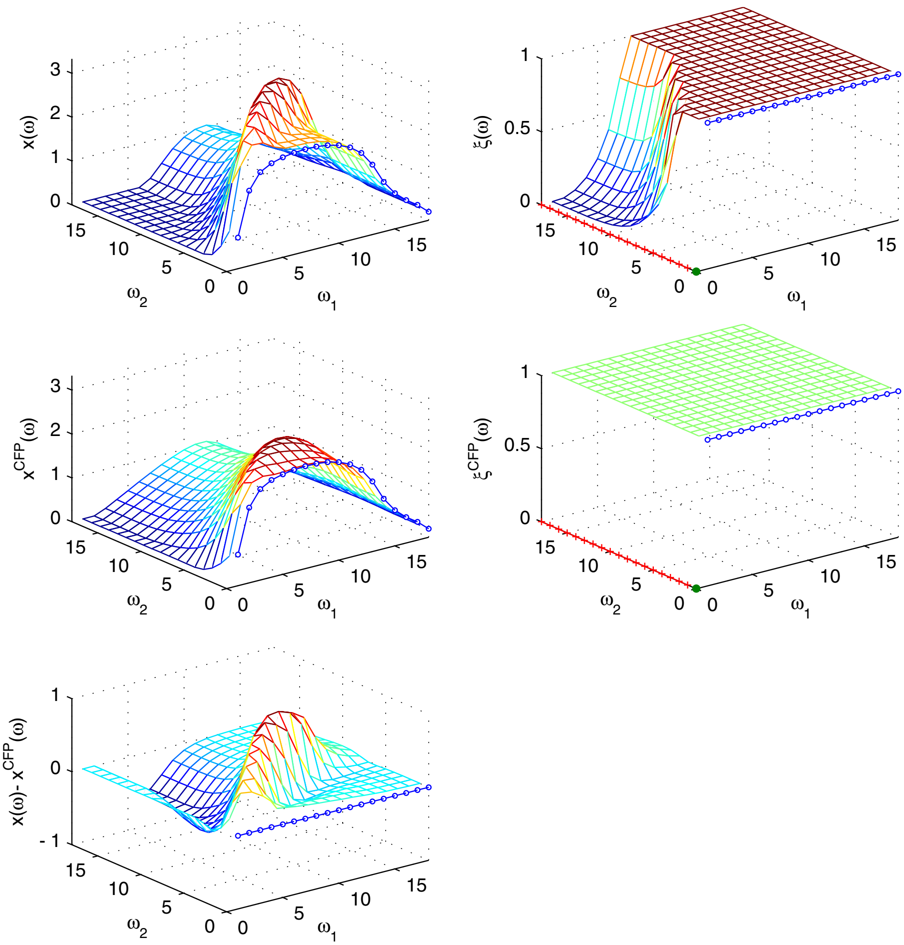


Fig. 6 Predatory investment. Baseline scenario policy functions $\boldsymbol{x}(\cdot)$ and $\boldsymbol{\xi}(\cdot)$ for $\bar{\phi} = 20$ and $\bar{\phi}^e = \infty$ (*top panels*). Counterfactual scenario policy functions $\boldsymbol{x}^{\text{CFP}}(\cdot)$ and $\boldsymbol{\xi}^{\text{CFP}}(\cdot)$ for $\bar{\phi} = -\infty$ and $\bar{\phi}^e = \infty$ (*middle panels*). Difference between investment policy functions (*bottom panel*). $PI = 4.023$

exit over some long time horizon (in practice, 50 periods). Our predatory investment summary statistic is

$$PI = \sum_{t=0}^{49} \sum_{\boldsymbol{\omega} \in \{1,\ldots,M\}^2} \beta^t \mu_{PI}^t(\boldsymbol{\omega})(d_{PI}(\boldsymbol{\omega}) + d_{PI}(\boldsymbol{\omega}^{[2]})),$$

where $d_{PI}(\boldsymbol{\omega}) = \max\{\xi(\boldsymbol{\omega})x(\boldsymbol{\omega}) - \xi^{\text{CFP}}(\boldsymbol{\omega})x^{\text{CFP}}(\boldsymbol{\omega}), 0\}$ is the expected additional investment undertaken by incumbent firm 1 in state $\boldsymbol{\omega}$ because of the opportunity to induce exit.[10] First, as investment is undertaken conditional on remaining active, incumbent firm 1's expected investment in duopolistic state $\boldsymbol{\omega}$ is $\xi(\boldsymbol{\omega})x(\boldsymbol{\omega})$ in the baseline scenario and $x^{\mathrm{CFP}}(\boldsymbol{\omega})$ in the counterfactual scenario because $\xi^{\mathrm{CFP}}(\boldsymbol{\omega}) = 1$ for $\boldsymbol{\omega} \in \{1, \ldots, M\}^2$. Second, in Fig. 6, in the subset of states in which the follower gives up in the baseline scenario, the follower invests *less* in the baseline scenario than in the counterfactual scenario. The max operator in the definition of $d_{PI}(\boldsymbol{\omega})$ excludes such differences, which cannot be attributed to predation. Third, we start the computation of the transient distributions $\boldsymbol{\mu}^t_{PI}(\cdot)$ at a distribution over the subset of states in which predatory investment occurs. In particular, we set the probability that the industry is in state $\boldsymbol{\omega}$ in period 0 to be

$$\mu^0_{PI}(\boldsymbol{\omega}) = \begin{cases} \dfrac{d_{PI}(\boldsymbol{\omega}) + d_{PI}(\boldsymbol{\omega}^{[2]})}{\sum_{\boldsymbol{\omega}\in\{1,\ldots,M\}^2}(d_{PI}(\boldsymbol{\omega}) + d_{PI}(\boldsymbol{\omega}^{[2]}))} & \text{if } \boldsymbol{\omega} \in \{1, \ldots, M\}^2, \\ 0 & \text{if } \boldsymbol{\omega} \notin \{1, \ldots, M\}^2. \end{cases}$$

if $d(\boldsymbol{\omega}) \neq 0$ for some $\boldsymbol{\omega} \in \{1, \ldots, M\}^2$. Otherwise, we set $PI = 0$.

The left panels of Fig. 8 present PI for $(\delta, \alpha) \in [0, 1] \times [0, 25]$, $(\bar{\phi}, \bar{\phi}^e) \in [-5, 300] \times [-2, 120]$, and $\beta \in [0.925, 0.99]$, holding all other parameters fixed at the values of the baseline parameterization explored in Fig. 6. There is a wide range of parameterizations at which firms engage in extensive predatory investment—at least as much as in the example in Fig. 6. The extent of predation is determined by the ease with which the leader is able to induce the follower to exit. If the leader can induce the follower to exit only by achieving a very large lead, then it engages in relatively little predation because once it achieves a large lead, its dominance of the industry is likely whether it predates or not. However, if the leader can induce the follower to exit by achieving a small lead, then it engages in predation to a greater extent, in order to achieve the industry dominance that is not yet assured.

In the top panel of Fig. 8, we see that the extent of predation is increasing in the rate of depreciation and decreasing in the effectiveness of investment because such changes increase the cost of quality and therefore make it easier for the leader to induce the follower to exit. However, for a sufficiently high rate of depreciation, the extent of predation decreases because even the leader's investment incentives become very weak. The middle panel shows that there is a non-monotonic relationship between the scrap value and the extent of predation, and that the setup cost has little influence over predation. An increase in the scrap value leads to increased predation because it increases the benefit of exiting the industry. However, beyond a certain threshold (approximately $\bar{\phi} = 53$), increasing the scrap value reduces the extent of predation because it induces not only the follower but also the leader to exit the industry. This weakens the leaders incentive to invest and, accordingly, to engage in predatory investment. The bottom panel shows that the extent of

[10] If there are multiple baseline equilibria and/or multiple counterfactual equilibria for a given parameterization, we average over all possible pairs of baseline and counterfactual equilibria.

predation is relatively constant but decreasing slowly in the discount factor up to a certain threshold (approximately $\beta = 0.975$) beyond which there is no predation whatsoever. Increasing the discount factor increases the value of remaining active and therefore reduces the follower's incentive to exit. For a sufficiently high discount factor, the follower never exits—irrespective of how far behind it falls—and therefore the leader has no incentive to predate.

Limit investment We analogously study limit investment by comparing firms' policies in two scenarios: in the baseline scenario, the setup cost is moderate so that entry is possible but not certain ($\bar{\phi}^e = 22$); the counterfactual scenario differs from the baseline scenario only in that entry never occurs ($\bar{\phi}^e = \infty$). To generate a clean example of limit investment, we set the scrap value to be low enough so that exit never occurs ($\bar{\phi} = -\infty$) in both scenarios. By comparing these scenarios, we see how the opportunity to prevent entry and thus prevent the industry from becoming a perpetual duopoly affects an incumbent monopolist's incentives.

Figure 7 presents the policy functions $\boldsymbol{x}(\cdot)$ and $\boldsymbol{\xi}(\cdot)$ for the baseline scenario in the top panels and the policy functions $\boldsymbol{x}^{\text{CFL}}(\cdot)$ and $\boldsymbol{\xi}^{\text{CFL}}(\cdot)$ for the counterfactual scenario in the middle panels. In state $(\omega_1, 0)$, firm 2 is the potential entrant and via symmetry its behavior is identical to that of firm 1 in state $(0, \omega_1)$. In Fig. 7, we therefore graph policy functions in states $(\omega_1, 0)$ and $(0, \omega_1)$, for $\omega_1 \in \{1, \ldots, 18\}$. The left panels depict the investment policy function of an incumbent monopolist and the right panels depict the entry probability of a potential entrant. In the baseline scenario (top panels), entry occurs if the incumbent monopolist's quality is sufficiently low ($\omega_2 \leq 5$). Limit investment can be seen at $\omega_1 = 5$ and in neighboring states, where the incumbent monopolist significantly increases its investment relative to the counterfactual scenario (middle panels), realizing that an increase in its quality can prevent entry.

We devise a limit investment summary statistic that is an analogue of the predatory investment summary statistic. For a given equilibrium, it reflects the expected net present value of the additional investment undertaken by an incumbent monopolist because of the opportunity to prevent entry over some long time horizon:

$$LI = \sum_{t=0}^{49} \sum_{\boldsymbol{\omega} \in \{1,\ldots,M\} \times \{0\}} \beta^t \mu_{LI}^t(\boldsymbol{\omega}) d_{LI}(\boldsymbol{\omega}),$$

where $d_{LI}(\boldsymbol{\omega}) = \max\{\xi(\boldsymbol{\omega})x(\boldsymbol{\omega}) - \xi^{\text{CFL}}(\boldsymbol{\omega})x^{\text{CFL}}(\boldsymbol{\omega}), 0\}$, and

$$\mu_{LI}^0(\boldsymbol{\omega}) = \begin{cases} \dfrac{d_{LI}(\boldsymbol{\omega})}{\sum_{\boldsymbol{\omega} \in \{1,\ldots,M\} \times \{0\}} d_{LI}(\boldsymbol{\omega})} & \text{if } \boldsymbol{\omega} \in \{1, \ldots, M\} \times \{0\}, \\ 0 & \text{if } \boldsymbol{\omega} \notin \{1, \ldots, M\} \times \{0\}, \end{cases}$$

if $d(\boldsymbol{\omega}) \neq 0$ for some $\boldsymbol{\omega} \in \{1, \ldots, M\} \times \{0\}$; otherwise, we set $LI = 0$. For all $\boldsymbol{\omega} \in \{1, \ldots, M\} \times 0$, $\xi(\boldsymbol{\omega}) = 1$ and $\xi^{\text{CFL}}(\boldsymbol{\omega}) = 1$.

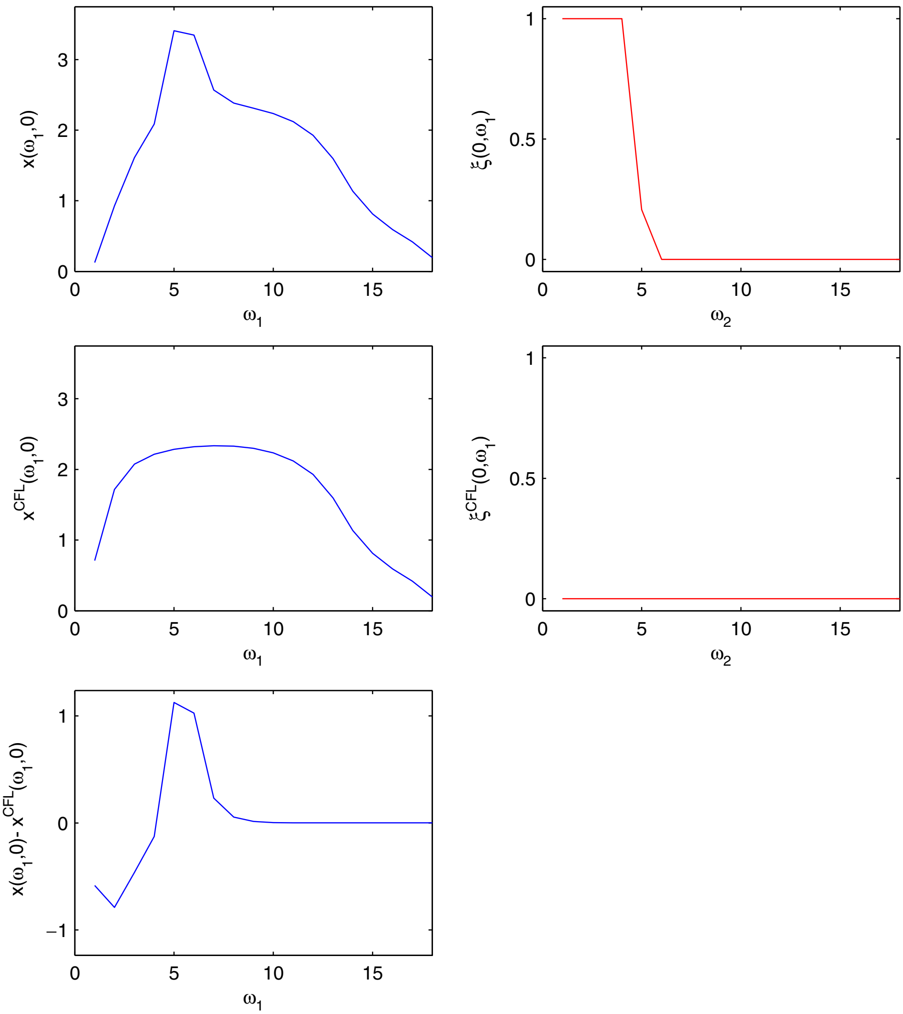


Fig. 7 Limit investment. Baseline scenario policy functions $\boldsymbol{x}(\cdot, 0)$ and $\boldsymbol{\xi}(0, \cdot)$ for $\bar{\phi} = -\infty$ and $\bar{\phi}^e = 22$ (*top panels*). Counterfactual scenario policy functions $\boldsymbol{x}^{\mathrm{CFL}}(\cdot, 0)$ and $\boldsymbol{\xi}^{\mathrm{CFL}}(0, \cdot)$ for $\bar{\phi} = -\infty$ and $\bar{\phi}^e = \infty$ (*middle panels*). Difference between investment policy functions (*bottom panel*). $LI = 3.841$

The right panels of Fig. 8 present LI for the same subsets of the parameter space for which we compute PI, holding all other parameters fixed at the values of the baseline parameterization explored in Fig. 7. Firms engage in extensive limit investment for a wide range of parameterizations. In the top panel, we see that as the rate of depreciation (effectiveness of investment) increases (decreases), it becomes more difficult for the monopolist to sustain its quality and it compensates by engaging in more limit investment. In the middle panel, there is little to no limit investment in a large subset of the

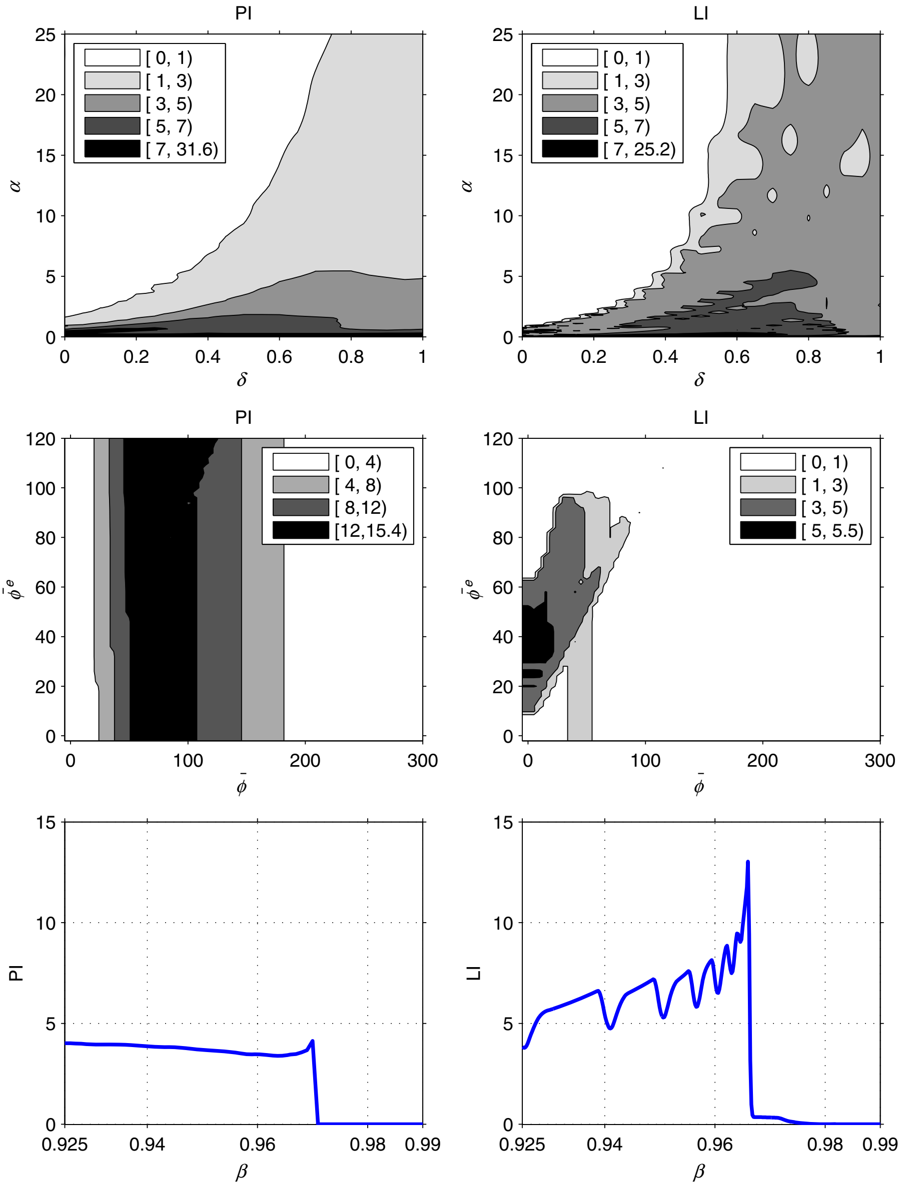


Fig. 8 Predatory investment (*left panels*) and limit investment (*right panels*) summary statistics

parameter space because, in this subset, the potential entrant either never enters or always enters. Moreover, we find that the extent of limit investment tends to decrease in the scrap value because as the scrap value increases, the probability that the monopolist exits in the imminent future increases and this weakens its investment incentives. The bottom panel shows that the extent of limit investment is increasing in the discount factor. As the discount factor increases, the expected net present value of cash flows that would accrue to the monopolist if it sustained its monopoly increases. This induces the monopolist to engage in more limit investment. However, for a sufficiently high discount factor, the potential entrant enters with certainty because the value of being an incumbent becomes sufficiently high. It follows that the monopolist cannot prevent entry and therefore does not engage in any limit investment whatsoever.[11]

6 Multiple equilibria

Pakes and McGuire (1994) do not find multiple equilibria of the quality ladder model; on the basis of this, they reason that the model does not admit multiple equilibria (p. 570). However, in systematically exploring the equilibrium correspondence using the homotopy method, we have uncovered several instances of multiplicity.

Figure 9 shows the number of equilibria that we have identified for $(\delta, \alpha) \in [0, 1] \times [0, 25]$ and $(\bar{\phi}, \bar{\phi}^e) \in [0, 40] \times [0, 40]$.[12] As can be seen, we have found a small region of the parameter space in which there are up to nine equilibria and several regions in which there are three equilibria. The multiplicity of equilibria arises as the homotopy algorithm traces out S-shaped paths—just as in the example in Fig. 1.

We explore the three equilibria that arise at $\bar{\phi} = 20$ and $\bar{\phi}^e = 22$ in more detail. We do not plot the policy functions for these equilibria because they are qualitatively similar to those presented in the bottom panels of Fig. 3. Table 2 presents the incumbent firm's probability of remaining active and the potential entrant's entry probability for state $(3, 0)$, where the differences between equilibria are most prominent. In state (3,0), firm 2 is the potential entrant and via symmetry its behavior is identical to that of firm 1 in state (0,3). Table 2 therefore presents $\xi(\boldsymbol{\omega})$ for both of these states. We see that a higher probability of the incumbent firm remaining in the industry is matched by a lower probability of the potential entrant entering, and vice versa.

All three equilibria lead to the same asymmetric (monopolistic) long-run industry structure; for each, the modal states are $(15, 0)$ and $(0, 15)$. This is not surprising, as we have already seen that a qualitatively similar equilibrium yields a very asymmetric long-run industry structure (see the bottom panels of Fig. 4). However, due to the differences between the policy functions near the origin and along the diagonal of the state space, differences in the short-run industry structures do arise, as demonstrated by the summary statistics in

[11] The slight non-monotonicities in the right panels of Fig. 8 arise because as we move through the parameter space, we move from equilibria where limit investment is concentrated in one state to equilibria where it is spread out over a small subset of states, as in Fig. 7. The latter type of equilibrium yields a lower limit investment summary statistic.

[12] We have not found any multiplicity of equilibria for $\beta \in [0.925, 0.99]$.

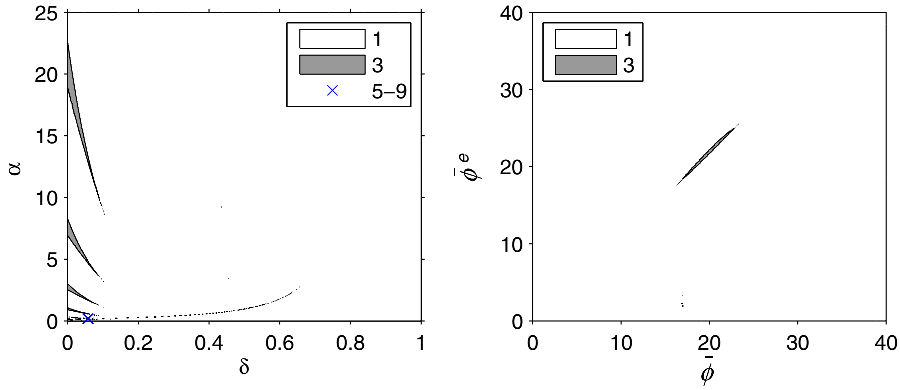


Fig. 9 Number of equilibria

Table 2. For each of the three equilibria, Table 2 presents the expected number of entering, exiting, and active firms in period 10, given that the industry starts from state $(\omega^e, \omega^e) = (4, 4)$ in period 0. While there is relatively little variation in the expected number of active firms across the equilibria, there is greater variation in the expected number of entering and exiting firms; both are highest for equilibrium A and lowest for equilibrium C. That is, there is a variation in *churn* across the equilibria.

In a companion paper (Borkovsky et al. 2010), we use the homotopy method to explore a quality ladder model that does not allow for entry and exit. Interestingly, in the model that allows for entry and exit, multiple equilibria arise for parameterizations for which we did not find multiple equilibria in the model that does not allow for entry and exit. This suggests that entry and exit may be a source of multiplicity in the Ericson and Pakes (1995) framework. In particular, in the left panel of Fig. 9, the regions of multiplicity at $\delta < 0.11$ and $\alpha \geq 1$ coincide with the regions of multiplicity that we find in Borkovsky et al. (2010). These equilibria are qualitatively similar to those we find in Borkovsky et al. (2010)—which provides an example—in that incumbent firms never exit and exhibit similar investment behavior. At $\delta \geq 0.11$, we find regions of multiplicity only for the model that allows for entry and exit. The equilibria in these regions are qualitatively similar to those in the example discussed

Table 2 Incumbent firm's probability of remaining active and potential entrant's entry probability in state (3, 0), and summary statistics for period 10 given initial state (4, 4)

	MPE A	MPE B	MPE C
$\xi(3, 0)$	0.030	0.694	1.000
$\xi(0, 3)$	0.975	0.403	0.103
Expected # of entering firms	0.055	0.034	0.016
Expected # of exiting firms	0.068	0.047	0.030
Expected # of active firms	1.126	1.142	1.138

above and differ from those at $\delta < 0.11$ and $\alpha \geq 1$ in that incumbent firms exit with positive probability.

In sum, while we find multiplicity in the Pakes and McGuire (1994) quality ladder model, it is hardly as dramatic as in other models (Besanko et al. 2010b, c); the differences between equilibria tend to be small and may matter little in practice.

7 Concluding remarks

We conduct the first comprehensive exploration of the equilibrium correspondence of the Pakes and McGuire (1994) quality ladder model. We uncover a variety of interesting economic phenomena.

We find that the industry structure that arises is determined by the cost and benefit of achieving or maintaining any given quality level. The more costly and/or less beneficial it is to achieve or maintain a given quality level, the more a leader invests in striving to induce the follower to give up; the more quickly the follower does so; and the more asymmetric is the industry structure that arises.

We also find that equilibria in the Pakes and McGuire (1994) model are often characterized by predatory and limit investment. As this model is a relatively straightforward application of the Ericson and Pakes (1995) framework, it is likely that such behaviors arise is other models in this framework as well. Besanko et al. (2010a) provide a detailed discussion of predatory pricing in a dynamic stochastic game.

Exploring the equilibrium correspondence using the homotopy method allows us to systematically search for multiple equilibria. We find several instances of multiplicity. Furthermore, we find multiple equilibria for parameterizations of the model for which we did not find multiple equilibria in the model that does not allow for entry and exit, suggesting that entry and exit can be a source of multiplicity in the Ericson and Pakes (1995) framework.

Besides systematically exploring the equilibrium correspondence, the homotopy method has other uses (see Section 3.2). First, all-solutions homotopies can be used to compute all equilibria of games (Sommese and Wampler 2005). Both all-solutions homotopies and artificial homotopies have been applied to static games (see Herings and Peeters 2010 for a survey) and may be useful for dynamic games as well. Second, the homotopy method may be useful for structural estimation; if all equilibria of a model can be computed, then one can estimate an equilibrium selection rule along with the primitives of the model (Bajari et al. 2008, 2010; Grieco 2011). Moreover, using the homotopy method, one can bound the range of outcomes that may occur after a policy intervention. Finally, Doraszelski and Escobar (2010) show that in dynamic stochastic games with a finite number of states and actions, the homotopy method can be used to single out the equilibrium that is likely to be played after a policy intervention. Hence, even if computing all equilibria of a dynamic stochastic game proves difficult, the homotopy method can be useful in conducting policy experiments.

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References

Aguirregabiria, V., & Mira, P. (2007). Sequential estimation of dynamic discrete games. *Econometrica, 75*(1), 1–54.
Bajari, P., Benkard, L., & Levin, J. (2007). Estimating dynamic models of imperfect competition. *Econometrica, 75*(5), 1331–1370.
Bajari, P., Hahn, J., Hong, H., & Ridder, G. (2008). A note on semiparametric estimation of finite mixtures of discrete choice models. Forthcoming *International Economic Review*.
Bajari, P., Hong, H., & Ryan, S. (2010). Identification and estimation of discrete games of complete information. *Econometrica, 78*(5), 1529–1568.
Besanko, D., & Doraszelski, U. (2004). Capacity dynamics and endogenous asymmetries in firm size. *Rand Journal of Economics, 35*(1), 23–49.
Besanko, D., Doraszelski, U., & Kryukov, Y. (2010a). The economics of predation: What drives pricing when there is learning-by-doing? Working paper, Northwestern University, Evanston.
Besanko, D., Doraszelski, U., Kryukov, Y., & Satterthwaite, M. (2010b). Learning-by-doing, organizational forgetting, and industry dynamics. *Econometrica, 78*(2),453–508.
Besanko, D., Doraszelski, U., Lu, L., & Satterthwaite, M. (2010c). Lumpy capacity investment and disinvestment dynamics. *Operations Research, 58*(4), 1178–1193.
Bischof, C., Khademi, P., Mauer, A., & Carle, A. (1996). ADIFOR 2.0: Automatic differentiation of Fortran 77 programs. *IEEE Computational Science and Engineering, 3*(3), 18–32.
Borkovsky, R. (2010). *The timing of version releases in R&D-intensive industries: A dynamic duopoly model*. Working paper, University of Toronto, Toronto.
Borkovsky, R., Doraszelski, U., & Kryukov, S. (2010). A user's guide to solving dynamic stochastic games using the homotopy method. *Operations Research, 58*(4), 1116–1132.
Cabral, L., & Riordan, M. (1997). The learning curve, predation, antitrust, and welfare. *Journal of Industrial Economics, 45*(2), 155–169.
Caplin, A., & Nalebuff, B. (1991). Aggregation and imperfect competition: On the existence of equilibrium. *Econometrica, 59*(1), 26–59.
Chen, J., Doraszelski, U., & Harrington, J. (2009). Avoiding market dominance: Product compatibility in markets with network effects. *Rand Journal of Economics, 49*(3), 455–485.
Doraszelski, U. and Escobar, J. (2010). A theory of regular markov perfect equilibria in dynamic stochastic games: Genericity, stability, and purification. *Theoretical Economics, 5*, 369–402.
Doraszelski, U., & Judd, K. (2011). Avoiding the curse of dimensionality in dynamic stochastic games. Forthcoming *Quantitative Economics*.
Doraszelski, U., & Markovich, S. (2007). Advertising dynamics and competitive advantage. *Rand Journal of Economics, 38*(3), 1–36.
Doraszelski, U., & Satterthwaite, M. (2010). Computable Markov-perfect industry dynamics. *Rand Journal of Economics, 41*(2), 215–243.
Dubé, J., Hitsch, G., & Manchanda, P. (2005). An empirical model of advertising dynamics. *Quantitative Marketing and Economics, 3*, 107–144.

Ericson, R., & Pakes, A. (1995). Markov-perfect industry dynamics: A framework for empirical work. *Review of Economic Studies, 62*, 53–82.
Farias, V., Saure, D., & Weintraub, G. (2010). An approximate dynamic programming approach to solving dynamic oligopoly models. Working Paper, Columbia University, New York.
Ferris, M., Judd, K., & Schmedders, K. (2007). Solving dynamic games with Newton's method. Working Paper, University of Wisconsin, Madison.
Fudenberg, D., & Tirole, J. (1986). A "signal-jamming" theory of predation. *Rand Journal of Economics, 17*(3), 366–376.
Goettler, R., & Gordon, B. (2011). Does AMD spur Intel to innovate more? Working paper, University of Chicago, Chicago.
Gowrisankaran, G. (1999). A dynamic model of endogenous horizontal mergers. *Rand Journal of Economics, 30*(1), 56–83.
Gowrisankaran, G., & Holmes, T. (2004). Mergers and the evolution of industry concentration: Results from the dominant firm model. *Rand Journal of Economics, 35*(3), 561–582.
Grieco, P. (2011). Discrete games with flexible information structures: An application to local grocery markets. Working Paper, Pennsylvania State University, University Park.
Herings, P., & Peeters, R. (2010). Homotopy methods to compute equilibria in game theory. *Economic Theory, 42*(1), 119–156.
Judd, K. (1998). *Numerical methods in economics*. Cambridge: MIT Press.
Laincz, C., & Rodrigues, A. (2008). The impact of cost-reducing spillovers on the ergodic distribution of market structures. Working Paper, Drexel University, Philadelphia.
Langohr, P. (2004). Competitive convergence and divergence: Position and capability dynamics. Working Paper, Bureau of Labor Statistics, Washington, DC.
Markovich, S. (2008). Snowball: A dynamic oligopoly model with network externalities. *Journal of Economic Dynamics and Control, 32*, 909–938.
Markovich, S., & Moenius, J. (2009). Winning while losing: Competition dynamics in the presence of indirect network effects. *International Journal of Industrial Organization, 27*(3), 333–488.
McKelvey, R., McLennan, A., & Turocy, T. (2006). Gambit: Software tools for game theory. Technical Report, California Institute of Technology, Pasadena.
Milgrom, P., & Roberts, J. (1982). Predation, reputation, and entry deterrence. *Journal of Economic Theory, 27*, 280–312.
Narajabad, B., & Watson, R. (2011). The dynamics of innovation and horizontal differentiation. *Journal of Economic Dynamics and Control, 35*, 825–842.
Ordover, J., & Willig, R. (1981). An economic definition of predation: Pricing and product innovation. *Yale Law Journal, 91*, 8–53.
Pakes, A., Gowrisankaran, G., & McGuire, P. (1993). Implementing the Pakes–McGuire algorithm for computing Markov perfect equilibria in Gauss. Working Paper, Yale University, New Haven.
Pakes, A., & McGuire, P. (1994). Computing Markov-perfect Nash equilibria: Numerical implications of a dynamic differentiated product model. *Rand Journal of Economics, 25*(4), 555–589.
Pakes, A., & McGuire, P. (2001). Stochastic algorithms, symmetric Markov perfect equilibrium, and the "curse" of dimensionality. *Econometrica, 69*(5), 1261–1281.
Pakes, A., Ostrovsky, M., & Berry, S. (2007). Simple estimators for the parameters of discrete dynamic games (with entry/exit examples). *Rand Journal of Economics, 38*(2), 373–399.
Pesendorfer, M., & Schmidt-Dengler, P. (2008). Asymptotic least squares estimators for dynamic games. *Review of Economic Studies, 75*(3), 901–928.
Santos, C. (2009). Solving dynamic games by discretizing the state distribution. Working Paper, University of Alicante, Alicante.
Snider, C. (2009). Predatory incentives and predation policy: The American Airlines case. Working Paper, UCLA, Los Angeles.
Sommese, A., & Wampler, C. (2005). *The numerical solution of systems of polynomials arising in engineering and science*. Singapore: World Scientific Publishing.
Song, M. (2010). A dynamic analysis of cooperative research in the semiconductor industry. Forthcoming *International Economic Review*.
Verschelde, J. (1999). Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation. *ACM Transcations on Mathematical Software, 25*(2), 251–276.

Watson, L., Sosonkina, M., Melville, R., Morgan, A., & Walker, H. (1997). Algorithm 777: HOMPACK90: A suite of Fortran 90 codes for globally convergent homotopy algorithms. *ACM Transcations on Mathematical Software, 23*(4), 514–549.

Weintraub, G., Benkard, L., & Van Roy, B. (2010). Computational methods for oblivious equilibrium. *Operations Research, 58*(4), 1247–1265.

Zangwill, W., & Garcia, C. (1981). *Pathways to solutions, fixed points, and equilibria*. Englewood Cliffs: Prentice Hall.