

# Components of Bull and Bear Markets: Bull Corrections and Bear Rallies

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Existing methods of partitioning the market index into bull and bear regimes do not identify market corrections or bear market rallies. In contrast, our probabilistic model of the return distribution allows for rich and heterogeneous intraregime dynamics. We focus on the characteristics and dynamics of bear market rallies and bull market corrections, including, for example, the probability of transition from a bear market rally into a bull market versus back to the primary bear state. A Bayesian estimation approach accounts for parameter and regime uncertainty and provides probability statements regarding future regimes and returns. We show how to compute the predictive density of long-horizon returns and discuss the improvements our model provides over benchmarks. This article has online supplementary materials.

KEY WORDS: Long-horizon returns; Markov switching; Predictive density.

## 1. INTRODUCTION

There is a widespread belief among investors, policy makers, and academics that low-frequency trends do exist in the stock market. Traditionally these positive and negative low-frequency trends have been labeled as bull and bear markets, respectively. We propose a model that provides answers to typical questions such as “Are we in a bull market or a bear market rally?” or “Will this bull market correction become a bear market?” We propose a latent four-state Markov-switching (MS-4) model for weekly stock returns. Our focus is on modeling the component states of bull and bear market regimes to identify and forecast bull, bull correction, bear, and bear rally states.

Traditional methods of identifying bull and bear markets are based on an ex post assessment of the peaks and troughs of the price index. Formal dating algorithms based on a set of rules for classification are found, for example, in Gonzalez et al. (2005), Lunde and Timmermann (2004), and Pagan and Sossounov (2003). Some of this work is related to the dating methods used to identify turning points in the business cycle Bry and Boschan (1971). A drawback is that a turning point can only be identified several observations after it occurs. Ex post dating methods cannot be used for statistical inference on returns or for investment decisions that require more information from the return distribution, such as changing risk assessments. For adequate risk management and investment decisions, we need a probability model for returns and one for which the distribution of returns changes over time.

Stock markets are perceived to have a cyclical pattern that can be captured with regime-switching models. For example, Hamilton and Lin (1996) related business cycles and stock market regimes, and Chauvet and Potter (2000) and Maheu and McCurdy (2000a) used a duration-dependent MS parameterization

to analyze properties of bull and bear market regimes extracted from aggregate stock market returns. Lunde and Timmermann (2004) studied duration dependence after sorting stock returns into either a bull or bear market by using their dating algorithm. Ntantamis (2009) explored potential explanatory variables for stock market regimes' duration. Applications that explore the implications of nonlinearities due to regimes switches for asset allocation and/or predictability of returns include Turner, Startz, and Nelson (1989); van Norden and Schaller (1997); Maheu and McCurdy (2000b); Perez-Quiros and Timmermann (2001); Ang and Bekaert (2002); and Guidolin and Timmermann (2007).

In a related literature that investigated cyclical patterns in a broader class of assets, Guidolin and Timmermann (2005) used a three-state regime-switching model to identify bull and bear markets in monthly U.K. stock and bond returns and to analyze implications for predictability and optimal asset allocation. Guidolin and Timmermann (2006) added an additional state to model the nonlinear joint dynamics of monthly returns associated with small- and large-cap stocks and long-term bonds. Kim, Nelson, and Startz (1998) found that three states in the variance of monthly stock returns provides a good fit.

Higher data frequencies tend to obscure any structure in the conditional mean in favor of the conditional variance dynamics. Our objective is to use higher-frequency weekly data and to provide a real-time approach to identifying phases of the market that relate to investors' perceptions of primary and secondary trends in aggregate stock returns. Existing approaches do not explicitly model bull market corrections and bear market rallies.

The bear and bear rally states govern the bear regime; the bull correction and bull states govern the bull regime. Therefore each regime consists of two primitive states. The model can accommodate short-term reversals (secondary trends) within each regime of the market. For example, in the bull regime, it is possible to have a series of persistent negative returns (a bull correction), despite the fact that the expected long-run return (primary trend) is positive in that regime. Analogously, bear markets often exhibit persistent rallies that are subsequently reversed as investors take the opportunity to sell with the result that the average return in that regime is still negative.

It is important to note that our additional states allow for both intraregime and interregime transitions. A bear rally is allowed to move back to the bear state or to exit the bear regime by moving to a bull state. Likewise, a bull correction can move back to the bull state or exit the bull regime by transitioning to a bear state. A bull regime can be characterized by a combination of bull states and bull corrections. Similarly, a bear regime can consist of several episodes of the bear state and the bear rally state, exactly as many investors feel we observe in the data. Because the realization of states in a regime will differ over time, bull and bear regimes can be heterogeneous over time. These important intraregime and interregime dynamics are absent in the existing literature.

Our Bayesian estimation approach accounts for parameter and regime uncertainty and provides probability statements regarding future regimes and returns. An important contribution of this article is to show how to compute the predictive density for cumulative returns over long horizons. Model comparison based on the density of long-horizon returns is very informative in differentiating models and shows the importance of a richer specification of bull and bear phases.

Applied to 125 years of data, our model provides superior identification of trends in stock prices. One important difference with our specification is that the richer dynamics in each regime, facilitated by our four-state model, allow us to extract bull and bear markets in higher-frequency data. A two-state Markov-switching (MS-2) model applied to higher-frequency data results in too many switches between the high- and low-return states. It is incapable of extracting the low-frequency trends in the market. In high-frequency data, it is important to allow for short-term reversals in the regime of the market. Relative to a two-state model, we find that market regimes are more persistent and there is less erratic switching. According to Bayes factors, our four-state model of bull and bear markets is strongly favored over alternatives, including an unrestricted four-state model.

Bull regimes have an average duration of just less than 5 years, while the duration of a bull correction is 4 months on average and a bear rally is just over half a year. The cumulative return mean of the bull market state is 7.88% but bull corrections offset this by 2.13% on average. Average cumulative return in the bear market state is  $-12.4\%$  but bear market rallies counteract that steep decline by yielding a cumulative return of 7.1% on average. Note that these states are combined into bull and bear market regimes in heterogeneous patterns over time yielding an average cumulative return in the bull market regime of 33% (average duration of 5 years), while that for the bear market

regime is about  $-10\%$  (average duration of 1.5 years). Although the average cumulative return in the bear rally state is not much less than in the bull market state, the latter state's standard deviation is 50% less. This result highlights the importance of also considering assessments of volatility associated with the alternative states, for example, when identifying bear market rallies versus bull markets.

The model identifies in real time a transition from a bull market correction to a bear market in early October 2008. The bear rally and bull correction states are critical to modeling turning points between regimes; our results show that most transitions between bull and bear regimes occur through these states. This is consistent with investors' perceptions. Further, we find asymmetries in intraregime dynamics, for example, a bull market correction returns to the bull market state more often than a bear market rally reverts to the bear state. These are important features that the existing literature on bull and bear markets ignores.

In contrast to many MS applications that estimate a model and then label states, *ex post*, as bull and bear markets, we start with a model that imposes prior restrictions corresponding to practitioner descriptions of the phases of the market, including bull corrections and bear rallies. The article then shows that the states of the market that are identified are realistic and useful, not only from the perspective of describing market dynamics, but also for forecasting long-horizon return densities. Bear rallies and bull corrections have important implications for cumulative returns. Compared with alternatives, including an unrestricted four-state model, our four-state model provides superior density forecasts for long-horizon returns. We also show the importance of the model in a Value-at-Risk (VaR) application.

The next section describes the data and models. The latter summarizes the benchmark two-state model and develops our proposed four-state specification. Estimation and model comparison are discussed in Section 3. Section 4 presents results including parameter estimates, probabilistic identification of the market states and regimes including real-time bear market forecasts, VaR forecasts, and out-of-sample long-horizon density forecasts. Section 5 concludes. The online appendix collects additional results.

## 2. DATA AND MODELS

We begin with 125 years of daily capital gain returns on a broad market equity index. Our source for the period 1926–2008 inclusive is the value-weighted return excluding dividends associated with the CRSP (Center for Research in Security Prices) S&P 500 (Standard & Poor 500) index. The 1885:02-1925 daily capital gain returns are courtesy of Bill Schwert (see Schwert 1990). For 2009–2010, we use the daily rates of change of the S&P 500 index level (SPX) obtained from Reuters.

Returns are converted to daily continuously compounded returns from which we construct weekly continuously compounded returns by summing daily returns from Wednesday close to Wednesday close of the following week. If a Wednesday is missing, we use Tuesday close. If the Tuesday is also missing, we use Thursday. Weekly realized variance (RV) is computed as the sum of daily (intra-week) squared returns.

Table 1. Weekly return statistics (1885–2010)<sup>a</sup>

<i>N</i>	Mean	Standard deviation	Skewness	Kurtosis	J–B <sup>b</sup>
6498	0.085	2.40	−0.49	11.2	18475.5*

<sup>a</sup>Continuously compounded returns.

<sup>b</sup>Jarque–Bera normality test: \**p*-value = 0.00000.

Weekly returns are scaled by 100 so they are percentage returns. Unless otherwise indicated, henceforth returns refer to weekly continuously compounded returns expressed as a percentage. We have 6498 weekly observations covering the period February 25, 1885, to January 20, 2010. Summary statistics are shown in Table 1.

We now briefly review a benchmark two-state model, our proposed four-state model, and some alternative specifications of the latter used to evaluate robustness of our best model.

### 2.1 Two-State Markov-Switching Model

The concept of bull and bear markets suggests cycles or trends that get reversed. Since those regimes are not observable, as discussed in Section 1, two-state latent-variable MS models have been applied to stock market data. A two-state first-order Markov model can be written as

$$r_t | s_t \sim N(\mu_{s_t}, \sigma_{s_t}^2), \tag{2.1}$$

$$p_{ij} = p(s_t = j | s_{t-1} = i), \tag{2.2}$$

where  $i = 1, 2$  and  $j = 1, 2$ . We impose  $\mu_1 < 0$  and  $\mu_2 > 0$  so that  $s_t = 1$  is the bear market and  $s_t = 2$  is the bull market.

Modeling of the latent regimes, regime probabilities, and state transition probabilities allows explicit model estimation and inference. In addition, in contrast to dating algorithms or filters, forecasts are possible. Investors can base their investment decisions on the posterior states or the whole forecast density.

### 2.2 MS-4 to Allow Bull Corrections and Bear Rallies

Consider the following general  $K$ -state first-order MS model for returns:

$$r_t | s_t \sim N(\mu_{s_t}, \sigma_{s_t}^2), \tag{2.3}$$

$$p_{ij} = p(s_t = j | s_{t-1} = i), \tag{2.4}$$

where  $i = 1, \dots, K$  and  $j = 1, \dots, K$ . We explore a four-state model,  $K = 4$ , to focus on modeling potential phases of the aggregate stock market. Without any additional restrictions, we cannot identify the model and relate it to market phases. Therefore, we consider the following restrictions. First, the states  $s_t = 1, 2$  are assumed to govern the bear market; *we label these two states as the bear regime*. The states  $s_t = 3, 4$  are assumed to govern the bull market; *these two states are labeled the bull regime*. Each regime has two states that allows for positive and negative periods of price growth within each regime. In particular, we impose

$$\text{Bear regime} \quad \begin{cases} \mu_1 < 0 & (\text{bear market state}), \\ \mu_2 > 0 & (\text{bear market rally}), \end{cases}$$

$$\text{Bull regime} \quad \begin{cases} \mu_3 < 0 & (\text{bull market correction}), \\ \mu_4 > 0 & (\text{bull market state}). \end{cases} \tag{2.5}$$

This structure can capture short-term reversals in market trends. Each state can have a different variance and can accommodate autoregressive heteroscedasticity in returns. In addition, conditional heteroscedasticity within each regime can be captured.

Consistent with the two states in each regime, the full transition matrix is

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} & 0 & p_{14} \\ p_{21} & p_{22} & 0 & p_{24} \\ p_{31} & 0 & p_{33} & p_{34} \\ p_{41} & 0 & p_{43} & p_{44} \end{pmatrix}. \tag{2.6}$$

This structure allows for several important features that are excluded in the MS models in the literature that have fewer states. First, a bear regime can feature several episodes of the bear state and bear rally state, exactly as many investors feel we observe in the data. Similarly, the bull regime can be characterized by a combination of bull states and bull corrections. Because the realization of states in a regime will differ over time, both bull and bear regimes can be heterogeneous over time. For instance, based on returns, a bear regime lasting five periods made of the states

$$s_t = 1, s_{t+1} = 1, s_{t+2} = 1, s_{t+3} = 2,$$

$$s_{t+4} = 2, s_{t+5} = 2, s_{t+6} = 4$$

will look very different from

$$s_t = 1, s_{t+1} = 1, s_{t+2} = 1, s_{t+3} = 2,$$

$$s_{t+4} = 1, s_{t+5} = 1, s_{t+6} = 4.$$

A second important contribution is that a bear rally is allowed to move either into the bull state or back to the bear state; analogously, a bull correction can move to a bear state or back to the bull state. We restrict the movement of a bull (bear) regime to a bear rally (bull correction) state for identification. These important interregime and intraregime dynamics are absent in the existing literature and, as we show, are supported by the data.

The unconditional probabilities associated with  $\mathbf{P}$  can be solved (Hamilton 1994)

$$\boldsymbol{\pi} = (\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\mathbf{e}, \tag{2.7}$$

where  $\mathbf{A}' = [\mathbf{P}' - \mathbf{I}, \boldsymbol{\iota}]$  and  $\mathbf{e}' = [0, 0, 0, 0, 1]$  and  $\boldsymbol{\iota} = [1, 1, 1, 1]'$ .

Using the matrix of unconditional state probabilities given by (2.7), we impose the following conditions on long-run returns in the bear and bull regimes, respectively,

$$E[r_t | \text{bear regime}, s_t = 1, 2] = \frac{\pi_1}{\pi_1 + \pi_2} \mu_1 + \frac{\pi_2}{\pi_1 + \pi_2} \mu_2 < 0, \tag{2.8}$$

$$E[r_t | \text{bull regime}, s_t = 3, 4] = \frac{\pi_3}{\pi_3 + \pi_4} \mu_3 + \frac{\pi_4}{\pi_3 + \pi_4} \mu_4 > 0. \tag{2.9}$$

We do not impose any constraint on the variances.

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Equations (2.5) and (2.6), along with Equations (2.8) and (2.9), serve to identify bull and bear regimes in the MS-4 model. The bull (bear) regime has a long-run positive (negative) return. Each market regime can display short-term reversals that differ from their long-run mean. For example, a bear regime can display a bear market rally (temporary period of positive returns), even though its long-run return is negative, and similarly for the bull market.

### 2.3 Other Models for Robustness Checks

Besides the two-state and four-state model, we consider several other specifications and provide model comparisons among them. The dependencies in the variance of returns are the most dominant feature of the data. This structure may adversely dominate dynamics of the conditional mean. The following specifications are included to investigate this issue.

- (1) Restricted four-state model. This is identical to the four-state model in Section 2.2 except that inside a regime the return innovations are homoscedastic, that is,  $\sigma_1^2 = \sigma_2^2$  and  $\sigma_3^2 = \sigma_4^2$ . In this case, the variance within each regime is restricted to be constant, although the overall variance of returns can change over time due to switches between regimes.
- (2) MS with decoupled conditional mean and variance. In this model, the mean and variance dynamics are decoupled and directed by independent latent Markov chains. This is a robustness check to determine to what extent the variance dynamics might be driving the regime transitions. This specification has the following structure:

$$r_t | s_t, w_t \sim N(\mu_{s_t}, \sigma_{w_t}^2), \tag{2.10}$$

$$p_{ij} = p(s_t = j | s_{t-1} = i), \quad i, j = 1, \dots, K, \tag{2.11}$$

$$q_{ij} = p(w_t = j | w_{t-1} = i), \quad i, j = 1, \dots, L. \tag{2.12}$$

We focus on the case  $K = 4$  and  $L = 4$ , again to allow us to capture at least four phases of cycles for aggregate stock returns. The conditional means have the same restriction imposed as the MS-4; to identify the conditional variances,  $\sigma_1 > \sigma_2 > \sigma_3 > \sigma_4$  is imposed.

- (3) A generalized autoregressive conditional heteroscedasticity (GARCH) (1,1) model

$$r_t | I_{t-1} \sim N(\mu, \sigma_t^2), \tag{2.13}$$

$$\sigma_t^2 = \omega + \alpha(r_{t-1} - \mu)^2 + \beta\sigma_{t-1}^2, \tag{2.14}$$

where  $I_{t-1} = \{r_1, \dots, r_{t-1}\}$ . This model captures volatility dynamics but does not model changes in the conditional mean.

- (4) MS-4 with  $p_{24} = p_{31} = 0$  that restricts the moves between bull and bear regime to be from state 1 to 4 and state 4 to 1.
- (5) An unrestricted four-state model labeled as MS-4 unrestricted. In this case, all 16 elements of  $\mathbf{P}$  are estimated and no restrictions are put on the state or regime means. To identify the model,  $\sigma_1 > \sigma_2 > \sigma_3 > \sigma_4$  is imposed.

## 3. ESTIMATION AND MODEL COMPARISON

### 3.1 Estimation

In this section, we discuss Bayesian estimation for the most general model introduced in Section 2.2 assuming there are  $K$  states,  $k = 1, \dots, K$ . The other models are estimated in a similar way with minor modifications.

There are three groups of parameters  $\mathbf{M} = \{\mu_1, \dots, \mu_K\}$ ,  $\mathbf{\Sigma} = \{\sigma_1^2, \dots, \sigma_K^2\}$ , and the elements of the transition matrix  $\mathbf{P}$ . Let  $\theta = \{\mathbf{M}, \mathbf{\Sigma}, \mathbf{P}\}$  and given data  $I_T = \{r_1, \dots, r_T\}$ , we augment the parameter space to include the states  $\mathbf{S} = \{s_1, \dots, s_T\}$  so that we sample from the full posterior  $p(\theta, \mathbf{S} | I_T)$ . Assuming conditionally conjugate priors  $\mu_i \sim N(m_i, n_i^2)$ ,  $\sigma_i^{-2} \sim G(v_i/2, w_i/2)$ , and each row of  $\mathbf{P}$  following a Dirichlet distribution, allows for a Gibbs sampling approach following Chib (1996). Gibbs sampling iterates on sampling from the following conditional densities given startup parameter values:  $\mathbf{S} | \mathbf{M}, \mathbf{\Sigma}, \mathbf{P}$ ;  $\mathbf{M} | \mathbf{\Sigma}, \mathbf{P}, \mathbf{S}$ ;  $\mathbf{\Sigma} | \mathbf{M}, \mathbf{P}, \mathbf{S}$ , and  $\mathbf{P} | \mathbf{M}, \mathbf{\Sigma}, \mathbf{S}$ .

Sequentially sampling from each of these conditional densities results in one iteration of the Gibbs sampler. Dropping an initial set of draws to remove any dependence from startup values, the remaining draws  $\{\mathbf{S}^{(j)}, \mathbf{M}^{(j)}, \mathbf{\Sigma}^{(j)}, \mathbf{P}^{(j)}\}_{j=1}^N$  are collected to estimate features of the posterior density. Simulation-consistent estimates can be obtained as sample averages of the draws. For example, the posterior mean of the state dependent mean is estimated as  $\frac{1}{N} \sum_{j=1}^N \mu_k^{(j)}$  for  $k = 1, \dots, K$  and are simulation-consistent estimates of  $E[\mu_k | I_T]$ .

The first sampling step of  $\mathbf{S} | \mathbf{M}, \mathbf{\Sigma}, \mathbf{P}$  involves a joint draw of all the states. Chib (1996) showed that this can be done by a so-called forward and backward smoother. The second and third sampling steps are straightforward and use results from the linear regression model. Given the conjugate Dirichlet prior on each row of  $\mathbf{P}$ , the final step is to sample  $\mathbf{P} | \mathbf{M}, \mathbf{\Sigma}, \mathbf{S}$  from the Dirichlet distribution Geweke (2005).

An important byproduct of Gibbs sampling is an estimate of the smoothed state probabilities  $p(s_t | I_T)$ , which can be estimated as  $p(s_t = i | I_T) = \frac{1}{N} \sum_{j=1}^N \mathbf{1}_{s_t=i}(\mathbf{S}^{(j)})$  for  $i = 1, \dots, K$ .

At each step, if a parameter draw violates any of the prior restrictions in (2.5), (2.6), (2.8), and (2.9), then it is discarded. For the four-state model, we set the independent priors as  $\mu_1 \sim N(-0.7, 1)$ ,  $\mu_2 \sim N(0.2, 1)$ ,  $\mu_3 \sim N(-0.2, 1)$ ,  $\mu_4 \sim N(0.3, 1)$ ,  $\sigma_i^{-2} \sim G(0.5, 0.05)$ ,  $i = 1, 2, 3, 4$ ,  $\{p_{11}, p_{12}, p_{14}\} \sim \text{Dir}(8, 1.5, 0.5)$ ,  $\{p_{21}, p_{22}, p_{24}\} \sim \text{Dir}(1.5, 8, 0.5)$ ,  $\{p_{31}, p_{33}, p_{34}\} \sim \text{Dir}(0.5, 8, 1.5)$ ,  $\{p_{41}, p_{43}, p_{44}\} \sim \text{Dir}(0.5, 1.5, 8)$ . These priors are informative but cover a wide range of empirically relevant parameter values.

### 3.2 Model Comparison

If the marginal likelihood can be computed for a model, it is possible to compare models based on Bayes factors. Nonnested models can be compared as well as specifications with a different number of states. Note that the Bayes factor compares the out-of-sample prediction record of models and penalizes over-parameterized models that do not deliver improved predictions. This is referred to as an Ockham's razor effect. See Kass and Raftery (1995) for a discussion on the benefits of Bayes factors.

For the general MS model with  $K$  states, the marginal likelihood for model  $\mathcal{M}_i$  is defined as

$$p(\mathbf{r}|\mathcal{M}_i) = \int p(\mathbf{r}|\mathcal{M}_i, \boldsymbol{\theta})p(\boldsymbol{\theta}|\mathcal{M}_i)d\boldsymbol{\theta}, \quad (3.1)$$

which integrates out parameter uncertainty;  $p(\boldsymbol{\theta}|\mathcal{M}_i)$  is the prior and

$$p(\mathbf{r}|\mathcal{M}_i, \boldsymbol{\theta}) = \prod_{t=1}^T f(r_t|I_{t-1}, \boldsymbol{\theta}) \quad (3.2)$$

is the likelihood which has  $S$  integrated out according to

$$f(r_t|I_{t-1}, \boldsymbol{\theta}) = \sum_{k=1}^K f(r_t|I_{t-1}, \boldsymbol{\theta}, s_t = k)p(s_t = k|\boldsymbol{\theta}, I_{t-1}). \quad (3.3)$$

The term  $p(s_t = k|\boldsymbol{\theta}, I_{t-1})$  is available from the Hamilton filter. Due to the prior restrictions, we use Chib and Jeliazkov (2001) to compute the marginal likelihood. For details on this, see this article and Chib (1995).

A log-Bayes factor between model  $\mathcal{M}_i$  and  $\mathcal{M}_j$  is defined as

$$\log(BF_{ij}) = \log(p(\mathbf{r}|\mathcal{M}_i)) - \log(p(\mathbf{r}|\mathcal{M}_j)). \quad (3.4)$$

Kass and Raftery (1995) suggested interpreting the evidence for  $\mathcal{M}_i$  versus  $\mathcal{M}_j$  as not worth more than a bare mention for  $0 \leq \log(BF_{ij}) < 1$ , positive for  $1 \leq \log(BF_{ij}) < 3$ , strong for  $3 \leq \log(BF_{ij}) < 5$ , and very strong for  $\log(BF_{ij}) \geq 5$ .

### 3.3 Predictive Density

An important feature of our probabilistic approach is that a predictive density of future returns can be computed by integrating out all uncertainty regarding states and parameters. We first discuss the predictive density of the one-period-ahead return. Then we show how to compute the predictive density of multiperiod returns (cumulative log-returns). Model comparison based on the density of multiperiod returns is a new result and, as we show, very informative in differentiating models.

The predictive density for future returns based on current information at time  $t$  is computed as

$$p(r_{t+1}|I_t) = \int f(r_{t+1}|I_t, \boldsymbol{\theta})p(\boldsymbol{\theta}|I_t)d\boldsymbol{\theta}, \quad (3.5)$$

which involved integrating out both state and parameter uncertainty using the posterior distribution  $p(\boldsymbol{\theta}|I_t)$ . From the Gibbs sampling draws  $\{\mathbf{S}^{(j)}, \mathbf{M}^{(j)}, \boldsymbol{\Sigma}^{(j)}, \mathbf{P}^{(j)}\}_{j=1}^N$  based on data  $I_t$ , we approximate the predictive density as

$$\begin{aligned} p(\widehat{r_{t+1}}|I_t) &= \frac{1}{N} \sum_{i=1}^N \sum_{k=1}^K f(r_{t+1}|\boldsymbol{\theta}^{(i)}, s_{t+1} = k) \\ &\times p(s_{t+1} = k|s_t^{(i)}, \boldsymbol{\theta}^{(i)}), \end{aligned} \quad (3.6)$$

where  $f(r_{t+1}|\boldsymbol{\theta}^{(i)}, s_t = k)$  follows  $N(\mu_k^{(i)}, \sigma_k^{2(i)})$  and  $p(s_{t+1} = k|s_t^{(i)}, \boldsymbol{\theta}^{(i)})$  is the transition probability.

The probability of a future state  $s_{t+1}$  can also be easily estimated by simulating from the distribution  $p(s_{t+1}|s_t^{(i)}, \boldsymbol{\theta}^{(i)})$  a state  $s_{t+1}^{(j)}$  for each state and parameter draw  $s_t^{(i)}, \boldsymbol{\theta}^{(i)}$ . The proportion of draws for which  $s_{t+1} = k$  is an estimate of  $P(s_{t+1} = k|I_t)$ .

The predictive density can also be computed for cumulative returns over long horizons. Define the  $h$ -period return as  $r_{t,h} = \sum_{i=1}^h r_{t+i}$ . Then the predictive density for the multiperiod return is

$$\begin{aligned} p(r_{t,h}|I_t) &= \int \left[ \prod_{j=1}^{h-1} f(r_{t+j}|\boldsymbol{\theta}, s_{t+j}) \right] \\ &\times f\left(r_{t,h} - \sum_{j=1}^{h-1} r_{t+j} \mid \boldsymbol{\theta}, s_{t+h}\right) \\ &\times \left[ \prod_{j=2}^h p(s_{t+j}|s_{t+j-1}, \boldsymbol{\theta}) \right] p(s_{t+1}|I_t, \boldsymbol{\theta})p(\boldsymbol{\theta}|I_t) \\ &\times d\boldsymbol{\theta}ds_{t+1} \dots ds_{t+h}dr_{t+1} \dots dr_{t+h-1}. \end{aligned} \quad (3.7)$$

This integrates out all parameter uncertainty and future state uncertainty as well as all possible sample paths of returns  $\{r_{t+1}, \dots, r_{t+h-1}\}$  such that  $r_{t,h} = \sum_{i=1}^h r_{t+i}$ . The predictive density can be approximated as

$$p(\widehat{r_{t,h}}|I_t) = \frac{1}{N} \sum_{i=1}^N f\left(r_{t,h} - \sum_{j=1}^{h-1} r_{t+j}^{(i)} \mid \boldsymbol{\theta}^{(i)}, s_{t+h}^{(i)}\right).$$

Given  $\boldsymbol{\theta}^{(i)}, s_t^{(i)}$  from the posterior, we simulate out  $\{s_{t+j}^{(i)}, r_{t+j}^{(i)}\}_{j=1}^{h-1}$  and  $s_{t+h}^{(i)}$  from the MS-4 and evaluate the sum above at the data  $r_{t,h}$ . Similar calculations are used for the other models we compare with.

Evaluating the predictive density at the data  $r_{t,h}$  gives the predictive likelihood. Models can be compared on their out-of-sample performance for observations  $\tau_1$  to  $\tau_2$ ,  $\tau_1 < \tau_2$  based on the cumulative log predictive likelihood,

$$\sum_{t=\tau_1}^{\tau_2} \log(p(\widehat{r_{t,h}}|I_t)). \quad (3.8)$$

To compute this, the model must be reestimated at each point in the out-of-sample period. As with the marginal likelihood, better models will have a larger cumulative log predictive likelihood value. A model with a larger value is better able to account for the data.

## 4. RESULTS

### 4.1 Parameter Estimates and Implied Distributions

Model estimates for the MS-2 model are given in Table 2. State 1 has a negative conditional mean along with a high conditional variance, whereas state 2 displays a high conditional mean with a low conditional variance. Both regimes are very persistent. These results are consistent with the sorting of bull and bear regimes in Maheu and McCurdy (2000a) and Guidolin and Timmermann (2005).

Estimates for our proposed four-state model (MS-4) are given in Table 3. All parameters are precisely estimated indicating that the data are quite informative. Recall that states  $s_t = 1, 2$  capture the bear regime, while states  $s_t = 3, 4$  capture the bull regime. Each regime contains a state with a positive and a negative conditional mean. We label states 1 and 2 as the bear and bear

Table 2. MS-2 model estimates

	Mean	Median	Standard deviation	0.95 DI
$\mu_1$	-0.46	-0.46	0.14	(-0.73, -0.20)
$\mu_2$	0.20	0.20	0.02	(0.16, 0.25)
$\sigma_1$	4.42	4.42	0.13	(4.18, 4.69)
$\sigma_2$	1.64	1.64	0.02	(1.59, 1.69)
$p_{11}$	0.94	0.94	0.01	(0.92, 0.96)
$p_{22}$	0.99	0.99	0.002	(0.98, 0.99)

NOTE: This table reports the posterior mean, median, standard deviation, and 0.95 density intervals (DI) for model parameters.

rally states, respectively; states 3 and 4 are the bull correction and bull states, respectively.

Consistent with the MS-2 model, volatility is highest in the bear regime. In particular, the highest volatility occurs in the bear regime in state 1. This state also delivers the lowest average return. The highest average return and lowest volatility is in state 4, which is part of the bull regime. The bear rally state ( $s_t = 2$ ) delivers a conditional mean of 0.23 and conditional standard deviation of 2.63. However, this mean is lower and the volatility higher than the bull positive growth state ( $s_t = 4$ ). Analogously, the bull correction state ( $s_t = 3$ ) has a larger conditional mean ( $-0.13 > -0.94$ ) and smaller volatility ( $2.18 < 6.01$ ) than the bear state 1.

State 4 provides the most favorable risk-return trade-off followed by states 2, 3, and 1. Note that the bull state 4 delivers a larger expected return with a standard deviation that is 50% smaller than the bear rally (state 2). Even though the bear rally delivers a positive expected return, that return is much more variable than in the bull state.

Table 3. MS-4 model estimates

	Mean	Median	Standard deviation	95% DI
$\mu_1$	-0.94	-0.92	0.27	(-1.50, -0.45)
$\mu_2$	0.23	0.23	0.10	(0.04, 0.43)
$\mu_3$	-0.13	0.12	0.08	(-0.31, -0.01)
$\mu_4$	0.30	0.29	0.04	(0.22, 0.38)
$\sigma_1$	6.01	5.98	0.35	(5.41, 6.77)
$\sigma_2$	2.63	2.61	0.18	(2.36, 3.08)
$\sigma_3$	2.18	2.19	0.12	(1.94, 2.39)
$\sigma_4$	1.30	1.30	0.04	(1.20, 1.37)
$p_{11}$	0.921	0.923	0.020	(0.877, 0.955)
$p_{12}$	0.076	0.074	0.020	(0.042, 0.120)
$p_{14}$	0.003	0.001	0.004	(3e-6, 0.013)
$p_{21}$	0.015	0.014	0.007	(0.005, 0.031)
$p_{22}$	0.966	0.967	0.009	(0.945, 0.980)
$p_{24}$	0.019	0.018	0.006	(0.009, 0.034)
$p_{31}$	0.010	0.009	0.003	(0.004, 0.017)
$p_{33}$	0.939	0.943	0.018	(0.899, 0.965)
$p_{34}$	0.051	0.048	0.017	(0.027, 0.088)
$p_{41}$	0.001	0.0003	0.0007	(6e-7, 0.002)
$p_{43}$	0.039	0.037	0.012	(0.024, 0.067)
$p_{44}$	0.960	0.963	0.012	(0.933, 0.976)

NOTE: This table reports the posterior mean, median, standard deviation, and 0.95 density intervals (DI) for model parameters.

Table 4. Unconditional state probabilities

	Mean	0.95 DI
$\pi_1$	0.070	(0.035, 0.117)
$\pi_2$	0.157	(0.073, 0.270)
$\pi_3$	0.304	(0.216, 0.397)
$\pi_4$	0.469	(0.346, 0.579)

NOTE: This table reports the posterior mean and 0.95 density intervals (DI) associated with the posterior distribution for  $\pi$  from Equation (2.7).

All states display high persistence ( $p_{ii}$  is high for all  $i$ ). However, the transition probabilities display some asymmetries. For example, the probability of a bear rally moving back to the bear state 1 ( $p_{21} = 0.015$ ) is a little lower than changing regime to a bull market ( $p_{24} = 0.019$ ). On the other hand, the probability of a bull correction returning to a bull market ( $p_{34} = 0.051$ ) is considerably higher than changing regime to the bear state ( $p_{31} = 0.010$ ).

What do the dynamics of the MS-4 model imply for investors? First, the probability of a positive return in any state is high with  $P(r_t > 0 | s_t) = 0.44, 0.53, 0.48,$  and  $0.59$ , respectively, for states  $s_t = 1, 2, 3, 4$ . Second, investing in any state eventually results in future returns being positive. For instance, the  $H = \arg \min\{E(r_{t+H} | s_t) > 0\}$  is 37 ( $s_t = 1$ ), 1 ( $s_t = 2$ ), 15 ( $s_t = 3$ ), and 1 ( $s_t = 4$ ). Therefore, investors concerned with long investment horizons or obtaining a high probability of a positive return may find investing in states 1 and 3 desirable.

Table 4 reports the unconditional probabilities for the states. On average the market spends 0.157 of time in a bear rally while 0.304 in a bull correction. The most time is spent in the bull growth state 4. The unconditional probability of the bull regime is 0.773.

A comparison of the regime statistics implied by the parameter estimates for the MS-2 and MS-4 models is given in Table 5. The expected duration of regimes is much longer in the four-state model. That is, by allowing heterogeneity within a regime in our four-state model, we switch between bull and bear markets less frequently. For instance, in a MS-2 parameterization, the bull market has a duration of only 82.6 weeks, about 18 months, while the richer MS-4 model has a bull duration of just less than 5 years. As we will see in the following, there is much more switching between regimes in the MS-2 model.

In the two-state model, the expected return and variance are fixed within a regime. For the bear regime in the MS-2 model, the expected variance is  $E[\text{var}(r_t | s_t = 1)] = 19.6$ . In contrast, the average variance for each regime in the four-state model can be attributed to changes in the conditional mean as well as to the average conditional variance of the return innovations. For instance, the average variance of returns in the bear regime can be decomposed as  $\text{var}(r_t | s_t = 1, 2) = \text{var}(E[r_t | s_t] | s_t = 1, 2) + E[\text{var}(r_t | s_t) | s_t = 1, 2] = 0.31 + 16.1$ , with a similar result for the bull regime. For the bull and bear phase, the mean dynamics account for a small share, 2% of the total variance.

The MS-2 model assumes normality in both market regimes, while the MS-4 shows that the data are at odds with this assumption. Skewness is present in bear markets, while excess kurtosis is found in both bull and bear regimes. Overall the bear market

Table 5. Posterior regime statistics for MS-2 and MS-4 models

	MS-2	MS-4
Bear mean	-0.46 (-0.73, -0.20)	-0.13 (-0.367, -0.005)
Bear duration	18.2 (13.2, 25.0)	77.8 (44.4, 134.6)
Bear standard deviation	4.42 (4.18, 4.69)	4.04 (3.51, 4.73)
Bear variance from $\text{var}(E[r_t s_t] s_t = 1, 2)$	0.00	0.31 (0.07, 0.68)
Bear variance from $E[\text{var}(r_t s_t) s_t = 1, 2]$	19.6 (17.5, 22.0)	16.1 (12.1, 22.0)
Bear skewness	0	-0.42 (-0.68, -0.20)
Bear kurtosis	3	5.12 (4.37, 5.93)
Bull mean	0.20 (0.16, 0.25)	0.13 (0.07, 0.18)
Bull duration	82.6 (59.1, 115.9)	256.0 (123.5, 509.6)
Bull standard deviation	1.64 (1.59, 1.69)	1.71 (1.59, 1.83)
Bull variance from $\text{var}(E[r_t s_t] s_t = 3, 4)$	0.00	0.04 (0.02, 0.09)
Bull variance from $E[\text{var}(r_t s_t) s_t = 3, 4]$	2.69 (2.54, 2.85)	2.89 (2.47, 3.30)
Bull skewness	0	0.04 (-0.11, 0.16)
Bull kurtosis	3	3.77 (3.51, 4.03)

NOTE: This table reports the posterior mean and 0.95 density interval for regime statistics.

deviates more from a normal distribution; it has thicker tails and captures more extreme events.

Table 6 summarizes features of the MS-4 parameterization for both the regimes and their component states derived from the posterior parameter estimates. The bear regime duration is

77.8 weeks, much shorter than the bull regime duration of 256.0 weeks. The average cumulative return in the bear (bull) regime is -9.94 (33.0). The volatility in the bear market is more than twice that in the bull market. The third panel provides a breakdown of cumulative return means in each of the component states of the market regimes. The bear rally yields a cumulative return of 7.10 on average that partially offsets the average decline of -12.4 in state 1. On the other hand, the bull correction has a cumulative return mean of -2.13 that diminishes the average cumulative return of 7.88 in state 4. Note that these states are combined into bull and bear market regimes in heterogeneous patterns over time yielding the statistics for regimes summarized in the first two panels of Table 6.

Although the stock market spends most of the time in the bull regime (states 3 and 4), in terms of individual states, it is state 2 that has the longest duration but this state is visited less often so its unconditional probability is low relative to the others ( $\pi_2 = 0.157$  from Table 4).

### 4.2 Model Comparisons

We conduct formal model comparisons based on the marginal likelihoods reported in Table 7. The constant mean and variance model performs the worst (has the lowest marginal likelihood). The next model is the MS-2 followed by the restricted MS-4,  $\sigma_1^2 = \sigma_2^2, \sigma_3^2 = \sigma_4^2$  (model 1 in Section 2.3). Next is the decoupled model with four states in the conditional mean and an independent four states in the conditional variance (model 2 in Section 2.3). Following this is a GARCH(1,1) model (model 3 in Section 2.3) and the MS-4 with  $p_{24} = p_{31} = 0$  (model 4 in Section 2.3). Following this is an unrestricted four-state model labeled as MS-4 unrestricted (model 5 in Section 2.3). The last specification is our preferred MS-4 model and the final column of the table presents log-Bayes factor for this model against the alternatives.

The log-Bayes factor between the MS-2 and the MS-4 in the conditional mean restricted to have only a two-state conditional

Table 6. Posterior state statistics for the MS-4 model

	Mean	Median	Standard deviation	95% DI
Bear mean	-0.13	-0.11	0.10	(-0.367, -0.005)
Bear duration	77.8	74.0	23.1	(44.4, 134.6)
Bear cumulative return	-9.94	-8.28	7.89	(-29.6, -0.41)
Bear standard deviation	4.04	4.01	0.31	(3.51, 4.73)
Bull mean	0.13	0.13	0.03	(0.07, 0.18)
Bull duration	256.0	235.6	100.9	(123.5, 509.6)
Bull cumulative return	33.0	30.0	14.9	(12.9, 70.3)
Bull standard deviation	1.71	1.71	0.06	(1.59, 1.83)
$s = 1$ : cumulative return	-12.4	-11.8	4.49	(-23.0, -5.45)
$s = 2$ : cumulative return	7.10	6.97	3.10	(1.47, 13.8)
$s = 3$ : cumulative return	-2.13	-2.07	1.09	(-4.46, -0.27)
$s = 4$ : cumulative return	7.88	7.75	1.67	(5.02, 11.6)
$s = 1$ : duration	13.5	13.0	3.63	(8.13, 22.2)
$s = 2$ : duration	31.2	30.1	8.39	(18.3, 51.0)
$s = 3$ : duration	17.9	17.4	4.80	(9.91, 28.8)
$s = 4$ : duration	27.2	26.9	6.75	(14.9, 41.4)

NOTE: This table reports posterior statistics for various population moments.

Table 7. Log marginal likelihoods: alternative models

Model	$\log f(Y   \text{Model})$	log-Bayes factor <sup>a</sup>
Constant mean with constant variance	-14924.1	1183.7
MS-2	-13903.3	162.9
MS-4 with restricted variances ( $\sigma_0^2 = \sigma_1^2$ , and $\sigma_2^2 = \sigma_3^2$ )	-13849.9	109.5
MS-4, 4-state mean and decoupled 4-state variance	-13754.4	14.0
GARCH(1,1)	-13809.1	68.7
MS-4 with $p_{24} = p_{31} = 0$	-13766.9	26.5
MS-4 unrestricted	-13747.2	6.8
MS-4	-13740.4	

<sup>a</sup>log-Bayes factor for MS-4 versus each model.

variance is large at  $53.4 = -13849.9 - (-13903.3)$ . This improved fit comes when additional conditional mean dynamics (going from states 2 to 4) are added to the basic MS-2 model. The best model is the MS-4 model. The log-Bayes factor in support of the four-state versus the two-state model is  $162.9 = -13740.4 - (-13903.3)$ . The zero restrictions in the transition matrix (2.6) are also strongly supported by the data. For instance, the log-Bayes factor is  $6.8 = -13740.4 - (-13747.2)$  in support of our MS-4 model compared with an unrestricted four-state model. The evidence is also against additional restrictions on  $\mathbf{P}$

such as  $p_{24} = p_{31} = 0$  relative to our MS-4 parameterization with  $\mathbf{P}$  matrix (2.6). The data favor the more flexible transitions of our MS-4 between bull and bear regimes.

Our preferred four-state model also dominates the more flexible model with four states for conditional mean and four independent states for the conditional variance (decoupled four-state model). Similarly our MS-4 model improves upon the GARCH(1,1) model.

Overall, there is very strong evidence that the four-state specification of Section 2.2 provides the best fit to weekly returns.

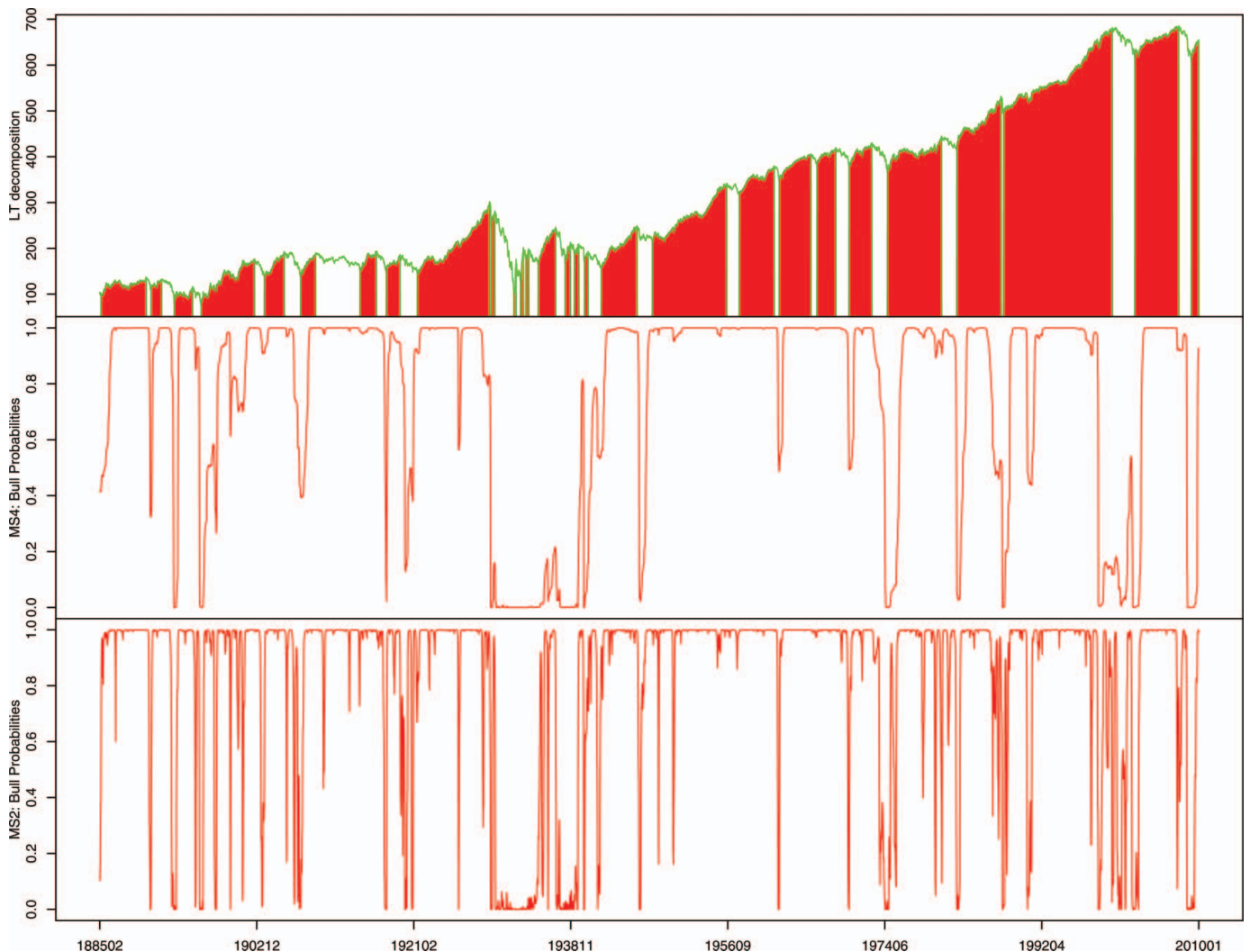


Figure 1. LT algorithm, MS-4, and MS-2. The online version of this figure is in color.



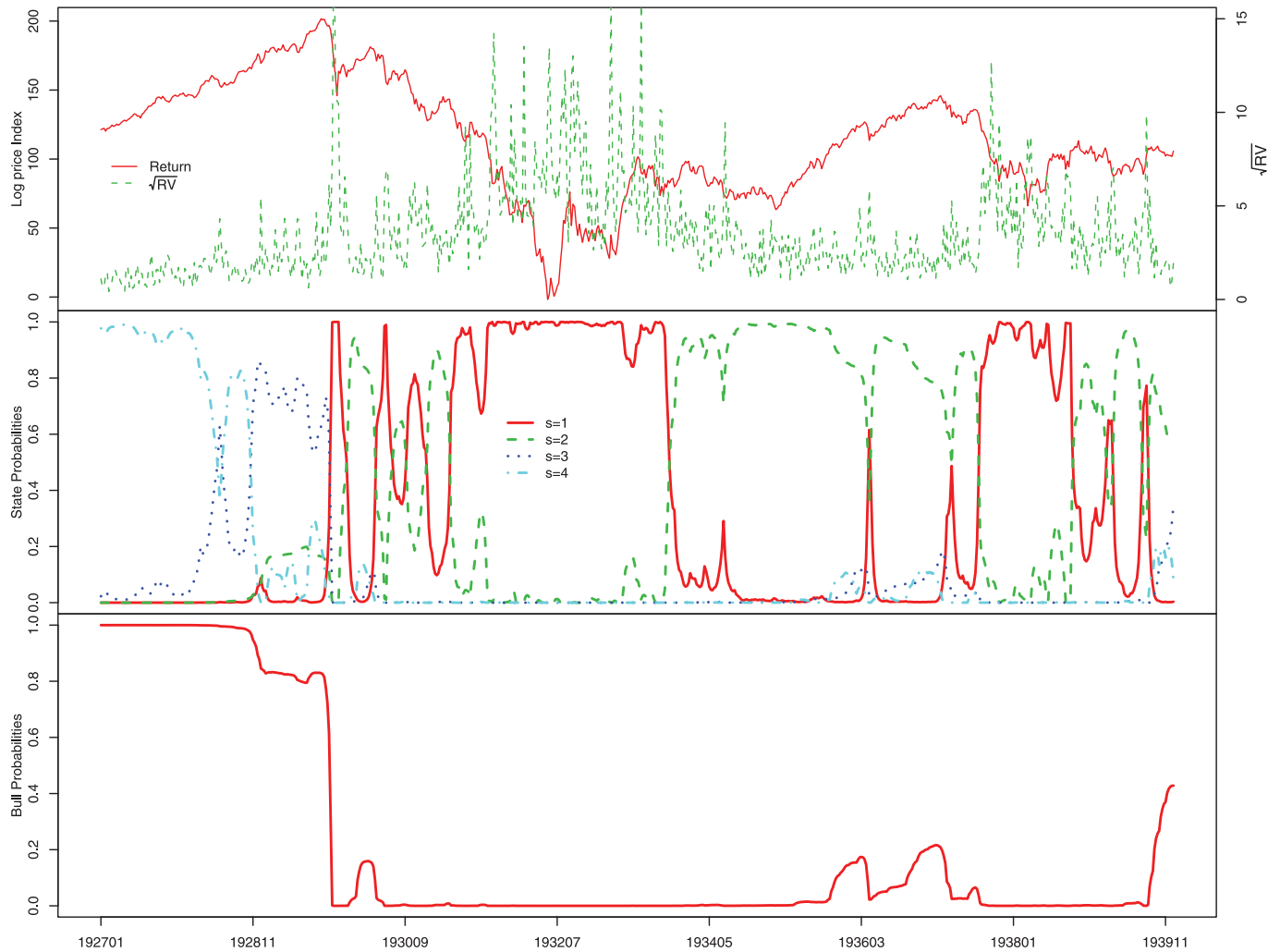


Figure 2. MS-4, 1927–1939. The online version of this figure is in color.

The comparisons also show that this improved fit comes from improved fit to both the conditional mean and variance. Not only does our MS-4 model provide a better economic characterization of differences in stock market cycles but also the model statistically dominates other alternatives.

In the online appendix, the Lunde and Timmermann (2004) (LT) dating algorithm is used as a lens to view both the S&P 500 data and data simulated from our preferred MS-4 model. Although our model provides a richer four-state description of bull and bear markets, it does account for all of the data statistics associated with a simpler two-state view of the market using the LT dating algorithm.

#### 4.3 Identification of Historical Turning Points in the Market

The dating of the market regimes using the LT dating algorithm are found in the top panel of Figure 1. Below this panel is the smoothed probability of a bull market,  $p(s_t = 3|I_T) + p(s_t = 4|I_T)$  for the MS-4 model. The final plot in Figure 1 is the smoothed probability of a bull market,  $p(s_t = 2|I_T)$  from the two-state model. The four-state model produces less erratic shifts between market regimes, closely matches the trends in prices, and generally corresponds to the dating algorithm. The

two-state model is less able to extract the low-frequency trends in the market. In high frequency data, it is important to allow intraregime dynamics, such as short-term reversals.

Note that the success of our model should not be based on how well it matches the results from dating algorithms. Rather this comparison is done to show that the latent-state MS models can identify bull and bear markets with similar features to those identified by conventional dating algorithms. Beyond that, the MS models presented in this article provide a superior approach to modeling stock market trends as they deliver a full specification of the distribution of returns along with latent market dynamics. Such an approach permits out-of-sample forecasting, which we turn to in Section 4.4.

Illustrating the 1927–1939 subperiod, Figure 2 displays the log-price and the realized volatility (square root of RV) in the top panel, the smoothed states of the MS-4 model in the second panel, and the posterior probability of the bull market,  $p(s_t = 3|I_T) + p(s_t = 4|I_T)$ , in the last panel.

Just before the crash of 1929, the model identifies a bull correction state. The transition from a bull to bear market occurs as a move from a bull market state to a bull correction state and then into the bear regime. For the week ending October 16, 1929, there was a return of  $-3.348$  and the market transitioned from the bull correction state into the bear market state with

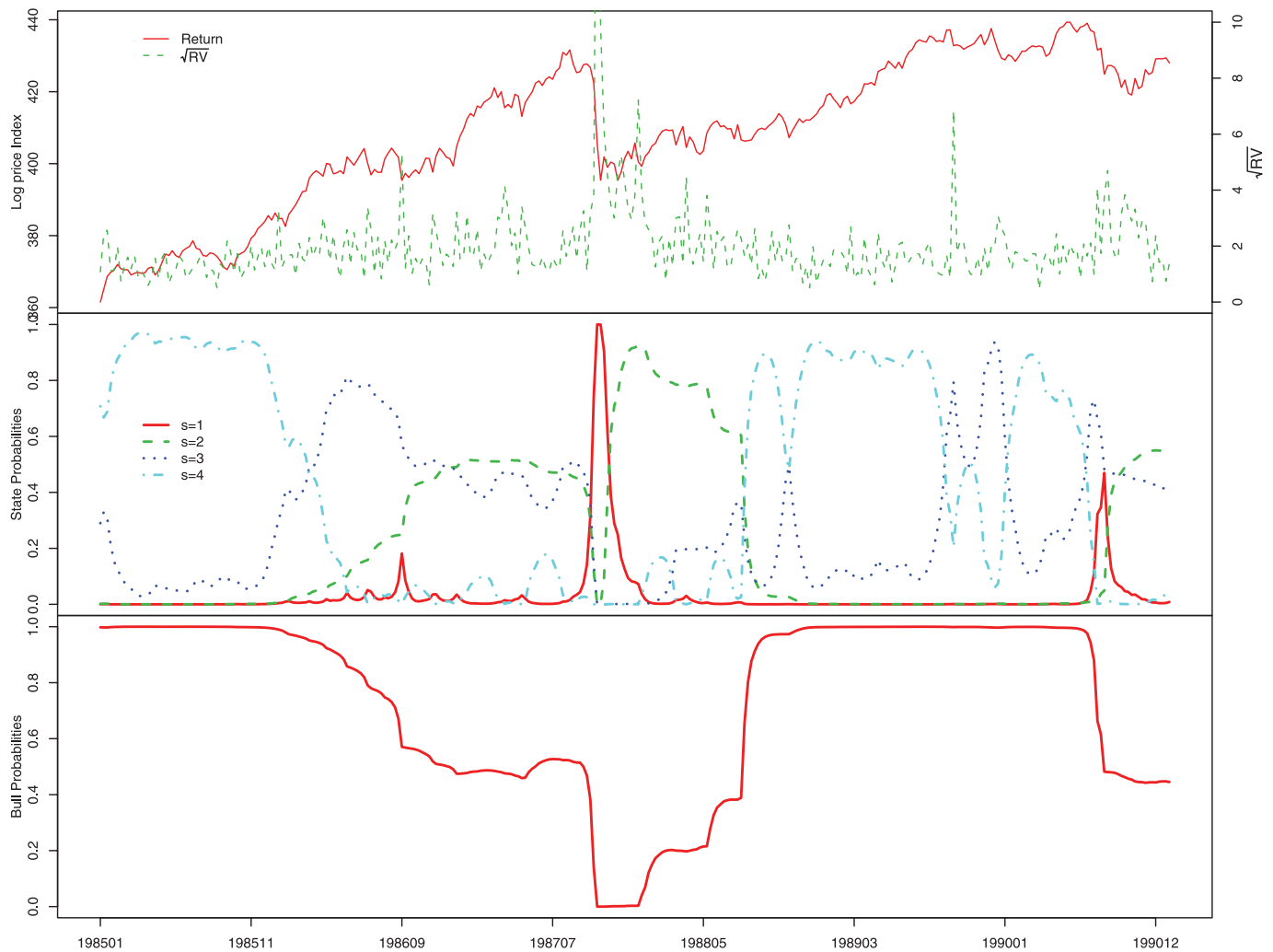


Figure 3. MS-4, 1985–1990. The online version of this figure is in color.

$p(s_t = 1 | I_T) = 0.63$ . This is further reinforced so that the next 5 weeks have essentially probability 1 for state 1.

As this figure shows, the remainder of this subperiod is decisively a bear market, but displays considerable heterogeneity in that there are several short-lived bear rallies. The high levels of realized volatility coincide with the high volatility in the bear market states. Periods of somewhat lower volatility are associated with the bear rally states. A strong bear rally begins in late November 1933 and lasts until August 25, 1937, at which time there is a move back into the bear market state. Realized volatility increases with this move into state 1.

For the 1985–1990 subperiod, prior to the 1987 crash, there is a dramatic run-up in stock prices with generally low volatility, as illustrated in the top panel of Figure 3. It is interesting to note that the model shows a great deal of uncertainty about the state of the market well before the crash. In the first week of October, just before the crash, the most likely state is the bull correction with  $p(s_t = 3 | I_T) = 0.37$ . The bear state that starts the following week lasts for about 5 weeks after which a strong bear rally quickly emerges as of the week ending November 18, 1987. It is the bear rally state that exits into a bull market during the week of August 17, 1988. Prices resume their strong increase until they plateau with a bull correction beginning the week of October 4, 1989.

Finally, we use our model to investigate recent market activity in Figure 4. The bull market state turned into a bull correction in mid-July 2007, which persisted until an abrupt move into the bear market state in early September 2008. This transition was accompanied by a dramatic increase in realized volatility. According to our model, the bear market became a bear market rally in the third week of March 2009 where it stayed until mid-November 2009 when it moved into the bull market state. As noted earlier, the positive trend in returns during a bear market rally does not get interpreted as a bull market until the market volatility declines to levels more typical of bull markets. See the online appendix for the 1980–1985 period.

There are several important points revealed by this discussion. First, bear (bull) markets are persistent but are made of many regular transitions between states 1 and 2 (3 and 4). Second, in each of the examples, the move between regimes occurs through either the bear rally or the bull correction state. In other words, these additional dynamics are critical to fully capturing turning points in stock market cycles. This is also borne out by our model estimates. The most likely route for a bear market to go to a bull market is through the bear rally state. Given that a bull market has just started, the probability is 0.9342 that the previous state was a bear rally (i.e.,  $p(s_t = 2 | s_{t+1} = 4, s_t = 1 \text{ or } 2) \propto \frac{p_{24} \pi_2}{\pi_1 + \pi_2}$ ) and only 0.0658 that it was a bear state. Similarly, given that a bear

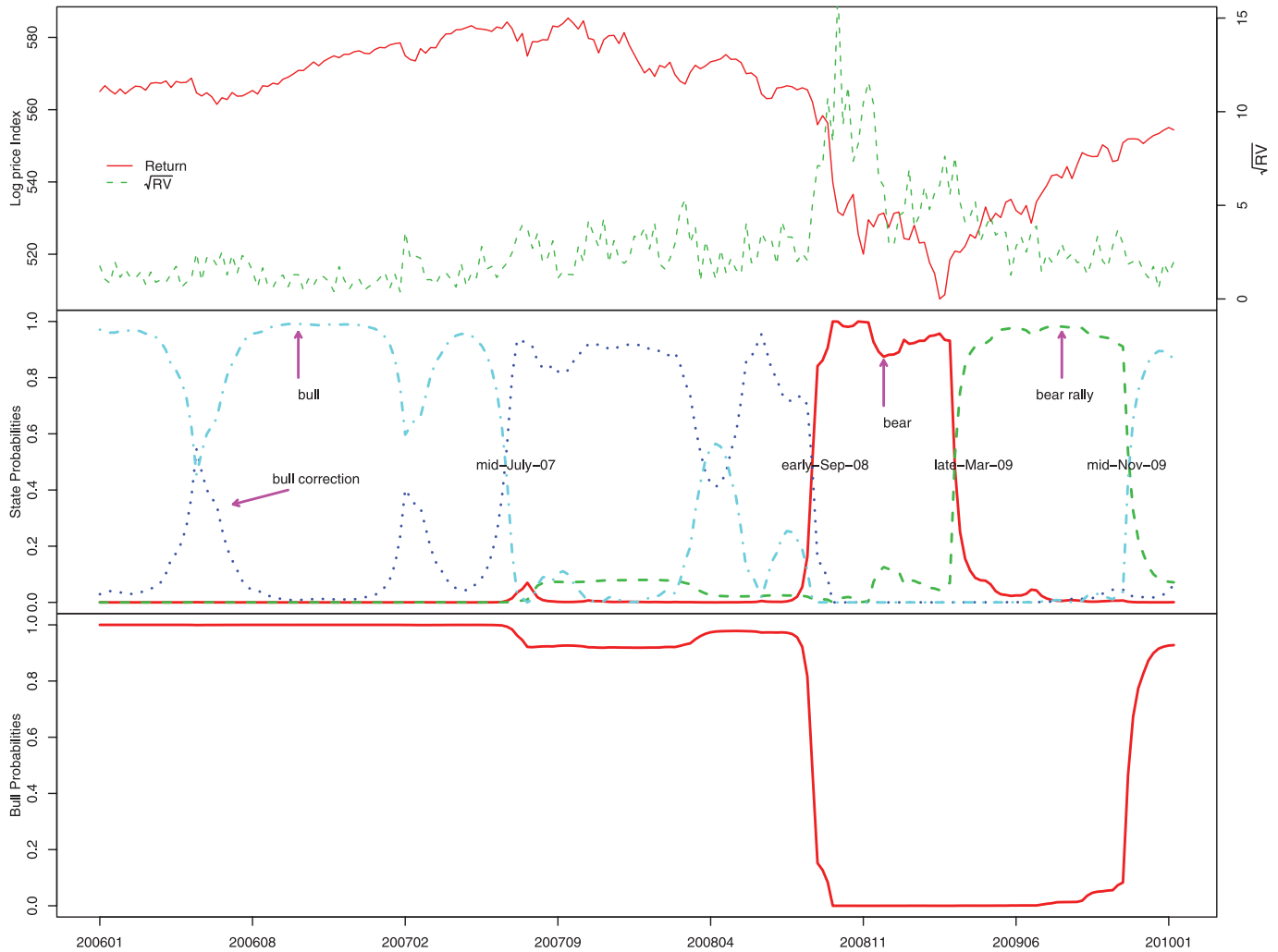


Figure 4. MS-4, 2006–2010. The online version of this figure is in color.

market has just started, the probability is 0.8663 that the previous state was a bull correction, and only 0.1337 that it was a bull state. The following subperiod descriptions provide examples of this richer specification of turning points plus frequent reversals within a regime.

#### 4.4 Density Forecasts

The different phases of the market according to the MS-4 model should have important implications for the forecast density of returns. In this section, we discuss two illustrations.

An industry standard measure of potential portfolio loss is the VaR that measures the tail area of the return distribution.  $VaR_{(\alpha),t}$  is defined as the 100 $\alpha$  percent quantile of the portfolio value or return distribution given information at time  $t - 1$ . We compute  $VaR_{(\alpha),t}$  from the predictive density of the MS-4 model as  $p(r_t < VaR_{(\alpha),t} | I_{t-1}) = \alpha$ . Given a correctly specified model, the probability of a return of  $VaR_{(\alpha),t}$  or less is  $\alpha$ .

To compute the VaR from the MS-4 model, we do the following. First,  $N$  draws from the predictive density are taken as follows: draw  $\theta$  and  $s_{t-1}$  from the Gibbs sampler, a future state  $\tilde{s}_t$  is simulated based on  $\mathbf{P}$  and  $\tilde{r}_t | \tilde{s}_t \sim N(\mu_{\tilde{s}_t}, \sigma_{\tilde{s}_t}^2)$ . The details

are discussed in Section 3.3. From the resulting draws, the  $\tilde{r}_t$  with rank  $[N\alpha]$  is an estimate of  $VaR_{(\alpha),t}$ .

The first panel of Figure 5 displays the conditional VaR from January 3, 2007, to January 20, 2010, predicted by the MS-4 model, as well as that implied by the normal benchmark for  $\alpha = 0.05$ . At each point, the model is estimated based on information up to  $t - 1$ . Similarly, the benchmark,  $N(0, \sigma^2)$ , sets  $\sigma^2$  to the sample variance using  $I_{t-1}$ .

The normal benchmark overestimates the VaR for the early part of this subsample but starts to understate it at times, beginning in mid-2007, and then severely under estimates in the last few months of 2008. The MS-4 model provides a very different  $VaR_{(0.05),t}$  over time because it takes into account the predicted regime, as indicated by the middle and bottom panels of Figure 5, which show forecasts of the states and regimes, respectively. Note that the potential losses, shown in the top panel, increase considerably in September and October 2008 as the model identifies a move from a bull to a bear market.

The density for cumulative returns over long horizons will be sensitive to the different states in the MS-4 model and if they are empirically important should result in better density forecasts. To investigate this, we compute the predictive likelihood

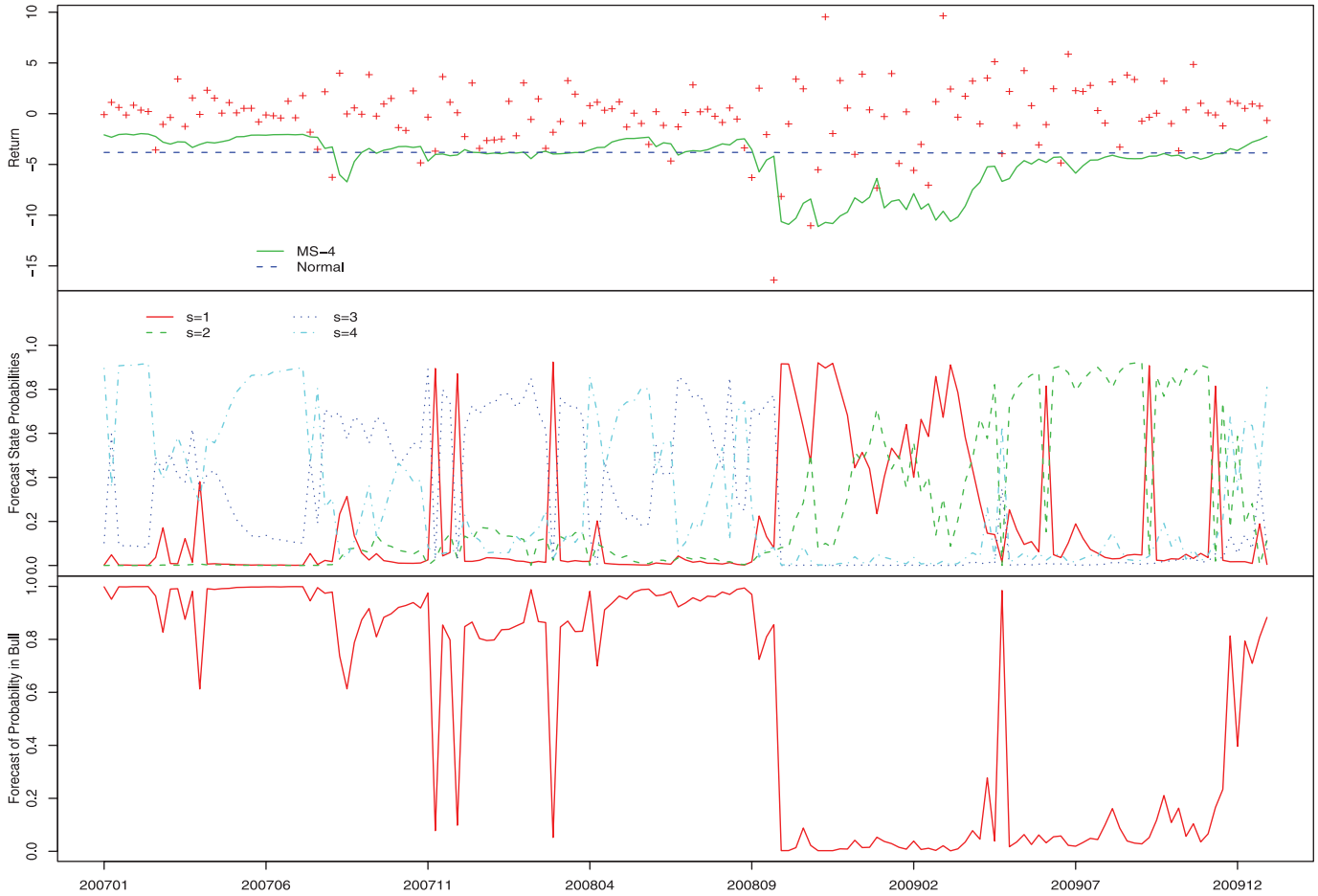


Figure 5. Value-at-risk from MS-4 and benchmark normal distribution. The online version of this figure is in color.

according to Section 3.3 for a range of investment horizons for the time period November 12, 1913, to January 20, 2010. Along with our MS-4 model, several other specifications are included in Table 8. We can compare the models based on their relative log-probabilities for long-horizon returns just as we would normally compare models using the log-Bayes factor in Section 3.2.

The GARCH(1,1) model provides the most accurate density forecasts for 1-month-ahead returns, while our MS-4 model dominates for all other horizons. For example, the difference in the log-predictive likelihoods for our MS-4 specification versus GARCH(1,1) is very large at 3 months (60), 6 months (117), and 12 months (210). Our MS-4 parameterization also dominates the

unrestricted four-state model at all horizons beyond 1 month. These long-horizon density forecasts provide very strong support for the four phases of the stock market that our MS-4 model captures. In summary, our model accurately identifies the phases of bull and bear markets that lead to competitive long-horizon out-of-sample density forecasts.

This out-of-sample application also gives us an opportunity to assess in real time when our model identified a move into the bear regime. We now consider the identification process that would have been historically available to investors using the model forecasts. This will differ from the previous results as we are using a smaller sample and updating estimates as new data arrives.

Table 8. Long-horizon return density forecasts

$h = \text{weeks}$	1 month 4	3 months 13	6 months 26	12 months 52
MS-4	-14428.8	<b>-17610.5</b>	<b>-19535.0</b>	<b>-21652.7</b>
MS-2	-14575.7	-17796.9	-19725.7	-22454.5
GARCH(1,1)	<b>-14341.6</b>	-17670.6	-19652.2	-21862.7
MS-4, 4-state mean and decoupled 4-state variance	-14408.7	-17626.4	-19582.6	-21827.5
MS-4 with $p_{24} = p_{31} = 0$	-14457.1	-17624.2	-19569.6	-21608.7
MS-4 unrestricted	-14421.5	-17625.1	-19561.0	-21720.7

NOTE: This table reports the log-predictive density of  $h$ -period returns:  $\sum_{i=T_1}^{T_2} \log(p(r_{t,h}|I_t, \text{Model}))$  where  $r_{t,h} = \sum_{i=1}^h r_{t+i}$ . Out-of-sample data: November 12, 1913, to January 20, 2010. This gives 4997 (1 month), 4988 (3 months), 4975 (6 months), and 4949 (12 months) observations. Bolded numbers indicate the largest column entries.

The second and third panel of Figure 5 report the predictive mean of the states and regimes, respectively. Prior to 2008, forecasts of the bull states occur the most, including some short episodes of bull corrections. In the first week of October 2008, the probability of a bull regime drops from 0.85 to essentially zero and remains there for some time. In other words, the model in real time detects a turning point in the first week of October 2008 from the bull to the bear regime. The first half of the bear regime that follows is characterized by the bear state, while the second half is largely classified as a bear rally.

Toward the end of our sample, there is a move from the bear market rally state to a bull market. In real time, in early December 2009, the model forecasts a move from the bear rally to the bull market state. For the week ending December 9, we have  $p(s_t = 1|I_{t-1}) = 0.02$ ,  $p(s_t = 2|I_{t-1}) = 0.17$ ,  $p(s_t = 3|I_{t-1}) = 0.14$ , and  $p(s_t = 4|I_{t-1}) = 0.67$ . The evidence for a bull market regime gradually strengthens; the last observation in our sample, January 20, 2010, has probabilities 0.01, 0.11, 0.07, and 0.81 for states 1, 2, 3, and 4, respectively, with the bull market state being the most likely.

## 5. CONCLUSION

This article proposes a new MS-4 model to identify the components of bull and bear market regimes in weekly stock market data. Bull correction and bull states govern the bull regime; bear rally and bear states govern the bear regime. Our probability model fully describes the return distribution while treating bull and bear regimes and their component states as unobservable.

A bear rally is allowed to move back to the bear state or to exit the bear regime by moving to a bull state. Likewise, a bull correction can move back to the bull state or exit the bull regime by transitioning to a bear state. This implies that regimes can feature several episodes of their component states. For example, a bull regime can be characterized by a combination of bull states and bull corrections. Similarly, a bear regime can consist of several episodes of the bear state and the bear rally state. Because the realization of states in a regime will differ over time, bull and bear regimes can be heterogeneous over time. This richer structure, including both intraregime and interregime dynamics, results in a richer characterization of market cycles.

Probability statements on regimes and future returns are available. Our model strongly dominates other alternatives. Model comparisons show that the four-state specification of bull and bear markets is strongly favored over several alternatives including a two-state model and an unrestricted four-state model, as well as various alternative specifications for variance dynamics. For example, relative to a two-state model, there is less erratic switching so that market regimes are more persistent.

We find that bull corrections and bear rallies are empirically important for out-of-sample forecasts of turning points and VaR predictions. Our model provides superior density forecasts of long-horizon returns.

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