An Index Number Theory of Accounting Cost Variances

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Both management accountants and economists claim to be able to measure production and cost efficiency (or, alternatively, inefficiency) at the firm level. Yet, the tools each group brings to bear on efficiency measurement appear to be markedly different. Accountants measure efficiency via accounting cost variances such as price, efficiency, mix, and yield variances.[1] On the other hand, economists employ economic indices such as Farrell technical and allocative efficiency indices, Russell technical efficiency indices, and the Konus price index.[2] It is therefore quite natural to inquire whether there is some relationship between these two broad approaches for measuring efficiency, or whether they are conceptually independent. Of course, it may well be that the tools brought to bear on efficiency measurement depend on the objective for undertaking such measurements, which may be different for the two professions. But even if it were true that these professions have different objectives in measuring firm efficiency, nevertheless it is quite conceivable that each may find the other's efficiency measurement tools useful if only their interrelationships were made apparent. Furthermore, it seems reasonable to assume that both professions are interested in the same objective, namely, efficiency measurement for cost control purposes.

The purpose of this paper is to demonstrate that there are certain important economic efficiency indices that can be readily computed from the data generated by a standard cost accounting system for a broad class of production technologies. These economic indices, further defined below, are important to the accounting profession because they possess axiomatic or economic underpinnings that accounting efficiency measures such as mix and yield variances do not possess, and yet their computation requires no additional informational inputs. Furthermore, these economic indices are related to accounting measures in that they can be shown to be approximate

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1. See Horngren (1982), for example.

2. See Konus (1939), Farrell (1957), Fare and Lovell (1978), Kopp (1981a,b), Zieschang (1983), and Russell (1985).

(after suitable transformation) overall price and efficiency variances—for the family of production technologies under consideration.

This paper is constructed as follows: Section I defines the Farrell technical efficiency index and the Konus price index and shows why these indices are superior to mix and yield variances as efficiency indicators. Section II demonstrates that the Farrell and Konus indices can be computed solely by reference to the informational inputs available from a standard cost accounting system for the class of semi-translog production technologies. This section also proves that the Farrell technical efficiency index and the Konus price index approximate (and are approximated by) suitably transformed efficiency and price variances, respectively, for these technologies. Section III illustrates the concepts of the earlier sections by a numerical example. Section IV briefly concludes the paper.

Economic versus Accounting Efficiency Indices

The Farrell Technical Efficiency Index

The most well-known and best analyzed of the economic efficiency indices is the Farrell technical efficiency index introduced to the accounting literature by Mensah (1982). Figure 1 illustrates this index for a one output–two input firm. The firm produces some output quantity y with the input vector $X^A = (X_1^A, X_2^A)$. Alternatively, the firm could have produced the same output y with any one of the input combinations along the isoquant II'. The input vector X^A is clearly inefficient since any other combination of the inputs along the isoquant II' (say Z, for example) could have produced the same original quantity using less of the inputs.

The issue addressed by Farrell (1957) was how to go about measuring the inefficiency of X^A relative to the isoquant II'. His solution was to measure this inefficiency with reference to the actual input mix, that is, with reference to the ray from the origin OX^A. More specifically, Farrell defined technical efficiency to be the ability of the firm to transform inputs into outputs efficiently *without* reference to input prices. In Figure 1, Farrell technical efficiency is measured by the distance from X^A to Z along the ray OX^A or equivalently by the index OZ/OX^A.[3] Therefore, this efficiency indicator is measured with reference to the actual inputs used and the production tech-

3. A third equivalent alternative is to measure the cost differential, at standard prices, between the cost of producing with the input combination at Z and the input combination at X^A. Although this puts a dollar figure on the Farrell measure, it is still a technical index since the point Z is defined by the technology alone and not by some cost-minimization calculus.

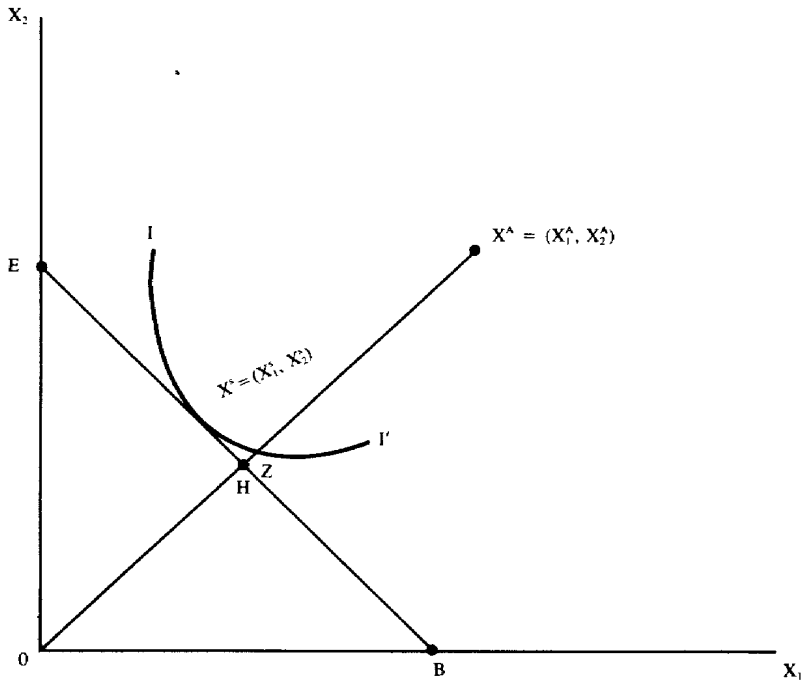

FIGURE 1
Farrell Technical and Allocative Efficiencies

nology (i.e., the isoquant). Prices, to repeat, play no part in determining the technical efficiency measure.

One potential criticism of the Farrell technical efficiency index is the arbitrariness of using the ray OX^A (the actual input mix) as the reference for measuring the distance between X^A and the isoquant II'. There are, after all, an infinite number of rays from a point (X^A) to a curve (II') so that there are a potentially infinite number of technical efficiency measures.[4] Also, there would appear to be no compelling reason to measure the firm's inefficiency relative to the actual input mix. As Haseldine (1967) has shown, mix and yield variances are measured with reference to the standard input mix.[5] Although neither variance is a pure technical efficiency measure, since both are determined by standard input prices, they are nevertheless efficiency indicators and they are measured relative to an alternative ray from the origin.[6] Given, then, the multiplicity of potential technical efficiency measures, why single out the Farrell technical efficiency index for special treatment?

The problem of multiple indices is not unique to efficiency measures. In the first quarter of this century, Irving Fisher (1922) listed several hundred cost-of-living indices and discussed how one might go about trying to distinguish among them. One method for discriminating among indices is on axiomatic grounds. One posits a minimal set of self-evident axioms that an efficiency index should satisfy and then chooses for the premiere index the one(s) that satisfies those axioms.[7] In fact, it can be shown under fairly weak conditions that the Farrell technical efficiency index satisfies the following three axioms:[8]

1. A technical efficiency index should specify when an input vector is on the isoquant, that is, when the vector is technically efficient and when it is not. For example, the Farrell technical efficiency index

4. This is the basis for the Russell technical efficiency measures analyzed by Fare and Lovell. Interestingly enough, Russell himself finds the Farrell index more compelling.

5. See also Figure 2 in Mensah (1982).

6. They are not technical efficiency measures because they are determined by the standard mix, and the standard mix is determined in turn by the minimum cost of production at standard input prices. See point P_{SM} in Figure 2 of Mensah.

7. This is how the economics literature has approached the issue. See the references in footnote 2. It is difficult to discriminate among technical efficiency indices on other bases since technical efficiency is not dependent on some cost-optimization framework.

8. See Fare and Lovell, Kopp (1981b), and Russell. The condition for the Farrell technical efficiency index to satisfy these three conditions is for the production technology to be strictly monotonic, meaning that the marginal product of each input is positive. Fare and Lovell posit four conditions that the Farrell technical efficiency measure satisfies. However, Russell has shown that three of these conditions imply the fourth.

is equal to one if and only if the input vector is on the isoquant (and less than one otherwise).

2. A technical efficiency index should be monotonic in the inputs; that is, increasing one input while holding the others constant should result in a lower efficiency indicator.
3. A technical efficiency index should be homogeneous of degree (plus or minus one) in the inputs. Thus, if the inputs are doubled, the index should show twice as much inefficiency. For example, the Farrell technical efficiency index is homogeneous of degree minus one so that doubling the inputs halves the index.

These axioms, or at least the first two, are sufficiently weak and intuitive that it might seem that almost any efficiency index will satisfy them. However, as we shall now demonstrate, neither the mix nor the yield variance satisfies these axioms in general.[9] Let us consider the mix variance and the first axiom. To satisfy this axiom, the mix variance should be equal to zero if and only if the input vector is on the isoquant. What can be shown is that if the actual mix for each input equals the standard mix, then this variance is zero. But a zero variance is neither necessary nor sufficient for technical efficiency. To see why, suppose the firm is operating on point X^T on the isoquant in Figure 2, where the standard mix is the ray from the origin through the point X^S. Here the firm is technically efficient, X^T is on the isoquant, but the mix variance is not zero (actual and standard mix are not equal) so that a zero mix variance is not necessary for technical efficiency. A zero mix variance is not sufficient either, since one can easily concoct examples for which the unfavorable mix variance for one input cancels out the favorable mix variance of another input, thus yielding an overall zero mix variance despite the fact that the input vector is not on the isoquant.[10] Similar arguments show that a zero yield variance is neither necessary nor sufficient for the firm to be operating efficiently, that is, on the isoquant.[11]

9. The mix variance is defined by:

$$\sum_i \left(\sum_K AQ_K\right) SP_i \, (AM_i - SM_i)$$

where i and k denote the input, SP_i is the standard price, AQ_K is the actual quantity, AM_i is the actual mix, and SM_i is the standard mix. The yield variance is defined by:

$$\sum_i SM_i \, SP_i \left(\sum_K AQ_K - \sum_K SQ_K\right)$$

where SQ_K is the standard quantity.

10. Suppose $i = K = 1,2$, $SP_1 = SP_2 = 1$, $AQ_1 = AQ_2 = 1$, $AM_1 = AM_2 = .5$, $SM_1 = .6$, $SM_2 = .4$ for the Cobb-Douglas isoquant $.8 = L^{.6}M^{.4}$ where the subscript 1 denotes the L input and the subscript 2 denotes the M input. AQ_1 and AQ_2 are not technically efficient (not on the isoquant) since this combination would produce 1 unit output rather than .8 if the firm were operating efficiently. The mix variance for this example is zero.

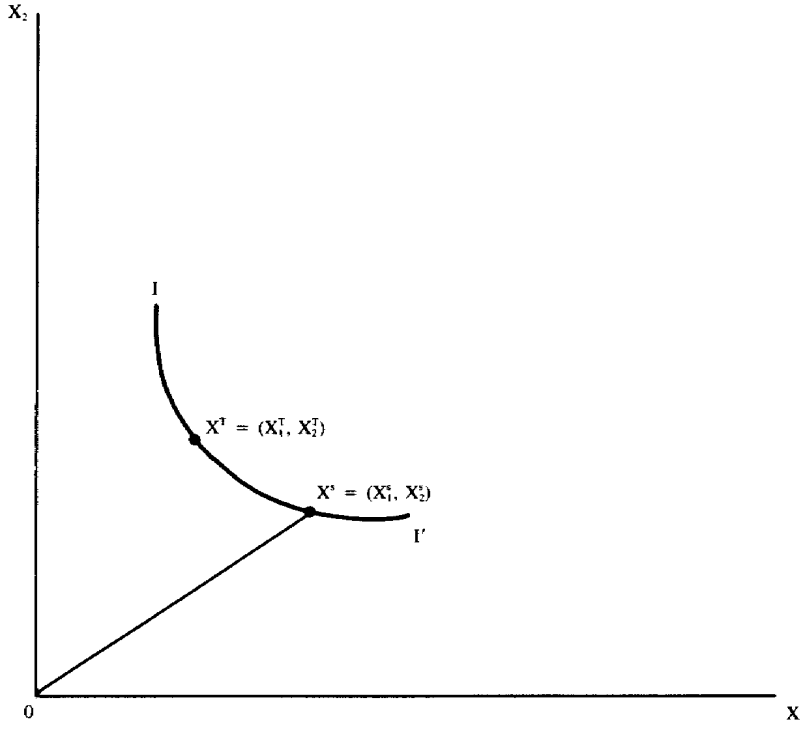

FIGURE 2
Allocative Efficiency

Although the yield variance does satisfy the second axiom, the mix variance does not. Increasing one input while holding the others constant can be shown trivially to increase the yield variance monotonically. In the case of the mix variance, however, increasing the usage of one input while holding the others constant increases the actual mix as well as the actual usage of this input and reduces the actual mix of the other inputs. Thus, if for the input in question the previous actual mix was less than the standard mix, increasing this input's usage could well reduce rather than increase the overall mix variance.[12]

Finally, it is easy to show that doubling all inputs does not double the value of the yield variance.[13]

In summary, unlike the Farrell technical efficiency index, neither the mix nor the yield variance satisfies three fairly weak axioms that one would expect a technical efficiency measure to satisfy. It is especially damaging that neither variance indicates in general when the firm is operating efficiently (that is, on the isoquant), which is presumably what an efficiency indicator is all about.

The Konus Price Index

In addition to technical efficiency, Farrell also defined what he called allocative efficiency. Farrell allocative efficiency measures the firm's ability to choose the input mix that allocates factor inputs to their highest value use.[14] This allocative efficiency Farrell measured as the index OH/OZ along the ray OX^A in Figure 1 or equivalently as the difference in the cost of production at Z and the minimum cost of production at X^S—EB is the isocost line. The advantage of this index is that, together with the Farrell technical efficiency index, it provides a breakdown of the flexible budget variance

11. The yield variance is zero if and only if the sum of the actual quantities of all the inputs is equal to the sum of the standard quantities of all the inputs. This condition in general bears no relationship to whether or not the firm is operating on the isoquant. In any case, it is easy to come up with examples where the firm is operating on its isoquant and the yield variance is not zero, and vice versa.

12. Consider the example in footnote 10, where the mix variance is zero. Now increase AQ_1 from 1 to 1.5. Then $AM_1 = .6$ and $AM_2 = .4$ and again the mix variance is zero, contradicting the monotonicity property.

13. This is easy to see from the definition of the yield variance in footnote 9.

14. Formally, the firm is efficient in the allocative sense if:

$$\frac{MP_i}{MP_j} = \frac{W_i{}^S}{W_j{}^S} \text{ for all inputs } i \neq j$$

where MP_i denotes the marginal product of input i and $W_i{}^S$ is the standard price of intput i. If the equality is not satisfied for any two inputs, the firm is inefficient (that is, not minimizing costs).

into a technical nonprice efficiency component and an allocative price efficiency component.[15]

The Farrell allocative efficiency index, unlike its technical counterpart, is somewhat problematic. First, the former has no known axiomatic underpinnings. Second, it has no basis in any economic-optimization framework. To see what we mean by this, reconsider Figure 2 in which the firm produces at X^T—there is no technical inefficiency here since X^T is on the isoquant—and where X^S is the cost-minimizing (standard) input vector. The question that immediately arises is, Why is the firm producing at X^T rather than X^S? In Farrell's scheme of things, X^T is an arbitrary point. The firm simply produces inefficiently at X^T. How it got there is unknown. An alternative scenario, which is consistent with both an optimization framework and the way the accountant (or economist) is likely to see it, is that X^T is the cost-minimizing input vector at actual input prices in contrast with X^S, which is the cost-minimizing input vector at standard input prices.[16] Therefore, the cost differential between producing at X^T and producing at X^S represents the cost of improperly forecasting input prices—a sort of price variance. In the economics literature this cost differential is called a Konus price index or a true "cost-of-living" index.[17] Although neither the Konus price index nor the Farrell allocative efficiency index has axiomatic underpinnings, the former is consistent with the basic economic assumption that firms try to minimize the cost of production. Allocative inefficiency then arises from imperfect foresight about future input prices rather than from some unknowable inefficiency.

The Konus price index also has implications for cost control in a world of changing prices. As argued cogently by Mensah, in a world of changing input prices, the production manager should be held responsible for the impact of changing (relative) input prices on optimal input usage.[18] This is because the greater the difference between actual relative input prices and standard relative input prices, the greater the *ex post* cost of production will be. Since the Konus price index measures the impact of changing input prices on the cost of production, it is more suitable for cost control purposes

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15. This is so since $\frac{OH}{OZ} \times \frac{OZ}{OX^A}$ is simply the flexible-budget variance in index form. Alternatively, the flexible-budget variance in Figure 1 is the cost differential between X^A and Z plus the cost differential between Z and X^S.

16. This point was initially made in Marcinko and Petri's (1984) comment on the Mensah paper.

17. See, for example, Deaton and Muellbauer (1980, Ch. 7).

18. Even if the production manager is not held responsible for changing input prices, management would still probably like to know the opportunity cost of not substituting inputs optimally when there are input price changes.

than is the Farrell allocative efficiency index. But there is one caveat here. The Konus price index is a function of absolute rather than relative input prices, so the production manager cannot be held responsible for the entire change in the index. (It should be noted that the same problem also occurs with respect to the Farrell allocative efficiency index, which is also a function of absolute prices.)[19] Fortunately, as we shall see, the Konus price index can be broken down into two components: an absolute price effect for which the production manager should not be held responsible and a relative price component for which he should.

It is worth noting, as emphasized by Mensah, that neither the yield nor the mix variance can be used to motivate the production manager to minimize cost when relative input prices are changing. This is because neither of these variances is a function of actual input prices, let alone actual relative input prices.

Figure 3 summarizes this section of the paper. Again, X^A is the vector of actual inputs, X^S is the vector of standard inputs, and X^T is the optimum vector of inputs at actual prices. The distance from X^A to Z is the Farrell technical efficiency index, which measures the firm's ability to efficiently transform inputs into outputs without reference to input prices. Since prices are not relevant, the choice of a technical efficiency index is difficult if not impossible to make on economic-optimization grounds. Rather, it was shown that, unlike accounting efficiency measures such as mix and yield variances, Farrell technical efficiency satisfies certain very basic axioms that one presumes efficiency measures should satisfy. The Farrell technical efficiency index can then be used to motivate production managers to be technically efficient in transforming inputs into outputs.

The cost differential between production at Z and production at X^S is the Farrell allocative efficiency index. Although, together with the Farrell technical index, it provides for a complete decomposition of the flexible budget variance, the Farrell allocative efficiency index is problematic. First, it has no axiomatic or economic-optimization underpinnings. Second, it seems to have no implications for cost control purposes. On the other hand, the Konus price index, which is the cost differential between X^S and X^T, is more meaningful.[20] The latter is based on an economic-optimization framework. Also, the Konus price index can be decomposed into absolute and relative input price components where the relative price component measures

19. This problem was also noted by Mensah; hence, the three-part decomposition in his Table 4.

20. The point Z in Figure 3 is not arbitrary from the point of view of a technical efficiency measure since it is determined ultimately by the axioms of the Farrell technical efficiency index. On the other hand, from the point of view of allocative efficiency defined as the impact of changing (relative) input prices on input substitutability, the point Z is irrelevant and it is X^T that matters.

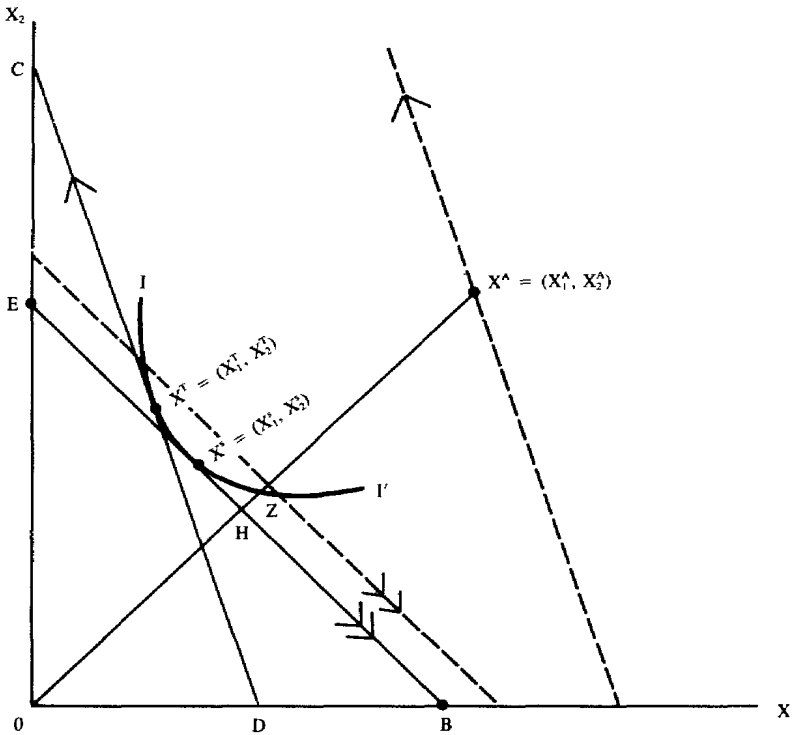

FIGURE 3
Farrell Technical Efficiency and the Konus Price Index

the impact of changing relative input prices on optimal input adjustments. Therefore, the relative price component of the Konus price index can be used to motivate production managers to minimize cost in a world of changing relative input prices. It was also noted that neither the mix nor the yield variance could serve this function since they are not a function of actual input prices.

II. Farrell and Konus Indices as Approximate Accounting Measures

The previous discussion on the "optimality" of the Farrell and Konus indices still leaves a number of important issues unresolved. First of all, from our previous analysis and also from Mensah's example—it would seem that in order to measure Farrell technical efficiency one needs to know the firm's technology. In his discussion of this index, Mensah assumes the firm has a Cobb-Douglas technology. This would appear to call for informational inputs unavailable in a standard cost accounting system. If measuring Farrell technical efficiency requires additional information, should we not be concerned with whether the benefits of the index outweigh the cost of computing the index? Perhaps it is better to stick with less costly conventional accounting efficiency measures. Second, while the previous section made it clear that our economic efficiency indices are unrelated to mix and yield variances, it did not tell us what relationships, if any, these economic indices bear to accounting indices in general. Third, we know that mix and yield variances, on the one hand, and the Farrell efficiency indices, on the other, decompose the flexible-budget variance into intuitively meaningful components. What decomposition, if any, obtains for the Farrell technical efficiency index and the Konus price index?

The purpose of this section is to resolve these issues for a family of production technologies called semi-translog.[21] To talk sensibly about this family of production technologies, we need some notation. Specifically, we shall analyze an M output–N input firm producing a vector of outputs:

$$y = (y_1, \ldots, y_M) = (y_1, \bar{y}) \tag{1}$$

with a vector of inputs:

$$X = (X_1, \ldots, X_N) \tag{2}$$

The vector:

21. The semi-translog is a special case of the translog flexible functional form made famous by Christensen, Jorgenson, and Lau (1973, 1975). See also Barlev and Callen (forthcoming).

$$W = (W_1, \ldots, W_N) \tag{3}$$

denotes input prices. The superscript A will be used to denote actual inputs and actual input prices, whereas the superscript S will be used to denote standard inputs and standard input prices whenever it is important to distinguish between the actual and the standard.
The function:

$$y_1 = F(\bar{y}, X) \tag{4}$$

is the firm's production function and:

$$C(y, W) \tag{5}$$

is the firm's cost function. Standard inputs X^S are defined in terms of the firm's production technology and standard input prices W^S. Formally, the standard inputs are assumed to be the solution to the cost-minimization program:

$$C(y, W^S) = \min_X \{W^S \cdot X: F(\bar{y}, X) \geq y_1\} \tag{6}$$

where . denotes the inner product of two vectors ($W^S \cdot X^S = \sum_{i=1}^{N} W_i^S X_i^S$). Rather than working with the production function, it turns out to be more convenient to work with what is called a distance function. The firm's distance function D is defined by:

$$\begin{aligned} D(y, X) &= \max_{\delta}\{\delta: F(\bar{y}, X/\delta) \geq y_1\} \\ &= [\min_{\lambda}\{\lambda: F(y, \lambda X) \geq y_1\}]^{-1} \end{aligned} \tag{7}$$

where · denotes the inner product of two vectors
Rather than working with the production function, it turns out to be more convenient to work with what is called a distance function. The firm's distance function D is defined by:
The distance function is simply an alternative to the production function for describing the firm's technology.[22] The convenience of the distance function for our purposes lies in the fact that the inverse of the distance function (see Equation (7)) is the Farrell technical efficiency index. Mathematically, the inverse of the distance function measures the minimum amount by which the inputs can be reduced radially and still produce the same output vector y. Geometrically, the inverse of the distance function is the index OZ/OX^A in Figure 1.

22. See Diewert (1981) on the correspondence between production functions and distance functions.

With these preliminaries in mind, we are now in a position to define a semi-translog distance function. Formally, the semi-translog distance function takes the form:

$$\ln D(y,X) = \alpha_o + \sum_{n=1}^{N} \beta_n \ln X_n + \sum_{i=1}^{M} \sum_{n=1}^{N} \gamma_{in} \ln y_i \ln X_n + \sum_{i=1}^{M} \alpha_i \ln y_i + \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \alpha_{ij} \ln y_i \ln y_j \tag{8}$$

where the α's, β's, and γ's are parameters. One can view the semi-translog assumption in two different ways. One can view the semi-translog as a family of production technologies, which includes the generalized Cobb-Douglas as a special case. The importance of the semi-translog is that it does not overly limit the elasticities of substitutions among inputs and outputs, unlike the Cobb-Douglas, for example, which implicitly assumes that all elasticities of substitution are equal to one.[23] Thus, the semi-translog is much more likely to encompass many real-life production technologies. Alternatively, one can view the semi-translog as an approximation to any arbitrary production technology where outputs are approximated quadratically and inputs are approximated (log) linearly. The implication here is that although we may not know the firm's true underlying technology, we can at least try to approximate it with a semi-translog form.[24]

Having defined the distance function and a semi-translog form, we are now in a position to prove an important theorem relating Farrell technical efficiency and the efficiency (usage) variance.

Theorem 1: If the firm's distance function is semi-translog, the overall efficiency variance, normalized by standard costs, approximates the logarithm of the inverse Farrell technical efficiency index.

The proof of this theorem and the others in the paper can be found in the appendix. What the proof shows is that the inverse of the Farrell technical efficiency index is *exactly* equal to:

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23. The elasticity of substitute measures the extent to which any two inputs or outputs can be substituted for each other holding all else constant. An elasticity of substitution between inputs of one means they are fairly substitutable for each other. A good example is the natural gas transmission industry, which corresponds to a Cobb-Douglas technology. In this technology, compressors and linepipe are fairly substitutable for each other. See Callen (1978), for example, on this issue.

24. Whether the semi-translog is a good approximation is an empirical question. Some relevant results have come to light recently. Mountain and Hsiao (1987), on analyzing energy substitutions, find many industries to be adequately described by Cobb-Douglas, CES, or linear technologies, all of which are special cases of the semi-translog. In fact, these parsimonious structures are often better descriptions and predictors than more complex flexible-functional forms such as the full translog.

$$\sum_{k=1}^{N} \frac{W_k^S X_k^S}{W^S \cdot X^S} \ln \frac{X_k^A}{X_k^S} \tag{9}$$

which, in turn, can be approximated by:

$$\sum_{k=1}^{N} \frac{W_k^S (X_k^A - X_k^S)}{W^S \cdot X^S} \tag{10}$$

using the approximation:

$$\ln \frac{X_k^A}{X_k^S} \cong \frac{X_k^A - X_k^S}{X_k^S}$$

This theorem has two important ramifications for the issues mentioned earlier. First, Equation (9) shows that the Farrell technical efficiency index can be computed with reference to just the data generated by a conventional standard accounting system. In other words, for the family of semi-translog production technologies, the Farrell technical efficiency index can be computed exactly by knowing only the actual and standard inputs and the actual and standard input prices. Second, the theorem shows that the Farrell technical efficiency index is in fact related to accounting measures in that it is an approximate overall efficiency variance. (The approximation is to the *normalized* efficiency variance since the Farrell measure is an index.) How good the approximation is depends on how close the actual inputs are to the standard inputs.[25] The closer the actual is to the standard for each input, the better the approximation. It should be emphasized that the approximation is not required for computing the Farrell technical efficiency index, but only to show how the Farrell technical efficiency index is related to other conventional accounting efficiency measures.

In a similar fashion, the Konus price index can be related to the overall price variance provided the firm's cost function takes on the semi-translog form:

$$\ln C(y,W) = \alpha_o' + \sum_{n=1}^{N} \beta_n' \ln W_n + \sum_{i=1}^{M} \alpha_i' \ln y_i \tag{11}$$
$$+ \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \alpha_{ij}' \ln y_i \ln y_j + \sum_{i=1}^{M} \sum_{n=1}^{N} \gamma_{in}' \ln y_i \ln W_n$$

where the α''s, β''s, and γ''s are parameters.

This then leads to the next theorem.

Theorem 2: If the firm's cost function is semi-translog, then the overall

25. This is the well-known approximation of the log relative $\ln(1+r) \cong r$. The smaller is r, the better the approximation.

price variance, weighted by standard inputs and normalized by standard costs, approximates the logarithm of the Konus price index. Again, the proof of Theorem 2 shows that the Konus price index can be computed exactly by:

$$\sum_{k=1}^{N} \frac{W_k^S X_k^S}{W^S \cdot X^S} \ln \frac{W_k^A}{W_k^S} \tag{12}$$

using only the informational inputs generated by a conventional cost accounting system. Furthermore, Equation (12) and, hence, the Konus price index are the approximate (normalized) price variance:[26]

$$\sum_{k=1}^{N} \frac{X_k^S (W_k^A - W_k^S)}{W^S \cdot X^S} \tag{13}$$

The closer the actual input prices are to the standard input prices, the better this approximation will be.

Similar to the decomposition of the flexible budget variance into either mix and yield variances or Farrell efficiency indices, we show in Theorem 3 that the (normalized) flexible budget variance can be decomposed *approximately* into a Farrell technical efficiency index and a Konus price index.

Theorem 3. Assuming the production technology of the firm is semi-translog, the overall flexible budget variance, normalized by standard costs, approximates the sum of the logarithms of the inverse Farrell technical efficiency index and the Konus price index. The approximation will be better the closer are the actual costs of each input to its standard costs.

We mentioned earlier that only part of the Konus index is relevant for motivating production managers. To separate out the impact of relative price changes on optimal input substitutions in the Konus index—absolute price changes have no implications for input substitutions[27]—we can simply divide the Konus index into a conventional Laspeyres price index:

$$\frac{W^A \cdot X^S}{W^S \cdot X^S} \tag{14}$$

The rationale for this is that the Laspeyres index measures absolute price changes and abstracts from input substitution activity due to relative price changes. Thus, the Laspeyres index overstates the true impact of price

26. In Equation (13), the variance is defined in terms of standard inputs rather than actual inputs. This has no specific implications except perhaps to suggest that price variances should be measured relative to standard inputs rather than actual inputs.

27. Doubling all input prices, for example, leaves the optimal input combinations unchanged.

variances on production costs.[28] The Konus index, on the other hand, is sensitive to input-substitution activity because it is a function of a cost-minimization calculus at both standard and actual prices. Thus, the Konus price index can be decomposed into two components:

$$\sum_{K=1}^{N} \frac{W_K^S X_K^S}{W^S . X^S} \ln \frac{W_k^A}{W_K^S} = \left[\sum_{K=1}^{N} \frac{W_K^S X_K^S}{W^A . X^S} \ln \frac{W_K^A}{W_K^S} \right] X \left[\frac{W^A . X^S}{W^S . X^S} \right] \quad (15)$$

where only the inverse of the first component on the right-hand side of Equation (15) is relevant for motivating production managers.[29]

III. An Example

To help make the ideas in the previous section more concrete, we consider a simple numerical example. Specifically, our firm is assumed to be once more a one output–two input firm with Cobb-Douglas production function:

$$y = X_1^{\alpha} X_2^{\beta} \quad (16)$$

Using the definition of the distance function in Equation (7), it is easy to show that D takes on the form:[30]

$$D(y,X_1,X_2) = [y\, X_1^{-\alpha} X_2^{-\beta}]^{-\frac{1}{\alpha+\beta}} \quad (17)$$

Also, from standard production theory it can be shown that the firm's cost function is:[31]

$$C(y,W_1W_2) = \left(\frac{\alpha+\beta}{\beta}\right) \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} W_1^{\frac{\alpha}{\alpha+\beta}} W_2^{\frac{\beta}{\alpha+\beta}} y^{\frac{1}{\alpha+\beta}} \quad (18)$$

Similarly, the standard inputs can be shown to be defined by:

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28. The Laspeyres is never less than the Konus price index. On the relationship between the two, see Deaton and Muellbauer (pp. 170–173).

29. Alternatively, we can look at the issue as decomposing the Laspeyres into a Konus, which has already taken into account input substitution possibilities and another index which in fact measures the cost savings (in index terms) of having substituted inputs optimally when relative prices changed. This index is the inverse of the first term on the right-hand side of Equation (15).

30. $D(y,X_1, X_2) = [\min_g \{g: (gX_1)^a(gX_2)^b \geq y\}]^{-1}$

$= [\min_g \{g: g \geq (yX_1^{-a} X_2^{-b})^{\frac{1}{a+b}} \}]^{-1}$

$= [y\, X_1^{-a} X_2^{-b}]^{\frac{-1}{a+b}}$

31. See Silberberg (1978).

$$X_1{}^S = \frac{\beta W_1{}^S}{\alpha W_2{}^S}^{\frac{-\beta}{\alpha+\beta}} \frac{1}{y^{\alpha+\beta}} \tag{19}$$

$$X_2{}^S = \frac{\beta W_1{}^S}{\alpha W_2{}^S}^{\frac{\alpha}{\alpha+\beta}} \frac{1}{y^{\alpha+\beta}} \tag{20}$$

For purposes of the example, let the subscript 1 denote machine-hours and the subscript 2 labor-hours. Furthermore, we assume that the firm produces 10 units of output with 110 machine-hours and 100 labor-hours. The parameters are assumed to take on the values:

α = .3
β = .2
W_1^S = \$3.50 per machine-hour
W_2^S = \$2.50 per labour-hour

These parameter values yield the input standards (From equations (19) and (20)) X_1^s = 102.80 machine-hours and X_2^s = 95.95 labor-hours. Assuming no price variances for the moment, the exact Farrell technical efficiency index is equal to .9444 and the Konus index is trivially equal to one.[32] Using the overall efficiency variance (of \$35.325 unfavorable) to approximate the results, yields a Farrell technical efficiency measure of .9428, an approximation error of less than .2 percent.

If the labor input has an unfavorable price variance of 50 cents per labor-hour (all else the same), then the Konus price index is 1.0756 when computed exactly. When approximated via the price variance, the Konus price index is 1.0833, an approximation error of less than 1 percent. The Farrell technical efficiency index is unaffected by the price variance.[33]

Since only the labor input has an unfavorable price variance, the price change must be a relative price change. Suppose, however, the labor input has an unfavorable price variance of \$1.50 and the machine input has an unfavorable price variance of \$1.00. Then the Konus price index is 1.4033 when computed exactly. Since the price variances include absolute as well as relative price changes, the production manager should not be held responsible for all of the Konus index. Rather, from Equation (15), the manager should be held responsible only for an index value of 1.0058, which represents the cost inefficiency caused by the production manager in not

32. The exact Farrell index can be computed as $D(y, X^A)^{-1}$ from Equation (17) or by using Equation (9).

33. For this example, the normalized flexible-budget variance is .1423, whereas the logarithms of the inverse Farrell technical efficiency index and the Konus price index sum to .1301, an approximation error of about 9 percent.

adjusting inputs to take into account changes in relative input prices. In other words, of the more than 40 percent increase in the Konus index, less than 1 percent is due to the cost of not adjusting inputs optimally to changes in relative input prices.[34] The remainder, like the conventional price variance, is not in the production manager's bailiwick.

IV. Conclusion

This paper has attempted to relate certain economic efficiency indices to conventional accounting efficiency measures. In particular, we showed that the Farrell technical efficiency index and the Konus price index are preferable efficiency measures to mix and yield variances on either axiomatic or economic grounds. Furthermore, assuming the firm's technology belongs to the general semi-translog class, we were able to show that the Farrell technical efficiency index and the Konus price index are approximate (transformed and normalized) overall efficiency and price variances, respectively. More importantly, we demonstrated that the Farrell technical efficiency index and the Konus price index could be computed exactly for these technologies by reference to the information produced by any standard cost accounting system. Thus, computing these indices does not involve additional costs beyond those already borne by having a cost accounting system in the first place.

The broad implications of our analysis is that economic and accounting efficiency measures are interrelated and that the accounting profession is likely to find economic measures useful for cost control purposes. Future research should attempt to determine if there are other economic indices that are similarly related to accounting measures. It would also be of interest to know if such relationships exist for other classes of technologies. Indeed, it may turn out that the specific relationship between an economic index and an accounting measure depends on the firm's technology, or at least a broad class of technologies.

APPENDIX

Proof of the Theorems

The following lemmas, cited without proof, are needed to prove the theorems:

34. The reason the figure is so low is that most of the change in the Konus is due to the increase in absolute rather than relative prices.

Lemma 1: (Diewert's Quadratic Approximation Lemma for (Semi-)Translog Forms)

$$\text{Let } \ln f(X) = a_o + \sum_{i=1}^{N} a_i \ln X_i + \sum_{j=1}^{N}\sum_{k=1}^{N} a_{jk} \ln X_j \ln X_k$$

Then, $\ln f(X^1) - \ln f(X^\circ) = \frac{1}{2}[\hat{X}^1 \nabla_x \ln f(X^1) + \hat{X}^\circ \nabla_x \ln f(X^\circ)].[\ln X^1 - \ln X^\circ]$ where $\nabla_x f = (\frac{\partial f}{\partial X_1},\ldots,\frac{\partial f}{\partial X_n})$ for X^1, $X^\circ \gg 0_N$ and $\hat{X}^\circ$ and $\hat{X}^1$ are X° and X^1 diagonalized into matrices (0_N is the zero vector).[35] Proof: See Diewert (1976, pp. 117–119).[36]

Lemma 2: Assuming that F is differentiable at the point $(\bar{y}, X^s)$; X^s, $W^s \gg 0_N$; and $X^s.\nabla_x F(\bar{y}, X^s) > 0$, then $\nabla_x D(y, X^s) = W^s/W^s, X^s$. Proof: See Caves, Christensen, and Diewert (1982, pp. 1396–1397).

Lemma 3: (Shephard's Lemma)

If $C(y,w)$ is the firm's indirect cost function, then $\nabla_w C(y,w) = X^S$, where X^S solves the cost minimization program (1). Proof: See Shephard (1970, pp. 169–170).

In the proofs below, the conditions of Lemma 2 are assumed to hold.

Theorem 1: If the firm's distance function is semi-translog, the overall efficiency variance, normalized by standard costs, approximates the logarithm of the Inverse Farrell technical efficiency index.

Proof:

$$\ln D(y, X^A)$$

$$= \ln D(y, X^A) - \ln D(y, X^s)$$

since $D(y, X^s) = 1$, X^s being on the isoquant

$$= \frac{1}{2}[\hat{X}^s \nabla_x \ln D(y, X^s) + \hat{X}^A \nabla_x \ln D(y, X^A)].[\ln X^A - \ln X^s]$$

using Lemma 1

$$= \hat{X}^s \nabla_x \ln D(y, X^s).[\ln X^A - \ln X^s] \quad \text{since}$$

$$\frac{1}{2}[\hat{X}^A \nabla_x \ln D(y, X^A) - \hat{X}^s \nabla_x \ln D(y, X^s)].[\ln X^A - \ln X^s] = 0$$

for the semi-translog[37]

$$= \hat{X}^s \frac{\nabla_x D(y, X^s)}{D(y, X^s)}.[\ln X^A - \ln X^s]$$

$$= \hat{X}^s \frac{W^s}{W^s . X^s}.[\ln X^A - \ln X^s]$$

35. $X^2 \gg 0_N$ means that each component of the vector X^2 is positive.

36. The X's in this lemma are not necessarily inputs but could be inputs and/or outputs.

using Lemma 2 and the fact that $D(y, X^s) = 1$

$$= \sum_{k=1}^{N} \left[\frac{W_k^s X_k^s}{W^s . X^s} \ln \frac{X_k^A}{X_k^s} \right]$$

$$\cong \sum_{k=1}^{N} \frac{W_k^s (X_k^A - X_k^s)}{W^s X^s}$$

using the approximation $\ln \frac{X_k^A}{X_k^s} \cong \frac{X_k^A - X_k^s}{X_k^s}$.

Theorem 2: If the firm's cost function is semi-translog, then the overall price variance, weighted by standard inputs and normalized by standard costs, approximates the logarithm of the Konus price index.

Proof: $\ln \frac{C(y, W^A)}{C(y, W^s)}$

$$= \ln C(y, W^A) - \ln C(y, W^s)$$

$$= \frac{1}{2} [\hat{W}^A \nabla_w \ln C(y, W^A) + \hat{W}^s \nabla_w \ln C(y, W^s)]. [\ln W^A - \ln W^s]$$

using Lemma 1

$$= \hat{W}^s \nabla_w \ln C(y, W^s). [\ln W^A - \ln W^s] \quad \text{since}$$

$$\frac{1}{2} [\hat{W}^A \nabla_w \ln C(y, W^A) - \hat{W}^s \nabla_w \ln C(y, W^s)]. [\ln W^A - \ln W^s] = 0$$

for the semi-translog

$$= \hat{W}^s \frac{\nabla_w C(y, W^s)}{C(y, W^s)} . [\ln W^A - \ln W^s]$$

$$= \hat{W}^s \frac{X^s}{C(y, W^s)} . [\ln W^A - \ln W^s]$$

by Lemma 3

37. This is because for the semi-translog:

$$\hat{X}^S A_x \ln D(y, X^s) = \hat{X}^A A_x \ln D(y, X^A)$$

$$= [\ldots, b_h + \int_{i=1}^{M} q_{ih} \ln y_i, \ldots].$$

Note that $A_x \ln D(y,X)$ is a vector with terms of the form:

$$\frac{b_L + \int_{i=1}^{M} q_{iL} \ln y_i}{X_L} \text{ since } D(y,X) \text{ is semi-translog.}$$

$$= \sum_{k=1}^{N} \left[\frac{W_k^s X_k^s}{W^s \cdot X^s} \ln \frac{W_k^A}{W_k^s} \right]$$

$$\cong \sum_{k=1}^{N} \frac{X_k^s (W_k^A - W_k^s).}{W^s \cdot X^s}$$

Theorem 3. Assuming the production technology of the firm is semi-translog, the overall flexible budget variance, normalized by standard costs, approximates the sum of the logarithms of the inverse Farrell technical efficiency index and the Konus price index.

Proof: $\ln D(y, X^A) + \ln \dfrac{C(y, W^A)}{C(y, W^s)}$

$$= \sum_{k=1}^{N} \left[\frac{W_k^s X_k^s}{W^s \cdot X^s} \ln \frac{X_k^A}{X_k^s} \right] + \sum_{k=1}^{N} \left[\frac{W_k^s X_k^s}{W^s \cdot X^s} \ln \frac{W_k^A}{W_k^s} \right]$$

by the proofs of Theorems 1 and 2

$$= \sum_{k=1}^{N} \left[\frac{W_k^s X_k^s}{W^s \cdot X^s} \ln \frac{X_k^A \cdot W_k^A}{X_k^s W_k^s} \right]$$

$$\cong \sum_{k=1}^{N} \frac{X_k^A W_k^A - X_k^s W_k^s}{W^s \cdot X^s}.$$

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