Journal of Business & Economic Statistics, July 1994, Vol. 12, No. 3

Duration-Dependent Transitions in a Markov Model of U.S. GNP Growth

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Hamilton's nonlinear Markovian filter is extended to allow state transitions to be duration dependent. Restrictions are imposed on the state transition matrix associated with a τ-order Markov system such that the corresponding first-order conditional transition probabilities are functions of both the inferred current state and also the number of periods the process has been in that state. High-order structure is parsimoniously summarized by the inferred duration variable. Applied to U.S. postwar real GNP growth rates, we obtain evidence in support of nonlinearity, asymmetry between recessions and expansions, and duration dependence for recessions but not for expansions.

KEY WORDS: Nonlinear asymmetric cycles; Regime switches; Time-varying transition probabilities.

1. INTRODUCTION

In recent years there has been a growing interest in the econometric modeling of nonlinear temporal dependence in time series data. Indeed the possible nonlinearity and asymmetry of business cycles is an old topic in economics. For example, Terasvirta and Anderson (1991) referenced Keynes (1936), who argued that contractions are more violent but also more short-lived than expansions. The possibility of asymmetric nonlinear cyclical processes raises important issues for econometric estimation and testing.

Cyclical processes such as gross national product (GNP) have traditionally been modeled as linear stationary processes. This approach includes autoregressive moving average processes around a deterministic time trend (e.g., Campbell and Mankiw 1987; Nelson and Plosser 1982), linear unobserved-components models (e.g., Harvey 1985; Watson 1986), and cointegration specifications (e.g., King, Plosser, Stock, and Watson 1991). Obviously, linear models impose restrictions on the conditional densities of the variable in question. An interesting question is whether or not such restrictions are at odds with the data in particular applications.

One approach to answering this question has been to test a linear process against a nonspecified nonlinear alternative. For example, the Brock, Dechert, and Scheinkman (1987) statistic (hereafter referred to as the BDS statistic) can be used to test the null hypothesis that a time series is iid against a variety of alternatives that exhibit nonrandom structure. Although such structure could be linear or nonlinear, the test does have power against nonlinear alternatives (Brock, Hsieh, and LeBaron 1991) in contrast to some traditional tests for persistence.

Alternatively, many authors have proceeded in various directions from the null of linearity by characterizing particular features of the potentially nonlinear dynamics. In the context of modeling business-cycle indicators, these approaches to testing for and or modeling potential nonlinearities and asymmetries have included nonparametric tests for steepness of contractions versus expansions (Neftci 1984), nonparametric or seminonparametric approaches to evaluating asymmetry in the conditional distributions of GNP growth rates and employment (Brunner 1992; Hussey 1992), chaos models (Brock and Sayers 1988; Frank, Gencay, and Stengos 1988), threshold autoregressive models (Potter 1991), smooth transition autoregressive models (Terasvirta and Anderson 1991), regime-switching models with constant transition probabilities (Hamilton 1989a,b; Hansen 1991a; Lam 1990), regime-switching models with time-varying transition probabilities (Diebold, Lee, and Weinbach in press; Filardo 1992a,b; Ghysels 1992), duration models (Diebold and Rudebusch 1990; Diebold, Rudebusch, and Sichel 1993; Sichel 1991), and duration models with seasonal hazard rates (De Toldi, Gourieroux, and Monfort 1992).

This article extends Hamilton's (1989) model to allow state transitions to be duration dependent. In particular, we impose restrictions on the state transition matrix associated with a τ-order system such that the corresponding first-order conditional (time-varying) transition probabilities are functions of both the inferred current state and also the number of periods the process has been in that state. Because any effects of long lags of the Markov states are summarized by the inferred duration variable, this structure exploits information concerning state temporal dependence but is much more parsimonious than an unrestricted high-order Markov process. It may be useful to note that our model is a particular parameterization of a semi-Markov process. In the latter, one collapses the higher-order structure into a conditional holding time distribution.

Our nonlinear filter and smoother extends that of Hamilton to allow duration dependence. We are grateful to a referee for noting that this technical advance is related to that of Lam (1990), who generalized the Hamilton filter to allow for an autoregressive process without a unit root so that the Markov trend specification can be applied to GNP data in levels. In Lam's algorithm, dependence of GNP levels on all past lags of Markov states is captured by an additional Markovian state variable, which is the sum of past Markov states since the beginning of the sample. In our case, high-order dependence of states is summarized by the duration variable. In Lam's algorithm, however, the transition probabilities are constant, whereas ours are duration dependent.

Quasi-maximum likelihood estimation (QMLE) allows inference concerning nonlinearity, asymmetry, and state dependence of parameters associated with the first two conditional moments for various business-cycle indicators. Applied to duration dependence, our approach complements recent works that use a hazard-function approach. For example, Diebold and Rudebusch (1990) used nonparametric tests of the conformity of half-cycle and full-cycle lengths to the exponential distribution that would be implied by a constant hazard or no duration dependence. Sichel (1991) employed a parametric hazard function that nests the constant hazard; Diebold et al. (1993) used an exponential-quadratic hazard that is proposed to balance parsimony and flexibility.

Our extension of Hamilton's (1989) constant-transition-probability Markov-switching model to allow duration-dependent transition probabilities can also be compared (see Sec. 2) with the recent literature that extends the Hamilton model by allowing observable economic fundamentals (Diebold et al. in press; Filardo 1992a,b) or seasonal indicators (Ghysels 1992) to affect the state transition probabilities.

Comparing these Markov-switching nonlinear models with a linear AR null involves some nonstandard conditions that affect the asymptotic distributions of typical tests. In particular, the transition probabilities are not identified under the null, and the scores associated with parameters of interest under the alternative may be identically 0 under the null. Model comparison under these nonstandard conditions has been analyzed by several recent works (e.g., Davies 1987; Garcia 1992a,b; Hansen 1991a,b). Using the critical values of Garcia (1992b), which are based on the distribution theory of Hansen (1991b), our evidence suggests that, for this sample, the linear specification of real GNP growth rates can be rejected in favor of our Markov-switching alternative, which has duration-dependent transition probabilities.

Allowing regime switches between two states for the mean growth rate, as in the work of Hamilton (1989), our results provide evidence in support of asymmetry between contractions and expansions. This is indicated by the difference in the transition probabilities associated with these states and strong evidence of duration dependence associated with postwar recessions but not for expansions. Therefore, we reject the first-order Markov specification with constant transition probabilities in favor of a Markov-switching specification with duration-dependent state transition probabilities.

As discussed by Hamilton (1989), exploiting any nonlinear structure can be particularly important for optimal forecasts. In the case of the nonlinear Markov filter, the estimated parameters can be used in conjunction with the observable series to infer the probability of being in a particular state at a specific time. As emphasized by Hamilton, this output (either smoothed or conditional on current information) can be used to indicate turning points in the cycle. As such, this model could contribute to the range of tools being used in the leading-indicators literature [see the collection of papers edited by Lahiri and Moore (1991) for examples of recent advances in this area]. We compare our predictor of business-cycle turning points with the National Bureau of Economic Research (NBER) dating of U.S. business-cycle peaks and troughs. We also plot the expected half-cycle durations implied by our estimated model.

In Section 2 we discuss our extensions to the Hamilton (1989) model. The Appendix provides technical details, including a description of the unconditional probabilities and the associated transition matrix, the filter we use to estimate the nonlinear model and to infer probabilities concerning the unobservable states, and initialization of the filter. In Section 3 our model is applied to the real GNP series from Hamilton and our results are compared to those for his first-order model and to those for a linear AR model. These results include parameter and robust-standard-error estimates, model-comparison test statistics, plots of duration dependence, and plots of inferred probabilities concerning the observable state and expected half-cycle durations at each point of the sample. Section 4 provides some concluding comments.

2. A STOCHASTIC REGIME-SWITCHING MODEL FOR CYCLICAL PROCESSES

2.1 The Stochastic Setting

Consider a time series denoted y_t generated by the stochastic process

$$y_t = \mu(S_t) + \sum_{i=1}^{q} \phi_i(S_{t-i})\big(y_{t-i} - \mu(S_{t-i})\big) + \sigma(S_t)v_t, \qquad v_t \sim N(0,1). \quad (1)$$

The state operative at time t is indexed by the discrete-valued variable S_t. Assume a k-state Markov process for the state vector S_t. The true state is unobservable (is hidden or embedded) and must be inferred from the observations on the series y from time 0 to time t.

2.2 The Hamilton Model

Hamilton (1989) applied a filter that draws on the information contained in the observable series to make probabilistic inferences about the historical sequence of states $\{S_t\}_0^N$. The filter iterates over the length of the time series sample, producing, as a by-product, the likelihood that is maximized to obtain estimates of the vector of parameters that define the model. He set the number of states $k = 2$ and assumed a first-order Markov process for state transitions. The latter assumption implies that the transition probabilities are completely defined by the current ($t-1$) state. That is, using uppercase S to refer to the random state variable and lowercase to refer to a particular realization, $P(S_t = s_t \mid S_{t-1} = s_{t-1}) = P(S_t = s_t \mid S_{t-1} = s_{t-1}, \ldots, S_0 = s_0)$. Hamilton also assumed that the state transition probabilities are constant over the sample.

2.3 Time-Varying State Transition Probabilities

As noted previously, several recent works have extended the Hamilton model by incorporating time-varying state transition probabilities. Diebold et al. (in press) and Filardo (1992a,b) allowed the state transition probabilities to evolve as logistic functions of observable economic fundamentals, whereas Ghysels (1992) conditioned on seasonal indicators.

In particular, Diebold et al. (in press) provided an EM algorithm for estimation of the parameters of a model for which the transition probabilities evolve as logistic functions of $x_{t-1}\beta_s$, in which the conditioning vector x_{t-1} contains economic variables that might influence the transition probabilities. In this case, Hamilton's $P(S_t = s_t \mid S_{t-1} = s_{t-1})$ is extended to $P(S_t = s_t \mid S_{t-1} = s_{t-1}, x_{t-1})$. In his model and empirical applications, Filardo (1992b) conditioned on leading economic indicators at time t. Filardo (1992a) extended his time-varying transition-probability specification to a bivariate process.

2.4 Higher-Order Markov Processes

As suggested by Hamilton (1989), one way of increasing the information to his filter is to apply a higher-order transition matrix. One difficulty with estimating unrestricted high-order Markov models, however, is that the number of transition parameters, and thus the order of computational magnitude, grows rapidly with the order of the Markov process. This results in a substantial reduction in the degrees of freedom and eventual intractability. Some restrictions would be desirable.

2.4.1 Duration-Dependent Transition Probabilities. It is possible that time series that appear cyclical, or exhibit long swings, might have higher-order embedded Markov processes with state transition probabilities that exhibit duration dependence. We extend the Hamilton model by allowing the state transition probabilities to be functions of both the inferred current state *and* the associated number of periods the system has been in the current state. We refer to the latter as duration dependence and summarize it by the integer-valued random variable D_{t-1}.

In our case, the conditional state transition probabilities can be written as $P(S_t = s_t \mid S_{t-1} = s_{t-1}, D_{t-1} = d)$. That is, unlike Hamilton's specification, state transition probabilities are not completely defined by the current state. In particular, we extend his first-order Markov specification to a τ-order ($1 \leq \tau \leq N$) model, where N is the length of the sample for the time series being modeled. To focus on duration dependence, any effects of long lags of the Markov states are collapsed into the inferred duration variable D_{t-1}. In other words, restrictions are imposed on the ($2\tau \times 2\tau$) transition matrix such that it is very sparse.

Therefore, unlike Hamilton's (1989) specification, which imposes constant state transition probabilities over the sample, our conditional state transition probabilities are allowed to vary as a function of the number of periods (duration) that the system has been in a particular inferred state. On the other hand, unlike the extensions of Hamilton discussed in Section 2.3, our duration-dependent transition probabilities, $P(S_t = s_t \mid S_{t-1} = s_{t-1}, D_{t-1} = d)$, are inferred from the observable series $\{y_{t-1}, y_{t-2}, \ldots, y_0\}$ rather than being parameterized as functions of additional observable economic fundamentals. Although simpler in that respect, our specification incorporates higher-order effects to focus on state temporal dependence in cyclical data.

It remains to parameterize our conditional state transition probabilities $P(S_t = s_t \mid S_{t-1} = s_{t-1}, D_{t-1} = d)$. There are certain necessary properties and other desirable properties that such functions should exhibit. First, because they represent probabilities their values must always lie in the interval (0, 1). Second, their sum over all s_t must be equal to one for each t. Third, it would be useful for testing purposes if the functional specification nested a first-order Markov model with constant transition probabilities by being independent of D_{t-1} for given parameter values. Fourth, it is desirable that the specification be flexible so that it is capable of capturing a broad range of possible duration structures.

As is common in parameterizing probabilities or rates (e.g., Diebold et al. in press; Filardo 1992a, b), these objectives can be accommodated by a logistic functional form. For the two-state case ($i = 0, 1$), this results in the following functional specification for our characterization of the probabilities associated with changes of regime:

$$\begin{aligned} P(S_t = i \mid S_{t-1} = i,\ D_{t-1} = d) &\equiv P_{ii} \\ &= \exp(a(i) + b(i)d) \,/\, (1 + \exp(a(i) + b(i)d)) \\ &\qquad \text{if } d \leq \tau \qquad (2) \\ &= \exp(a(i) + b(i)\tau) \,/\, (1 + \exp(a(i) + b(i)\tau)) \\ &\qquad \text{if } d > \tau, \end{aligned}$$

$$P(S_t = j \mid S_{t-1} = i,\ D_{t-1} = d) \equiv P_{ij} = (1 - P_{ii}), \qquad (3)$$

where τ defines the memory of the process.

Therefore, the conditional probability matrix can be written as

$$P \equiv \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} = \begin{bmatrix} \dfrac{\exp(a(0) + b(0)d)}{1 + \exp(a(0) + b(0)d)} & 1 - \dfrac{\exp(a(0) + b(0)d)}{1 + \exp(a(0) + b(0)d)} \\ 1 - \dfrac{\exp(a(1) + b(1)d)}{1 + \exp(a(1) + b(1)d)} & \dfrac{\exp(a(1) + b(1)d)}{1 + \exp(a(1) + b(1)d)} \end{bmatrix}. \qquad (4)$$

That is, in the notation of Hamilton (1989) and others, for which the state $i = 0$ corresponds to recessions and the state $i = 1$ refers to expansions, P_{00} is the probability of being in a recession next period if in a recession this period, P_{01} is the probability of moving out of a recession, P_{11} is the probability of staying in an expansion, and P_{10} is the probability of moving out of an expansion.

Note that this parameterization of the conditional probabilities ensures that they lie in the interval (0, 1), sum to 1, and, if $b(i) = 0$ for all i, then $P(S_t = s_t \mid S_{t-1} = s_{t-1}, D_{t-1} = d) = P(S_t = s_t \mid S_{t-1} = s_{t-1})$ and the process collapses to a first-order Markov process identical to that assumed by Hamilton (1989).

The Appendix describes the filter we use to estimate the nonlinear model and to infer probabilities concerning the unobservable states, as well as initialization of the filter using unconditional probabilities. The Appendix also summarizes the algorithms used to compute those unconditional probabilities and provides a numerical example, computed at the parameter estimates given in Table 4, Section 3.2, and using the easily illustrated case of $\tau = 3$.

3. APPLICATION TO THE GROWTH RATES OF U.S. REAL GNP

3.1 Data and Empirical Models

Three models are compared using the data on postwar U.S. real GNP growth rates used by Hamilton (1989). These models include a linear (AR) specification, a first-order Markov model for regime switches with constant state transition probabilities, and our Markov model, which allows for duration dependence. Therefore, the test equations are nested versions of (1) with conditional probability matrices that are nested versions of (4).

The data are 100 times the change in the log of U.S. real GNP from the second quarter of 1951 to the fourth quarter of 1984. As in Hamilton (1989), the number of states k is set equal to 2 and q (the number of AR lags included in the mean) is set equal to 4. In this case, (1) can be rewritten as

$$y_t = \mu(S_t) + \sum_{i=1}^{4} \phi_i \left(y_{t-i} - \mu(S_{t-i})\right) + \sigma v_t, \qquad (5)$$

in which $\mu(S_t) = \alpha_0$ for $S = 0$ and $\mu(S_t) = \alpha_0 + \alpha_1$ for $S = 1$. That is, Hamilton assumed that the AR coefficients and the standard deviation are constant so that regime switches only shift the mean growth rate.

As demonstrated by Perron (1989), once one models discrete changes in regime, it may be possible to reject the unit-root specification. As noted previously, in this regard Lam (1990) generalized the Hamilton (1989) filter to allow for an AR process without a unit root and applied the Markov trend specification to GNP data in levels. Hansen (1991a) introduced a "modified Markov trend" model in which the states shift the intercept of growth rates rather than the mean. In that case, the $\phi_i(S_{t-i})$ and the $\mu(S_{t-i})$ in (1) are functions of S_{t-1} for all lags rather than functions of S_{t-i}.

Table 1. BDS Statistics for Real GNP Growth Rates

	ε = normalized SD	ε = normalized SD scaled by 1.2
Embedding dimension	BDS	BDS
3	2.29	2.34
5	1.90	2.52
7	2.32	3.44
9	2.30	3.68

NOTE: The BDS statistics are distributed as standard normal variates. The ε is chosen to be proportional to the standard deviation divided by its range.

Table 2. Linear Autoregressive Model

Parameter	Estimate	Robust standard error
α_0	.720	.103
ϕ_{01}	.310	.075
ϕ_{02}	.127	.090
ϕ_{03}	−.121	.082
ϕ_{04}	−.089	.078
σ	.983	.061

NOTE: $y_t = \alpha_0 + \sum_{i=1}^{4} \phi_{0i} y_{t-i} + \sigma v_t$. Value of the log-likelihood (excluding the constant) = −63.288.

3.2 Results

Table 1 presents some nonparametric evidence that the growth rates in this sample are not iid. Estimates for a linear fourth-order AR model are reported in Table 2. QMLE for the Hamilton (1989) specification are given in Table 3. That model specifies that the transition probabilities are constant and follow a first-order Markov process. Finally, Table 4 presents the estimates for the duration-dependent model for which the evolution of the states is determined by the conditional probability matrix parameterized as in (4).

Figure 1 plots the duration dependence associated with the model summarized in Table 4. Figure 2 presents the filtered probability that $S_t = 0$, the recession state, for each point in the sample according to the Hamilton model reported in Table 3. Figure 3 provides the corresponding probabilities for the model reported in Table 4 that allows duration dependence. These predictors of turning points are not smoothed. That is, they are ex ante in that they are conditional on current $(t-1)$ information. Finally, Figures 4 and 5 plot expected half-cycle durations conditional on the inferred state of recessions and expansions, respectively. As comparisons, the NBER dating of business-cycle peaks and troughs is marked with vertical lines from the horizontal axes of Figures 2 to 5.

Table 3. First-Order Markov Model: Constant Transition Probabilities

Parameter	Estimate	Robust standard error
α_0	−.359	.491
α_1	1.522	.474
ϕ_{01}	.014	.226
ϕ_{02}	−.058	.283
ϕ_{03}	−.247	.197
ϕ_{04}	−.213	.192
σ	.769	.122
$a(0)$	1.124	.688
$a(1)$	2.243	.296

NOTE: $y_t = \mu(S_t) + \sum_{i=1}^{4} \phi_{0i}(y_{t-i} - \mu(S_{t-i})) + \sigma v_t$, $\mu(S_t) = \alpha_0$ if $S_t = 0$, $\mu(S_t) = \alpha_0 + \alpha_1$ if $S_t = 1$, and $P_{00} = \exp(a(0))/(1 + \exp(a(0))) = .76$, $P_{11} = \exp(a(1))/(1 + \exp(a(1))) = .90$. Value of the log-likelihood (excluding the constant) = −60.882.

Table 4. Markov Model With Duration-Dependent Transition Probabilities

Parameter	*Estimate*	*Robust standard error*
α_0	−.448	.264
α_1	1.594	.273
ϕ_{01}	−.017	.105
ϕ_{02}	−.092	.107
ϕ_{03}	−.255	.094
ϕ_{04}	−.246	.103
σ	.761	.063
$a(0)$	6.516	2.055
$a(1)$	4.305	2.363
$b(0)$	−1.348	.296
$b(1)$	−.243	.282

NOTE: $y_t = \mu(S_t)+\sum_{i=1}^{4}\phi_{0i}(y_{t-i} - \mu(S_{t-i}))+\sigma v_t,\ v_t \sim N(0, 1),\ \mu(S_t) = \alpha_0$ if $S_t = 0$, and $\mu(S_t) = \alpha_0 + \alpha_1$ if $S_t = 1$.

$$P(S_t = j \mid S_{t-1} = i, D_{t-1} = d)$$
$$= \exp(a(i) + b(i)d)/(1 + \exp(a(i) + b(i)d)) \text{ for } i = j = 0, 1$$
$$= 1 - \exp(a(i) + b(i)d)/(1 + \exp(a(i) + b(i)d)) \text{ for } i \neq j.$$

Value of the log-likelihood (excluding the constant) = −55.860.

As discussed in Section 1, one can evaluate whether the GNP growth data exhibit (potentially nonlinear) dependence using the BDS test statistic. Table 1 presents a battery of BDS test statistics for alternative values of m and ε. The embedding dimension m refers to the dimension of the histories used to compute the correlation integral. The correlation integral measures the proportion of the m-dimensional points that are ε-close to each other according to the supnorm criterion. The ε is chosen to be proportional to the standard deviation of the series of real GNP growth rates divided (normalized) by its range. The proportionality factor in column 1 is unity and that in column 2 is 1.2. Although these BDS statistics are not independent, all but one have a p value of less than .05 indicating evidence against the null hypothesis that these GNP growth rates are iid. We now proceed to report estimation results for various models that incorporate linear or nonlinear temporal dependence.

Given correct specification of the first two moments, our QMLE of the parameters will generally be consistent and asymptotically normal (Bollerslev and Wooldridge 1992; Domowitz and White 1982). We compute standard errors that are robust to departures from the maintained conditional normality assumption using the diagonal of the matrix $(J^{-1}KJ^{-1})/T$, where J is the numerical approximation to the Hessian and K is the inner product of the score matrix, which we also evaluate numerically [Engle and Gonzalez-Rivera (1991) discussed potential loss of efficiency associated with QMLE and proposed a more efficient semiparametric approach]. Note that the scores used to compute K will reflect the smoothed probabilities as indicated by Garcia (1992a) and Hamilton (1993).

Table 2 reports estimates for a linear fourth-order AR model for the GNP growth rates. There is evidence of strong persistence associated with the first lag only.

Our estimates for the Hamilton specification (5) are given in Table 3. That model specifies that the state transition probabilities are constant and follow a first-order Markov process. The alternative regimes appear to be identifying a low growth state (conditional on $S_t = 0$, the growth rate, $\mu(0) = \alpha_0$, is estimated to be −.36% per quarter) and a high growth state (conditional on $S_t = 1$, $\hat{\mu}(1) = \hat{\alpha}_0 + \hat{\alpha}_1 = 1.16\%$ per quarter).

The robust t statistic associated with $\hat{\alpha}_1(\equiv \hat{\mu}(1) - \hat{\mu}(0))$ being significantly different from 0 is 3.21. As discussed in our introduction and in Section 3.3, the asymptotic distribution

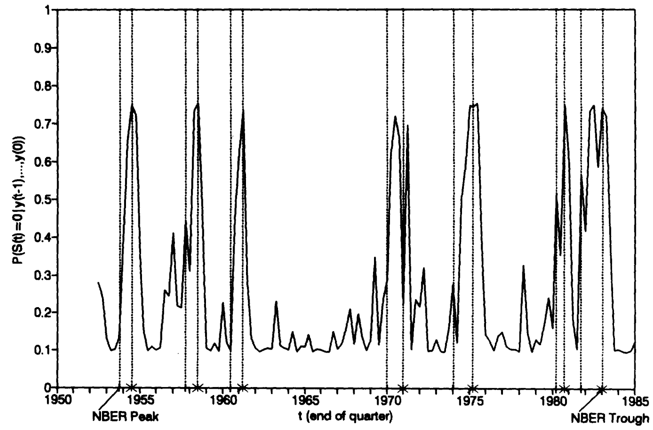

Figure 2. Inferred Probabilities That S(t) = 0: Hamilton Model.

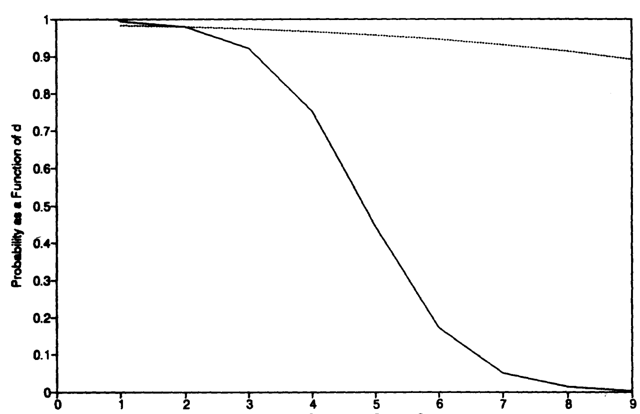

Figure 1. Duration-Dependent Transition Probabilities: - - - - - -, Expansion [P(S(t) = 1 | S(t − 1) = 1, D(t − 1) = d)]; ———, Recession [P(S(t) = 0 | S(t − 1) = 0, D(t − 1) = d)].

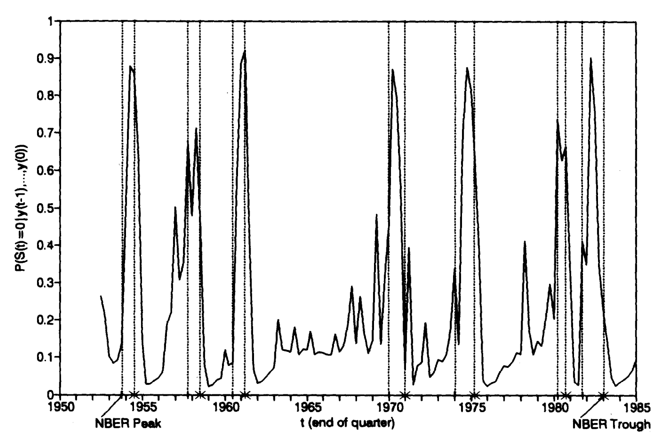

Figure 3. Inferred Probabilities That S(t) = 0: Duration-Dependent Model.

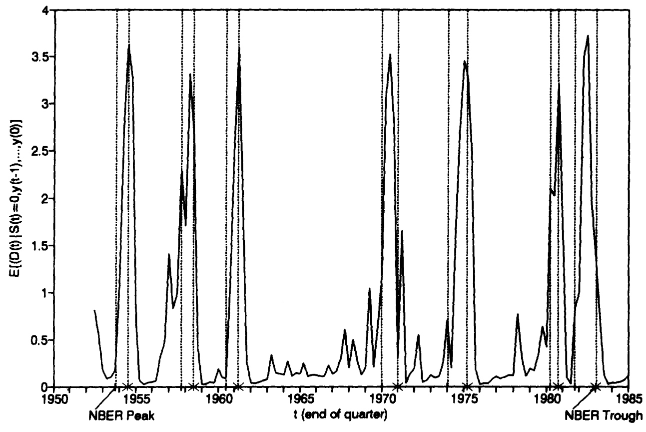

Figure 4. Expected Duration of Recessions.

for this test will be nonstandard. Using the critical values reported by Garcia (1992b), this t statistic has a p value of between .05 and .01. This suggests that we cannot reject the hypothesis that the growth rates are the same in the two states very convincingly. More on this follows.

The estimated constant transition probability associated with recessions ($P_{00} = .76$, $1 - P_{00} = .24$) is quite different from that associated with expansions ($P_{11} = .90$, $1 - P_{11} = .10$). These estimates imply shorter expected (constant) duration for recessions [$(1 - P_{00})^{-1} = 4.1$ quarters] than for expansions. Consequently, this nonlinear model suggests that the cycles are asymmetric.

Table 4 presents estimates for our model for which the evolution of the states is characterized by a duration-dependent process with conditional probability matrix parameterized as in (4). As discussed in Section 2, our specification nests the Hamilton first-order Markov model. For example, if $b(0) = 0$, there will be no duration dependence associated with recessions. The memory, τ, of the Markov process is set equal to nine quarters as a result of a grid search with the likelihood value as criterion.

Table 4 reports that the robust t statistic for $\hat{b}(0)$ is -4.55, which rejects that it is 0 and indicates a significant duration dependence for the probability of a transition out of recessions. In particular, as the recession ages, the probability of a transition into an expansion increases. The robust t statistic for $\hat{b}(1)$ is $-.86$, which suggests that, at least for $\tau = 9$ in this sample, there is no significant duration dependence associated with the probability of a transition out of expansions.

Graphic evidence of this asymmetry between recessions and durations is given in Figure 1. The inferred conditional probability that we stay in a recession appears to be strongly dependent on the number of quarters the system has been in a recession. Using an inferred probability of .5, the model predicts that on average there will be a move out of the recession after a duration of between four and five quarters. In contrast, U.S. postwar expansions do not exhibit nearly as strong duration dependence. These plots support the statistical inference based on the estimates reported in Table 4.

Figures 2 and 3 present, for each point in the sample, the inferred probability of being in the recession state [i.e., $P(S_t = 0 \mid y_{t-1}, \ldots, y_0)$] according to the models reported in Tables 3 (the Hamilton model) and 4 (the duration-dependent model), respectively. These predictors of turning points are conditional on ex ante $(t - 1)$ information rather than on contemporaneous (t) information and information available one year later $(t + 4)$ as in Hamilton's figure 1.

As a comparison, the NBER dating of business-cycle peaks and troughs is marked with vertical lines from the horizontal axes of these figures. Both models do quite well at indicating turning points. Using the same dating of information, however, the duration-dependent specification leads the Hamilton specification in predicting turning points and also exhibits more decisive probabilities associated with the inferred state.

Figures 4 and 5 plot the expected half-cycle durations, at each t, conditional on the inferred state of recessions and expansions, respectively. That is, for $i = 0, 1$,

$$E[D_t \mid S_t = i, y_{t-1}, \ldots, y_0] = \sum_{d=1}^{\infty} P(D_t = d \mid S_t = i, y_{t-1}, \ldots, y_0) \times d. \tag{6}$$

For example, Figure 5 indicates that changes in the expected half-cycle durations lead NBER turning points. This suggests that such a variable might also serve as a useful conditioning variable or leading indicator.

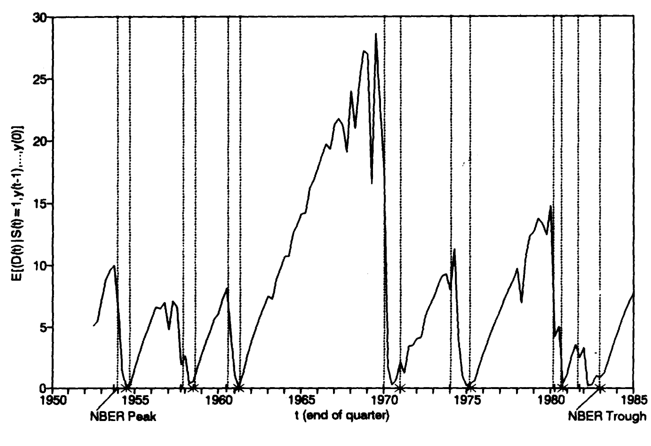

Figure 5. Expected Duration of Expansions.

3.3 Model Comparisons

Formal comparison of the two alternative nonlinear models with the linear AR null is difficult because the transition probabilities are not identified under the null and the scores associated with parameters of interest under the alternative will be identically 0 under the null for certain values of those parameters (e.g., $P_{11} = 1$). As discussed in the introduction, these nonstandard conditions affect the asymptotic distributions of typical tests. To compute asymptotic critical values under these conditions, Garcia (1992a,b) treated the parameters associated with the transition probabilities as nuisance parameters and bounded the unconditional probabilities away from the degenerate cases of 0 and 1.

Our likelihood ratio (LR) test statistic comparing the linear AR(4) model (Table 2) with the Hamilton first-order Markov specification (Table 3) is 4.812, but the LR test statistic comparing the linear AR(4) model with our duration-dependent specification (Table 4) is 14.856. Based on the appropriate critical values [Garcia 1992b, table 2] for the LR test statistic comparing a linear AR(4) model against a two-state Markov trend AR(4) model, we are unable to reject the linear model in favor of the first-order Markov model with constant transition probabilities (p value between .25 and .30). We *are* able however, to reject the linear model in favor of our model, which allows duration dependence (p value less than .01). Although this rejection of the linear model in favor of a duration-dependent parameterization of the state transition probabilities in a regime-switching model appears to be decisive, our results in this regard are preliminary. More work needs to be done with respect to any potential effect the additional nuisance parameters (introduced by our duration-dependence specification) might have on LR test statistics comparing nonlinear alternatives to a linear null.

In any event, we are able to reject the nonlinear first-order Markov specification for regime switches (Table 3) in favor of our nonlinear Markov model, which allows duration dependence (Table 4). The LR test statistic is 10.044, which has a p value of less than .01 according to the chi-squared distribution with 2 df. In addition, as discussed in the previous section, the robust t statistic for $\hat{b}(0)$ rejects that it is 0 and indicates significant positive duration dependence for the probability of a transition out of recessions. The robust t statistic for $\hat{b}(1)$ suggests that there is insignificant duration dependence associated with the probability of a transition out of expansions.

As discussed in Section 1, much recent work on duration dependence in GNP data has used hazard functions. Using nonparametric tests, Diebold and Rudebusch (1990) found the strongest evidence for duration dependence in prewar expansions. With respect to postwar data, they detected some weak evidence of positive duration dependence associated with postwar contractions but none for postwar expansions. This empirical evidence led them to conclude that the assumption of constant Markov transition probabilities by Hamilton (1989) is "particularly valid for expansions and perhaps less so for contractions, although the very small size of these samples may impair the power of the tests" (p. 613).

That concern about sample size no doubt motivated Diebold et al. (1993) to propose the exponential-quadratic hazard as balance between parsimony and flexibility. Their extensive empirical evidence includes the postwar U.S. experience for which they found strong positive duration dependence for contractions and no dependence for expansions.

As noted previously, our duration-dependent specification rejected Hamilton's constant transition model for his postwar sample. Therefore, our parametric approach produces evidence of duration dependence that is considerably stronger than that of Diebold and Rudebusch (1990) and closer to that of Diebold et al. (1993). The results are also similar, however, in that all three approaches detect an asymmetry with respect to duration dependence for recessions and expansions.

4. CONCLUDING COMMENTS

The econometric modeling of nonlinear temporal dependence in time series data involves many interesting challenges. This article takes one step in that direction by extending Hamilton's (1989) nonlinear Markov-switching model to allow duration dependence in the state transition probabilities. Explicit conditional probability representation of time duration in any given state allows inference concerning several hypotheses. For example, such dependence is one important source of asymmetries in the cycles that necessitate the use of nonlinear models.

QMLE allows inference concerning nonlinearity, asymmetry, and state dependence of parameters associated with the first two conditional moments for various business-cycle indicators. Our results, for a sample of postwar U.S. real GNP growth rates, provide evidence in support of nonlinearity and in particular of asymmetry between contractions and expansions. This asymmetry is indicated by the difference in the transition probabilities associated with these states and also evidence of considerably stronger (positive) duration dependence associated with postwar recessions than with expansions. This duration dependence result rejects the first-order Markov specification with constant transition probabilities for regime switches between two states in these data.

ACKNOWLEDGMENTS

We thank John Galbraith, René Garcia, Allan Gregory, Simon van Norden, Ed Petersen, Gregor Smith, and Hamish Taylor for useful comments and also James Hamilton for providing us with the data and code for his model. We also particularly thank the associate editor and two referees for very helpful suggestions. Financial support from the Social Sciences and Humanities Research Council of Canada is gratefully acknowledged.

APPENDIX: ESTIMATION ALGORITHM

A.1 A Nonlinear Filter

This appendix describes a nonlinear filter that extends that of Hamilton (1989) by relaxing the restrictions imposed by Hamilton's first-order Markovian filter and allowing the conditional state transition probabilities to be duration dependent. In particular, we impose restrictions on the transition matrix associated with a τ-order system such that the evolution of the corresponding first-order process depends not only on the current state but also on the number of periods the process has been in the current state.

A.2 Computing the τ-order Transition Matrix and Unconditional Probabilities

Given a τ-order system with two state variables, S and D, and assuming that $d_t \leq \tau$, the state vector at time t is

$$\chi_t = \big[(S_t = 0,\ D_t = 1)(S_t = 0,\ D_t = 2)\cdots (S_t = 0,\ D_t = \tau)(S_t = 1,\ D_t = 1)\cdots(S_t = 1,\ D_t = \tau)\big]' \qquad \text{(A.1)}$$

so that π, the vector of unconditional probabilities of χ, is $(2\tau \times 1)$ and the associated transition matrix, T, is $(2\tau \times 2\tau)$. Given T, one can solve for the unconditional probabilities π as the solution to

$$T'\pi = \pi \tag{A.2}$$

subject to

$$\pi'\iota = 1, \tag{A.3}$$

in which ι is a $(2\tau \times 1)$ vector of ones.

Using our parameterization of the conditional probability functions in (2) and (3), the sparse $(2\tau \times 2\tau)$ transition matrix, T, can be computed as follows:

For $j = 1$,

$$T_{i,j} = 1 - \frac{\exp\big(a(1) + b(1)(i-\tau)\big)}{1 + \exp\big(a(1) + b(1)(i-\tau)\big)} \quad \text{for } \tau < i \le 2\tau \tag{A.4}$$

$$= 0 \quad \text{otherwise.} \tag{A.5}$$

For $1 < j \le \tau$,

$$T_{i,j} = \frac{\exp\big(a(0) + b(0)i\big)}{1 + \exp\big(a(0) + b(0)i\big)} \quad \text{for } i = j - 1 \tag{A.6}$$

$$= 0 \quad \text{otherwise.} \tag{A.7}$$

For $j = \tau + 1$,

$$T_{i,j} = 1 - \frac{\exp\big(a(0) + b(0)(i)\big)}{1 + \exp\big(a(0) + b(0)(i)\big)} \quad \text{for } 1 \le i \le \tau \tag{A.8}$$

$$= 0 \quad \text{otherwise.} \tag{A.9}$$

For $\tau + 1 < j \le 2\tau$,

$$T_{i,j} = \frac{\exp\big(a(1) + b(1)(i-\tau)\big)}{1 + \exp\big(a(1) + b(1)(i-\tau)\big)} \quad \text{for } i = j - 1 \tag{A.10}$$

$$= 0 \quad \text{otherwise.} \tag{A.11}$$

Finally,

$$T_{\tau,\tau} = \frac{\exp\big(a(0) + b(0)\tau\big)}{1 + \exp\big(a(0) + b(0)\tau\big)};$$

$$T_{2\tau,2\tau} = \frac{\exp\big(a(1) + b(1)\tau\big)}{1 + \exp\big(a(1) + b(1)\tau\big)}. \tag{A.12}$$

For example, using the parameter estimates from our duration-dependent model reported in Table 4, and for ease of illustration setting $\tau = 3$, the sparse $(2\tau \times 2\tau)$ transition matrix T is

$$j\text{:} \quad i\text{:} \begin{bmatrix} .000 & .994 & .000 & .006 & .000 & .000 \\ .000 & .000 & .979 & .021 & .000 & .000 \\ .000 & .000 & .922 & .078 & .000 & .000 \\ .017 & .000 & .000 & .000 & .983 & .000 \\ .021 & .000 & .000 & .000 & .000 & .979 \\ .027 & .000 & .000 & .000 & .000 & .973 \end{bmatrix}$$

and the corresponding unconditional probabilities, associated with the state vector (A.1) with $\tau = 3$, are $\pi = [.0193 \quad .0191 \quad .2415 \quad .0193 \quad .0190 \quad .6817]'$.

A.3 Initialization of the Filter

Compute the unconditional joint probabilities

$$P(S_{q-1} = s_{q-1}, S_{q-2} = s_{q-2}, \ldots, S_0 = s_0, D_{t-1} = d), \tag{A.13}$$

using the unconditional probabilities computed in Section A.2, and the conditional transition probability functions from (2) and (3). For example,

$$\begin{aligned} P(S_4 = 1, S_3 = 1, S_2 = 1, S_1 = 1, D_4 = 7) &= P(S_0 = 1, D_0 = 2) \times P\big(S_1 = 1 \mid S_0 = 1, D_0 = 3\big) \\ &\times P\big(S_2 = 1 \mid S_1 = 1, D_1 = 4\big) \\ &\times P\big(S_3 = 1 \mid S_2 = 1, D_2 = 5\big) \\ &\times P\big(S_4 = 1 \mid S_3 = 1, D_3 = 6\big). \end{aligned}$$

We are grateful to a referee for very helpful suggestions concerning initialization of the filter.

A.4 Iterative Structure of the Filter

Step 1. In Step 1, use the input

$$P\big(S_{t-1} = s_{t-1}, \ldots, S_{t-q} = s_{t-q}, D_{t-1} = d \mid y_{t-1}, \ldots, y_0\big) \tag{A.14}$$

and compute

$$\begin{aligned} P\big(S_t = s_t, S_{t-1} = s_{t-1}, \ldots, S_{t-q} &= s_{t-q}, D_{t-1} = d \mid y_{t-1}, \ldots, y_0\big) \\ &= P\big(S_t = s_t \mid S_{t-1} = s_{t-1}, D_{t-1} = d\big) \\ &\times P\big(S_{t-1} = s_{t-1}, \ldots, S_{t-q} = s_{t-q}, \\ &\qquad D_{t-1} = d \mid y_{t-1}, \ldots, y_0\big), \end{aligned} \tag{A.15}$$

where the first term on the right side is the time t conditional probability matrix given by (4).

For the first iteration the input (A.14) comes from the initialization described in Section A.3. For subsequent iterations, that input is provided as output of the last step.

Step 2. From the output of Step 1, compute the joint conditional distribution

$$\begin{aligned} P(y_t, S_t = s_t, \ldots, S_{t-q} = s_{t-q}, D_{t-1} = d \mid y_{t-1}, \ldots, y_0) &= P(y_t \mid S_t = s_t, \ldots, S_{t-q} = s_{t-q}, y_{t-1}, \ldots, y_0) \\ &\times P(S_t = s_t, \ldots, S_{t-q} = s_{t-q}, \\ &\qquad D_{t-1} = d \mid y_{t-1}, \ldots, y_0). \end{aligned} \tag{A.16}$$

The second term on the right side is given by (A.15), and the first term is the state-dependent likelihood of y_t, which under the distributional assumption of conditional normality is given by

$$P(y_t \mid S_t = s_t, \ldots, S_{t-q} = s_{t-q}, y_{t-1}, \ldots, y_0)$$
$$= \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ -\frac{1}{2\sigma^2}\left[(y_t - \mu(S_t)) - \sum_{i=1}^{q} \phi_i(S_{t-i})(y_{t-i} - \mu(S_{t-i}))\right]^2 \right\}. \quad (A.17)$$

Step 3. Using (A.16), which is the output of Step 2, integrate out $S_t, \ldots, S_{t-q}$ and D_{t-1} to compute the conditional likelihood of y_t:

$$P(y_t \mid y_{t-1}, \ldots, y_0) = \sum_{s_t} \cdots \sum_{s_{t-q}} \sum_{d} P(y_t, S_t = s_t, \ldots, S_{t-q} = s_{t-q}, D_{t-1} = d \mid y_{t-1}, \ldots, y_0), \quad (A.18)$$

from which the log-likelihood function is formed as

$$L = \sum_{t}^{T} \log P(y_t \mid y_{t-1}, \ldots, y_0), \quad (A.19)$$

which may be maximized numerically to form estimates of the parameter set

$$\Omega \equiv (\mu, \phi, \sigma, P_{ii}), \qquad i = 0, 1. \quad (A.20)$$

Finally, to create the necessary input for the next iteration of the filter, we must obtain

$$P(S_t = s_t, \ldots, S_{t-q+1} = s_{t-q+1}, D_t = d | y_t, \ldots, y_o). \quad (A.21)$$

This is accomplished in two steps.

Step 4. First, we must add y_t to the right side of the conditional operator in the output of Step 2 to obtain the conditional probability

$$P(S_t = s_t, \ldots, S_{t-q} = s_{t-q}, D_{t-1} = d \mid y_t, \ldots, y_0) = \frac{P(y_t, S_t = s_t, \ldots, S_{t-q} = s_{t-q}, D_{t-1} = d \mid y_{t-1}, \ldots, y_0)}{P(y_t \mid y_{t-1}, \ldots, y_0)}, \quad (A.22)$$

which is the output from Step 2 standardized by the output of Step 3.

Step 5. Now, using the output of Step 4, compute

$$P(S_t = s_t, \ldots, S_{t-q+1} = s_{t-q+1}, D_t = x \mid y_t, \ldots, y_0). \quad (A.23)$$

For $1 < x \leq \tau$,

$$P(S_t = s_t, \ldots, S_{t-q+1} = s_{t-q+1}, D_t = x \mid y_t, \ldots, y_0)$$
$$= \sum_{s_{t-q}} P(S_t = s_t, \ldots, S_{t-q} = s_{t-q}, D_{t-1} = x - 1 \mid y_t, \ldots, y_0) \quad \text{for } S_t = S_{t-1} \quad (A.24)$$
$$= 0 \quad \text{for } S_t \neq S_{t-1}. \quad (A.25)$$

For $x = 1$, that is

$$P(S_t = s_t, \ldots, S_{t-q+1} = s_{t-q+1}, D_t = 1 \mid y_t, \ldots, y_0)$$
$$= \sum_{d} \sum_{s_{t-q}} P(S_t = s_t, \ldots, S_{t-q} = s_{t-q}, D_{t-1} = d \mid y_t, \ldots, y_0) \quad \text{for } S_t \neq S_{t-1} \quad (A.26)$$
$$= 0 \quad \text{for } S_t = S_{t-1}. \quad (A.27)$$

Using (A.23) as input, proceed to the first step of the next iteration and continue until convergence is obtained.

[*Received May 1993. Revised October 1993.*]

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