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3

Data Envelopment Analysis: Partial Survey and Applications for Management Accounting

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Abstract: Controlling cost in the organization is an essential element of the managerial accounting function. One important factor for controlling costs is the ability to measure the *ex post* efficiency performance of the firm (or department) either over time or by comparison to other firms (departments) in the industry. Data Envelopment Analysis (DEA) is a practical yet theoretically powerful tool for measuring *ex-post* efficiency performance. Moreover, since the application of DEA requires only physical input-output data—market prices are irrelevant—it is applicable to not-for-profit as well as profit-seeking organizations.

This paper provides an intuitive description of the DEA methodology and shows how DEA is related to neo-classical production theory. DEA is then applied to an example in order to illustrate the simplicity of application. The DEA literature is partially surveyed and potential applications to management accounting emphasized. The strengths and weaknesses of DEA are analyzed.

Measuring the *ex post* efficiency of the firm is important for cost control purposes. Data Envelopment Analysis (DEA) is an empirical procedure, originally developed by Charnes et al. [1978], for estimating the relative efficiency of a group of firms operating in the same industry, or of departments within the firm or of the firm itself over time. DEA is a comparatively simple procedure even by comparison with alternative efficiency measures such as conventional variance analysis. There are two reasons for this. First, DEA requires only the minimum of informational inputs, namely, input-output *quantity* data. Neither input nor output prices are needed for the analysis. Second, DEA ultimately translates into a linear programming problem. Given the many linear programming packages available for P.C.'s, solving a DEA problem requires little computational expertise.

The relative ease with which DEA can be implemented does not mean that the procedure is ad hoc. On the contrary, DEA is deeply rooted in neo-classical production function theory. Essentially, DEA estimates the firm's production function by using linear (or log-linear) approximations to map the envelope (frontier) of the input-output data. Firms which operate on the envelope are relatively efficient whereas firms operating off the envelope are relatively inefficient. Unlike most common econometric techniques for

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estimating production or cost functions, DEA neither estimates some *average* production function relationship nor does it assume that all firms satisfy first-order cost minimization conditions.[1] On the contrary, DEA estimates frontier production functions and allows for firm inefficiencies. Moreover, unlike most econometric methods which presuppose particular functional forms for the production function, DEA is a non-parametric technique.

Since only quantities, and not prices, are the informational requirements for DEA, it is equally applicable to not-for-profit institutions as well as profit-seeking organizations. In fact, DEA has been applied to determining the relative efficiency of such diverse industries as hospitals [Banker et al., 1986], schools [Bessent and Bessent, 1980; Charnes et al., 1981; Bessent et al., 1982], police services [Parks, 1983], and banking [Sherman and Gold, 1985].

This introduction to and survey of DEA is structured as follows. Section I describes the basics of DEA. This section also applies DEA to a simple example to illustrate the procedure. Section II describes the more recent advances in DEA. Section III partially surveys applications of DEA to various non-profit and profit-seeking institutions and attempts to generalize these applications to management accounting. Section IV concludes the paper by analyzing the strengths and weaknesses of DEA.

I. THE DEA PROCEDURE

Consider the problem of measuring the relative efficiency of n firms which operate in the same industry. Each firm j utilizes m factors of production $X_{1j},\ldots, X_{mj}$ to produce S products and/or services $Y_{1j},\ldots, Y_{Sj}$. Each factor of production has an implicit price (opportunity cost) of V_i i=1...,m and each product or service has an implicit price of U_r, r=1,...,S. These are not market prices nor are they specified by the analyst. Rather, as seen further on, these implicit prices are outputs of the analysis.

Efficiency of the firm is measured by the index

$$\frac{\sum_{r=1}^{S} U_r Y_{ro}}{\sum_{i=1}^{m} V_i X_{io}}$$

where the subscript o denotes the particular firm whose efficiency is currently being measured. The numerator of this index is the total value of the firm's outputs whereas the denominator is the total value of the firm's inputs using the implicit prices for valuation purposes. Alternatively, one can view this index as a Total Factor Productivity measure where the numerator is an index of the firm's outputs and the denominator is an index of the firm's inputs with weights U_r and V_i, respectively.[2]

[1]There are fairly recent econometric techniques for estimating frontier production functions which allow for potential inefficiency. See the citations in footnote 13 below. These techniques are much more difficult to implement than DEA and are relatively problematic as argued further below.

[2]On Total Factor Productivity, see Barlev and Callen [1986].

Let us now proceed to measure the efficiency of one of the j=1,...,n firms. This firm is denoted by the subscript o. DEA measures the efficiency of firm o by the fractional programming problem:

$$\underset{U_r, V_i}{\text{Maximize}} \quad \frac{\sum_{r=1}^{s} U_r Y_{ro}}{\sum_{i=1}^{m} V_i X_{io}} \tag{A}$$

Subject to:

$$\frac{\sum_{r=1}^{s} U_r Y_{rj}}{\sum_{i=1}^{m} V_i X_{ij}} \leq 1 \quad j=1,\ldots n$$

$$U_r, V_i > 0 \quad r=1,\ldots,S \quad i=1,\ldots,m$$

The objective function of program (A) is the efficiency index of firm o whose relative efficiency is currently being measured. There are n constraints involving the efficiency indices of all n firms in the industry including that of firm o. Each such constraint requires the firm's efficiency index to be no greater than one, for all firms j=1,...,n. These n constraints are necessary since no firm can be more than 100 percent efficient. The variables in this program are the implicit prices U_r, V_i which are constrained to be positive for all r=1,...,S and i=1,...,m. Since inputs and outputs are necessarily non-negative, each efficiency index has a lower bound of zero.

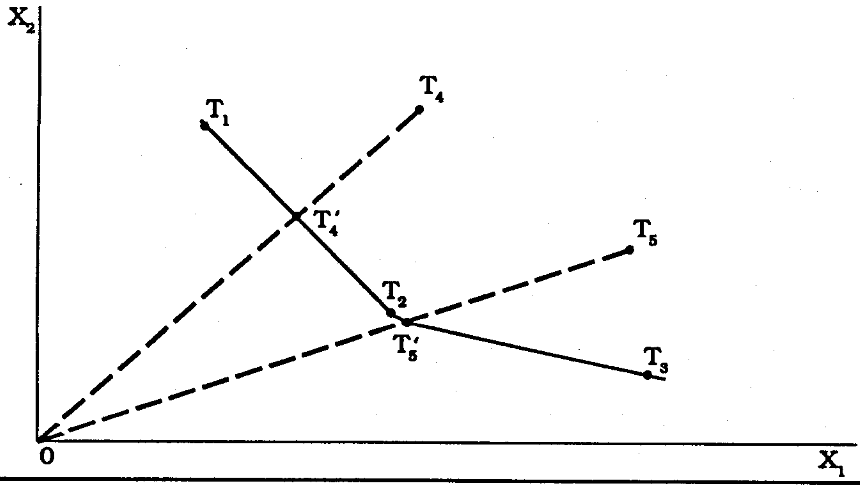
The objective of program (A) is to choose a set of positive implicit prices which maximize the efficiency index of firm o such that no firm—including firm o—is more than 100 percent efficient at these same set of prices. To appreciate this objective function, it must be understood that DEA measures efficiency in a relative sense. Specifically, firm o is inefficient if, by comparison to other firms in the industry, firm o could have reduced input usage with no reduction in outputs or produced more outputs with no reduction in inputs or both reduced inputs and increased outputs. Although this definition of efficiency is solely in terms of input-output quantities and prices appear to play no part, efficiency measurement and prices are closely related. Suppose, for example, that firm 1 uses the same level of inputs as say firm 15 and produces the same level of outputs except that firm 1 produces one less unit of output Y_1. Firm 1 is obviously inefficient by comparison to firm 15. Moreover, *for all* positive implicit input-output prices, firm 1's efficiency index is going to be less than that of firm 15 and, hence, must be less than one. In particular, by choosing implicit prices which maximize firm 1's efficiency index, firm 1's efficiency index will be less than one as it should be. Conversely, suppose firm 1 and firm 15 employ the same level of inputs and produce the same level of outputs except that firm 1 produces 4 units of Y_1 and 5 units of Y_2 whereas firm 15 produces 5 units of Y_1 and 4 units of Y_2. Both firms are equally efficient relative to each other and the objective of program (A) will show this. This is because the prices which maximize the firm's efficiency index will place a higher value on Y_2 by comparison to Y_1 when firm 1's efficiency is being measured whereas the objective of program (A) will place a higher value on Y_1 by comparison to Y_2 when firm 15's efficiency is being measured. More generally, the objective of choosing a set of prices which makes firm o's efficiency index as large as possible (but with no firm being more than 100 percent efficient), determines whether firm o is efficient or inefficient relative to other firms in the industry.

Program (A) was solved for firm o where firm o is one of the n firms in the industry, say firm j. However, this solution only establishes if firm j is efficient relative to the other n–1 firms in the industry. To determine if firm k, k≠j, is also efficient, one replaces the data for firm j with the data for firm k in the objective function. There is no need to change the constraints since these constraints refer to all n firms in the industry. Thus, n programs are solved separately to determine the relative efficiency of all firms in the industry.

DEA can be illustrated diagrammatically. Assume all firms produce the same output quantities with the two inputs X_1, X_2. Figure 1 shows the input combinations for five firms (n=5) labelled T_1 to T_5. The (piecewise linear) envelope for these five data points is the estimated isoquant T_1, T_2, T_3. Relative to this estimated isoquant, the input combinations T_4 and T_5 are inefficient since they use more of the inputs than would the other firms (1, 2 and 3) or some convex combination of them. Effectively, DEA measures inefficiency by the distance from the point of production to the isoquant along the ray from the origin. Thus, the inefficiency of T_4 is measured by the index $\frac{OT_4'}{OT_4}$ where T_4' lies on the estimated isoquant.

Similarly, the inefficiency of T_5 is measured by the index $\frac{OT_5'}{OT_5}$. Moreover, the lower the index, the greater the inefficiency. This index, called Farrell technical efficiency by the economics literature, was introduced into the accounting literature by Mensah [1982].

Figure 1
DEA and Technical Efficiency



Although program (A) in its current formulation is a nonlinear fractional programming problem, it can be transformed into a linear programming format which is easily solvable by any linear programming package. Specifically, Charnes et al., [1978] have shown that program (A) can be transformed into the following linear program (B):

$$\underset{W_i,\, t_r}{\text{Maximize}} \quad \sum_{r=1}^{S} t_r Y_{ro} \tag{B}$$

Subject to:

$$-\sum_{r=1}^{S} t_r Y_{rj} + \sum_{i=1}^{m} W_i X_{ij} \geq 0 \quad j=1,\ldots,n$$

$$\sum_{i=1}^{m} W_i X_{io} = 1$$

$$W_i,\ t_r \geq \varepsilon \quad i=1,\ldots n \quad r=1,\ldots,S$$

All the variables are defined as above except that the implicit prices are now denoted W_i and t_r.[3]

ε is a small positive number which in effect constrains the implicit prices to be positive. The importance of this is that it eliminates the possibility that one of the inputs (or outputs) be considered a free good. Consider for example the envelope in Figure 2. Here the envelope is comprised of the efficient loci NPR as well as the inefficient loci MN and RS. If any one of the implicit prices could take on a zero value then an efficiency indicator of one would not necessarily mean that the envelope is truly efficient. For example, if the implicit price of input X_2 were zero, then any point on the inefficient locus MN would be considered efficient since it would map onto the envelope. To eliminate loci like MN or RS from the envelope, one need only constrain the implicit prices to be positive so that it is more costly to utilize any point on the locus MN than the point N itself.

This issue can also be seen by analyzing the dual of program (B), namely,

$$\underset{L_j,\, Z_i,\, Z_r^1}{\text{Maximize}} \quad L_0 - \varepsilon\left[\sum_{i=1}^{m} Z_i + \sum_{r=1}^{S} Z_r^1\right] \tag{C}$$

Subject to:

$$-\sum_{j=1}^{n} X_{ij} L_j + L_0 X_{io} - z_i = 0 \quad i=1,\ldots,m$$

$$\sum_{j=1}^{n} Y_{rj} L_j - z_r^1 = Y_{ro} \quad r=1,\ldots,S$$

$$L_j,\ z_i,\ z_r^1 \geq 0 \quad i=1,\ldots,m \quad j=1,\ldots,n \quad r=1,\ldots,S$$

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[3] It can be shown that

$$W_i = Z V_i \text{ and } t_r = Z U_r$$

where $Z^{-1} = \sum_{r=1}^{S} U_r Y_{ro}$.

On this transformation from fractional to linear programming, see Charnes et al. [1978].

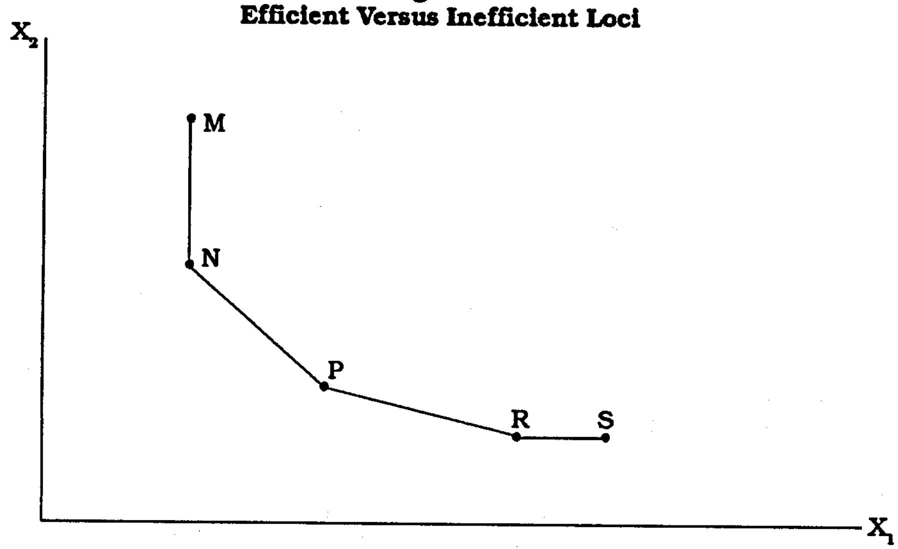
where the z_i and z_r^1 are slack variables and the L_j the dual (price) variables. Efficiency requires all the slack variables to be zero. Consider for example the first m constraints. If say the ith slack is positive, this means that firm o uses more of input i than would a convex combination of other firms in the industry, meaning that firm i is inefficient. Similarly, if slack r of the following S constraints is positive, then firm o is producing less of output r than would a convex combination of other firms and again firm o is inefficient. Positive implicit prices in the primal guarantee that if firm o is efficient, all of the dual slacks will be zero. As we shall see later, the dual is not simply an alternative to the primal but is useful in its own right.

AN EXAMPLE

This section shows how to measure the relative efficiency of three hospitals using DEA for a specific year's operations. These hospitals are assumed to have the same basic mission, that is, to be in the same industry, but need not necessarily be of the same comparable size. The data are hypothetical but illustrative. For descriptive simplicity, inputs are defined to be capital measured in machine-hours (X_1) and labor measured in labor-hours (X_2). Outputs are defined by the number of patient days for inpatients below age 14 (Y_1), patient days for inpatients aged between 14 and 65 (Y_2) and patient days for inpatients aged above 65 (Y_3).[4] These output definitions follow the studies of Conrad and Strauss [1983] and Banker et al. [1986]. In their case, however, inputs are defined somewhat differently. Specifically, they define capital as the number of hospital beds. Also, la-

[4]As pointed out by Conrad and Strauss [1933], the definition of output has been a constant problem in this industry. The definition used here focuses on the impact of age cohorts on hospital costs and is based on available Medicare reports. Output definitions based on well-offness indices would be more useful but are simply not available.

Figure 2
Efficient Versus Inefficient Loci



bor-hours in their study are disaggregated into three categories: (1) nursing services, (2) ancillary services (including operating room, anesthesiology, laboratory and x-ray labor) and (3) administrative and general services (including dietary and housekeeping labor).

Table 1 provides the relevant data for the year in question:

Table 1

Hospital	X_1	X_2	Y_1	Y_2	Y_3
1	5	14	9	4	16
2	8	15	5	7	10
3	7	12	4	9	13

The efficiency of hospital 2 is determined via program (B).[5] Substituting the data from Table 1 into program (B), with the subscript o referring to hospital 2, yields

$$\text{Maximize} \quad 5\,t_1 + 7\,t_2 + 10\,t_3 \tag{0}$$

Subject to:

$$-9\,t_1 - 4\,t_2 - 16\,t_3 + 5\,W_1 + 14\,W_2 \geq 0 \tag{1}$$

$$-5\,t_1 - 7\,t_2 - 10\,t_3 + 8\,W_1 + 15\,W_2 \geq 0 \tag{2}$$

$$-4\,t_1 - 9\,t_2 - 13\,t_3 + 7\,W_1 + 12\,W_2 \geq 0 \tag{3}$$

$$8\,W_1 + 15\,W_2 = 1 \tag{4}$$

$$W_i,\, t_r \geq .00001 \quad i = 1,2 \quad r = 1,2,3$$

Using a standard linear programming package, we obtain the following solution output:

Objective Function Value
0.7733030

VARIABLE	VALUE	ROW	DUAL PRICES
$(Y_3)\ t_3$	0.000010	1	0.261538
$(Y_2)\ t_2$	0.053328	2	0.000000
$(Y_1)\ t_1$	0.079982	3	0.661538
$(X_2)\ W_2$	0.066661	4	0.773383
$(X_1)\ W_1$	0.000010		

The objective function value is less than one so that hospital 2 is inefficient. The column headed VARIABLE denotes the implicit prices and in parentheses the input or output with which each price is associated. The column headed VALUE denotes the optimal values of the implicit prices.

The DUAL PRICES column (the solution to the dual) will be more informative to the management accountant than the VALUE column. The dual prices inform us as to the specific hospitals with respect to which

[5] We arbitritrarily set $\varepsilon = .00001$ in the program. Any other small positive value for ε would do. However, it should be recognized that the optimal solution is a function of the particular ε chosen. It also pays to solve the program with $\varepsilon = 0$ to see what impact a positive ε has on the optimal solution. In most cases even with $\varepsilon = 0$, the envelope will be efficient. Only on rare occasions is it really important for $\varepsilon > 0$.

hospital 2 is judged inefficient. In particular, non-zero dual prices indicate the basis vector against which hospital 2 is being compared. To understand this, refer again to Figure 1. The firm denoted by T_4 is inefficient relative to (the convex combination of) firms T_1 and T_2. Thus T_1 T_2 is the basis vector against which firm T_4 is being compared. Similarly, T_2 T_3 is the basis vector against which firm T_5's efficiency is being compared. In our example, since there are only three hospitals by assumption, it is fairly certain that the basis vector for hospital 2 consists of hospital 1 and hospital 3, although the basis vector could consist of just hospital 1 or just hospital 3. To determine the basis vector for hospital 2, that is, those hospitals against which hospital 2's efficiency is being measured, one looks first at the column labelled ROW. Each number in this column corresponds to the same number in the constraints of the linear program. Thus, 1 in the ROW column refers to constraint (1) in the linear program. Furthermore constraint (1) in the linear program is associated with hospital 1 since constraint (1) is made up of a linear combination of the inputs and outputs of hospital 1 (see Table 1). Similarly, 2 in the ROW column refers to constraint (2) in the linear program which is the constraint for hospital 2. Since the dual prices for hospital 1 (ROW number 1) and hospital 3 (ROW number 3) are non-zero, this indicates that hospitals 1 and 3 are efficient and comprise the basis vector for hospital 2.

There are two reasons why the management accountant would like to know the basis vector for hospital 2. First, this knowledge may help to indicate why hospital 2 is inefficient. Perhaps, the accountant knows that hospitals 1 and 3 have better trained staff or that their location is the cause for their efficiency. If the management accountant is aware of these qualitative factors, he is in a better position to advise how to deal sensibly with the inefficiency of hospital 2. Second, by utilizing the basis vector, the management accountant can compute via the dual prices, the amount of resources which could have been saved and/or outputs increased had hospital 2 been efficient. Specifically, hospital 1 and 3's input-output vectors are averaged using the dual prices as weights, yielding

$$.261538 \begin{bmatrix} 16 \\ 4 \\ 9 \\ 14 \\ 5 \end{bmatrix} + .661538 \begin{bmatrix} 13 \\ 9 \\ 4 \\ 12 \\ 7 \end{bmatrix} = \begin{bmatrix} 12.785 \\ 7.000 \\ 5.000 \\ 11.600 \\ 5.938 \end{bmatrix}$$

This weighted vector is what hospital 2 could have produced had it been efficient. Comparing this vector to hospital 2's actual input-output vector yields:

$$\begin{bmatrix} 12.785 \\ 7.000 \\ 5.000 \\ 11.600 \\ 5.938 \end{bmatrix} - \begin{bmatrix} 10.000 \\ 7.000 \\ 5.000 \\ 15.000 \\ 8.000 \end{bmatrix} = \begin{bmatrix} 2.785 \\ 0.000 \\ 0.000 \\ -3.400 \\ -2.062 \end{bmatrix}$$

Thus, had it operated efficiently, hospital 2 could have used 3.4 less labor-hours and 2.062 less machine-hours and still managed to produce an additional 2.785 patient days for patients over 65. This data is important to the management accountant because it tells him the true opportunity cost of hospital 2's inefficiency in terms of the specific resources wasted and the outputs not produced.

The information contained in the above table can also be used to compute marginal rates of substitution between inputs and marginal rates of transformation between outputs.[6] In particular, the marginal rate of substitution between inputs X_1 and X_2 is the implicit price ratio W_1/W_2 whereas the marginal rate of transformation between outputs f and g is the implicit price ratio $\frac{t_f}{t_g}$. For example, the marginal rate of transformation between outputs Y_2 and Y_1 in the above example is $\frac{.053328}{.079982}=0.67$. This means that a unit reduction in output Y_2 allows for a 0.67 unit increase in output Y_1 at the optimum. This is the sort of data the management accountant might wish to pass on to the production manager in order to help minimize production costs.

For future reference in this paper, we present the dual program for the above example, which takes the form

$$\text{Minimize } L_0 - .00001\,[z_1 + z_2 + z_1^1 + z_2^1 + z_3^1] \qquad (1)'$$

Subject to:

$$-5L_1 - 8L_2 - 7L_3 + 8L_0 - z_1 = 0 \qquad (2)'$$

$$-14L_1 - 15L_2 - 12L_3 + 15L_0 - z_2 = 0 \qquad (3)'$$

$$9L_1 + 5L_2 + 4L_3 - z_1^1 = 5 \qquad (4)'$$

$$4L_1 + 7L_2 + 9L_3 - z_2^1 = 7 \qquad (5)'$$

$$16L_1 + 10L_2 + 13L_3 - z_3^1 = 10 \qquad (6)'$$

$$L_j,\ z_i,\ z_r^1 \geq 0 \qquad j = 1,2,3 \quad i = 1,2 \quad r = 1,2,3$$

The solution output to this problem is:

Objective Function Value
0.7733030

Variable	**Value**
L_1	0.261539
L_2	0.000000
L_3	0.661538
z_1	0.248206
z_2	0.000000
z_1^1	0.000000
z_2^1	0.000000
z_3^1	2.784616

[6]The marginal rate of substitution is the rate at which one input can be reduced while another input is increased and still maintain the same level of production. Similarly, the marginal rate of transformation is the rate at which one output can be reduced while another output is increased and still use the same amount of resources.

In summary, this example has thus far illustrated the following uses of DEA:

(1) To determine if a particular firm or institution is efficient relative to others in the same industry. Hospital 2 was judged inefficient given an efficiency index less than one.

(2) To determine relative to which specific firms (institutions) in the industry is the firm in question inefficient. In the example, hospital 2 was deemed inefficient relative to (a convex combination of) hospitals 1 and 3.

(3) To determine the amount of resources which could have saved and/or the amount by which outputs could have been increased if the firm (institution) in question had been efficient. In our example, hospital 2 could have used 3.4 less labor-hours and 2.062 less machine-hours and still produced an additional 2.785 patient days for patients over 65 (in addition to the other outputs).

(4) To determine the marginal rates of substitution between inputs and marginal rates of transformation between outputs. In the hospital example, the marginal rate of transformation between the outputs Y_2 and Y_1 was found to be 0.67 so that a reduction in Y_2 allows for 0.67 unit increase in Y_1 at the optimum.

II. EXTENSIONS OF THE BASIC DEA MODEL

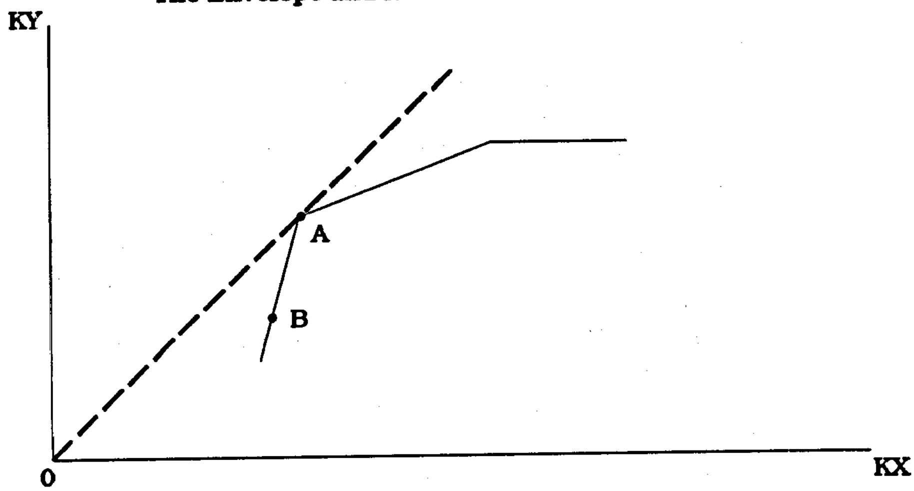
The basic DEA model, as pointed out earlier, is based upon neoclassical production theory. However, the model makes some fairly strong assumptions about the industry production technology which weakens the scope of its potential application. As a consequence, much theoretical work has been expended in trying to relax the assumptions of the basic model while at the same time maintaining the basic linear programming format with its ease of computation. *Inter alia*, the basic DEA model assumes (1) constant returns to scale, (2) constant marginal productivity, and (3) proportional wastage of all inputs and outputs if the firm is inefficient. As we shall see in this section, it is possible to relax each one of these assumptions; but for each assumption relaxed, one obtains an alternative DEA program. Moreover, except for the returns to scale variable—for which the same program can be used whether the firm exhibits constant, decreasing, or increasing returns to scale—DEA cannot determine which program is the correct one to use. That sort of information must come from *a priori* sources. For example, as we shall see, in order to incorporate increasing marginal productivity, the log-linear form of DEA must be used. For some industries, in fact, it is well known that the production technology exhibits increasing marginal productivity. The natural gas pipeline industry is a case in point [Callen, 1978]. Thus, if one is to investigate the efficiency of pipelines, it makes sense to use the log-linear form of DEA rather than the conventional form. Similarly, if there is reason to believe *a priori* that inefficiencies, to the extent that they exist, are caused by one input rather than all inputs proportionally (for example a poorly trained labor force), it makes more sense to use a non-radial form of DEA rather than the conventional form. The original DEA measure of efficiency was radial in nature so that efficiency was measured by reference to a ray from the origin to the production vector being examined (Figure 1). This means that inefficiencies are assumed to occur proportionally to all inputs and outputs. *Non-radial* measures allow for alternative assumptions about wastage. One such non-radial measure is presented further on.

There may be applications for which it may not be crucial which form of DEA is used. For example, if DEA is being used to rank the efficiency of the firm (department) relative to other firms (departments) in the industry (firm)—this might be the case if managers are rewarded in terms of the ranking of the efficiency index rather than its value. However, it may be that different DEA programs yield identical rankings. Some evidence that this can happen is provided by Callen and Falk [1989]. Since in general it is likely to be important which form of DEA is used, the remainder of this section describes alternatives to the basic DEA program.

Although DEA is primarily an empirical technique, much progress has been made in understanding the economic underpinnings of the model. Banker et al. [1984] have shown that the standard DEA framework is based upon four assumed properties of the production possibility set. Two of these properties, namely convexity and the ray unboundedness postulate, have subsequently been relaxed by the literature. The convexity property assumes that given any two producible input-output combinations, the average is also producible. The ray unboundedness postulate is equivalent to assuming that the production possibility set exhibits constant returns to scale throughout. In particular, this implies that if the vector (X,Y) is efficient then so is any other vector (KX,KY) where K >0.

The ray unboundedness assumption is most problematic. Suppose the true production possibility set does not exhibit constant returns to scale throughout as in Figure 3. Now, both points A and B are efficient because they are on the frontier of the production set. However, a ray from the origin to point A will dominate all points below the ray. This ray is the estimated envelope of the production set and relative to this estimated envelope, point B is deemed inefficient.

Figure 3
The Envelope and Non-Constant Returns to Sale



K is any positive constant

Banker et al. show that the constant returns to scale assumption can be relaxed with little change to the basic DEA framework. More formally, they show that for all degrees of scale economies (constant, decreasing or increasing) firm efficiency can be measured by the linear program.

$$\text{Maximize} \quad \sum_{r=1}^{S} t_r Y_{ro} - t_o \tag{D}$$

Subject to:

$$-\sum_{n=1}^{S} t_r Y_{rj} + \sum_{i=1}^{m} W_i X_{ij} + t_o \geq 0 \quad j=1,\ldots,n$$

$$\sum_{i=1}^{m} W_i X_{io} = 1$$

$$W_i,\ t_r \geq \varepsilon \quad i=1,\ldots,n \quad r=1,\ldots,S$$

where t_o is a new variable unconstrained in sign. In fact, the optimum value of t_o, t_o^*, can be used to determine the (local) degree of scale economies at the production vector being analyzed (X_o, Y_o). If $t_o^* = 0$, constant returns to scale obtains; $t_o^* < 0$ means increasing returns to scale; and $t_o^* > 0$ means decreasing returns to scale. Thus, by adding the new unconstrained variable t_o to the objective function and some of the constraints, DEA can be used to evaluate the efficiency of (X_o, Y_o) even with non-constant returns to scale.

The degree of (local) scale economies can be determined by referring to the original DEA program in its dual form. Specifically, the degree of local returns to scale is equal to the sum of the L_i variables $i=1,\ldots, n$ in the dual program (C) above. If this sum is less than one, decreasing returns to scale obtains at (X_o, Y_o); if it is equal to one constant returns to scale obtains; and if it is greater than one increasing returns to scale obtains. Considering our hospital example again, we see that the $\sum_{i=1}^{n} L_i = .923077$. Thus, decreasing returns to scale obtains for hospital 2.

For both of the efficient hospitals (1 and 3) we get $\sum_{i=1}^{n} L_i = 1$. As shown by Banker [1984] this result is actually quite general. If the production possibility set exhibits constant returns somewhere, then efficient points must exhibit constant returns. The intuition is that an efficient point is one for which all productivity gains due to increasing returns have been exploited, but decreasing returns have not yet set in. Such a point is referred to as the most productive scale size (m.p.s.s) because it yields the largest average productivity for a given mix of inputs and outputs.[7] In effect, all efficient points are necessarily m.p.s.s. In the event that the production possibility set exhibits either decreasing or increasing returns to scale throughout, then so will efficient points. In such a case an m.p.s.s. simply doesn't exist. The notion of m.p.s.s. is an important piece of economic data. A relatively low m.p.s.s. indicates that decreasing returns set in early

[7]It is important to constrain the mix, otherwise the average product in a multiple-input multiple-output firm is ill-defined since each potential mix defines a different average.

whereas a relatively high m.p.s.s. indicates that increasing returns prevail for large-scale sizes.

Banker also shows that for any production point (X_o, Y_o), the point

$$\frac{L_o^*}{\sum_{i=1}^{n} L_i^*} X_o, \frac{Y_o}{\sum_{i=1}^{n} L_i^*} \tag{6}$$

is m.p.s.s. where the * denotes the optimal values of the variables L_i in the dual program. If we consider hospital 2 again, $L_o^* = .77303$ and

$\sum_{i=1}^{n} L_i^* = .923077.$

Thus, the vector

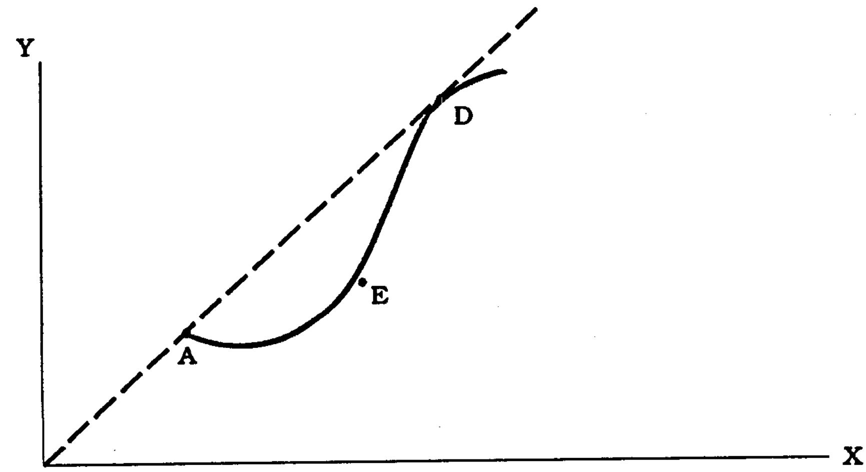
	X_1	X_2	Y_1	Y_2	Y_3
Hospital 2	6.7	12.6	5.4	7.6	10.8

is m.p.s.s. Of course, for the efficient hospitals 1 and 3, $L_o^* = 1$ and $\sum_{i=1}^{n} L_i^* = 1$ so that they are m.p.s.s. to begin with. An application which uses the m.p.s.s. concept is provided in the next section of this paper.

As shown by Banker and Maindiratta [1986], the convexity assumption in standard DEA analysis can also be relaxed, or to be more exact, replaced by geometric convexity. Unlike standard convexity, geometric convexity allows for cases of increasing marginal productivity. Consider Figure 4 below. The solid line shows the true production function. Both points D and E are efficient. However, point E also exhibits increasing marginal productivity. Standard DEA based on standard convexity would presuppose the envelope to be on the segment AD and deem point E to be inefficient although it is truly efficient.

The major effect of replacing standard convexity by geometric convexity is that the production frontier is estimated by piecewise Cobb-Douglas (log linear) envelopes rather than by piecewise linear envelopes. The Cobb-

Figure 4
The Envelope and Increasing Marginal Productivity



Douglas curvature allows for non-constant rates of technical substitution and transformation as well as increasing marginal products. Thus, DEA with geometric convexity is a potentially important generalization of standard DEA analysis.

The DEA approach with geometric convexity can also be cast in a linear programming framework. Formally, the efficiency of the production vector (X_o, Y_o) can be determined by the program

$$\text{Minimize} \quad \sum_{i=1}^{m} W_i \hat{X}_{io} - \sum_{r=1}^{S} t_r \hat{Y}_{ro} + t_o \qquad \text{(E)}$$

subject to:

$$-\sum_{r=1}^{S} t_r \hat{Y}_{rj} + \sum_{i=1}^{m} W_i \hat{X}_{ij} + t_o \geq 0 \quad j=1,\ldots,n$$

$$\sum_{r=1}^{S} t_r = 1$$

$$W_i,\ t_r \geq \varepsilon \quad i=1,\ldots,m \quad r=1,\ldots,S$$

where carets denote logarithms.

Banker and Maindiratta point out that the solution to this program also yields (i) marginal rates of substitution $(W_i^*/X_{io}) / (W_k^*/X_{ko})$, (ii) marginal rates of transformation $(t_r^*/Y_{ro}) / (t_k^*/Y_{ko})$, (iii) marginal product of input i for output r $(W_i^*/X_{io}) / (t_r^*/Y_{ro})$ and (iv) economies of scale $\sum_{i=1}^{m} W_i^*$.

Estimating the efficiency of hospital 2 in our example using this new form of DEA yields the program

$$\text{Minimize } (\ln 8)\, W_1 + (\ln 15)\, W_2 - (\ln 5)\, t_1 - (\ln 7)\, t_2 - (\ln 10)\, t_3 + t_o \qquad (7)$$

Subject to:

$$(\ln 5)\, W_1 + (\ln 14)\, W_2 - (\ln 9)\, t_1 - (\ln 4)\, t_2 - (\ln 16)\, t_3 + t_o \geq 0 \qquad (8)$$

$$(\ln 8)\, W_1 + (\ln 15)\, W_2 - (\ln 5)\, t_1 - (\ln 7)\, t_2 - (\ln 10)\, t_3 + t_o \geq 0 \qquad (9)$$

$$(\ln 7)\, W_1 + (\ln 12)\, W_2 - (\ln 4)\, t_1 - (\ln 9)\, t_2 - (\ln 13)\, t_3 + t_o \geq 0 \qquad (10)$$

$$t_1 + t_2 + t_3 = 1$$

$$W_i,\ t_r \geq .00001 \quad i = 1,2 \quad r = 1,2,3$$

The solution to this program is:

Objective Function Value
0.015007440

Variable	Value
W_1	0.000010
W_2	0.000010
t_1	0.499993
t_2	0.499997
t_3	0.000010
t_o	1.794965

Exponentiating the objective function value and taking the reciprocal, yields the efficiency index 0.9851.[8] Thus, although hospital 2 is still inefficient by comparison to 1 and 3, the degree of inefficiency is much less than would be the case from standard DEA analysis. This is not surprising given our earlier discussion of geometric convexity and, especially, of Figure 4. Also, hospital 2 now exhibits almost constant returns to scale rather than decreasing returns as before. The marginal rate of substitution between Y_1 and Y_2 is virtually the same as before. Therefore, if the true production function leans more towards a Cobb-Douglas curvature than linear envelopes, hospital 2 is far more efficient than the conventional analysis has shown. Unfortunately, however, without a priori knowledge of hospital production functions, one cannot specify which form of DEA is appropriate and, hence, the extent of hospital 2's inefficiency. This is one of the weaknesses of the DEA procedure.

Another extension to the basic DEA model involves utilizing piecewise Cobb-Douglas envelopes with a non-radial efficiency measure. Thus, Charnes et al. [1982, 1983] suggest a multiplicative measure of efficiency measurement. In fractional programming format, this approach is

$$\text{Maximize} \quad \left(e^{p_1} \prod_{r=1}^{S} Y_{ro}^{u_r}\right) \Big/ \left(e^{p_2} \prod_{i=1}^{m} X_{io}^{V_i}\right) \tag{F}$$

Subject to:

$$\left(e^{p_1} \prod_{r=1}^{S} Y_{rj}^{u_r}\right) \Big/ \left(e^{p_2} \prod_{i=1}^{m} X_{ij}^{V_i}\right) \le 1 \quad j=1,\dots,n$$

$$U_r, V_i \ge \varepsilon \quad p_1, p_2 \ge 0 \quad r=1,\dots,S \quad i=1,\dots,m$$

As we mentioned above, previous DEA measures of efficiency were radial in nature so that efficiency was measured by reference to a ray from the origin to the production vector being examined.[9] The multiplicative measure is non-radial. A radial measure assumes that any inefficiencies (wastage) occur proportionally to all inputs and outputs. This is a strong assumption.[10] Non-radial measures, on the other hand, make alternative assumptions about wastage but what these are is difficult to specify at times. Figure 5 illustrates the potential difference between a radial and non-radial measure. $I_1 I_2$ is an isoquant. X^A is the vector of inputs used to produce the output denoted by $I_1 I_2$. Farrell technical efficiency, upon which DEA is based, measures the inefficiency of X^A along the ray OX^A to the efficient-vector Q in Figure 5. An alternative non-radial measure might measure the inefficiency of X^A along the line UX^A to the efficient vector W.[11] Which measure is preferable depends upon the nature of wastage in this firm. If all inputs are being wasted proportionally, then the Farrell measure is preferable. If it is input 2 which is primarily being wasted then the alternative non-radial measure may be preferable.

[8]Since the index is in log form, exponentiation is required. Also, since the objective is to minimize rather than to maximize, it is the reciprocal which is the efficiency index.

[9]Also called Farrell technical efficiency or Shephard's [1970] inverse distance function.

[10]In fact, this assumption has been strongly criticized by Kopp [1981] and Marcinko and Petri [1984].

[11]See Kopp [1981].

The multiplicative efficiency measure can also be formulated in linear programming terms, namely,

$$\text{Maximize} \quad \sum_{r=1}^{s} t_r \hat{Y}_{ro} - \sum_{i=1}^{m} W_i \hat{X}_{io} + t_o \tag{G}$$

Subject to:

$$-\sum_{r=1}^{s} t_r \hat{Y}_{rj} + \sum_{i=1}^{m} W_i \hat{X}_{ij} - t_o \geq 0 \quad j=1,\dots,n$$

$$W_i,\ t_r \geq \varepsilon . \quad r=1,\dots,S \quad i=1,\dots,m$$

Using program (G) to evaluate the efficiency of hospital 2 in our example yields

$$\text{Maximize} \quad (\ln 5)\ t_1 + (\ln 7)\ t_2 + (\ln 10)\ t_3 - (\ln 8)\ W_1 - (\ln 15)\ W_2 + t_o \tag{11}$$

Subject to:

$$-(\ln 9)\ t_1 - (\ln 4)\ t_2 - (\ln 16)\ t_3 + (\ln 5)\ W_1 + (\ln 14)\ W_2 - t_o \geq 0 \tag{12}$$

$$-(\ln 5)\ t_1 - (\ln 7)\ t_2 - (\ln 10)\ t_3 + (\ln 8)\ W_1 + (\ln 15)\ W_2 - t_o \geq 0 \tag{13}$$

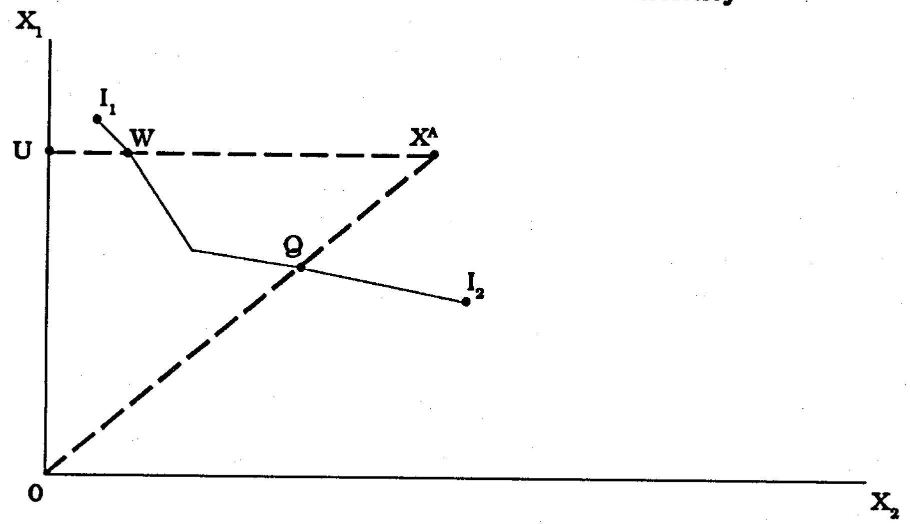
$$-(\ln 4)\ t_1 - (\ln 9)\ t_2 - (\ln 13)\ t_3 + (\ln 7)\ W_1 + (\ln 12)\ W_2 - t_o \geq 0 \tag{14}$$

$$W_1,\ t_r \geq .00001 \quad i = 1,2 \quad r = 1,2,3$$

This program yields an efficiency index of 0.9999906 for hospital 2. Although hospital 2 is inefficient relative to hospitals 1 and 3, the inefficiency is almost neglible. Thus, the impact of a non-radial multiplicative measure on top of piecewise Cobb-Douglas envelopes, is to make hospital 2 in our example look even more efficient than before.

In summary, this section has illustrated three extensions of the basic DEA program. The first extension (program D) analyzed efficiency for firms which exhibit increasing or decreasing returns to scale as well as the case of constant returns to scale. The second extension (program E) analyzed efficiency in the case where the firm's technology exhibits increasing

Figure 5
Radial Versus Non-Radial Technical Efficiency



marginal productivity. Finally, a non-radial extension of DEA (programs F and G) was illustrated for cases wherein wastage does not occur proportionally for all inputs and outputs.

III. APPLICATIONS

This section surveys some of the applications of DEA in order to provide some idea of the scope of this technique. In addition, the lessons to be learned from these studies for management accounting are discussed.

DEA was initially applied by Charnes and Cooper [1980] and Charnes et al. [1981] to assess the effectiveness of "Program Follow-Through." This was a large-scale experiment in public education to help disadvantaged children from kindergarten to grade three. The program was undertaken by 70 selected schools throughout the United States. To determine the effectiveness of this program, Charnes et al. compared the relative efficiency of schools with Program Follow-Through (PFT) against a matched pair sample of Non-Follow-Through (NFT) schools. Outputs were defined by reading scores, mathematical scores and self-esteem scores from educational achievement tests at the end of grade three. Inputs were defined by the educational level of the mother, highest education of a family member, number of parent visits to the school, time spent with the child on school related topics and the number of teachers at the school.

Using DEA, efficiency frontiers (as in Figure 1) were computed separately for each of the NFT and PFT schools. Neither frontier was found to dominate the other implying that for some range of the inputs, NFT schools were more efficient whereas for another range of the inputs PFT schools were more efficient. Not satisfied with this inconclusive result, an inter-envelope efficiency measure was also computed. The idea here is to derive the efficiency frontier for all 140 schools in both categories. To determine the inter-envelope, first the data of those PFT schools judged inefficient relative to other PFT schools were adjusted using the m.p.s.s. formula (equation (6)) so that they too were on the efficiency frontier of the PFT schools.[12] In other words, after the adjustment, all PFT schools would be on the efficiency frontier for PFT schools. Similarly, the data of all the inefficient NFT schools were adjusted so that they too lay on the frontier of the NFT schools so that no NFT school was inefficient relative to the other NFT schools. Finally, after these adjustments, DEA was performed on all of the data together to yield one efficiency frontier for all the schools. Here the results were unambiguous. While many of the NFT schools were inefficient relative to the overall efficiency frontier, virtually all of the PFT schools were efficient providing evidence for the effectiveness of Program Follow-Through.

The methodology of this study is also applicable to management accounting in situations where the industry has been subjected to a radical policy change and the management accountant is asked to evaluate the impact of this change on costs. For example, suppose the management accountant is asked to evaluate the impact of deregulation on the efficiency

[12] This can be accomplished by taking the inefficient production point (X_o, Y_o) and adjusting it by the L_1 dual variables as in equation (6). In our hospital example as shown earlier, hospital 2 can be made efficient if X_1=6.7, X_2=12.6, Y_1=5.4, Y_2=7.6 and Y_3=10.8.

of an airline's operations. He could use DEA to evaluate his firm's efficiency both prior to and after deregulation. In addition, to determine the impact of deregulation on the industry generally, he could use the m.p.s.s. formula to place inefficient firms on their frontiers, that is the frontier prior to regulation and the frontier after deregulation. After this, DEA could be performed on all of the data simultaneously (each firm is likely to appear twice) to see if firm efficiency prior to deregulation dominated or was dominated by firm efficiency after regulation.

In another educational study, Bessent et al. [1987] used DEA to evaluate the relative efficiency of 241 schools in the Houston Independent School District. Outputs were measured by a composite score from a comprehensive achievement test in each of grades three and six. Inputs were defined by 12 characteristics of the student body and the resources available to the school. DEA was then performed on all schools in the district to determine their relative efficiency. The validity of the DEA results was assessed by an informal test whereby the General Superintendent and his staff identified outstanding schools and trouble schools in the district before looking at the DEA results. DEA provided a 100 percent correct classification of the informal ranking in that the outstanding (i.e., efficient) and trouble (inefficient) schools identified by DEA were identical to those identified by the General Superintendent. In addition, the reasons suggested for the schools' status by the superintendent and his staff coincided generally with the magnitude of the DEA slack variables.

Sherman and Gold [1985] used DEA to determine the relative efficiency of 14 branches of a savings bank. Because of the large number (17) of potential transaction services offered by the bank, Sherman and Gold, in consultation with bank management, decided to limit the number of outputs to four category types. These types were defined by the complexity and resources demanded by the underlying transactions. For example, loan applications and new passbook loans were put into the most resource consuming and complex category. On the other hand, deposits and checks cashed were placed together in the least resource consuming category. Inputs were defined by labor hours employed, office space as measured by rent paid, and supplies. The results showed 6 of the 14 branches to be relatively inefficient. When the results were shown to bank management they pointed out that the most inefficient branch had already been earmarked for termination. In two other cases, management maintained that the small sizes of the branches and the resultant scale diseconomies were the reasons for inefficiency. In other cases, management was unaware of the inefficiency and placed the issue on the agenda for future analysis by internal auditors.

Both the latter studies use the DEA efficiency measure to determine which parts of the organization (school or branch) are efficient and what are the opportunity costs of being inefficient. While these issues are important for the management accountant, they are not the whole story as the two studies indicate. In particular, these studies illustrate that efficiency cannot be measured in a vacuum. Managerial input is crucial for evaluating the efficiency indices and for making decisions based upon them. The management accountant must also necessarily involve management in evaluating the indices since management is likely to have the knowledge necessary for pinpointing the potential causes of the inefficiencies.

These studies also show how DEA can be used, like cost variances, as a monitoring device to focus on those potentially problematic branches or departments which should be evaluated more fully.

Parks [1983] used DEA to evaluate the relative efficiency of 469 municipal police agencies. Inputs were defined by the number of sworn officers, civilians and vehicles available to each agency. Outputs were defined by the number of patrol units deployed to respond to requests for service and the number of crimes cleared by arrest in a year. After determining the relative efficiency of each unit, Parks related the efficiency score to alternative factors not included among the DEA variables. Thus, Parks related the DEA score to the proportion of the serviced population living in families whose income was below the poverty line, unionization of the agency, age of the police agency, and personnel policies of the agency. Although these other factors showed a weak correlation with the efficiency measures, Parks' analysis suggested that certain personnel and structural issues—such as more civilianization of the police force and reduction in the number of divisions—might improve efficiency.

This study shows how one might try to relate efficiency to other quantitative and qualitative variables in order to explain the value obtained for the efficiency index. This methodology is likely to prove useful for the cost accountant who works for a non-profit organization where the cost behavior of the organization cannot be related directly to inputs and outputs and where inputs and outputs may be ill-defined.

Banker et al. [1986] evaluated the relative-efficiency of 14 hospitals in North Carolina using both DEA and econometric techniques. The econometic approach modifies the first-order conditions to allow for technical inefficiency.[13] Using a flexible function form translog cost function, share equations were derived and estimated using a non-linear maximum likelihood technique.[14] Outputs were defined as in our example above in terms of the number of patient days for different age cohorts. Inputs were defined by the amount of nursing services, ancillary services, administrative and general services and capital. With the exception of scale economies, both approaches to efficiency measurement tended to yield similar results. However, while the econometric approach could not reject constant returns to scale, DEA found both local increasing and decreasing returns. In particular, DEA found that decreasing returns tended to set in early for hospitals with a larger proportion of elderly (Medicare) patients, a result well known from the medical literature.[15] Banker et al. argued that, because the econometric approach is based on aggregate data, the economies of scale in one hospital may well be offsetting the diseconomies of another yielding constant returns on average. They also found that the econometric efficiency estimates, in contrast to DEA, bore no discernible relationship to the observed capacity utilization of the hospital.

[13]There are currently a number of econometric techniques for measuring inefficiency with frontier production functions. See Schmidt and Lovell [1979, 1980], Schmidt and Lin [1984] and, more recently, Kumbhakar [1987, 1988].

[14]See Barlev and Callen [1986] and the references cited therein on estimating share equations derived from flexible functional form production (cost) functions.

[15]As cited by Banker et al. [1986].

Finally, although no study was found to illustrate the issue, DEA could be used to reward managers. For example, *one* input into evaluating the performance of bank branch managers could be the efficiency index of that branch relative to other branches. What may be crucial here is the relative ranking of the branch rather than the actual value taken by the index.

IV. CONCLUSION

DEA is a technique for determining the relative efficiency of both profit and non-profit institutions. DEA is easy to apply because input-output quantity data are the only informational requirements and because efficiency is computed by a linear programming formulation.

Unlike extant econometric techniques for estimating efficiency, DEA need not specify—within limits—a particular functional form for the production technology.[16] Unlike econometric techniques, DEA does not impose the requirement of differentiability of the production frontier. This is especially crucial where the firm faces capacity constraints.[17] Unlike econometric techniques, the focus of DEA is on the individual decision-making unit, either the firm or the department. Thus, (local) economies of scale are determined for the individual firm rather than for the industry as a whole. By aggregating and averaging over many firms, the economies of scale estimate derived econometrically may not even be a meaningful number. Because DEA requires only physical input-output quantity data, there is less chance of confusing technical with allocative inefficiency.[18] Input and output prices are central to most econometric techniques for measuring efficiency and hence are likely to confuse the two.

In addition to these apparent strengths of DEA, there are weaknesses as well. Most troublesome is the absence of a statistical testing methodology of the efficiency index. Does an index of 0.9 mean the firm is inefficient or, perhaps, this number is not statistically different from an index value of one? It is not entirely correct to say that DEA does not presuppose a specific functional form for the production technology. After all, the underlying envelopes may be piecewise linear or piecewise Cobb-Douglas. Is there some way of testing whether one is preferred to the other or is the management accountant forced, as we argued earlier, to search for *a priori* information as to which procedure is applicable? The issue is important since whether one uses linear or Cobb-Douglas envelopes can affect the value of the index substantially. What impact does the aggregation of the inputs or outputs have on the results? In the Sherman and Gold study, 12 outputs were aggregated and placed into 4 categories. Intuition suggests that the more disaggregated the data, the more likely any given firm will prove to be efficient. How distortive is the aggregation bias?

DEA analysis to date has tended to ignore the issue of technological change. Firms are generally compared cross-sectionally under the assumption that all utilize the same technology. But suppose one has both

[16]The econometrics approach does not necessarily specify a specific function, but it does specify a family of functions. See Berndt and Khaled [1979].

[17]See Charnes et al. [1982].

[18]On allocative versus technical efficiency, see Mensah [1982].

cross-sectional and time series data. How should these be combined in a DEA problem? In theory, if one could specify the two technologies, then DEA could be applied to each set of firms separately, just as Charnes, Cooper and Rhodes did in the case of the NFT and PFT schools. But for many applications, such a simple dichotomy of the data would not be known *a priori*.[19]

[19]Although this potential problem with DEA applies to the econometric approach as well, it is easier to endogenize technological change in a flexible functional form analysis. See, for example, Aivazian et al. [1987].

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